Idempotent Mathematics and Interval Analysis

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Abstract

A brief introduction into Idempotent Mathematics and an idempotent version of Interval Analysis are presented. Some applications are discussed.

Key words: Idempotent Mathematics, Interval Analysis, idempotent semiring, idempotent linear algebra

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1 Introduction

Many problems in the optimization theory and other fields of mathematics appear to be linear over semirings with an idempotent addition (the so-called *idempotent superposition principle* [1] which is a natural analog of the wellknown superposition principle in Quantum Mechanics). The corresponding approach is developed systematically as *Idempotent Mathematics* or *Idempotent Analysis*, a branch of mathematics which has been growing vigorously last time (see, e.g., [1] – [7]).

Moreover, there exists a correspondence between interesting, useful, and important constructions and results concerning the field of real (or complex) numbers and similar constructions dealing with various idempotent semirings. This correspondence can be formulated in the spirit of the well known N. Bohr's correspondence principle in Quantum Mechanics [4]. We discuss idempotent analogs of some basic ideas, constructions, and results in traditional calculus and functional analysis; also, we show that the correspondence principle is a powerful heuristic tool to apply unexpected analogies and ideas borrowed from different areas of mathematics (see, e.g., [1] - [5]).

The theory is well advanced and includes, in particular, new integration theory, new linear algebra, spectral theory, and functional analysis. Its applications include various optimization problems such as multicriteria decision making, optimization on graphs, discrete optimization with a large parameter (asymptotic problems), optimal design of computer systems and computer media, optimal

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organization of parallel data processing, dynamic programming, applications to differential equations, numerical analysis, discrete event systems, computer science, discrete mathematics, mathematical logic, etc. (see, e.g., [2] - [8] and references therein).

In this paper we present an idempotent analog of *Interval Analysis* (see, e.g., [9], [10]). The idempotent interval (and, more generally, set-valued) arithmetics appears to be remarkably simpler than its traditional analog. We stress that this construction provides another example of heuristic power of the idempotent correspondence principle.

This paper is organized as follows. In section 2 we give a short heuristic introduction into Idempotent Mathematics. Section 3 contains definitions of basic concepts of idempotent arithmetics and several important examples. In sections 4-6 we consider the notion of linearity in Idempotent Analysis and discuss its computer applications. In section 7 we discuss a set-valued generalization of algebraic operations of Idempotent Analysis. The interval extension of an idempotent semiring, already considered in [11], is constructed in section 8. In section 9 we consider a simple application of this structure to linear algebra.

2 Dequantization and idempotent correspondence principle

Let \mathbb{R} be the field of real numbers and \mathbb{R}_+ be the subset of all non-negative numbers. Consider the following change of variable:

$$u \mapsto w = h \ln u$$

where $u \in \mathbb{R}_+ \setminus \{0\}$, h > 0; thus $u = e^{w/h}$, $w \in \mathbb{R}$. Denote by **0** the additional element $-\infty$ and by S the extended real line $\mathbb{R} \cup \{0\}$. The above change of variable has a natural extension D_h to the whole S by $D_h(0) = 0$; also, we denote $D_h(1) = 0 = 1$.

Denote by S_h the set S equipped with the two operations \oplus_h (generalized addition) and \odot_h (generalized multiplication) such that D_h is a homomorphism of $\{\mathbb{R}_+, +, \cdot\}$ to $\{S, \oplus_h, \odot_h\}$. This means that $D_h(u_1 + u_2) = D_h(u_1) \oplus_h D_h(u_2)$ and $D_h(u_1 \cdot u_2) = D_h(u_1) \odot_h D_h(u_2)$, i.e., $w_1 \odot_h w_2 = w_1 + w_2$ and $w_1 \oplus_h w_2 = h \ln(e^{w_1/h} + e^{w_2/h})$. It is easy to prove that $w_1 \oplus_h w_2 \to \max\{w_1, w_2\}$ as $h \to 0$.

Denote by \mathbb{R}_{\max} the set $S = \mathbb{R} \cup \{\mathbf{0}\}$ equipped with operations $\oplus = \max$ and $\odot = +$, where $\mathbf{0} = -\infty$, $\mathbf{1} = 0$ as above. Algebraic structures in \mathbb{R}_+ and S_h are isomorphic; therefore \mathbb{R}_{\max} is a result of a deformation of the structure in \mathbb{R}_+ .

We stress the obvious analogy with the quantization procedure, where h is the analog of the Planck constant. In these terms, \mathbb{R}_+ (or \mathbb{R}) plays the part of a 'quantum object' while \mathbb{R}_{\max} acts as a 'classical' or 'semiclassical' object that arises as the result of a *dequantization* of this quantum object.

Likewise, denote by \mathbb{R}_{\min} the set $\mathbb{R} \cup \{0\}$ equipped with operations $\oplus = \min$ and $\odot = +$, where $\mathbf{0} = +\infty$ and $\mathbf{1} = 0$. Clearly, the corresponding

dequantization procedure is generated by the change of variables $u \mapsto w = -h \ln u$.

Consider also the set $\mathbb{R} \cup \{0, 1\}$, where $0 = -\infty$, $1 = +\infty$, together with the operations $\oplus = \max$ and $