Constructive Aspects of
Algebraic Euclidean Field Theory

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Abstract

This paper is concerned with constructive and structural aspects of euclidean field theory. We present a C*-algebraic approach to lattice field theory. Concepts like block spin transformations, action, effective action, and continuum limits are generalized and reformulated within the C*-algebraic setup. Our approach allows to relate to each family of lattice models a set of continuum limits which satisfies reflexion positivity and translation invariance which suggests a guideline for constructing euclidean field theory models. The main purpose of the present paper is to combine the concepts of constructive field theory with the axiomatic framework of algebraic euclidean field theory in order to separate model independent aspects from model specific properties.

Contents

1 Introduction

To begin with, we explain why euclidean field theory is of interest when constructive purposes are concerned. Furthermore, we briefly explain the basic notions which we are dealing with. In the second part of this section, we give an overview of the content of this paper by illustrating our main concepts and ideas.

1.1 Motivation

The techniques of euclidean field theory are powerful tools in order to construct quantum field theory models. Compared to the method of canonical quantization in Minkowski space, which, for instance, has been used for the construction of $P(\phi)_2$ and Yukawa$_2$ models [13, 14, 16, 25, 26], the methods of euclidean field theory simplify the construction of interactive quantum field theory models.

The existence of the ϕ_3^4 model as a Wightman theory has been established by using euclidean methods [7, 28, 21]. In the contrary the methods of canonical quantization are much more difficult to handle and lead by no means as far as euclidean techniques do. Only the proof of the positivity of the energy has been carried out within the hamiltonian framework [13, 15].

Motivated by the considerations above, a C*-algebraic version of the Osterwalder-Schrader reconstruction scheme has been worked out in [24]. The starting point of the analysis in [24] is a so called *euclidean field*. Within the present paper, we consider a particular class of euclidean fields, namely those which are statistical mechanics. These particular euclidean fields are called *euclidean statistical mechanics*. We point out that within the subsequent considerations all physical motivations and interpretations are concerned with statistical mechanical systems and not with the quantum field theory model which can be reconstructed from it. The axioms which we propose in [24] for an euclidean field theory are motivated by an analogous point of view as it has been used for the Haag-Kastler axioms [17].

In order to set up our language and the notions we are going to use, we briefly introduce and explain the mathematical formulation of the concept of statistical mechanics from a C*-algebraic point of view.

We apologize for being very formal within this part of the present section, but one aspect of our basic philosophy is to realize the physical notions and concept, we are dealing with, in terms of clear mathematical objects.

In order to describe a statistical mechanics, we consider a C*-algebra $\mathfrak{A}$ where the self adjoint elements describe observations related to the system under considerations. Each observable can be *localized* within open regions $\mathscr{U}$ of a topological space X. This region is related to particular *properties* of the corresponding quantity which can be measured in a certain experiment. For instance, one may think of a stochastic process, where observations (events) can be localized within a time interval $I \subset \mathbb{R}_+$, i.e. in this case the topological space $X = \mathbb{R}_+$ is simply the set of positive real numbers.

As a mathematical realization of the notion *statistical mechanics*, we propose the following list of axioms:

SM1:   Let $\mathcal{K}$ be a collection of open sets in X. The first ingredient of a statistical mechanics is a net of C*-subalgebras $\underline{\mathfrak{A}}$

$$\underline{\mathfrak{A}} : \mathcal{K} \ni \mathcal{U} \longmapsto \mathfrak{A}(\mathcal{U}) \subset \mathfrak{A}$$

which is inclusion preserving, i.e.

$$\mathcal{U} \subset \mathcal{U}_1 \quad \Rightarrow \quad \mathfrak{A}(\mathcal{U}) \subset \mathfrak{A}(\mathcal{U}_1) \quad .$$

A region $\mathcal{U} \in \mathcal{K}$ can be regarded as a set of *properties* which the observables in $\mathfrak{A}(\mathcal{U})$ have in common.

SM2:   In order to describe the dynamics and symmetries of the system, we consider a group G, which acts continuously on X, and a group homomorphism

$$\gamma \in \mathrm{Hom}(G, \mathrm{Aut}\,\mathfrak{A})$$

from G into the automorphism group of $\mathfrak{A}$. We require that γ acts partially covariantly, i.e.

$$\gamma_g \mathfrak{A}(\mathcal{U}) \quad = \quad \mathfrak{A}(g\mathcal{U})$$

for each $(g, \mathcal{U}) \in G \times \mathcal{K}$ with $g\mathcal{U} \in \mathcal{K}$.

SM3:   In addition to that, if for $\mathcal{U}, \mathcal{U}_1 \in \mathcal{K}$ the set $\mathcal{U}$ is a proper subset of $X \backslash \mathcal{U}_1$, then the algebras $\mathfrak{A}(\mathcal{U})$ and $\mathfrak{A}(\mathcal{U}_1)$ are statistically independent (see [23]). Roughly speaking, two observations which have no properties in common do not disturb each other.

SM4:   Finally, we consider a state ω is a state on the C*-algebra $\mathfrak{A}$ which is G-invariant, i.e. $\omega \circ \gamma_g = \omega$ for each $g \in G$. The state ω describes a basic distribution of events and the set of physically admissible states of the system under consideration is the norm closed convex hull $\mathscr{F}_\omega$ of the set of states

$$\left\{ \omega_v : a \mapsto \frac{\langle\, \omega \,,\, v^* a\, v \,\rangle}{\langle\, \omega \,,\, v^* v \,\rangle} \;\middle|\; v \in \mathfrak{A} : \langle\, \omega \,,\, v^* v \,\rangle \neq 0 \right\}$$

which is called the *folium* generated by ω. It is required that the GNS-representation of ω is faithful which is a sensible condition since, if the GNS-representation π_ω is not faithful, then the ideal $\mathfrak{J}_\omega = \pi_\omega^{-1}(0)$ is irrelevant when physical aspects are concerned. Without changing the physical content of the system under consideration we can replace the algebra $\mathfrak{A}$ by the quotient C*-algebra $\mathfrak{A}/\mathfrak{J}_\omega$.

The tuple $\Lambda = (\underline{\mathfrak{A}}, \gamma, \omega, X, G, \mathcal{K})$ which fulfills the axioms *SM1 - SM4* is called a *statistical mechanics*. If $\mathfrak{A}$ is an abelian C*-algebra, then we call Λ a *classical statistical mechanics*.

For later purpose, it is convenient to introduce the notion of a *subsystem* of a statistical mechanics. A statistical mechanics

$$\Lambda_1 \quad = \quad (\underline{\mathfrak{A}}_1, \gamma_1, \omega_1, X_1, G_1, \mathcal{K}_1)$$

is called a *subsystem of* Λ $(\Lambda_1 \prec \Lambda)$ if the following conditions are fulfilled:

SU1:   The inclusions $X_1 \subset X$, $G_1 \subset G$, and $\mathcal{K}_1 \subset \mathcal{K}$ are valid, i.e. if one restricts ones considerations to a subsystem, then the symmetry of the underlying system can be broken.

SU2:   The dynamics of a subsystem has to be compatible with the dynamics of the underlying theory. There exists a C*-subalgebra $\mathfrak{B} \subset \mathfrak{A}$ and a surjective *-homomorphism $\rho : \mathfrak{B} \to \mathfrak{A}_1$ and for each $g \in G_1$ and for each $\mathscr{U} \in \mathcal{K}_1$ the following relations hold true:

$$\gamma_g(\mathfrak{B}) \;=\; \mathfrak{B}$$

$$\gamma_{1,g} \circ \rho \;=\; \rho \circ \gamma_g |_{\mathfrak{B}}$$

$$\rho^{-1}(\mathfrak{A}_1(\mathscr{U})) \;\subset\; \mathfrak{A}(\mathscr{U}) \;.$$

SU3:   Each state of the subsystem which is physically admissible, should be related to a state of the underlying theory. Hence one requires that for each state $\varphi_1 \in \mathscr{F}_{\omega_1}$ there exists a state $\varphi \in \mathscr{F}_\omega$ such that

$$\varphi_1 \circ \rho \;=\; \varphi |_{\mathfrak{B}} \;.$$

Two statistical mechanics Λ, Λ_1 are *equivalent* if Λ_1 is a subsystem of Λ and vice versa.

In general, the *-homomorphism ρ is not faithful, which can be interpreted in physical terms: Relations between observables within the subsystem are tested by states in $\mathscr{F}_{\omega_1}$. Within the underlying theory a larger set of states $\mathscr{F}_\omega$ can be prepared and therefore relations between observables, which hold for the subsystem, can be violated within the underlying one.

It is clear that to each localizing region $\mathscr{U} \in \mathcal{K}$ we can assign a subsystem in a natural manner, namely

$$\Lambda_{\mathscr{U}} \;:=\; (\underline{\mathfrak{A}}_{\mathcal{K}(\mathscr{U})}, \gamma|_{G(\mathscr{U})}, \omega|_{\mathfrak{A}(\mathscr{U})}, \mathscr{U}, G(\mathscr{U}), \mathcal{K}(\mathscr{U})) \;\prec\; \Lambda$$

where $G(\mathscr{U}) \subset G$ is the stabilizer subgroup of $\mathscr{U}$ and $\mathcal{K}(\mathscr{U})$ contains all sets $\mathscr{U}_1 \in \mathcal{K}$ with $\mathscr{U}_1 \subset \mathscr{U}$.

We are now prepared to introduce the notion of euclidean statistical mechanics. Let $\mathcal{K}^d$ be the set of open bounded convex subsets of $\mathbb{R}^d$. A *euclidean statistical mechanics* is a statistical mechanics

$$(\underline{\mathfrak{A}}, \alpha, \omega, \mathbb{R}^d, \mathrm{E}(d), \mathcal{K}^d)$$

where the state ω fulfills the axioms:

E1:   The state ω is euclidean invariant, i.e. $\omega \circ \alpha = \omega$.

E2: The state ω is reflexion positive: Let $e \in S^{d-1}$ be an euclidean time-direction and let Σ_e be the hyper-plane which is orthogonal to e. The euclidean time reflexion $\theta_e : \mathbb{R}^d \to \mathbb{R}^d$ is the reflexion

$$x \longmapsto \theta_e(x) \quad = \quad x - 2(e \cdot x)\, e$$

where $y \cdot x$ is the canonical scalar product in $\mathbb{R}^d$. We consider the anti-linear involution

$$j_e \quad := \quad \alpha_{\theta_e} \circ * \in \operatorname{Aut}\mathfrak{A}$$

and we require that

$$\langle\, \omega\,, j_e(a)\,a\,\rangle \quad \geq \quad 0$$

for each $a \in \mathfrak{A}(\mathbb{R}_+ e + \Sigma_e)$.

E3: The state ω fulfills a regularity condition, namely for each $a, b, c \in \mathfrak{A}$ the map

$$g \quad \longmapsto \quad \langle\, \omega\,, a\alpha_g(b)c\,\rangle$$

is continuous.

By considering euclidean statistical mechanics, the property *SM2* is then called *euclidean covariance* and the statistical independence in *SM3* is called *locality* [24].

The problem of constructing non-trivial examples which fulfill the axioms *E1-E3* is rather difficult to handle. Up to now, the known examples for euclidean field theory models which are *not* related to free field theory models are examples in $d < 4$ space-time dimensions. The question whether there are interesting models in $d \geq 4$ dimensions is still open.

One possible procedure, which is often used within the framework of constructive field theory, is to start from a family of lattice field theory models which can be regarded as statistical mechanics in our sense (see [9] and references given there). As a tool to control the continuum limit, block spin transformations are used to relate models, which belong to a given lattice, with models on a finer lattices. This method has been applied to scalar field theories [12] as well as to the treatment of gauge theories [2], for instance. But even if a suitable continuum limit exists in the sense of [9, 12, 2], then this does not imply that the axioms *E1-E3* are fulfilled. Since one works here with cubic lattices, it is extremely difficult to prove the rotation invariance of the model which is indeed a crucial property for passing from a euclidean field theory to a quantum field theory in Minkowski space. One nice idea, which works at least in $d = 2$ dimensions and which makes use of the facts developed in [2], is presented in [18]. We also refer the reader to [9] where this problem is also mentioned.

Within this paper we also work with cubic lattices and the problem of rotation invariance is discussed within a forthcoming paper. Concerned with this simplification, we study statistical mechanics

$$\Lambda \quad = \quad \left(\underline{\mathfrak{A}}, \alpha, \omega, \mathbb{R}^d, \mathbb{Q}_b^d, \mathscr{K}^d\right) \ ,$$

with $\mathbb{Q}_b^d := \cup_n b^{-n}\mathbb{Z}^d$, $b \in \mathbb{N}$, i.e. the net $\underline{\mathfrak{A}}$ is translationally covariant with respect to a dense subgroup $\mathbb{Q}_b^d \subset \mathbb{R}^d$ of rational translations. The axioms *E1-E3* for the state ω are also substituted by weaker properties *WE1-WE3*:

WE1: The state ω is translationally invariant, i.e. $\omega \circ \alpha = \omega$.

WE2: The state ω is reflexion positive with respect to the directions e_k, where e_k is the unit vector with components $(e_k)_l = \delta_{kl}$.

WE3: The state ω fulfills a regularity condition, namely for each $a, b, c \in \mathfrak{A}$ the map

$$\mathbb{Q}_b^d \ni g \longmapsto \langle \omega, a\alpha_g(b)c \rangle$$

is continuous.

We call the tuple Λ a *weak euclidean statistical mechanics* if it satisfies the axioms *WE1-WE3* and the pair $(\underline{\mathfrak{A}}, \alpha)$ is called a *weak euclidean net of C*-algebras*.

We expect that the axioms for a weak euclidean statistical mechanics are not sufficient to construct a Haag-Kastler net within a vacuum representation from these data. Nevertheless, a weak euclidean statistical mechanics can be treated as a physical system by its own right.

1.2 Overview

After we have introduced the general concepts and notations in the previous section, we outline here the basic ideas and concepts which are developed within this paper in a concrete manner.

We consider the lattice of the discretized torus $\Sigma_0(n) = b^{-n^0}\mathbb{Z}^d/b^{n^1}\mathbb{Z}^d$ where $b \in \mathbb{N}$ is odd and $n = (n^0, n^1) \in \mathbb{Z}^2$ is a pair of integer numbers. The corresponding sets of q-cubes are denoted by $\Sigma_q(n)$, $q \leq d$. The set of q-cubes of the dual lattice is denoted by $\Sigma_q^*(n)$ and we use the symbol $*$ for the isomorphism which maps $\Sigma_{d-q}^*(n)$ onto $\Sigma_q(n)$ and vice versa. We introduce a partial ordering on $\mathbb{Z}^2$: We write $n \prec n_1$ for $n^j \leq n_1^j$, $j = 0, 1$.

For a given lattice, we build the C*-algebra of bounded continuous functions [1]

$$\mathfrak{A}_{(n,\mathbb{R})} := \mathscr{C}_b\left(\mathbb{R}^{\Sigma_d(n)}\right) .$$

The algebra $\mathfrak{A}_{(n,\mathbb{R})}$ contains subalgebras $\mathfrak{A}_{(n,\mathbb{R})}(\mathscr{U})$ which are related to an open convex sets $\mathscr{U} \subset [-b^{n^1}, b^{n^1}]^d$ in euclidean space, namely a function $a \in \mathfrak{A}_{(n,\mathbb{R})}$ is localized in $\mathscr{U}$ if it only depends upon the variables $u(\Delta), \phi(\Delta) \subset \mathscr{U}$, where ϕ is an appropriate chart from the torus $\mathbb{R}^d/b^{n^1}\mathbb{Z}^d$ into $\mathbb{R}^d$.

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[1]For a C*-algebra A we write $\mathfrak{A}_{(n,X[A])} = \mathfrak{A}_{(n,A)} := \otimes_{\Delta \in \Sigma_d(n)}\{\Delta\} \times A$ where $X[A]$ denotes the spectrum of A.
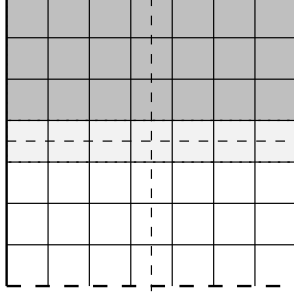
Figure 1: The bold as well as the bold dashed lines are indentified. Let $\mu = 1$ be the vertical direction, then operators which are localized in $\mathbb{R}_+ e_1 + \Sigma_{e_1}$ depend on grey and light grey shaded cubes. The reflexion $j_{(n,1)}$ maps the grey shaded region onto the unshaded one and the light grey shaded *time zero* layer is stable under $\dot{j}_{(n,1)}$.

As an example for a lattice field theory model we consider a lattice action functional of the form

$$\mathbf{s}_{(\lambda,n)}(u) \;=\; \lambda_0(n) \sum_{\Gamma \in \Sigma_{d-1}(n)} *\mathrm{d} * u(\Gamma)$$

$$+ \sum_{\Delta \in \Sigma_d(n)} \sum_{l=1}^{L} \lambda_l(n)\, u(\Delta)^{2l} \tag{1}$$

which induces a state $\eta_{(\lambda,n)}$ on $\mathfrak{A}_{(n,\mathbb{R})}$ by defining

$$\langle\, \eta_{(\lambda,n)}, a \,\rangle \;=\; \mathbf{z}_{(\lambda,n)}^{-1} \int \mathrm{d}u \; \exp\!\left(-\mathbf{s}_{(\lambda,n)}(u)\right) a(u)$$

where the partition function $\mathbf{z}_{(\lambda,n)}$ is for normalization. In order to formulate the important properties of the states $\eta_{(\lambda,n)}$, we look at particular automorphisms on $\mathfrak{A}_{(n,\mathbb{R})}$. The group $b^{-n^0}\mathbb{Z}^d$ acts on the set of cubes $\Sigma_d(n)$ in a natural manner and for each $g \in b^{-n^0}\mathbb{Z}^d$ we introduce an automorphism $\beta_{(n,g)}$ on $\mathfrak{A}_{(n,\mathbb{R})}$ by the prescription

$$\beta_{(n,g)} a(u) \;:=\; a\left(u \circ g\right) \;.$$

Moreover, the euclidean time reflexions $\theta_\mu = \theta_{e_\mu}$, $\mu = 1 \cdots d$, also act on $\Sigma_d(n)$ and we define anti-automorphisms j_μ

$$j_{(n,\mu)} a(u) \;:=\; \bar{a}\left(u \circ \theta_\mu\right) \;.$$

It can be proven that the states $\eta_{(\lambda,n)}$ are invariant under the automorphisms $\beta_{(n,g)}$ and that they are reflexion positive, i.e.

$$\langle\, \eta_{(\lambda,n)}, j_{(n,\mu)}(a)a \,\rangle \;\geq\; 0$$

for each operator a which is localized in $\mathbb{R}_+ e_\mu + \Sigma_{e_\mu}$. Figure 1 illustrates the situation for $d = 2$ dimensions.

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Let $\mathscr{K}_n^d$ be the collection of all open convex sets in $[-b^{n^1}, b^{n^1}]^d$ and let $\mathfrak{J}_{(n,\eta)}$ be the kernel of the GNS-representation of η_n. The prescription

$$\underline{\mathfrak{A}}_{(n,\mathbb{R}|\eta)} : \mathscr{K}_n^d \ni \mathscr{U} \longmapsto \mathfrak{A}_{(n,\mathbb{R}|\eta)}(\mathscr{U}) := \mathfrak{A}_{(n,\mathbb{R})}(\mathscr{U})/\mathfrak{J}_{(n,\eta)}$$

yields a concrete example for a classical statistical mechanics, namely the tuple

$$\Lambda_n = \left(\underline{\mathfrak{A}}_{(n,\mathbb{R}|\eta)}, \beta_n, \eta_n, \mathbb{R}^d, b^{-n^0}\mathbb{Z}^d, \mathscr{K}_n^d\right) .$$

In the subsequent, we call Λ_n a *lattice field theory* if the state η_n is $b^{-n^0}\mathbb{Z}^d$ invariant and reflexion positive.

Continuum limits for lattice field theories: As already mentioned, in order to control the continuum limit of lattice field theory models the concept of block spin transformations turned out to be a useful tool. For a review of the basic ideas, we refer the reader to [9] and references given there. We reformulate the basic concepts of block spin transformations from a C*-algebraic point of view. Each configuration $u \in \mathbb{R}^{\Sigma_d(n+k)}$, $k \in \mathbb{N}^2$, can be identified with a configuration $p_{(n,n+k)}u \in \mathbb{R}^{\Sigma_d(n)}$ by an averaging procedure. Usually, the averaging map $p_{(n,n+k)}$ is defined by the block average

$$\left(p_{(n,n+k)}u\right)(\Delta_0) := b^{-dk^0} \sum_{\Delta \subset \Delta_0} u(\Delta) . \tag{2}$$

A simplified version of a block spin transformation can by obtained by setting

$$\left(p_{(n,n+k)}u\right)(\Delta_0) := u\left(\Delta_{(n,n+k|\Delta_0)}\right) \tag{3}$$

where $\Delta_{(n,n+k|\Delta_0)}$ is the unique cube contained in Δ_0 which contains the point $*\Delta_0$ in the dual lattice (See Figure 2 for illustration).
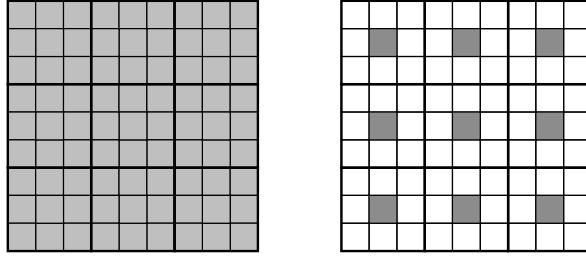
Figure 2: The left figure illustrates the block average, given by Equation (2), the right figure illustrates the block spin transformation, given by Equation (3). The bold lines belong to the coarser lattice, and the average is taken over the grey shaded cubes, respectively.

The block spin transformations can be used to identify operators in $\mathfrak{A}_{(n,\mathbb{R})}$ with operators in $\mathfrak{A}_{(n+k,\mathbb{R})}$, namely

$$\iota_{(n+k,n)}a := a \circ p_{(n,n+k)}$$

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defines a faithful *-homomorphism form $\mathfrak{A}_{(n,\mathbb{R})}$ into $\mathfrak{A}_{(n+k,\mathbb{R})}$. In contrary to the common literature, we distinguish here between *block spin* transformations and *renormalization group* transformations. One important feature of block spin transformations is that localizing regions are preserved, i.e. $\iota_{(n+k,n)}\mathfrak{A}_{(n,\mathbb{R})}(\mathscr{U}) \subset \mathfrak{A}_{(n+k,\mathbb{R})}(\mathscr{U})$. Hence there is no scaling involved as block spin transformations are concerned. On the other hand, renormalization group transformations identify operators which are localized in $\mathscr{U}$ with operators, localized in a scaled region $\lambda\mathscr{U}$. An overview of the basic ideas of renormalization group transformations applied to constructive field theory can be found in [12, 3] and references given there. The general concept of renormalization group transformations from an axiomatic point of view is presented in [5, 4] and related work.

By looking at algebraic properties, in Section 2.2 the general concept of block spin transformation is introduced within the C*-algebraic setting. As we shall describe in Section 2.3, by performing the C*-inductive limit, one constructs from a given family of block spin transformations $\iota = (\iota_{(n,n_0)})_{n_0 \prec n}$ and from the lattice algebras $\mathfrak{A}_{(n,\mathbb{R})}$ a C*-algebra $\mathfrak{A}_{(\iota,\mathbb{R})}$ which can be regarded as the C*-algebra for the continuum model. One obtains a net of C*-algebras

$$\underline{\mathfrak{A}}_{(\iota,\mathbb{R})} : \mathscr{U} \longrightarrow \mathfrak{A}_{(\iota,\mathbb{R})}(\mathscr{U})$$

on which the dense subgroup $\mathbb{Q}_b^d = \cup_{n \in \mathbb{N}} b^{-n}\mathbb{Z}^d \subset \mathbb{R}^d$ acts covariantly by automorphisms $\beta_{(\iota,g)}$ and thus this yields a weak euclidean net of C*-algebras $(\underline{\mathfrak{A}}_{(\iota,\mathbb{R})}, \beta_\iota)$.

One aim of this paper is to analyze the space of $\mathbb{Q}_b^d$-invariant and reflexion positive states $\mathfrak{S}_{(\iota,\mathbb{R})}$ on $\mathfrak{A}_{(\iota,\mathbb{R})}$. The application of block spin transformations to states leads to a net of invariant reflexion positive states

$$\left(\eta_{n+k} \circ \iota_{(n+k,n)}\right)_{k \in \mathbb{N}^2}$$

on $\mathfrak{A}_{(n,\mathbb{R})}$ which has, according to compactness arguments, weak limit points. We denote this weak limit points by $\varphi_n := \mathbf{E}[\xi \otimes \eta]_n$, where ξ labels a limit point, more precisely, ξ is a measure on the space $\bar{\mathbb{Z}}^2 \backslash \mathbb{Z}^2$, where $\bar{\mathbb{Z}}^2$ is the spectrum of the C*-algebra of bounded functions on $\mathbb{Z}^2$. The consistency condition

$$\varphi_{n+k} \circ \iota_{(n+k,n)} \quad = \quad \varphi_n$$

is fulfilled and hence there is a unique state $\varphi \in \mathfrak{S}_{(\iota,\mathbb{R})}$ on the C*-inductive limit $\mathfrak{A}_{(\iota,\mathbb{R})}$ such that

$$\varphi \circ \iota_n \quad = \quad \varphi_n$$

where ι_n is the embedding of $\mathfrak{A}_{(n,\mathbb{R})}$ into $\mathfrak{A}_{(\iota,\mathbb{R})}$. For a given family of lattice field theory models

$$\left(\underline{\mathfrak{A}}_{(n,\mathbb{R}|\eta)}, \beta_n, \eta_n, \mathbb{R}^d, b^{-n^0}\mathbb{Z}^d, \mathscr{K}_n^d\right)_{n \in \mathbb{Z}^2}$$

we symbolize the corresponding set of *continuum limits* by $\mathfrak{S}_{(\iota,\mathbb{R})}[\eta] \subset \mathfrak{S}_{(\iota,\mathbb{R})}$. Each continuum limit $\varphi \in \mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$ gives rise to a classical statistical mechanics

$$\Lambda \quad = \quad \left(\underline{\mathfrak{A}}_{(\iota,\mathbb{R}|\varphi)}, \beta_\iota, \varphi, \mathbb{R}^d, \mathbb{Q}_b^d, \mathscr{K}^d\right)$$

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where the net $\underline{\mathfrak{A}}_{(\iota,\mathbb{R}|\varphi)}$ is given by

$$\mathfrak{A}_{(\iota,\mathbb{R}|\varphi)} : \mathscr{U} \longmapsto \mathfrak{A}_{(\iota,\mathbb{R})}(\mathscr{U})/\mathfrak{J}_{(\iota,\varphi)}$$

and $\mathfrak{J}_{(\iota,\varphi)}$ is the kernel of the GNS-representation of φ.

The self adjoint operators in $\mathfrak{A}_{(\iota,\mathbb{R}|\varphi)}$ correspond to observations with respect to the full energy momentum range. By setting $\varphi_n := \varphi \circ \iota_n$, each lattice field theory

$$\Lambda_n \quad = \quad \left(\underline{\mathfrak{A}}_{(n,\mathbb{R}|\varphi)}, \beta_n, \varphi_n, \mathbb{R}^d, b^{-n^0}\mathbb{Z}^d, \mathscr{K}_n^d \right)$$

is a proper subsystem of Λ which corresponds to observations within the energy momentum range $[b^{-n^1}, b^{n^0}]$ and we regard Λ_n as an *effective theory* in which observations within the energy momentum range $[0, b^{-n^1}] \cup [b^{n^0}, \infty)$ are not admissible. The $\mathbb{Q}_b^d$ covariance of the effective theory is broken and only the $b^{-n^0}\mathbb{Z}^d$ covariance remains.

At this point, we have to emphasize that our considerations essentially rely on the C*-algebraic point of view. The advantage in comparison to non-C*-based approaches (see for example [9]) is that we always get continuum limits no matter how our input data $\eta = (\eta_n)_{n \in \mathbb{Z}^2}$ are chosen. In particular, by looking at the family $\eta_\lambda = (\eta_{(\lambda,n)})_{n \in \mathbb{Z}^2}$ of scalar field theory models, given by Equation (1), we get continuum limits for arbitrary couplings $(\lambda_l(n))_{n \in \mathbb{Z}^2}$, $l = 0 \cdots L$. Even in case of a perturbatively non-renormalizable model, it makes sense to study the set of continuum limits.

On the other hand, the fact that there are weak limit points is not sufficient for concluding the existence of interesting models. Therefore, the problem which occur here is to get detailed information about the states in $\mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$. At this point, we introduce a rough classification of families of states by considering the possible limit points of a given family η.

(1) For a given family η every limit point in $\mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$ is a character which is the most trivial case.

(2) There is another uninteresting case, namely each state in $\mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$ is ultra local, i.e. each state $\varphi \in \mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$ has no correlation for two operators $a_j \in \mathfrak{A}_{(\iota,\mathbb{R})}(\mathscr{U}_j)$, $j = 1, 2$, which are localized in disjoint regions $\mathscr{U}_1 \cap \mathscr{U}_2 = \emptyset$:

$$\langle \varphi, a_1 a_2 \rangle \quad = \quad \langle \varphi_1, a_1 \rangle \langle \varphi_2, a_2 \rangle$$

for suitable states φ_j on $\mathfrak{A}_{(\iota,\mathbb{R})}(\mathscr{U}_j)$, $j = 1, 2$. This implies that, if the corresponding theory in Minkowski space exists, then it is the constant field. The notion of ultra local (scalar) fields is explained in [19]. In particular an application of the measures, constructed in [1], to euclidean field theory leads to ultra local models.

(3) There exists a limit point $\varphi \in \mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$ which is not ultra local.

By looking at our example of scalar fields, the case *(3)* can be subdivided into two further cases:

(3.1) Let $\varphi \in \mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$ be a non-ultra local state, then it is equivalent to a gaussian state.

(3.2) There exists a limit point $\varphi \in \mathfrak{S}_{(\iota, \mathbb{R})}[\eta]$ which is not ultra local and which is not equivalent to a gaussian state.

We have to mention at this point that for many examples case *(1)* can be excluded. One now asks the following question:

Question: Can we decide, by studying the family of states η, whether the case *(3)* is valid or not?

In order to show the existence of states in $\mathfrak{S}_{(\iota, \mathbb{R})}$, which are not ultra local, we propose the following strategy: For a cube $\Delta \in \Sigma_d(n)$ and for an operator $a \in \mathscr{C}_{\mathrm{b}}(\mathbb{R})$ we define the function $\Phi_n(\Delta, a)$ by

$$\Phi_n(\Delta, a)(u) \quad := \quad a\big(u(\Delta)\big) \ .$$

Let $\Delta_1, \Delta_2 \in \Sigma_d(n)$ be two disjoint cubes $\Delta_1 \cap \Delta_2 = \emptyset$. Find a continuous bounded positive function $h \in \mathscr{C}_{\mathrm{b}}(\mathbb{R})$ and a family of states η such that there exists a constants $c^{\pm}_{(n, h, \Delta_1, \Delta_2)} > 0$ with

$$c^{+}_{(n, h, \Delta_1, \Delta_2)} \quad \geq \quad \big| \big\langle \, \mathbf{c}_{[\eta_{n+k} \circ \iota_{(n+k, n)}]}, \Phi_n(\Delta_1, h) \otimes \Phi_n(\Delta_2, h) \, \big\rangle \big|$$

$$> \quad c^{-}_{(n, h, \Delta_1, \Delta_2)}$$

for large k. Here we define for any state ω its correlation by

$$\big\langle \, \mathbf{c}_{[\omega]}, a \otimes b \, \big\rangle \quad := \quad \big\langle \, \omega, ab \, \big\rangle - \big\langle \, \omega, a \, \big\rangle \big\langle \, \omega, b \, \big\rangle \ .$$

Since the bound is uniform in k, there exists a state $\varphi \in \mathfrak{S}_{(\iota, \mathbb{R})}[\eta]$ such that the correlation of $\varphi_n = \varphi \circ \iota_n$ fulfills the bounds

$$c^{+}_{(n, h, \Delta_1, \Delta_2)} \quad \geq \quad \big| \big\langle \, \mathbf{c}_{[\varphi_n]}, \Phi_n(\Delta_1, h) \otimes \Phi_n(\Delta_2, h) \, \big\rangle \big|$$

$$> \quad c^{-}_{(n, h, \Delta_1, \Delta_2)}$$

which implies that φ is not ultra local. From the invariance properties of φ_n we conclude that this bound holds for each pair of cubes which can be obtained by applying a transformation $g \in b^{-n^{\mathrm{o}}} \mathbb{Z}^d$ to (Δ_1, Δ_2). Hence the constant $c_{(n, \Delta_1, \Delta_2)}$ only depends on the orbit of (Δ_1, Δ_2) under the action of $b^{-n^{\mathrm{o}}} \mathbb{Z}^d$. Let $d(\Delta_1, \Delta_2)$ be the distance of the cubes (Δ_1, Δ_2) and let us assume that the upper bound $c^{+}_{(n, h, \Delta_1, \Delta_2)}$ has the form

$$c^{+}_{(n, h, \Delta_1, \Delta_2)} \quad = \quad K_{(n, h)} \exp\left(-\frac{d(\Delta_1, \Delta_2)}{\ell(n, h)} \right)$$

with two constants $K_{(n, h)}, \ell(n, h)$, then the constant $\ell(n, h)$ plays the role of the correlation length. A proposal how to tackle the problem of estimating correlations is given in Appendix A.

Action, effective action, and continuum limits: We assume now that
somebody has already constructed a weak euclidean statistical mechanics

$$\Lambda \;\;=\;\; \left(\underline{\mathfrak{A}}_{(\iota,\mathbb{R}|\omega)},\, \beta_\iota,\, \omega,\, \mathbb{R}^d,\, \mathbb{Q}_b^d,\, \mathscr{K}^d\right) \;\;.$$

Then it is natural to ask whether one can construct new theories out of Λ
by a suitable deformation procedure. Remember that $\mathfrak{A}_{(\iota,\mathbb{R}|\omega)}$ denotes the
C*-algebra

$$\mathfrak{A}_{(\iota,\mathbb{R}|\omega)} \;\;:=\;\; \mathfrak{A}_{(\iota,\mathbb{R})}\big/\mathfrak{J}_{(\iota,\omega)}$$

where $\mathfrak{J}_{(\iota,\omega)}$ is the kernel of the GNS-representation of $\omega \in \mathfrak{S}_{(\iota,\mathbb{R})}$. The
basic idea is to perturb each of the subsystems

$$\Lambda_n \;\;=\;\; \left(\underline{\mathfrak{A}}_{(n,\mathbb{R}|\omega)},\, \beta_n,\, \omega_n,\, \mathbb{R}^d,\, b^{-n^0}\mathbb{Z}^d,\, \mathscr{K}_n^d\right)$$

separately, by replacing each of the states ω_n by appropriate states $\eta_n \in \mathscr{F}_{\omega_n}$. If we assume that $\eta_n^{(k)} := \eta_{n+k} \circ \iota_{(n+k,n)}$ is contained in $\mathscr{F}_{\omega_n}$ for
each $k \in \mathbb{N}^2$, then we obtain for each $n \in \mathbb{Z}^2$ and for each $k \in \mathbb{N}^2$ a
subsystem

$$\Lambda_n^{(k)} \;\;:=\;\; \left(\underline{\mathfrak{A}}_{(n,\mathbb{R}|\eta_n^{(k)})},\, \beta_n,\, \eta_n^{(k)},\, \mathbb{R}^d,\, b^{-n^0}\mathbb{Z}^d,\, \mathscr{K}_n^d\right) \;\;\prec\;\; \Lambda_n \;\;.$$

which is, in particular, a subsystem of Λ. There are also examples for which
the theories $\Lambda_n^{(k)} \cong \Lambda_n$ are equivalent for each k. Formally, the relation
$\Lambda_n^{(k)} \cong \Lambda$ may be no longer valid in the continuum limit $k, n \to \infty$. More
precisely, for a continuum limit $\varphi \in \mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$ the corresponding theory

$$\Lambda^{(\varphi)} \;\;=\;\; \left(\underline{\mathfrak{A}}_{(n,\mathbb{R}|\varphi)},\, \beta_\iota,\, \varphi,\, \mathbb{R}^d,\, \mathbb{Q}_b^d,\, \mathscr{K}^d\right)$$

is, however, *not equivalent* to the theory where we have started from.

But one may ask whether the subsystem $\Lambda_n^{(\varphi)}$, which corresponds to the
energy momentum range $[b^{-n^1}, b^{n^0}]$, is a subsystem of Λ_n. This question
is related to the existence of an effective action [12]. The states η_n under
consideration are of the form

$$\langle\, \eta_n, a\, \rangle \;\;=\;\; \int_{\mathbb{R}^{\Sigma_d(n)}} \mathrm{d}\omega_n(u)\; \mathbf{v}_n(u)\; a(u)$$

and we call the family of functions $\mathbf{v} = (\mathbf{v}_n)_{n \in \mathbb{Z}^2}$ an *action*.

Our notion of action is slightly different to the one which one usually
finds in the literature where in comparison the negative logarithm $-\ln \mathbf{v}_n$
is usually called the action. In order to distinguish these notions we call
$-\ln \mathbf{v}_n$ the *action functional* with respect to n. For example, choose ω_n to
be the gaussian part and $\mathbf{v}_n$ to be the interaction part (see [3]). Within our
analysis, we also consider examples where ω_n is an ultra local state and $\mathbf{v}_n$
contains the next neighbor coupling.

From a given action $\mathbf{v}$, we obtain a new family of functions by

$$\mathbf{e}_{(\omega)}^{(k)}(\mathbf{v})_n(u) \;\;:=\;\; \int \mathrm{d}\omega_{(n+k)}(u')\; \mathbf{k}_{(\omega,n,n+k)}(u,u')\; \mathbf{v}_{n+k}(u')$$

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where the kernel $\mathbf{k}_{(\omega,n,n+k)}(u,u')$ is determined by the condition

$$\int \mathrm{d}\omega_n(u)\,\mathrm{d}\omega_{(n+k)}(u')\,\mathbf{k}_{(\omega,n,n+k)}(u,u')\,a(u')$$

$$= \int \mathrm{d}\omega_{(n+k)}(u')\,a(u')\ .$$

We call $\mathrm{e}^{(k)}_{(\omega)}(\mathbf{v})$ the *effective action* with respect to the action $\mathbf{v}$. For a fixed cut-off $n \in \mathbb{Z}^2$ the operation of $\mathrm{e}^{(k)}_{(\omega)}$ corresponds to a substitution by the underlying lattice theory on $\Sigma_d(n)$ by a lattice theory, also defined on $\Sigma_d(n)$, arising from a lattice theory on $\Sigma_d(n+k)$ by integrating out the corresponding high energy degrees of freedom (See [12]).

In Section 3, we discuss in a more general context conditions for $\mathbf{v}$ under which there exists a family of measurable functions $\mathbf{v}' = (\mathbf{v}'_n)_{n\in\mathbb{Z}^2}$ ($\mathbf{v}'_n$ is ω_n-measurable) such that

$$\langle\,\varphi,\iota_n(a)\,\rangle \;=\; \mathbf{z}^{-1}_{(\omega,\mathbf{v}',n)}\int \mathrm{d}\omega_n(u)\,\mathbf{v}'_n(u)\,a(u) \qquad (4)$$

holds for a continuum limit $\varphi \in \mathfrak{S}_{(\iota,\mathbb{R})}[\eta]$. In this case, $\Lambda^{(\varphi)}_n$ is a subsystem of Λ_n since the folium $\mathscr{F}_{\varphi_n}$ is contained in $\mathscr{F}_{\omega_n}$.

In Section 3.4 we formulate a sufficient condition (*multiplicative renormalizability*) for an action $\mathbf{v}$ which allows to construct a new action $\mathbf{v}'$ from $\mathbf{v}$ such that $\mathbf{v}'$ satisfies the fix point equation $\mathrm{e}^{(k)}_{(\omega)}(\mathbf{v}') = \mathbf{v}'$ and therefore Equation (4) (Proposition 3.7). We have to emphasize here that the existence of $\mathbf{v}'$ does not exclude the case *(2)* of ultra locality. In order to conclude that one deals with an interesting model one has to study $\mathbf{v}'$ in more detail.

To illustrate the notion *multiplicative renormalizability*, an ultra local example for a multiplicatively renormalizable action is also presented in Section 3.4 and Appendix B deals with a larger class of examples.

A large variety of lattice models: The C*-algebraic point of view suggests to study a large class of lattice field theories among which there are examples which are rather different from the usual lattice field theory models, like $P(\phi)_d$ for instance. To some extend they can be regarded as generalized spin models.

The abelian C*-algebra $\mathscr{C}_{\mathrm{b}}(\mathbb{R})$, which we have used for illustration in the previous paragraphs, can easily be replaced by any C*-algebra A in particular by a σ-finite von Neumann algebra M acting on a Hilbert space K. As usual, we denote by M' the commutant of M, i.e. the set of bounded operators on K which commute with all operators in M.

As input data for the construction of lattice models we choose

(1) a von Neumann algebra M, acting on a Hilbert space K, and a vector Ω, which is cyclic and separating for M,

(2) a family of positive operators $\mathbf{w} \in (M' \,\overline{\otimes}\, M')^{\mathbb{Z}^2}$ such that

$$[\mathbf{w}_n \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{w}_n] = 0$$

for each $n \in \mathbb{Z}^2$.

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Figure 3: The left figure illustrates the operator $\Phi_n(\Delta, a)$ where Δ is the shaded cube. The figure on the right illustrates operators $\Phi_n(\Gamma, w)$, $w = \sum_i a_i \otimes b_i$ which couple nexted neighbor cubes. Here Γ is the face which the gray shaded cubes have in common.

The algebra $\mathfrak{A}_{(n,M)} = \overline{\otimes}_{\Delta \in \Sigma_d(n)} \{\Delta\} \times M$ is simply the von Neumann tensor product of M over $\Sigma_d(n)$ and the vector $\Omega_n := \otimes_{\Delta \in \Sigma_d(n)} \{\Delta, \Omega\}$ is cyclic and separating for $\mathfrak{A}_{(n,M)}$. For a cube $\Delta \in \Sigma_d(n)$ and an operator $a \in M$ we denote by $\Phi_n(\Delta, a)$ operator in $\mathfrak{A}_{(n,M)}$ which is a tensor product of operators in M where at Δ the factor a appears and the unit 1 else (See Figure 3). For a hypercube $\Gamma \in \Sigma_{d-1}(n)$ there are two unique cubes Δ_0, Δ_1 such that $\Delta_0 \cap \Delta_1 = \Gamma$ and we put $\Phi_n(\Gamma, a \otimes b) := \Phi_n(\Delta_0, a)\Phi_n(\Delta_1, b)$. We introduce a state η_n on $\mathfrak{A}_{(n,M)}$ by

$$\langle \eta_n, a \rangle \quad := \quad \mathbf{z}_{(\Omega, n, \mathbf{w})}^{-1} \left\langle \Omega_n, \prod_{\Gamma \in \Sigma_{d-1}(n)} \Phi_n(\Gamma, \mathbf{w}_n)\, a\, \Omega_n \right\rangle$$

where the partition function $\mathbf{z}_{(\Omega, n, \mathbf{w})}$ is for normalization. If the vector

$$\Psi_{(\Omega, n, \mathbf{w})} \quad := \quad \prod_{\Gamma \in \Sigma_{d-1}(n)} \Phi_n(\Gamma, \mathbf{w}_n)^{1/2} \Omega_n$$

is cyclic and separating for $\mathfrak{A}_{(n,M)}$ for each $n \in \mathbb{Z}^2$, then η_n is faithful and we obtain for each $n \in \mathbb{Z}^2$ a lattice field theory model

$$\Lambda_n \quad := \quad \left(\underline{\mathfrak{A}}_{(n,M)}, \beta_\iota, \eta_n, \mathbb{R}^d, b^{-n^0}\mathbb{Z}^d, \mathscr{K}_n^d \right) \ .$$

For each $n \in \mathbb{N}$, we consider the state $\omega_n = \langle \Omega_n, (\cdot)\Omega_n \rangle$ and by an appropriate choice of block spin transformations ι the consistency condition $\omega_{n+k} \circ \iota_{(n+k,n)} = \omega_n$ is fulfilled and we obtain the corresponding continuum model

$$\left(\underline{\mathfrak{A}}_{(\iota,M)}, \beta_\iota, \omega, \mathbb{R}^d, \mathbb{Q}_b^d, \mathscr{K}^d \right) \ .$$

The state η_n is a perturbation of ω_n where the action is given by

$$\mathbf{v} : n \longmapsto \prod_{\Gamma \in \Sigma_{d-1}(n)} \Phi_n(\Gamma, \mathbf{w}_n) \ .$$

Each of the operators $\Phi_n(\Gamma, \mathbf{w}_n)$ induces a coupling of the two next neighbor cubes which have the face Γ in common (See Figure 3). If $\mathbf{w}_n$ is of the form $\mathbf{w}_n = \mathbf{h}_n \otimes \mathbf{h}_n$ the cubes are decoupled and the resulting theory is ultra local.

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The simplest non-trivial choice for $\mathbf{w}_n$ is $\mathbf{1} + \mathbf{h}_n \otimes \mathbf{h}_n$ for instance. More general, one can choose $\mathbf{w}$ in the following manner: Put

$$\mathbf{w}_n \quad := \quad \int_0^1 \mathbf{h}_n(s) \otimes \mathbf{h}_n(s)$$

where $\mathbf{h}_n \in \mathscr{C}^\infty([0,1], M')$ is a smooth function with $\mathbf{h}_n(s) > 0$ and $[\mathbf{h}(s_1), \mathbf{h}(s_2)] = 0$. If $\mathbf{h}_n(s) = \mathbf{h}_n$ is constant, then we would end up with an ultra local theory. Therefore we have to require that the derivative of $\mathbf{h}_n$ does not vanish. For this kind of examples, the effective action $\mathbf{e}_{(\omega)}^{(k)}(\mathbf{v})$ can be computed quite explicitly and our hope is that the corresponding continuum limits could be easier controlled than the continuum limits for $P(\phi)_d$-like models, for instance.

There is a further nice feature of models which correspond to such actions like $\mathbf{v}$. Particular correlation functions can be interpreted in terms of correlation functions of a different, some kind of dual, lattice field theory. In order to explain this, we introduce for a cube $\Delta \in \Sigma_d(n)$ and for each $s \in [0,1]^{\Sigma_{d-1}(n)}$ a normal state on M:

$$\langle\, \mathbf{E}_{(\mathbf{h},n|\Delta)}^{(s)}, a \,\rangle \quad := \quad [\mathbf{z}_{(\mathbf{h},n|\Delta)}^{(s)}]^{-1} \Big\langle\, \Omega,\, \prod_{\Gamma \in \partial\Delta} \mathbf{h}_n(s(\Gamma))\, a\, \Omega \,\Big\rangle$$

where $\mathbf{z}_{(h,n|\Delta)}(s)$ is for normalization. This yields a state $\hat{\eta}_{(n,h)}$ on the algebra $\hat{\mathfrak{A}}_{(n,[0,1])} := \mathscr{C}([0,1]^{\Sigma_{d-1}(n)})$ such that for operators $(a_j)_{j=1 \cdots k}$, $a_j \in M$, the correlation functions fulfill the relation

$$\Big\langle\, \eta_n,\, \prod_{j=1}^k \Phi_n(\Delta_j, a_j) \,\Big\rangle \quad = \quad \int \mathrm{d}\hat{\eta}_n(s) \prod_{j=1}^k \langle\, \mathbf{E}_{(n,\mathbf{h}|\Delta_j)}^{(s)}, a_j \,\rangle \quad.$$

The state $\hat{\eta}_{(h,n)}$ is given by

$$\langle\, \hat{\eta}_{(h,n)}, \hat{a} \,\rangle \quad = \quad \mathbf{z}_{(h,n)}^{-1} \int \prod_{\Gamma \in \Sigma_{d-1}(n)} \mathrm{d}s(\Gamma)\, \hat{\mathbf{v}}_n(s)\, \hat{a}(s)$$

and $\hat{\mathbf{v}}_n$ is given by

$$\hat{\mathbf{v}}_n(s) \quad = \quad \prod_{\Delta \in \Sigma_d(n)} \mathbf{z}_{(h,n|\Delta)}^{(s)} \quad.$$

Indeed, particular correlation functions of the model, which is given by the action $\mathbf{v}$, can be expressed in terms of correlation functions of a lattice model which is given by the action $\hat{\mathbf{v}}$ and whose corresponding field configurations are functions from the faces of cubes into the interval $[0,1]$. Hence some properties of the non-commutative lattice field theory models Λ_n can be investigated by studying commutative lattice models. This point of view may be helpful in order to construct non-ultra local models.

On the regularity condition WE3: However, the above discussion is not concerned with the the regularity condition *WE3*. By using a slightly different construction for the continuum C*-algebras we show in Section 4 how

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from a given family of invariant and reflexion positive states $\eta = (\eta_n)_{n \in \mathbb{N}}$ continuum limits can be constructed which fulfill all the axioms of a weak euclidean statistical mechanics.

Conclusion and outlook: We close our paper by the Section 5 *conclusion and outlook.*

2   Continuum limits for lattice field theory models

Within this section we develop a concept of *continuum limit* which can be applied to a large class of lattice field theory models. In Section 2.1, we introduce notation and conventions which we are going to use.

A general and model independent notion of block spin transformations is given in Section 2.2. Although the construction of C*-inductive limits is standard and can be found in many text books, we present a version of this procedure in Section 2.3. One reason is to keep the paper as self contained as possible. Furthermore, the notations and definitions which we introduce in Section 2.3, are used later to perform a procedure which is slightly different from taking the C*-inductive limit of a net of C*-algebras.

Finally, we present in Section 2.4 a general concept for continuum limits of lattice models.

2.1   Notation and conventions

We consider a C*-algebra A and for a given cutoff $n \in \mathbb{Z}^2$ we introduce the C*-algebra

$$\mathfrak{A}_{(n,A)} \quad := \quad \bigotimes_{\Delta \in \Sigma_d(n)} \{\Delta\} \times A \ .$$

To each subset $\mathscr{U} \subset [-b^{n_1}, b^{n_1}]^d$ a subalgebra

$$\mathfrak{A}_{(n,A)}(\mathscr{U}) \quad := \quad \bigotimes_{\Delta \in \Sigma_d(n,\mathscr{U})} \{\Delta\} \times A$$

can be assigned in a natural manner. The set $\Sigma_d(n, \mathscr{U})$ is defined as follows: We identify the set $\Sigma_0^o(n) := b^{-n^0}\mathbb{Z}^d \cap [-b^{n^1}, b^{n^1}]^d$ with a subset of the torus $\Sigma_0(n)$. The q-cubes $\Sigma_q^o(n)$ in $b^{-n^0}\mathbb{Z}^d \cap [-b^{n^1}, b^{n^1}]^d$ can also be identified with q-cubes in $\Sigma_q(n)$. The set $\Sigma_d(n, \mathscr{U})$ consists of all cubes Δ in $\Sigma_d^o(n)$ with $\Delta \subset \mathscr{U}$.

The group $b^{-n^0}\mathbb{Z}^d \subset \mathrm{E}(d)$ acts by automorphisms covariantly on the algebra $\mathfrak{A}_n(A)$. In other words, there exists a group homomorphism

$$\beta_n \in \mathrm{Hom}(b^{-n^0}\mathbb{Z}, \mathrm{Aut}\mathfrak{A}_{(n,A)})$$

such that for each $g \in b^{-n^0}\mathbb{Z}^d$ the equation

$$\beta_{(n,g)}\mathfrak{A}_{(n,A)}(\mathscr{U}) \quad = \quad \mathfrak{A}_{(n,A)}(g\mathscr{U})$$

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holds. The automorphism $\beta_{(n,g)}$ is simply given by

$$\beta_{(n,g)} \left[\bigotimes_{\Delta \in \Sigma_d(n)} \{\Delta, a(\Delta)\} \right] \quad := \quad \bigotimes_{\Delta \in \Sigma_d(n)} \{\Delta, a(g^{-1}\Delta)\} \ .$$

There is one important automorphism which corresponds to the euclidean time reflexion.

$$x \longmapsto \theta_\mu(x) \quad = \quad x - 2(e_\mu \cdot x)\, e_\mu$$

where $y \cdot x$ is the canonical scalar product in $\mathbb{R}^d$ and $e_\mu \in \mathbb{R}^d$ is the unit vector with components $(e_\mu)_\nu = \delta_{\mu\nu}$. For each $\mu = 1, \cdots, d$ we consider the anti-linear involution

$$j_{(n,\mu)} : \mathfrak{A}_{(n,A)} \longrightarrow \mathfrak{A}_{(n,A)}$$

which is given by

$$j_{(n,\mu)} \left[\bigotimes_{\Delta \in \Sigma_*(n)} \{\Delta, a(\Delta)\} \right] \quad := \quad \bigotimes_{\Delta \in \Sigma_d(n)} \{\Delta, a(\theta_\mu \Delta)^*\} \ .$$

Since b is odd, the set of d-cubes can be decomposed into a union of three disjoint set

$$\Sigma_d(n) \quad = \quad \Sigma_d(n,\mu,0) \cup \Sigma_d(n,\mu,+) \cup \Sigma_d(n,\mu,-)$$

where $\Sigma_d(n,\mu,0)$ is a layer of θ_μ-invariant d-cubes and $\Sigma_d(n,\mu,+)$ is mapped onto $\Sigma_d(n,\mu,-)$ via θ_μ. Therefore, operators of the form

$$a \quad = \quad \bigotimes_{\Delta \in \Sigma_d(n,\mu,0)} \{\Delta, a(\theta_\mu \Delta)\}$$

with $a(\Delta)^* = a(\Delta)$ are $j_{(n,\mu)}$-invariant. The algebra $\mathfrak{A}_{(n,A)}$ can be written as a tensor product

$$\mathfrak{A}_{(n,A)} \quad = \quad \mathfrak{A}_{(n,A)}(\mu,0) \otimes \mathfrak{A}_{(n,A)}(\mu,+) \otimes \mathfrak{A}_{(n,A)}(\mu,-)$$

where $\mathfrak{A}_{(n,A)}(\mu,0)$ is stable under $j_{(n,\mu)}$ and $\mathfrak{A}_{(n,A)}(\mu,+)$ is mapped onto $\mathfrak{A}_{(n,A)}(\mu,-)$ via $j_{(n,\mu)}$ (see Figure 2).

2.2 Block spin transformations: The general setup

In our context, block spin identify operators in $\mathfrak{A}_{(n,A)}$ with operators contained in a algebra $\mathfrak{A}_{(n_1,A)}$ which corresponds to a finer lattice, i.e. $n \prec n_1$. Let us state a list of axioms which characterizes the notion of block spin transformations.

Definiton 2.1 : A family of *-homomorphisms

$$\iota \quad = \quad \{\iota_{(n_1,n_0)} \in \mathrm{Hom}(\mathfrak{A}_{(n_0,A)}, \mathfrak{A}_{(n_1,A)}) | n_0 \prec n_1\}$$

is called a family of *block spin transformations* if it fulfills the following conditions:

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(1) Cosheaf condition: For each $n_0 \prec n_1 \prec n_2$:

$$\iota_{(n_2,n_1)} \circ \iota_{(n_1,n_0)} \quad = \quad \iota_{(n_2,n_0)} \quad .$$

(2) Locality: For each $n_0 \prec n_1$ and for each $\mathscr{U} \subset [-b^{n_0^1}, b^{n_0^1}]^d$:

$$\iota_{(n_1,n_0)} \mathfrak{A}_{(n_0,A)}(\mathscr{U}) \quad \subset \quad \mathfrak{A}_{(n_1,A)}(\mathscr{U}) \quad .$$

(3) Translation covariance: Let $\mathscr{U}_0, \mathscr{U}_1 \subset [-b^{n_0^1}, b^{n_0^1}]^d$ such that $\mathscr{U}_0 \cup g\mathscr{U}_0 \subset \mathscr{U}_1$ for some $g \in b^{-n_0^0}\mathbb{Z}^d \subset b^{-n_1^0}\mathbb{Z}^d$:

$$\iota_{(n_1,n_0)} \beta_{(n_0,g)} a \quad = \quad \beta_{(n_1,g)} \iota_{(n_1,n_0)} a$$

for each $a \in \mathfrak{A}_{(n_0,A)}(\mathscr{U}_0)$.

2.3   C*-inductive limits revisited

For a given family of block spin transformations ι we construct the C*-inductive limit $\mathfrak{A}_{(\iota,A)}$ of the net $n \mapsto \mathfrak{A}_{(n,A)}$. In order to carry through our subsequent analysis, we briefly describe the construction of $\mathfrak{A}_{(\iota,A)}$.

Step I.   Let $\mathscr{C}_{\mathrm{b}}(\mathbb{Z}^2, \mathfrak{A}_A)$ be the C*-algebra which is generated by bounded sections in the bundle $\mathfrak{A}_A : n \mapsto \mathfrak{A}_{(n,A)}$. We consider the closed two-sided ideal $\mathscr{C}_0(\mathbb{Z}^2, \mathfrak{A}_A)$ in $\mathscr{C}_{\mathrm{b}}(\mathbb{Z}^2, \mathfrak{A}_A)$, which is generated by sections $a : n \mapsto a_n$ for which the limit $\lim_{n\to\infty} \|a_n\| = 0$ vanishes. We build the quotient C*-algebra

$$\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2, \mathfrak{A}_A) \quad := \quad \mathscr{C}_{\mathrm{b}}(\mathbb{Z}^2, \mathfrak{A}_A) / \mathscr{C}_0(\mathbb{Z}^2, \mathfrak{A}_A) \quad .$$

In the following, $\mathbf{p}$ denotes the corresponding canonical projection onto the quotient.

Step II.   For a given family of block spin transformations ι, we denote by $\mathfrak{A}^o_{(\iota,A)}$ the C*-subalgebra in $\mathscr{C}_{\mathrm{b}}(\mathbb{Z}^2, \mathfrak{A}_A)$ which is generated by sections $a : n \mapsto a_n$ for which there exists $n_0 \in \mathbb{Z}^2$ and there exists $a_0 \in \mathfrak{A}_{(n_0,A)}$ such that

$$a_n \quad = \quad \iota_{(n,n_0)} a_0$$

for each $n_0 \prec n$. The C*-inductive limit of the pair (ι, A) now is given by

$$\mathfrak{A}_{(\iota,A)} \quad := \quad \mathbf{p}[\mathfrak{A}^o_{(\iota,A)}] \subset \mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2, \mathfrak{A}_A) \quad .$$

For each $n \in \mathbb{Z}^2$ we obtain a *-homomorphism $\iota_n : \mathfrak{A}_{(n,A)} \to \mathfrak{A}_{(\iota,A)}$ which identifies $\mathfrak{A}_{(n,A)}$ with a subalgebra in $\mathfrak{A}_{(\iota,A)}$. It is given by the prescription

$$\iota_n a \quad := \quad \mathbf{p}[n_1 \mapsto \iota_{(n_1,n)} a]$$

where the section $a_o = [n_1 \mapsto \iota_{(n_1,n)} a]$ is any representative such that $a_o(n_1) = \iota_{(n_1,n)} a$ for each $n \prec n_1$. It is obvious that the relation $\iota_n \circ \iota_{(n,n_0)} = \iota_{n_0}$ holds for $n_0 \prec n$.

The C*-algebra $\mathfrak{A}_{(\iota,A)}$ can be regarded as the continuum C*-algebra and it contains observables which correspond to observations at the full energy momentum range, whereas The C*-subalgebras $\iota_n(\mathfrak{A}_{(n,A)})$ contain only observables which correspond to observations for the energy momentum range $[b^{-n^1}, b^{n^0}]$.

We consider the dense subgroup $\mathbb{Q}_b^d := \cup_n b^{-n}\mathbb{Z}^d$ of the translation group $\mathbb{R}^d$. There exists a group homomorphism

$$\beta_\iota \in \mathrm{Hom}(\mathbb{Q}_b^d, \mathrm{Aut}\,\mathfrak{A}_{(\iota,A)})$$

which acts covariantly on $\mathfrak{A}_{(\iota,A)}$. For $g \in \mathbb{Q}_b^d$ we define

$$\beta_{(\iota,g)}\mathbf{p}[n \mapsto \iota_{(n,n_0)}a] \quad := \quad \mathbf{p}[n \mapsto \beta_{(n,g)}\iota_{(n,n_0)}a]$$

with $g \in b^{-l}\mathbb{Z}^d$ and $n^0 > l$. Let $\mathfrak{A}_{(\iota,A)}(\mathscr{U})$ be the C*-subalgebra which is generated by local operators in $\iota_n[\mathfrak{A}_{(n,A)}(\mathscr{U})]$ for some $n \in \mathbb{Z}^2$. Then we conclude from the construction of β_ι:

$$\beta_{(\iota,g)}\mathfrak{A}_{(\iota,A)}(\mathscr{U}) \quad = \quad \mathfrak{A}_{(\iota,A)}(g\mathscr{U}) \quad .$$

Thus the prescription

$$\underline{\mathfrak{A}}_{(\iota,A)} : \mathscr{U} \longmapsto \mathfrak{A}_{(\iota,A)}(\mathscr{U})$$

is a (weak) euclidean net of C*-algebras which is translationally covariant with respect to the group $\mathbb{Q}_b^d$.

2.4 On a general concept for continuum limits for lattice models

For each cutoff $n \in \mathbb{Z}^2$ we select a class of appropriate states on $\mathfrak{A}_{(n,A)}$. We denote by $\mathfrak{S}_{(n,A)}$ the set of all states $\eta \in \mathfrak{S}(\mathfrak{A}_{(n,A)})$ which satisfy the assumptions:

Invariance: For each $g \in b^{-n^0}\mathbb{Z}$:

$$\eta \circ \beta_{(n,g)} \quad = \quad \eta \quad .$$

Reflexion positivity: The sesqui-linear form

$$a \otimes b \longmapsto \langle \eta, j_{(n,\mu)}(a)b \rangle$$

is positive semi-definite on $\mathfrak{A}_{(n,A)}(\mu,+)$ for each $\mu = 1, \cdots, d$.

There are also anti-linear involutions $j_{(\iota,\mu)}$ acting on the C*-inductive limit $\mathfrak{A}_{(\iota,A)}$ according to the prescription:

$$j_{(\iota,\mu)}\mathbf{p}[n \mapsto \iota_{(n,n_0)}a] \quad := \quad \mathbf{p}[n \mapsto \iota_{(n,n_0)}j_{(n_0,\mu)}a]$$

$$= \quad \mathbf{p}[n \mapsto j_{(n,\mu)}\iota_{(n,n_0)}a] \quad .$$

Analogously to the definition, given above, we introduce the space $\mathfrak{S}_{(\iota,A)}$ of $\mathbb{Q}_b^d$-invariant and reflexion positive functionals on $\mathfrak{A}_{(\iota,A)}$.

Let $\Gamma(\mathbb{Z}^2,\mathfrak{S}_A)$ be the convex set of sections

$$\eta : \mathbb{Z}^2 \ni n \longmapsto \eta_n \in \mathfrak{S}_{(n,A)}$$

We identify $\mathfrak{S}_{(\iota,A)}$ with the corresponding subset in $\Gamma(\mathbb{Z}^2,\mathfrak{S}_A)$ by identifying $\omega \in \mathfrak{S}_{(\iota,A)}$: with the section

$$\omega : n \longmapsto \omega_n := \omega \circ \iota_n \quad.$$

For simplicity, we do not distinguish the state $\omega \in \mathfrak{S}_{(\iota,A)}$ and the corresponding section within our notation.

Proposition 2.2 : *There is a canonical surjective convex-linear map*

$$\mathbf{E} : \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2,\mathbb{C})] \otimes \Gamma(\mathbb{Z}^2,\mathfrak{S}_A) \longrightarrow \mathfrak{S}_{(\iota,A)} \quad.$$

Proof. For a state $\xi \in \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2,\mathbb{C})]$ and a section $\eta \in \Gamma(\mathbb{Z}^2,\mathfrak{S}_A)$ we define a new section $\mathbf{E}[\xi \otimes \eta]$ by

$$\langle\, \mathbf{E}[\xi \otimes \eta]_n\,, \mathbf{p}[n \mapsto \iota_{(n,n_0)}a]\,\rangle \quad := \quad \langle\, \xi, \mathbf{p}[\,n \mapsto \,\langle\, \eta_n, \iota_{(n,n_0)}a\,\rangle\,]\,\rangle \quad.$$

It is obvious that $\mathbf{E}$ is convex linear and that $\mathbf{E}[\xi \otimes \eta]$ fulfills the consistency condition

$$\mathbf{E}[\xi \otimes \eta]_n \circ \iota_{(n,n_0)} \quad = \quad \mathbf{E}[\xi \otimes \eta]_{n_0} \quad.$$

Let $\omega \in \mathfrak{S}_{(\iota,A)}$ be given, then we obtain by a straight forward computation

$$\mathbf{E}[\xi \otimes \omega] \quad = \quad \omega$$

for each $\xi \in \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2,\mathbb{C})]$. Thus $\mathbf{E}$ is surjective. Finally, the invariance and the reflexion positivity follow directly from the construction of $\mathbf{E}$.

∎

Remark:

(1) For a given family of lattice field theory models

$$\Lambda_n := \big(\underline{\mathfrak{A}}_{(n,A|\eta)}, \beta_n, \eta_n, \mathbb{R}^d, b^{-n^0}\mathbb{Z}^d, \mathscr{K}_n^d\big)_{n \in \mathbb{N}}$$

we introduce the set of continuum limits by

$$\mathfrak{S}_{(\iota,A)}[\eta] \quad := \quad \left\{ \mathbf{E}[\xi \otimes \eta] \;\middle|\; \xi \in \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2,\mathbb{C})] \right\} \; \subset \; \mathfrak{S}_{(\iota,A)} \quad.$$

(2) Proposition 2.2 suggests a guideline how to construct continuum limits from an appropriate family $(\Lambda_n)_{n \in \mathbb{N}}$ of lattice field theory models. For each continuum limit $\varphi \in \mathfrak{S}_{(\iota,A)}[\eta]$ the statistical mechanics

$$\big(\underline{\mathfrak{A}}_{(\iota,A|\varphi)}, \beta_\iota, \varphi, \mathbb{R}^d, \mathbb{Q}_b^d, \mathscr{K}^d\big)$$

fulfill the axioms of a weak euclidean statistical mechanics except the continuity requirement *WE3*. Hence we deal with a well posed problem, namely to analyze the properties of the states contained in $\mathfrak{S}_{(\iota,A)}[\eta]$ with respect to the properties of the section η.

3 Actions, effective actions, and continuum limits

This section is destined to introduce the notions action and effective action within the C*-algebraic setup. Section 3.1 is concerned with the problem of constructing from a given a weak euclidean statistical mechanics a new model by means of perturbations. For this purpose, we introduce the concept of action and effective action.

In particular, we study perturbations of ultra local models. We present in Section 3.2 a simple example for a family of block spin transformation which allows to compute some useful expressions quite explicitly.

In Section 3.3, we show that, for a given lattice, there is a large variety of reflexion positive invariant states.

A criterion for the existence of an effective action for continuum limits is formulated in Section 3.4.

3.1 Effective actions and continuum limits

To begin with, we consider for a weak euclidean statistical mechanics

$$\Lambda \quad = \quad \left(\underline{\mathfrak{A}}_{(\iota, A)}, \beta_\iota, \omega, \mathbb{R}^d, \mathbb{Q}_b^d, \mathscr{K}^d\right)$$

where A is a C*-algebra and ι a family of block spin transformations. The corresponding subsystems with respect to a finite lattice are

$$\Lambda_n \quad = \quad \left(\underline{\mathfrak{A}}_{(n, A)}, \beta_n, \omega_n, \mathbb{R}^d, b^{-n^0}\mathbb{Z}^d, \mathscr{K}_n^d\right) \quad \prec \quad \Lambda$$

with $\omega_n = \omega \circ \iota_n$. We are now interested in the problem of *deforming* the theory Λ in such a way that one obtains a new one.

We assume that the C*-algebras $\mathfrak{A}_{(n, A)}$, $\mathfrak{A}_{(\iota, A)}$, are von Neumann algebras, acting on separable Hilbert spaces $\mathcal{H}_n$, $\mathcal{H}$, and the states $\omega_n = \langle \Omega_n, (\cdot)\Omega_n \rangle, \omega = \langle \Omega, (\cdot)\Omega \rangle$, are induced by a vector $\Omega_n \in \mathcal{H}_n, \Omega \in \mathcal{H}$, respectively, which are cyclic and separating for the corresponding algebras. In order to study perturbations of the state ω, ι, we introduce the notion *action*.

Definiton 3.1 : We denote by $\mathscr{B}_{(\omega)}\left(\mathbb{Z}^2, \mathfrak{A}_A'\right)$ the set of all sections

$$\mathbf{v} : \mathbb{Z}^2 \ni n \ \longmapsto \ \mathbf{v}_n \in \mathfrak{A}_{(n, A)}'$$

for which $\mathbf{v}_n > 0$ for each $n \in \mathbb{Z}^2$ and for which the state $\eta_{(\omega, \mathbf{v}, n)}$, given by

$$\langle \eta_{(\omega, \mathbf{v}, n)}, a \rangle \quad := \quad \mathbf{z}_{(\omega, \mathbf{v}, n)}^{-1} \langle \omega_n, \mathbf{v}_n \ a \rangle \ ,$$

is contained in $\mathfrak{S}_{(n, A)}$. The section $\mathbf{v}$ is called *action* and $\mathbf{z}_{(\omega, \mathbf{v}, n)} = \langle \omega_n, \mathbf{v}_n \rangle$ is called the *partition function* with respect to the triple $(\omega, \mathbf{v}, n)$.

In oder to introduce the notion effective action, we consider for each $n_0 \prec n$ the normal conditional expectation

$$\mathbf{e}_{(\omega, n_0, n)} : \mathfrak{A}_{(n, A)}' \ \longrightarrow \ \mathfrak{A}_{(n_0, A)}'$$

which is determined by the condition

$$\langle\,\omega_n, b\,\iota_{(n,n_0)}(a)\,\rangle \quad = \quad \langle\,\omega_{n_0}, \mathbf{e}_{(\omega,n_0,n)}(b)a\,\rangle$$

for each $b \in \mathfrak{A}'_{(n,A)}$ and for each $a \in \mathfrak{A}_{(n_0,A)}$.

For a given action $\mathbf{v}$ and for $k \in \mathbb{N}^2$ we get a further action by

$$\mathbf{e}^{(k)}_{(\omega)}(\mathbf{v})_n \quad := \quad \mathbf{e}_{(\omega,n,n+k)}(\mathbf{v}_{n+k})$$

and $\mathbf{e}^{(k)}_{(\omega)}(\mathbf{v})$ is called the *effective action* with respect to k and $\mathbf{v}$. Note that $\mathbf{e}^{(k)}_{(\omega)}$ is a convex linear map from $\mathscr{B}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ into $\mathscr{B}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$. To carry through our subsequent analysis we select an appropriate class of actions in $\mathscr{B}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$.

Definiton 3.2 : We denote by $\Gamma^o_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ the linear space of sections

$$\mathbf{f} : n \longmapsto \mathbf{f}_n \in \mathfrak{A}'_{(n,A)}$$

for which the semi-norms

$$[[\mathbf{f}]]_{(\omega,n)} \quad = \quad \sup_{k \in \mathbb{N}^2} \|\mathbf{e}_{(\omega,n,n+k)}(\mathbf{f}_{n+k})\|$$

are finite for each $n \in \mathbb{Z}^2$. The closure with respect to the corresponding Fréchet topology is denoted by $\Gamma_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$. Furthermore, we introduce the convex subset

$$\mathscr{A}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A) \quad := \quad \mathrm{cls}[\Gamma^o_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A) \cap \mathscr{B}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)] \quad .$$

Remark:

(1) Note that the norms may increase with n, i.e.

$$[[\mathbf{f}]]_{(\omega,n)} \quad \leq \quad [[\mathbf{f}]]_{(\omega,n_1)}$$

for $n \prec n_1$.

(2) The maps $\mathbf{e}^{(k)}_{(\omega)}$ are continuous maps. For a fixed cut-off $n \in \mathbb{Z}^2$ the operation of $\mathbf{e}^{(k)}_{(\omega)}$ corresponds to a substitution by the underlying lattice theory on $\Sigma_d(n)$ by a lattice theory, also defined on $\Sigma_d(n)$, arising from a lattice theory on $\Sigma_d(n+k)$ by integrating out the corresponding high energy degrees of freedom. An action $\mathbf{v}$ which is stable under $\mathbf{e}^{(k)}_{(\omega)}$ for every $k \in \mathbb{N}^2$ can be interpreted as a *continuum limit*. As we shall see below, this can be justified by the fact that then the section $\eta_{(\omega,\mathbf{v})}$ is contained in $\mathfrak{S}_{(\iota,A)}$, i.e.

$$\eta_{(\omega,\mathbf{v},n)} \circ \iota_{(n,n_0)} \quad = \quad \eta_{(\omega,\mathbf{v},n_0)} \quad .$$

In order to point out the structure of the space $\Gamma_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ and the cone $\mathscr{A}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$, we summarize some facts in the proposition below.

Proposition 3.3 :

(i) *For each $\mathbb{Z}^2$-invariant state $\xi \in \mathfrak{S}[\mathscr{C}_\mathrm{a}(\mathbb{Z}^2, \mathbb{C})]$ there exists a continuous linear map*

$$\mathbf{e}_{(\omega,\xi)} : \Gamma_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A) \longrightarrow \Gamma_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$$

such that for each $k \in \mathbb{N}^2$ the following holds true:

$$\mathbf{e}_{(\omega)}^{(k)} \circ \mathbf{e}_{(\omega,\xi)} = \mathbf{e}_{(\omega,\xi)} \ .$$

(ii) *For each $\mathbf{v} \in \mathscr{A}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ the state $\eta_{(\omega,\mathbf{e}_{(\omega,\xi)}(\mathbf{v}))}$ is contained in $\mathfrak{S}_{(\iota,A)}$.*

Proof.

(i) For each $\mathbf{f} \in \Gamma_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ and for each $n \in \mathbb{Z}^2$ we obtain a bounded family of operators $(\mathbf{e}_{(\omega,n,n+k)}(\mathbf{f}_{n+k}))_{k \in \mathbb{N}^2}$ in $\mathfrak{A}'_{(n,A)}$ since the semi-norm $[[\mathbf{f}]]_{(\omega,n)}$ is finite. For any bounded family $(\mathbf{w}_k)_{k \in \mathbb{N}^2}$ and for a given state $\xi \in \mathfrak{S}[\mathscr{C}_\mathrm{a}(\mathbb{Z}^2, \mathbb{C})]$ we define a continuous linear map on the pre-dual $(\mathfrak{A}'_{(n,A)})_*$ by

$$\mathbf{w}_\xi : \varphi \longmapsto \big\langle \xi, \mathbf{p}[k \mapsto \langle \varphi, \mathbf{w}_k \rangle] \big\rangle$$

and hence $\mathbf{w}_\xi \in \mathfrak{A}'_{(n,A)}$. We define the map $\mathbf{e}_{(\omega,\xi)}$ according to

$$\mathbf{e}_{(\omega,\xi)}(\mathbf{f})_n := \mathbf{e}_{(\omega,n,\xi)}(\mathbf{f})$$

where $\mathbf{e}_{(\omega,n,\xi)}(\mathbf{f})$ is given by

$$\mathbf{e}_{(\omega,n,\xi)}(\mathbf{f}) : \varphi \longmapsto \big\langle \xi, \mathbf{p}[k \mapsto \langle \varphi, \mathbf{e}_{(\omega,n,n+k)}(\mathbf{f}_{n+k}) \rangle] \big\rangle \ .$$

We have for each $a \in \mathfrak{A}_{(n,A)}$:

$$\big\langle \omega_n, \mathbf{e}_{(\omega,n,n+k_0)} \mathbf{e}_{(\omega,n+k_0,\xi)}(\mathbf{f}) a \big\rangle$$

$$= \big\langle \omega_{n+k_0}, \mathbf{e}_{(\omega,n+k_0,\xi)}(\mathbf{f}) \iota_{(n+k_0,n)}(a) \big\rangle$$

$$= \big\langle \xi, \mathbf{p}[k \mapsto \langle \omega_{n+k+k_0}, \mathbf{f}_{n+k+k_0} \iota_{(n+k+k_0,n)}(a) \rangle] \big\rangle$$

$$= \big\langle \xi, \mathbf{p}[k \mapsto \langle \omega_{n+k}, \mathbf{f}_{n+k} \iota_{(n+k,n)}(a) \rangle] \big\rangle$$

$$= \big\langle \omega_n, \mathbf{e}_{(\omega,n,\xi)}(\mathbf{f}) a \big\rangle$$

which yields

$$\mathbf{e}_{(\omega,n,n+k_0)} \mathbf{e}_{(\omega,n+k_0,\xi)}(\mathbf{f}) = \mathbf{e}_{(\omega,n,\xi)}(\mathbf{f}) \ .$$

Finally we conclude

$$[[\mathbf{e}_{(\omega,\xi)}(\mathbf{f})]]_{(\omega,n)} = \sup_{k \in \mathbb{N}^2} \|\mathbf{e}_{(\omega,n,n+k)} \mathbf{e}_{(\omega,n+k,\xi)}(\mathbf{f})\|$$

$$= \sup_{k \in \mathbb{N}^2} \|\mathbf{e}_{(\omega,n,n+k)} \mathbf{e}_{(\omega,n+k,\xi)}(\mathbf{f})\|$$

$$= \|\mathbf{e}_{(\omega,n,\xi)}(\mathbf{f})\| \leq [[\mathbf{f}]]_{(\omega,n)}$$

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which proves *(i)*.

(ii) Let $\mathbf{v} \in \Gamma_{\omega}^{o}(\mathbb{Z}^2, \mathfrak{A}'_A) \cap \mathscr{B}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ be an action. For each $n \in \mathbb{Z}^2$ the state $\eta_{(\omega, \mathbf{v}, n)}$ is reflexion positive and $b^{-n^o}\mathbb{Z}^d$-invariant. For each $n \in \mathbb{Z}^2$ and for each $a \in \mathfrak{A}_{(n, A)}$ we compute for an action $\mathbf{v}'$ which is stable under $\mathbf{e}_{(\omega)}^{(k)}$ for each $k \in \mathbb{N}^2$: $\mathbf{e}_{(\omega)}^{(k)}(\mathbf{v}') = \mathbf{v}'$:

$$\langle\, \omega_{n+k}, \mathbf{v}'_{n+k}\iota_{(n+k,n)}a\,\rangle \;\; = \;\; \langle\, \omega_n, \mathbf{e}_{(\omega, n, n+k)}(\mathbf{v}'_{n+k})a\,\rangle$$

$$= \;\; \langle\, \omega_n, \mathbf{v}'_n a\,\rangle$$

which yields for $a = 1$:

$$\mathbf{z}_{(\omega, \mathbf{v}', n+k)} \;\; = \;\; \langle\, \omega_n, \mathbf{e}_{(\omega, n, n+k)}(\mathbf{v}'_{n+k})a\,\rangle$$

$$= \;\; \mathbf{z}_{(\omega, \mathbf{v}', n)}$$

and therefore

$$\mathbf{E}\big[\xi' \otimes \eta_{(\omega, \mathbf{v}')}\big] \;\; = \;\; \eta_{(\omega, \mathbf{v}')}$$

for each ξ', and thus we conclude for $\mathbf{v}' = \mathbf{e}_{(\omega, \xi)}(\mathbf{v})$:

$$\mathbf{E}\big[\xi \otimes \eta_{(\omega, \mathbf{v})}\big] \;\; = \;\; \mathbf{E}\big[\xi' \otimes \eta_{(\omega, \mathbf{e}_{(\omega, \xi)}(\mathbf{v}))}\big]$$

$$= \;\; \eta_{(\omega, \mathbf{e}_{(\omega, \xi)}(\mathbf{v}))}$$

which implies $\eta_{(\omega, \mathbf{e}_{(\omega, \xi)}(\mathbf{v}))} \in \mathfrak{S}_{(\iota, A)}$.

Let $(\mathbf{v}_i)_{i \in I}$ be a net in $\Gamma_{\omega}^{o}(\mathbb{Z}^2, \mathfrak{A}'_A) \cap \mathscr{A}(\mathbb{Z}^2, \mathfrak{A}'_A)$ which converges to $\mathbf{v}$ in $\mathscr{A}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$. For each $n \in \mathbb{Z}^2$ and for each $a \in \mathfrak{A}_{(n, A)}$ the map

$$\mathbf{T}_{(a, \omega, n)} : \mathbf{f} \longmapsto \langle\, \omega_n, \mathbf{e}_{(\omega, n, \xi)}(\mathbf{f})a\,\rangle$$

is a *continuous* linear functional on $\Gamma_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ which follows directly from the estimate

$$\big|\langle\, \omega_n, \mathbf{e}_{(\omega, n, \xi)}(\mathbf{f})a\,\rangle\big| \;\; \leq \;\; \|\mathbf{e}_{(\omega, n, \xi)}(\mathbf{f})\|\, \|a\|$$

$$\leq \;\; [[\mathbf{f}]]_{(\omega, n)}\, \|a\| \;\; .$$

Therefore we have

$$\langle\, \omega_n, \mathbf{e}_{(\omega, n, \xi)}(\mathbf{v})\beta_{(n, g)}a\,\rangle \;\; = \;\; \langle\, \omega_n, \mathbf{e}_{(\omega, n, \xi)}\big(\lim_{i \in I}\mathbf{v}_i\big)\beta_{(n, g)}a\,\rangle$$

$$= \;\; \lim_{i \in I}\langle\, \omega_n, \mathbf{e}_{(\omega, n, \xi)}(\mathbf{v}_i)\beta_{(n, g)}a\,\rangle$$

$$= \;\; \lim_{i \in I}\langle\, \omega_n, \mathbf{e}_{(\omega, n, \xi)}(\mathbf{v}_i)a\,\rangle$$

$$= \;\; \langle\, \omega_n, \mathbf{e}_{(\omega, n, \xi)}(\mathbf{v})a\,\rangle$$

which proves the $\mathbb{Q}_b^d$-invariance of $\eta_{(\omega, \mathbf{e}_{(\omega, \xi)}(\mathbf{v}))}$. The reflexion positivity follows by an analogous argument.

■

We formulate one important consequence of the proposition above by the following corollary:

Corollary 3.4 : *For each action* $\mathbf{v} \in \mathscr{A}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ *and for each continuum limit* $\eta \in \mathfrak{S}_{(\iota, A)}[\eta_{(\omega, \mathbf{v})}]$ *the states*

$$\eta \circ \iota_n \in \mathfrak{S}_{(n, A)} \cap (\mathfrak{A}_{(n, A)})_*$$

are normal for each $n \in \mathbb{Z}^2$.

Proof. For each operator $a \in \mathfrak{A}_{(n, A)}$ we have for a continuum limit $\eta = \mathbf{E}[\xi \otimes \eta_{(\omega, \mathbf{v})}]$

$$\langle \mathbf{E}[\xi \otimes \eta_{(\omega, \mathbf{v})}], \iota_n(a) \rangle \quad = \quad \langle \eta_{(\omega, n)}, \mathbf{e}_{(\omega, n, \xi)}(\mathbf{v}) a \rangle$$

which proves the normality.

■

Remark:

(1) A given action $\mathbf{v} \in \mathscr{A}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_A)$ can be used to *deform* the given theory Λ, namely for each $\mathbb{Z}^2$-invariant state $\xi \in \mathfrak{S}[\mathscr{C}_a(\mathbb{Z}^2, \mathbb{C})]$ we obtain a new theory

$$\Lambda^{(\xi, \mathbf{v})} \quad := \quad (\underline{\mathfrak{A}}_{(\iota, A | \eta)}, \beta_\iota, \eta, \mathbb{R}^d, \mathbb{Q}^d_b, \mathscr{K}^d)$$

with $\eta = \eta_{(\omega, \mathbf{e}_{(\omega, \xi)}(\mathbf{v}))}$ and the net $\underline{\mathfrak{A}}_{(\iota, A | \eta)}$ is given by

$$\underline{\mathfrak{A}}_{(\iota, A | \eta)} : \mathscr{U} \longmapsto \mathfrak{A}_{(\iota, A)}(\mathscr{U}) / \mathfrak{J}_{(\iota, \eta)} \quad .$$

(2) If $\mathfrak{A}_{(\iota, A)}$ is a factor of type III, then the theories $\Lambda^{(\xi, \mathbf{v})}$ and Λ are inequivalent if and only if η is not normal on $\mathfrak{A}_{(\iota, A)}$.

(3) If ω is a gaussian state, the statement of Corollary 3.4 can be regarded as a weaken version of the *local Fock property* [13, 26]. Whereas the local Fock property states that the restriction of the deformed state η is normal on each local algebra, Corollary 3.4 states that one also have to restrict to operators which correspond to a high energy momentum cut-off.

(4) We claim here that a necessary condition for $\mathbf{v}$ such that the states ω and η are disjoint is that the supreme

$$\sup_{n \in \mathbb{Z}^2} \| \mathbf{v}_n \|_{\mathfrak{A}'_{(n, A)}} \quad = \quad \infty$$

is infinite.

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3.2 Block spin transformations: Concrete examples

Let M be a von Neumann algebra acting on a Hilbert space K and let Ω be a cyclic and separating vector for M. We consider the von Neumann algebra

$$\mathfrak{A}_{(n,M)} \quad := \quad \overline{\bigotimes_{\Delta \in \Sigma_d(n)} \{\Delta\} \times M}$$

acting on the Hilbert space

$$\mathcal{H}_{(n,K)} \quad := \quad \bigotimes_{\Delta \in \Sigma_d(n)} \{\Delta\} \times K \quad.$$

The vector

$$\Omega_n \quad := \quad \bigotimes_{\Delta \in \Sigma_d(n)} \{\Delta, \Omega\}$$

is cyclic and separating for $\mathfrak{A}_{(n,M)}$.

For each $n_0 \prec n$ we define for each cube $\Delta_0 \in \Sigma_d(n_0)$ the cube

$$*\Delta_0 \subset \Delta_{(n,n_0|\Delta_0)} \subset \Delta_0$$

in $\Sigma_d(n)$ which is determined by the condition to contain the dual one cube $*\Delta_0$ of Δ_0 (See Figure 2). A faithful normal *-homomorphism from $\mathfrak{A}_{(n_0,M)}$ into $\mathfrak{A}_{(n,M)}$ is given by

$$\iota_{(n,n_0)} \quad := \quad \bigotimes_{\Delta_0 \in \Sigma_d(n_0)} \left[\mathbf{1}_{D(n,n_0|\Delta_0)} \otimes \{\Delta_{(n,n_0|\Delta_0)}, \mathrm{id}\} \right]$$

with $\mathbf{1}_D := \otimes_{\Delta \in D} \{\Delta, \mathbf{1}\}$ for a subset $D \subset \Sigma_d(n)$. Here the set $D(n,n_0|\Delta_0)$ of hypercubes is

$$D(n,n_0|\Delta_0) \quad := \quad \{\Delta \subset \Delta_0 | \Delta \neq \Delta_{(n,n_0|\Delta_0)}\} \quad.$$

The following proposition follows directly from the definitions, given above.

Proposition 3.5 : *The family $\iota = (\iota_{(n,n_0)})_{n_0 \prec n}$ is a family of block spin transformations.*

For each $n \in \mathbb{Z}^2$ we consider the normal state $\omega_n := \langle \Omega_n, (\cdot)\Omega_n \rangle$. One easily verifies that the section $\omega : n \mapsto \omega_n$ satisfies the consistency condition with respect to ι, i.e.

$$\omega_n \circ \iota_{(n,n_0)} \quad = \quad \omega_{n_0}$$

and therefore $\omega = \mathbf{E}[\xi \otimes \omega]$ is a state on the C*-inductive limit $\mathfrak{A}_{(\iota,M)}$, independent of the choice of ξ. This yields a statistical mechanics

$$\Lambda \quad := \quad (\underline{\mathfrak{A}}_{(\iota,M)}, \beta_\iota, \omega, \mathbb{R}^d, \mathbb{Q}_b^d, \mathcal{K}^d)$$

which fulfills the axioms *WEF1* and *WEF2*.

For each pair $n_0 \prec n$ there is a normal conditional expectation

$$\mathbf{e}_{(\omega, n_0, n)}$$

$$:= \bigotimes_{\Delta_0 \in \Sigma_d(n_0)} \left[\left[\bigotimes_{\Delta \in D(n, n_0 | \Delta_0)} \{\Delta, \langle \Omega, (\cdot)\Omega \rangle\} \right] \otimes \{\Delta_{(n, n_0 | \Delta_0)}, \mathrm{id}\} \right]$$

which maps $\mathfrak{A}'_{(n, M)}$ into $\mathfrak{A}'_{(n_0, M)}$ and one easily computes for $b \in \mathfrak{A}'_{(n, M)}$ and for $a \in \mathfrak{A}_{(n_0, M)}$:

$$\langle \omega_n, b\, \iota_{(n, n_0)}(a) \rangle = \langle \omega_{n_0}, \mathbf{e}_{(\omega, n_0, n)}(b)a \rangle .$$

3.3 Construction of invariant reflexion positive states

We are now interested in the space $\mathscr{B}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_M)$ of actions in order to perform deformations of the theory Λ which we have introduced in the previous section.

Let $a \in M$ be an operator and let $\Delta \in \Sigma_d(n)$ be a cube, then we write $\Phi_n(\Delta, a)$ for the corresponding element in $\mathfrak{A}_{(n, M)}$. For each face $\Gamma \in \Sigma_{d-1}(n)$ there are unique cubes $\Delta_0, \Delta_1 \in \Sigma_d(n)$ such that $\Gamma = \Delta_0 \cap \Delta_1$. We write: $\Phi_n(\Gamma, a \otimes b) = \Phi_n(\Delta_0, a)\Phi_n(\Delta_1, b)$ for $a, b \in M$. Let $w = (w_n)_{n \in \mathbb{N}} \subset M' \otimes M'$ be a family of positive operators such that

$$[w_n \otimes \mathbf{1}, \mathbf{1} \otimes w_n] = 0$$

for each $n \in \mathbb{Z}^2$. Then we introduce for $n \in \mathbb{Z}^2$ the positive operator

$$\mathbf{v}[w]_n := \prod_{\Gamma \in \Sigma_{d-1}(n)} \Phi_n(\Gamma, w_n) \in \mathfrak{A}'_{(n, M)}$$

and we obtain a section

$$\mathbf{v}[w] \in \Gamma(\mathbb{Z}^2, \mathfrak{A}_{M'}) .$$

Proposition 3.6 : *Given a family $w = (w_n)_{n \in \mathbb{N}} \subset M' \otimes M'$ of positive operators such that*

$$[\mathbf{1} \otimes w, w \otimes \mathbf{1}] = 0 ,$$

then the section $\mathbf{v}[w]$ is an action contained in $\mathscr{B}_{(\omega)}(\mathbb{Z}^2, \mathfrak{A}'_M)$.

Proof. For each $n \in \mathbb{Z}^2$ it is obvious, that the state $\eta_{(\omega, \mathbf{v}[w], n)}$ is $b^{-n^0}\mathbb{Z}^d$ invariant. Let $\Sigma_{d-1}(n, \mu, 0)$ be the subset in $\Sigma_{d-1}(n)$ which consists of all faces Γ of cubes in $\Sigma_d(n, \mu, 0)$ which intersect the hyperplane Σ_{e_μ}. We define the sets

$$\Sigma_{d-1}(n, \mu, \pm) := \{\Gamma \in \Sigma_{d-1}(n) \backslash \Sigma_{d-1}(n, \mu, 0) | \Gamma \subset \mathbb{R}_{\pm} e_\mu + \Sigma_{e_\mu}\}$$

and $\mathbf{v}[w]_n$ can be decomposed as follows:

$$\mathbf{v}[w]_n := \mathbf{v}[w, 0]_n \mathbf{v}[w, +]_n \mathbf{v}[w, -]_n$$

where $\mathbf{v}[w,\ell]_n$, $\ell = 0,\pm$ is given by

$$\mathbf{v}[w,\ell]_n \quad := \quad \prod_{\Gamma \in \Sigma_{d-1}(n,\mu,\ell)} \Phi_n(\Gamma, w_n) \quad .$$

The operators $\mathbf{v}[w,\pm]_n$ are contained in $\mathfrak{A}_{(n,M')}(\mu,0) \,\overline{\otimes}\, \mathfrak{A}_{(n,M')}(\mu,+)$ and we conclude for an operator $a \in \mathfrak{A}_{(n,M)}(\mu,0) \,\overline{\otimes}\, \mathfrak{A}_{(n,M)}(\mu,+)$

$$\mathbf{v}[w]_n \, j_{(n,\mu)}(a)\, a \quad = \quad \mathbf{v}[w,0]_n \, j_{(n,\mu)}(\mathbf{v}[w,+]_n a)\, \mathbf{v}[w,+]_n a$$

$$= \quad \mathbf{v}[w,0]_n^{1/2} \, j_{(n,\mu)}(\mathbf{v}[w,+]_n a)\, \mathbf{v}[w,+]_n a\, \mathbf{v}[w,0]_n^{1/2} \quad .$$

where we have used the fact that

$$j_{(n,\mu)}(\mathbf{v}[w,+]_n) \quad = \quad \mathbf{v}[w,-]_n$$

$$[\mathbf{v}[w,0]_n, \mathbf{v}[w,\pm]_n] \quad = \quad 0 \quad .$$

We put for $\ell = 0,\pm$

$$\Omega_{(n,\ell)} \quad := \quad \bigotimes_{\Delta \in \Sigma_d(n,\mu,\ell)} \{\Delta, \Omega\}$$

and we consider the conditional expectation

$$\mathbf{E}_{(\omega,n,\mu)}$$

$$:= \quad \langle\, \Omega_{(n,+)}, (\cdot)\Omega_{(n,+)} \,\rangle \otimes \langle\, \Omega_{(n,-)}, (\cdot)\Omega_{(n,-)} \,\rangle \otimes \left[\bigotimes_{\Delta \in \Sigma_d(n,\mu,0)} \{\Delta, \mathrm{id}\} \right] \quad .$$

We compute for operators $a_\pm \in \mathfrak{A}_{(n,\mathfrak{B}(K))}(\mu,\pm)$ and $b_\pm \in \mathfrak{A}_{(n,\mathfrak{B}(K))}(\mu,0)$:

$$\mathbf{E}_{(\omega,n,\mu)}\big((a_- \otimes b_-)(a_+ \otimes b_+)\big)$$

$$= \quad b_- b_+ \langle\, \Omega_{(n,+)}, a_+\Omega_{(n,+)} \,\rangle\langle\, \Omega_{(n,-)}, a_-\Omega_{(n,-)} \,\rangle$$

$$= \quad b_- b_+ \mathbf{E}_{(\omega,n,\mu)}(a_-)\mathbf{E}_{(\omega,n,\mu)}(a_+)$$

$$= \quad \mathbf{E}_{(\omega,n,\mu)}(a_- \otimes b_-)\mathbf{E}_{(\omega,n,\mu)}(a_+ \otimes b_+)$$

which implies

$$\mathbf{E}_{(\omega,n,\mu)}\big(\, \mathbf{v}[w]_n \, j_{(n,\mu)}(a)\, a \,\big)$$

$$= \quad \mathbf{v}[w,0]_n^{1/2} \, \mathbf{E}_{(\omega,n,\mu)}\big(\, j_{(n,\mu)}(\mathbf{v}[w,+]_n a)\, \big)\, \mathbf{E}_{(\omega,n,\mu)}(\mathbf{v}[w,+]_n a)\, \mathbf{v}[w,0]_n^{1/2}$$

$$= \quad \mathbf{v}[w,0]_n^{1/2} \, \mathbf{E}_{(\omega,n,\mu)}\big(\, \mathbf{v}[w,+]_n a \,\big)^* \, \mathbf{E}_{(\omega,n,\mu)}(\mathbf{v}[w,+]_n a)\, \mathbf{v}[w,0]_n^{1/2} \quad .$$

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Here we have used the fact that $\mathbf{E}_{(\omega,n,\mu)}$ is invariant under the euclidean time reflexion $j_{(n,k)}$. We conclude for $\Psi_n := \mathbf{v}[w,0]_n^{1/2}\Omega_{(n,0)}$

$$\langle\,\eta_{(\omega,n)},\mathbf{v}[w]_n\,j_{(n,\mu)}(a)\,a\,\rangle$$

$$=\quad\langle\,\Omega_{(n,0)},\,\mathbf{E}_{(\omega,n,\mu)}(\,\mathbf{v}[w]_n\,j_{(n,\mu)}(a)\,a\,)\Omega_{(n,0)}\,\rangle$$

$$=\quad\langle\,\Psi_n,\mathbf{E}_{(\omega,n,\mu)}(\,\mathbf{v}[w,+]_na\,)^*\,\mathbf{E}_{(\omega,n,\mu)}(\mathbf{v}[w,+]_na)\,\Psi_n\,\rangle$$

$$\geq\quad 0$$

which proves the reflexion positivity.

∎

3.4   Multiplicative renormalization

The main problem which arises here is to check that the set $\mathscr{A}_{(\omega)}(\mathbb{Z}^2,\mathfrak{A}_{M'})$ contains interesting elements. Let $\mathbf{v}\in\mathscr{B}_{(\omega)}(\mathbb{Z}^2,\mathfrak{A}_{M'})$ be any action. According to what we claim in Section 3.1, one has to deal with the following behavior for the partition function, provided one requires that $\|\mathbf{v}\| := \sup_{n\in\mathbb{Z}^2}\|\mathbf{v}_n\| < \infty$:

$$\lim_{n\in\mathbb{Z}^2}\mathbf{z}_{(\omega,\mathbf{v},n)}\quad=\quad 0\ ,$$

in order to obtain a deformed theory $\Lambda^{(\xi,\mathbf{v})}$ which is not equivalent to the underlying one.

Furthermore, one expects that for each $n\in\mathbb{Z}^2$

$$\lim_{k\in\mathbb{N}^2}\mathbf{e}_{(\omega,n,n+k)}(\mathbf{v}_{n+k})\quad=\quad 0$$

which yields $\mathbf{e}_{(\omega,\xi)}(\mathbf{v}) = 0$. In order to get a non-trivial limit we replace $\mathbf{v}$ by

$$\mathbf{r}_\omega\,\mathbf{v}:n\mapsto\mathbf{z}_{(\omega,\mathbf{v},n)}^{-1}\mathbf{v}_n$$

This implies for $\mathbf{r}_\omega\,\mathbf{v}$

$$\mathbf{z}_{(\omega,\mathbf{r}_\omega\mathbf{v},n)}\quad=\quad 1$$

for each $n\in\mathbb{Z}^2$ and therefore

$$1\quad\leq\quad\sup_{k\in\mathbb{N}^2}\|\mathbf{e}_{(\omega,n,n+k)}(\mathbf{r}_\omega\,\mathbf{v}_{n+k})\|\quad=\quad[[\mathbf{r}_\omega\mathbf{v}]]_{(\omega,n)}$$

provided the right hand side is finite. The semi-norms of the resulting fix-points $\mathbf{e}_{(\omega,\xi)}(\mathbf{r}_\omega\,\mathbf{v})$ are bounded from below by 1. The operation $\mathbf{r}_\omega$ can be regarded as *multiplicative renormalization*. Therefore it is natural to call the condition $\mathbf{r}_\omega\,\mathbf{v}\in\mathscr{A}_{(\omega)}(\mathbb{Z}^2,\mathfrak{A}_{M'})$ *multiplicative renormalizability* of $\mathbf{v}$.

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We first illustrate the notion multiplicative renormalization by an ultra-local example. Let $\mathbf{v} : n \mapsto \mathbf{v}_n$ be a section of the form

$$\mathbf{v}_n \quad = \quad \prod_{\Delta \in \Sigma_d(n)} \Phi_n(\Delta, w_n)$$

with $w_n \in M'$ and $\|w_n\| = 1$ for each $n \in \mathbb{Z}^2$. Then we easily compute

$$\mathbf{z}_{(\omega,\mathbf{v},n)} \quad = \quad \langle\, \Omega, w_n \Omega \,\rangle^{\tau(n)}$$

$$\mathbf{e}_{(\omega,n,n+k)}(\mathbf{v}_{n+k}) \quad = \quad \prod_{\Delta \in \Sigma_d(n)} \Phi_{n+k}(\Delta, w_{n+k}) \langle\, \Omega, w_{n+k}\Omega \,\rangle^{\tau(n+k)-\tau(n)} \quad .$$

For $\limsup \langle\, \Omega, w_n\Omega \,\rangle < 1$ the partition function $\mathbf{z}_{(\omega,\mathbf{v},n+k)}$ vanishes for $k \to \infty$. On the other hand we have

$$\mathbf{z}_{(\omega,\mathbf{v},n+k)}^{-1} \|\mathbf{e}_{(\omega,n,n+k)}(\mathbf{v}_{n+k})\| \quad \leq \quad \langle\, \Omega, w_{n+k}\Omega \,\rangle^{-\tau(n)}$$

with $\tau(n) := b^{d(n^0 + n^1)}$. By choosing $w_n = w$ with $\langle\, \Omega, w\Omega \,\rangle = \gamma < 1$, for instance, we conclude

$$1 \quad < \quad [[\mathbf{r}_\omega \mathbf{v}]]_{(\omega,n)} \quad \leq \quad \gamma^{-\tau(n)} \quad < \quad \infty$$

and $\mathbf{v}$ is multiplicatively renormalizable. An example for a multiplicatively non-renormalizable action can be obtained by choosing $(w_n)_{n \in \mathbb{Z}^2}$ in such a way that $\lim_n \langle\, \Omega, w_n\Omega \,\rangle = 0$.

From the physical point of view, perturbation of ω by ultra local action is quite uninteresting since the corresponding theory in Minkowski space, provided it exists, is then nothing else but the constant field. In the subsequent, we discuss conditions under which a non-ultra local action is multiplicatively renormalizable.

Let $h \in \mathscr{C}^\infty(\mathbb{Z}^2 \times [0,1], M')$ be an operator-valued function which is smooth in its second variable and for which $[h(n,s_1), h(n,s_2)] = 0$ for each $s_1, s_2 \in [0,1]$ and for which $\|h(n,s)\| \leq 1$. We introduce the following numbers in $\mathbb{R}_+ \cup \{\infty\}$ associated with h:

$$I_{(\omega,n)}(h) \quad := \quad \inf_{s_1 \cdots s_{2d}} \langle\, \Omega, h(n,s_1) \cdots h(n,s_{2d})\Omega \,\rangle$$

$$S_{(\omega,n)}(h) \quad := \quad \sup_{s_1 \cdots s_{2d}} \langle\, \Omega, h(n,s_1) \cdots h(n,s_{2d})\Omega \,\rangle$$

$$R_{(\omega,n)}(h) \quad := \quad \sup_{k \in \mathbb{N}^2} \left(\frac{S_{(\omega,n+k)}(h)}{I_{(\omega,n+k)}(h)} \right)^{\tau(n+k)} S_{(\omega,n+k)}(h)^{-\tau(n)}$$

and we define an action by

$$\mathbf{v}[h]_n \quad := \quad \prod_{\Gamma \in \Sigma_{d-1}(n)} \Phi_n\left(\Gamma, \int_0^1 \mathrm{d}s\, h(n,s) \otimes h(n,s)\right)$$

$$= \quad \int \prod_{\Gamma \in \Sigma_{d-1}(n)} \mathrm{d}s(\Gamma) \bigotimes_{\Delta \in \Sigma_d(n)} \left\{ \Delta, \prod_{\Gamma \in \partial\Delta} h(n, s(\Gamma)) \right\} \quad .$$

The proposition, given above states a sufficient condition for h such that $\mathbf{v}[h]$ can multiplicatively by renormalized.

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Proposition 3.7 : *Let $h \in \mathscr{C}^\infty\big(\mathbb{Z}^2 \times [0,1], M'\big)$ be given. If $R_{(\omega,n)}(h) < \infty$ is finite for every n, then the action $\mathbf{v}[h]$ is multiplicatively renormalizable, i.e. $\mathbf{r}_\omega \mathbf{v}[h] \in \mathscr{A}_{(\omega)}\big(\mathbb{Z}^2, \mathfrak{A}_{M'}\big)$.*

Proof. Computing the partition function gives

$$
\mathbf{z}_{(\omega,\mathbf{v}[h],n)} \quad = \quad \int \prod_{\Gamma \in \Sigma_{d-1}(n)} \mathrm{d}s(\Gamma) \prod_{\Delta \in \Sigma_d(n)} \left\langle \omega, \prod_{\Gamma \in \partial\Delta} h(n, s(\Gamma)) \right\rangle
$$

and according to our assumption the partition function $\mathbf{z}_{(\omega,\mathbf{v}[h],n)}$ satisfies the inequality

$$
I_{(\omega,n)}(h)^{\tau(n)} \quad \leq \quad \mathbf{z}_{(\omega,\mathbf{v}[h],n)} \quad \leq \quad S_{(\omega,n)}(h)^{\tau(n)} \quad .
$$

and we compute

$$
\mathbf{e}_{(\omega,n,n+k)}\big(\mathbf{v}[h]_{n+k}\big) \quad = \quad \int \prod_{\Gamma \in \Sigma_{d-1}(n+k)} \mathrm{d}s(\Gamma) \bigotimes_{\Delta_0 \in \Sigma_d(n)} \Bigg[
$$

$$
\times \prod_{\Delta \in D(n,n+k|\Delta_0)} \left\langle \omega, \prod_{\Gamma \in \partial\Delta} h(n+k, s(\Gamma)) \right\rangle
$$

$$
\times \left\{ \Delta_0, \prod_{\Gamma \in \partial\Delta_{(n,n+k|\Delta_0)}} h(n+k, s(\Gamma)) \right\} \Bigg]
$$

which implies for the norm

$$
\big\| \mathbf{e}_{(\omega,n,n+k)}\big(\mathbf{v}[h]_{n+k}\big) \big\| \quad \leq \quad S_{(\omega,n+k)}(h)^{\tau(n+k)-\tau(n)} \quad .
$$

This yields

$$
\mathbf{z}_{(\omega,\mathbf{v}[h],n+k)}^{-1} \big\| \mathbf{e}_{(\omega,n,n+k)}\big(\mathbf{v}[h]_{n+k}\big) \big\|
$$

$$
\leq \quad I_{(\omega,n+k)}(h)^{-\tau(n+k)} S_{(\omega,n+k)}(h)^{\tau(n+k)-\tau(n)}
$$

$$
\leq \quad \left(\frac{S_{(\omega,n+k)}(h)}{I_{(\omega,n+k)}(h)} \right)^{\tau(n+k)} S_{(\omega,n+k)}(h)^{-\tau(n)}
$$

and we obtain for the semi-norms the estimate

$$
[[\, \mathbf{v}[h] \,]]_{(\omega,n)} \quad \leq \quad R_{(\omega,n)}(h)
$$

and the result follows.

∎

4 Weak euclidean field theory models

As already mentioned, the previous sections are not concerned with the regularity condition *WE3*. In Section 4.1 we present a procedure which, in comparison to building the C*-inductive limit, leads to a euclidean net of C*-algebras on which the full euclidean group acts by automorphisms. In particular we show that the translations act norm continuously.

Section 4.2 is concerned with states which fulfill all axioms for weak euclidean statistical mechanics. We show that *each* section of states $\eta \in \Gamma(\mathbb{Z}^2, \mathfrak{S}_A)$ can be associated with a family of weak euclidean statistical mechanics.

4.1 Construction of a weak euclidean net of C*-algebras

For a given C*-algebra A, we consider for each $n \in \mathbb{Z}^2$ the tensor algebra $\mathfrak{T}_{(n,A)} := T(S(\mathbb{R}^d) \otimes \mathfrak{A}_{(n,A)})$ over the linear space $S(\mathbb{R}^d) \otimes \mathfrak{A}_{(n,A)}$. For a region $\mathscr{U} \subset \mathbb{R}^d$ we denote by $\mathfrak{T}_{(n,A)}(\mathscr{U})$ the *-subalgebra in $\mathfrak{T}_{(n,A)}$ which is generated by operators $f \otimes a$ with $a \in \mathfrak{A}_{(n,A)}(\mathscr{U}_0)$ and $\mathrm{supp}(f) + \mathscr{U}_0 \subset \mathscr{U}$. For each $n \in \mathbb{Z}^2$ there is a group homomorphism

$$\tau_n \in \mathrm{Hom}(\mathrm{E}(d), \mathrm{Aut}(\mathfrak{T}_{(n,A)}))$$

which is defined by

$$\tau_{(n,g)}(f \otimes a) \quad := \quad (f \circ g^{-1}) \otimes a \ .$$

It is obvious that the euclidean group $\mathrm{E}(d)$ acts covariantly on the net $\mathfrak{T}_{(n,A)} : \mathscr{U} \mapsto \mathfrak{T}_{(n,A)}(\mathscr{U})$.

For each $n \in \mathbb{Z}^2$ we now introduce a *-homomorphism Φ_n which maps $\mathfrak{T}_{(n,A)}$ into the C*-algebra of bounded $\mathfrak{A}_{(n,A)}$-valued functions on $\mathbb{R}^d$. The *-homomorphism

$$\Phi_n : \mathfrak{T}_{(n,A)} \longrightarrow \mathscr{C}_{\mathrm{b}}(\mathbb{R}^d, \mathfrak{A}_{(n,A)})$$

is given by

$$\Phi_n(f \otimes a) : x \longmapsto b^{-dn^0} \sum_{x' \in b^{-n^0}\mathbb{Z}^d} f(x' - x)\, \beta_{(n,x')}(a) \ .$$

We introduce the C*-algebra

$$\mathfrak{B}_{(n,A)} \quad = \quad \mathrm{cls}[\Phi_n(\mathfrak{T}_{(n,A)})] \subset \mathscr{C}_{\mathrm{b}}(\mathbb{R}^d, \mathfrak{A}_{(n,A)})$$

The norm on $\mathfrak{B}_{(n,A)}$ is denoted by $\| \cdot \|_n$. There is a natural group homomorphism

$$\alpha_n \in \mathrm{Hom}(\mathrm{E}(d), \mathrm{Aut}\mathscr{C}_{\mathrm{b}}(\mathbb{R}^d, \mathfrak{A}_{(n,A)}))$$

which is given by

$$(\alpha_{(n,g)}\mathbf{a})(y) \quad := \quad \mathbf{a}(g^{-1}y)$$

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and the *-homomorphism Φ_n is euclidean covariant

$$\alpha_{(n,g)} \circ \Phi_n \quad = \quad \Phi_n \circ \tau_{(n,g)} \quad .$$

In particular we obtain for $x \in b^{-n^0}\mathbb{Z}^d \subset \mathrm{E}(d)$:

$$\alpha_{(n,x)}\Phi_n(f \otimes a) \quad := \quad \Phi_n(f \otimes \beta_{(n,x)}a) \quad .$$

Proposition 4.1 : *The translation group $\mathbb{R}^d$ acts norm continuously on $\mathfrak{B}_{(n,A)}$ via α_n.*

Proof. It is sufficient to test the continuity on the generators $\Phi_n(f \otimes a)$. We compute

$$\left\| \Phi_n(f \otimes a) - \alpha_{(n,x)}\Phi_n(f \otimes a) \right\|_n$$

$$= \quad \sup_{y \in \mathbb{R}^d} \left\| b^{-dn^0} \sum_{y' \in b^{-n^0}\mathbb{Z}^d} (f(y'-y) - f(y'-y-x))\beta_{(n,y')}a \right\|_{\mathfrak{A}_{(n,A)}}$$

$$\leq \quad b^{-dn^0} \sum_{y' \in b^{-n^0}\mathbb{Z}^d} \sup_{y \in \mathbb{R}^d} |f(y'-y) - f(y'-y-x)| \|a\|_{\mathfrak{A}_{(n,A)}}$$

and since $f \in S(\mathbb{R}^d)$ we conclude

$$\lim_{x \to 0} \sup_{y \in \mathbb{R}^d} |f(y'-y) - f(y'-y-x)| \quad = \quad 0$$

and therefore

$$\lim_{x \to 0} \left\| \Phi_n(f \otimes a) - \alpha_{(n,x)}\Phi_n(f \otimes a) \right\|_n \quad = \quad 0$$

which proves the proposition.

∎

Instead of the C*-inductive limit $\mathfrak{A}_{(\iota,A)}$, we consider another C*-algebra in order to build continuum limits. We define $\mathfrak{B}_{(\iota,A)}$ to be the C*-subalgebra in $\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2, \mathfrak{B}_A)$ which is generated by elements of the form

$$\Phi_{(\iota,n)}(f \otimes a) \quad := \quad \mathbf{p}[k \mapsto \Phi_{n+k}(f \otimes \iota_{(n+k,n)}(a))] \quad .$$

with $a \in \mathfrak{A}_{(n,A)}$ and $n \in \mathbb{Z}^2$. The notion of local algebras $\mathfrak{B}_{(\iota,A)}(\mathscr{U})$ is obvious. We obtain a euclidean net of C*-algebras $(\underline{\mathfrak{B}}_{(\iota,A)}, \alpha)$ where the net is given by

$$\underline{\mathfrak{B}}_{(\iota,A)} : \mathscr{U} \longmapsto \mathfrak{B}_{(\iota,A)}(\mathscr{U})$$

and the euclidean group acts on $\mathfrak{B}_{(\iota,A)}$ as follows:

$$\alpha_g \mathbf{p}[n \mapsto \mathbf{a}_n] \quad := \quad \mathbf{p}[n \mapsto \alpha_{(n,g)}\mathbf{a}_n] \quad .$$

As a consequence of Proposition 4.1 we get:

Corollary 4.2 : *The pair $(\underline{\mathfrak{B}}_{(\iota,A)}, \alpha)$, where $\underline{\mathfrak{B}}_{(\iota,A)}$ is the net*

$$\underline{\mathfrak{B}}_{(\iota,A)} : \mathscr{U} \longmapsto \mathfrak{B}_{(\iota,A)}(\mathscr{U})$$

is a euclidean net of C-algebras and the translation group acts norm continuously on $\mathfrak{B}_{(\iota,A)}$.*

4.2 On the regularity condition for continuum limits

We denote by $\hat{\mathfrak{S}}_{(\iota,A)}$ the set of states on $\mathfrak{B}_{(\iota,A)}$ such that the triple $(\mathfrak{B}_{(\iota,A)},\alpha,\omega)$ is a weak euclidean statistical mechanics, i.e. it fulfills the axioms *WE1* to *WE3*, given in the introduction.

Theorem 4.3 : *There is a canonical convex-linear map*

$$\mathbf{F} : \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2, \mathbb{C})] \otimes \Gamma(\mathbb{Z}^2, \mathfrak{S}_A) \longrightarrow \hat{\mathfrak{S}}_{(\iota,A)} .$$

Proof. For a state $\xi \in \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2, \mathbb{C})]$ and a section $\eta \in \Gamma(\mathbb{Z}^2, \mathfrak{S}_A)$ we define $\mathbf{F}[\xi \otimes \eta]$ by

$$\langle\, \mathbf{F}[\xi \otimes \eta], \mathbf{p}[n \mapsto \mathbf{a}_n]\,\rangle \quad := \quad \langle\, \xi, \mathbf{p}[\, n \mapsto \,\langle\, \eta_n, \mathbf{a}_n(0)\,\rangle\,]\,\rangle .$$

where $\mathbf{a}_n$ is contained in $\mathscr{C}_{\mathrm{b}}(\mathbb{R}^d, \mathfrak{A}_{(n,A)})$. It is obvious that $\mathbf{F}$ is convex linear. In order to prove the translation invariance, we consider the correlation function

$$\langle\, \mathbf{F}[\xi \otimes \eta]\,,\, \prod_{j=1}^{k} \Phi_{(\iota,n_j)}(f_j \otimes a_j)\,\rangle \;=\; \Big\langle\, \xi, \mathbf{p}\Big[\, n \mapsto \, b^{-dn^0} \sum_{x_1 \cdots x_k \in b^{-n^0}\,\mathbb{Z}^d}$$

$$\times\; f_1(x_1) \cdots f_k(x_k)\,\langle\, \eta_n, \beta_{(n,x_1)}\iota_{(n,n_1)}(a_1) \cdots \beta_{(n,x_k)}\iota_{(n,n_k)}(a_k)\,\rangle\,\Big]\,\Big\rangle$$

Since each η_n is $b^{-n^0}\mathbb{Z}^d$-invariant, we conclude for $x \in \mathbb{Q}_b^d$:

$$\left\langle\ \mathbf{F}[\xi \otimes \eta]\ ,\alpha_x\left[\prod_{j=1}^{k}\Phi_{(\iota,n_j)}(f_j \otimes a_j)\right]\ \right\rangle$$

$$= \ \left\langle\ \mathbf{F}[\xi \otimes \eta]\ ,\ \prod_{j=1}^{k}\Phi_{(\iota,n_j)}(\tau_x f_j \otimes a_j)\ \right\rangle$$

$$= \ \left\langle\ \xi, \mathbf{p}\left[n \mapsto\ b^{-dn^0}\sum_{x_1\cdots x_k \in b^{-n^0}\mathbb{Z}^d} f_1(x_1 - x)\cdots f_k(x_k - x)\right.\right.$$

$$\times\ \left.\left.\langle\ \eta_n, \beta_{(n,x_1)}\iota_{(n,n_1)}(a_1)\cdots\beta_{(n,x_k)}\iota_{(n,n_k)}(a_k)\ \rangle\right]\ \right\rangle$$

$$= \ \left\langle\ \xi, \mathbf{p}\left[n \mapsto\ b^{-dn^0}\sum_{x_1\cdots x_k \in b^{-n^0}\mathbb{Z}^d} f_1(x_1)\cdots f_k(x_k)\right.\right.$$

$$\times\ \left.\left.\langle\ \eta_n, \beta_{(n,x)}[\beta_{(n,x_1)}\iota_{(n,n_1)}(a_1)\cdots\beta_{(n,x_k)}\iota_{(n,n_k)}(a_k)]\ \rangle\right]\ \right\rangle$$

$$= \ \left\langle\ \xi, \mathbf{p}\left[n \mapsto\ b^{-dn^0}\sum_{x_1\cdots x_k \in b^{-n^0}\mathbb{Z}^d} f_1(x_1)\cdots f_k(x_k)\right.\right.$$

$$\times\ \left.\left.\langle\ \eta_n, \beta_{(n,x_1)}\iota_{(n,n_1)}(a_1)\cdots\beta_{(n,x_k)}\iota_{(n,n_k)}(a_k)\ \rangle\right]\ \right\rangle$$

$$= \ \left\langle\ \mathbf{F}[\xi \otimes \eta]\ ,\ \prod_{j=1}^{k}\Phi_{(\iota,n_j)}(f_j \otimes a_j)\ \right\rangle$$

which implies that $\mathbf{F}[\xi \otimes \eta]$ is invariant under the dense subgroup $\mathbb{Q}_b^d$. Since the translation group acts norm continuously on $\mathfrak{B}_{(\iota,A)}$ the states $\mathbf{F}[\xi \otimes \eta]$ are invariant under the full translation group $\mathbb{R}^d$. In particular, the map

$$x\ \longmapsto\ \langle\ \mathbf{F}[\xi \otimes \eta], \mathbf{a}\,\alpha_x(\mathbf{b})\,\mathbf{c}\ \rangle$$

is continuous for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{B}_{(\iota,A)}$. Hence we have proven *WE1* and *WE3*. Let $n \mapsto \mathbf{a}_n \in \mathfrak{B}_{(n,A)}$ be a representative of $\mathbf{a} = \mathbf{p}[n \mapsto \mathbf{a}_n]$. If $\mathbf{a}$ is localized in $\mathbb{R}_+ e_k + \Sigma_{e_k}$, then $\mathbf{a}_n(0)$ is contained in $\mathfrak{A}_{(n,A)}(\mu,0) \otimes \mathfrak{A}_{(n,A)}(\mu,+)$, for n large enough. This implies

$$\langle\ \mathbf{F}[\xi \otimes \eta], j_k(\mathbf{a})\,\mathbf{a}\ \rangle\ =\ \langle\ \xi, \mathbf{p}[n \mapsto \langle\ \eta_n, j_{(\mu,n)}(\mathbf{a}_n(0))\mathbf{a}_n(0)\ \rangle]\ \rangle$$

$$\geq\ 0$$

according to the reflexion positivity of the η_ns. Thus *WE2* follows and the triple $(\mathfrak{B}_{(\iota,A)}, \alpha, \mathbf{F}[\xi \otimes \eta])$ is a weak euclidean field.

∎

Remark: For each section η we introduce the set of continuum limits

$$\hat{\mathfrak{S}}_{(\iota,A)}[\eta] \quad := \quad \{\mathbf{F}[\xi \otimes \eta] \mid \xi \in \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2, \mathbb{C})]\} \subset \hat{\mathfrak{S}}_{(\iota,A)} \quad.$$

The best situation is present if the block spin transformations ι are arranged in such a way that the group $\mathbb{Q}_b^d$ acts norm continuously on $\mathfrak{A}_{(\iota,A)}$. In this case the investigation of the set of continuum limits $\hat{\mathfrak{S}}_{(\iota,A)}[\eta]$ on $\mathfrak{B}_{(\iota,A)}$ is equivalent to the investigation of the set of continuum limits $\mathfrak{S}_{(\iota,A)}[\eta]$ on the C*-inductive limit algebra $\mathfrak{A}_{(\iota,A)}$. Since then we conclude for the correlation function

$$\langle\ \mathbf{F}[\xi \otimes \eta]\ ,\ \prod_{j=1}^k \Phi_{(\iota,n_j)}(f_j \otimes a_j)\ \rangle$$

$$=\quad \int \mathrm{d}x_1 \cdots \mathrm{d}x_k \prod_{j=1}^k f_j(x_j) \left\langle\ \mathbf{E}[\xi \otimes \eta]\ ,\ \prod_{j=1}^k \beta_{(\iota,x_j)} \iota_{n_j}(a_j)\ \right\rangle$$

and in particular we obtain for a consistent section $\eta \in \mathfrak{S}_{(\iota,A)}$:

$$\langle\ \mathbf{F}[\xi \otimes \eta]\ ,\ \prod_{j=1}^k \Phi_{(\iota,n_j)}(f_j \otimes a_j)\ \rangle$$

$$=\quad \int \mathrm{d}x_1 \cdots \mathrm{d}x_k \prod_{j=1}^k f_j(x_j) \left\langle\ \eta,\ \prod_{j=1}^k \beta_{(\iota,x_j)} \iota_{n_j}(a_j)\ \right\rangle$$

which is independent of ξ.

5   Conclusion and outlook

Concluding remarks: Some of the basic ideas and concepts which are used in order to construct euclidean field theory models are generalized by using the setup of algebraic euclidean field theory. We have introduced the notions block spin transformations, action, and effective action within a general model independent framework.

As described in Section 3 and Section 4, in the C*-algebraic approach to euclidean field theory the concept of continuum limits for lattice field theories arises in a very natural manner. To each section $\eta \in \Gamma(\mathbb{Z}^2, \mathfrak{S}_A)$, which is a family of lattice field theory models (these models can be chosen on each lattice $\Sigma_d(n)$ independently from each other), there always exists the corresponding set $\mathfrak{S}_{(\iota,A)}[\eta]$ of continuum limits.

Therefore, our point of view leads to a well posed problem. In order to prove the existence of non-trivial (weak) euclidean field theory models, one has to study the properties of the set of continuum limits with respect to the properties of the corresponding section η.

Outlook: It would be desirable to study the continuum limits, which arise from lattice models with an action (see Section 4) of the form

$$\mathbf{v}[h]_n \quad := \quad \prod_{\Gamma \in \Sigma_{d-1}(n)} \Phi_n(\Gamma, w)\ ,$$

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in more detail. As already mentioned in the introduction, one of the questions, which we want to investigate, is the following:

Question: Which are sufficient conditions for the family of operators $w = (w_n)_{n \in \mathbb{Z}^2} \subset M' \overline{\otimes} M'$ such that the set of continuum limits $\mathfrak{S}_{(\iota,A)}[\eta]$

(1) contains only characters (in case of abelian C*-algebras)?

(2) contains only ultra local states?

(3) contains at least one state which is not ultra local?

The states $\varphi \in \mathfrak{S}_{(\iota,A)}[\eta]$ are weak limit points and labeled by states ξ on the corona algebra $\mathscr{C}_a(\mathbb{Z}^2, \mathbb{C})$. The states ξ are not explicitly given, namely its existence is related to the Hahn-Banach extension theorem and therefore it relies on Zorn's lemma, however. In order to conclude properties for the continuum limits one has to think about which type of statements one can prove. For instance, one can use operators in $\mathfrak{A}_{(n,M)}$ to test properties of the states η_n like bounds of correlation functions.

In order to decide whether case *(3)* is valid, we propose to compute correlations

$$\langle\, \mathbf{c}_{[\eta_{n+k} \circ \iota_{(n+k,n)}]}, \Phi_n(\Delta_1, a) \otimes \Phi_n(\Delta_2, a) \,\rangle$$

for an appropriate choice of the operator $a > 0$. Then one has to arrange each operator w_n in such a way that the bound

$$|\langle\, \mathbf{c}_{[\eta_{n+k} \circ \iota_{(n+k,n)}]}, \Phi_n(\Delta_1, a) \otimes \Phi_n(\Delta_2, a) \,\rangle| \;>\; c_{(n,\Delta_1,\Delta_2,a)}$$

is fulfilled with a positive constant $c_{(n,\Delta_1,\Delta_2,a)}$ which only depends on n the cubes (Δ_1, Δ_2) and the operator a. Within Appendix A, we discuss a strategy how to deal with this problem.

In Section 4 the notion of effective action for continuum limits is discussed. Let $(X, \mathscr{P}, \omega_o)$ be a measure space with σ-algebra $\mathscr{P}$ and we consider the von Neumann algebra $M = \mathscr{L}^\infty(X, \mathscr{P}, \omega_o)$ and the states $\omega_n := \otimes_{\Delta \in \Sigma_d(n)}\{\Delta, \omega_o\}$. Let $\varphi \in \mathfrak{S}_{(\iota,M)}[\eta]$ be a continuum limit for which the effective action $\mathbf{v}$ exists, i.e.

$$\langle\, \varphi, \iota_n a \,\rangle \;=\; \int \mathrm{d}\omega_n \, \mathbf{v}_n \, a \;.$$

Then $\mathbf{v}_n$ is a $\otimes_{\Delta \in \Sigma_d(n)} \mathscr{P}$-measurable function.

Let X be a smooth orientable manifold, let $\mathscr{P}$ be the σ-Borel algebra and let ω_o be a volume form on X, then one can ask for a criterion for the section η such that the effective action $\mathbf{v}$ is a section of smooth functions $\mathbf{v}_n$ on $X^{\Sigma_d(n)}$. Within a coordinate chart $(\phi^\sigma)_{\sigma=0,\cdots,p}$, at $u_o \in X$ one can perform a Taylor expansion of the effective action functional $\mathbf{s}_n = -\ln \mathbf{v}_n$ at $u_{(o,n)} : \Delta \mapsto u_o$

$$\mathbf{s}_n \;=\; \sum_{k=0}^{K} \frac{1}{k!} \sum_{(\Delta_j,\sigma_j)} \phi^{(\Delta_1,\sigma_1)} \cdots \phi^{(\Delta_k,\sigma_k)} \partial_{(\Delta_1,\sigma_1)} \cdots \partial_{(\Delta_k,\sigma_k)} \mathbf{s}_n(u_{(o,n)})$$

$$+\quad \text{reminder}$$

where $\left(\phi^{(\Delta,\sigma)}\right)_{\Delta\in\Sigma_d(n),\sigma=0,\cdots,p}$ is the coordinate chart of $X^{\Sigma_d(n)}$ induced by $(\phi^\sigma)_{\sigma=0,\cdots,p}$. The *free part* $\mathbf{v}^{(0)}$ of $\mathbf{v}$ can be defined by

$$\mathbf{v}_n^{(0)} \quad = \quad \exp[-\langle\, \phi, \mathbf{A}_n\phi \,\rangle]$$

where the quadratic form $\mathbf{A}_n$ is given by

$$\langle\, \phi, \mathbf{A}_n\phi \,\rangle \quad = \quad \frac{1}{2} \sum_{(\Delta_1,\sigma_1),(\Delta_2,\sigma_2)} \phi^{(\Delta_1,\sigma_1)}\phi^{(\Delta_2,\sigma_2)}\partial_{(\Delta_1,\sigma_1)}\partial_{(\Delta_2,\sigma_2)}\mathbf{s}_n(u_0) \quad.$$

Since the sum over the pairs $(\Delta_1,\sigma_1), (\Delta_2,\sigma_2)$ may also contain cubes (Δ_1,Δ_2) which are not next neighbors, we expect that in general $\mathbf{v}^{(0)}$ is not an action.

Nevertheless, it makes sense to study the section of gaussian states $\eta^{(0)}$, where $\eta^{(0)}$ is a state on $\mathfrak{A}_{(n,T_{u_o}M)}$ and $T_{u_o}M$ is the von Neumann algebra $\mathscr{L}^\infty(T_{u_o}X)$ of Lebesgue measurable functions on the tangent space $T_{u_o}X$ at u_o. If we assume that $\mathbf{A}_n$ is a positive quadratic form, then we obtain for the characteristic functional

$$\langle\, \eta_n^{(0)}, \exp(\phi(f)) \,\rangle \quad = \quad \exp(-\langle\, f, \mathbf{A}_n^{-1}f \,\rangle) \quad.$$

This implies that the continuum limits in $\mathfrak{S}_{(\iota,T_{u_o}M)}[\eta^{(0)}]$ are (mixtures of) gaussian states. We propose to compare the set of continuum limits $\mathfrak{S}_{(\iota,T_{u_o}M)}[\eta^{(0)}]$ with the set of continuum limits $\mathfrak{S}_{(\iota,M)}[\eta]$ of the underlying section η in order to decide whether there are states $\varphi \in \mathfrak{S}_{(\iota,M)}[\eta]$ which describe a physical system with interaction phenomena.

Acknowledgment:

A   Criterion for the existence of non ultra local continuum limits

We use the following notation: We choose a von Neumann algebra M acting on K and a cyclic and separating vector $\Omega \in K$ and ω is the state $\omega := \langle \Omega, (\cdot)\Omega \rangle$. For a positive operator $w \in M' \overline{\otimes} M'$ $\Delta_1, \Delta_2 \in \Sigma_d(n)$ we introduce a correlation functional on $M \overline{\otimes} M$ by

$$\langle \mathbf{c}^{(\Delta_1,\Delta_2)}_{(\omega,w,n)}, a_1 \otimes a_2 \rangle \quad := \quad \langle \eta_{(\omega,w,n)}, \Phi_n(\Delta_1, a_1)\Phi_n(\Delta_2, a_2) \rangle$$

$$- \quad \langle \eta_{(\omega,w,n)}, \Phi_n(\Delta_1, a_1) \rangle \langle \eta_{(\omega,w,n)}, \Phi_n(\Delta_2, a_2) \rangle$$

where $\eta_{(\omega,w,n)}$ is the state which is given by

$$\langle \eta_{(\omega,w,n)}, a \rangle \quad := \quad \mathbf{z}^{-1}_{(\omega,w,n)} \Big\langle \eta_{(\omega,n)}, \prod_{\Gamma \in \Sigma_{d-1}(n)} \Phi_n(\Gamma, w)\, a \Big\rangle .$$

We now introduce particular classes of positive operators in $M' \overline{\otimes} M'$.

Definiton A.1 : For a constant $2 > c > 0$ and cubes $\Delta_1, \Delta_2 \in \Sigma_d(n)$ and projections $P_1, P_2 \in \mathrm{Proj}(M)$ we define the set

$$\mathscr{P}^{[\Delta_1,\Delta_2;P_1,P_2]}_{(c,n)} \quad := \quad \Big\{ w \in M' \overline{\otimes} M' \Big|$$

$$w > 0 \ \text{ and } \ |\langle \mathbf{c}^{[\Delta_1,\Delta_2]}_{(\omega,w,n)}, P_1 \otimes P_2 \rangle| > c \Big\} .$$

Remark: For each translation $g \in b^{-n_0}\mathbb{Z}^d$ we obtain the identity

$$\mathscr{P}^{[g\Delta_1,g\Delta_2;P_1,P_2]}_{(c,n)} \quad = \quad \mathscr{P}^{[\Delta_1,\Delta_2;P_1,P_2]}_{(c,n)} .$$

Furthermore, we have $\mathscr{P}^{[\Delta_1,\Delta_2;P,\mathbf{1}]}_{(c,n)} = \emptyset$ for each projection $P \in \mathrm{Proj}(M)$.

One easily computes the relation

$$\langle \mathbf{c}^{(\Delta_1,\Delta_2)}_{(\omega,w,n)}, (\mathbf{1} - P_1) \otimes P_2 \rangle \quad = \quad -\langle \mathbf{c}^{(\Delta_1,\Delta_2)}_{(\omega,w,n)}, P_1 \otimes P_2 \rangle$$

and therefore

$$\langle \mathbf{c}^{(\Delta_1,\Delta_2)}_{(\omega,w,n)}, P_1 \otimes P_2 \rangle \quad = \quad \langle \mathbf{c}^{(\Delta_1,\Delta_2)}_{(\omega,w,n)}, (\mathbf{1} - P_1) \otimes (\mathbf{1} - P_2) \rangle$$

and we obtain the identity

$$\mathscr{P}^{[\Delta_1,\Delta_2;P_1,P_2]}_{(c,n)} \quad = \quad \mathscr{P}^{[\Delta_1,\Delta_2;\mathbf{1}-P_1,P_2]}_{(c,n)}$$

$$= \quad \mathscr{P}^{[\Delta_1,\Delta_2;\mathbf{1}-P_1,P_2]}_{(c,n)}$$

$$= \quad \mathscr{P}^{[\Delta_1,\Delta_2;\mathbf{1}-P_1,\mathbf{1}-P_2]}_{(c,n)}$$

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By using the block spin transformation introduced in Section 3.1 we obtain

$$\iota_{(n+k,n)}\Phi_n(\Delta, P) \quad = \quad \Phi_{n+k}(\Delta_{(n+k,n|\Delta)}, P)$$

and we may define for each $k \in \mathbb{N}^2$ the sets

$$\mathscr{P}^{[\Delta_1, \Delta_2; P_1, P_2]}_{(c, n+k, n)} \quad := \quad \mathscr{P}^{[\Delta_{(n+k,n|\Delta_1)}, \Delta_{(n+k,n|\Delta_2)}; P_1, P_2]}_{(c, n)} \quad .$$

According to Proposition 3.6, a section

$$w : k \; \longmapsto \; w_k \in \mathscr{P}^{[\Delta_1, \Delta_2; P_1, P_2]}_{(c, n+k, n)} \tag{5}$$

yields an action $\mathbf{v}[h]$ by

$$\mathbf{v}[w]_{n+k} \quad = \quad \prod_{\Gamma \in \Sigma_{d-1}(n+k)} \Phi_{n+k}(\Gamma, w_k)$$

and therefore a section of reflexion positive invariant states η.

Proposition A.2 : *Let w be a section, given by Equation (5) and let η be the corresponding section of reflexion positive invariant states. Then the set of continuum limits $\mathfrak{S}_{(\iota, A)}[\eta]$ contains at least one state which is not ultra local, i.e. case (3) is valid (see Introduction).*

Proof. According to the definition of $\mathscr{P}^{[\Delta_1, \Delta_2; P_1, P_2]}_{(c, n+k, n)}$ we obtain the bound

$$c \quad < \quad \left| \langle\, \mathbf{c}_{[\eta_{n+k} \circ \iota_{(n+k, n)}]}, \Phi_n(\Delta_1, P_1) \otimes \Phi_n(\Delta_2, P_2)\, \rangle \right| \quad .$$

We define the subset $\mathbb{X}_+ \subset \mathbb{Z}^2$ to consist of all $n_1 \in \mathbb{Z}^2$ such that $n \prec n_1$ and

$$c \quad < \quad \langle\, \mathbf{c}_{[\eta_{n_1} \circ \iota_{(n_1, n)}]}, \Phi_n(\Delta_1, P_1) \otimes \Phi_n(\Delta_2, P_2)\, \rangle$$

and write $\mathbb{X}_- := \mathbb{Z}^2 \backslash \mathbb{X}_+$. Then there exists a character

$$\xi \in \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{X}_+, \mathbb{C})] \cup \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{X}_-, \mathbb{C})] \subset \mathfrak{S}[\mathscr{C}_{\mathrm{a}}(\mathbb{Z}^2, \mathbb{C})]$$

which implies

$$c \quad < \quad \left| \langle\, \mathbf{c}_{[\mathbf{E}[\xi \otimes \eta]_n]}, \Phi_n(\Delta_1, P_1) \otimes \Phi_n(\Delta_2, P_2)\, \rangle \right|$$

and the state $\mathbf{E}[\xi \otimes \eta]$ is not ultra local.

∎

In order to prove the existence of non-ultra local continuum limits one has to check the assumption of the following corollary:

Corollary A.3 : *If for a projection $P \in \mathrm{Proj}(M) \backslash \{\mathbf{1}\}$ and for each pair of cubes $\Delta_1, \Delta_2 \in \Sigma_d(n)$ there exists a constant $c[\Delta_1, \Delta_2, P] \in [2, 0)$ such that*

$$\mathscr{P}^{[\Delta_1, \Delta_2; P, P]}_{(c[\Delta_1, \Delta_2, P], n)} \quad \neq \quad \emptyset \quad ,$$

then there exists a non ultra local state in $\mathfrak{S}_{(\iota, M)}$.

B Multiplicatively renormalizable actions: An example

We consider the von Neumann algebra $M = \mathscr{L}^\infty([0,1])$ and a family of positive functions $h \in \mathscr{C}^\infty([0,1]^2)^{\mathbb{Z}^2}$. The action of the model under consideration is given by

$$\mathbf{v}[h]_n(u) \quad := \quad \int \prod_{\Gamma \in \Sigma_{d-1}(n)} \mathrm{d}s(\Gamma) \prod_{\Delta \in \Sigma_d(n)} \prod_{\Gamma \in \partial\Delta} h_n(u(\Delta), s(\Gamma))$$

and we introduce the function

$$\mathbf{H}^{(\Delta, I)}_{(h,n)}(s) \quad := \quad \int_I \mathrm{d}u \prod_{\Gamma \in \partial\Delta} h_n(u, s(\Gamma))$$

Let $y_n \in \mathscr{C}^\infty(\mathbb{R})$ be a smooth positive function with $y_n(s) \geq 1$ for each s. We choose $h_n(u,s) := \exp(u y_n(s))$ and by setting

$$\mathbf{y}_{(n,\Delta)}(s) \quad := \quad \sum_{\Gamma \in \partial\Delta} y(s(\Gamma))$$

we obtain:

$$\mathbf{H}^{(\Delta, [u_0, u_1])}_{(h,n)}(s) \quad = \quad \mathbf{y}_{(n,\Delta)}(s)^{-1}[\exp(u_1 \mathbf{y}_{(n,\Delta)}(s)) - \exp(u_0 \mathbf{y}_{(n,\Delta)}(s))] \quad.$$

By introducing $\mathbf{q}_n := \sup_s \mathbf{y}_{(n,\Delta)}(s)$ and $\mathbf{r}_n := \inf_s \mathbf{y}_{(n,\Delta)}(s)$ we conclude:

$$S_{(\omega,n)}(h) \quad = \quad \mathbf{q}_n^{-1}(\exp(\mathbf{q}_n) - 1)$$

$$I_{(\omega,n)}(h) \quad = \quad \mathbf{r}_n^{-1}(\exp(\mathbf{r}_n) - 1) \quad.$$

and for each $k \in \mathbb{N}^2$ we get

$$\left[\frac{S_{(\omega,n+k)}(h)}{I_{(\omega,n+k)}(h)}\right]^{\tau(n+k)} S_{(\omega,n+k)}(h)^{-\tau(n)} \quad = \quad \left[\frac{\mathbf{r}_{n+k}}{\mathbf{q}_{n+k}}\right]^{\tau(n+k)}$$

$$\times \quad \left[\frac{\exp(\mathbf{q}_{n+k}) - 1}{\exp(\mathbf{r}_{n+k}) - 1}\right]^{\tau(n+k)}$$

$$\times \quad \mathbf{q}_{n+k}^{\tau(n)}(\exp(\mathbf{q}_{n+k}) - 1)^{-\tau(n)} \quad.$$

The action $\mathbf{v}[h]$ is multiplicatively renornmalizable if the values of $\mathbf{q}_n$ and $\mathbf{r}_n$ can be arranged in such a way that the following holds true:

(1) There exists a constant $\mathbf{c} > 1$ such that

$$\mathbf{c} := \lim_{n \in \mathbb{Z}^2} \mathbf{q}_n = \lim_{n \in \mathbb{Z}^2} \mathbf{r}_n \quad.$$

(2) The supreme

$$\mathbf{S}_n \quad := \quad \sup_{k \in \mathbb{N}^2}\left[\frac{\exp(\mathbf{q}_{n+k}) - 1}{\exp(\mathbf{r}_{n+k}) - 1}\right]^{\tau(n+k)}$$

is finite for each $n \in \mathbb{Z}^2$.

Then one easily computes

$$1 \;\leq\; [[\, \mathbf{r}_\omega \mathbf{v}[h]\,]]_n \;\leq\; \mathrm{const}.\, \mathbf{S}_n \mathbf{c}^{\tau(n)} \left(\exp(\mathbf{c}) - 1\right)^{-\tau(n)} \; .$$

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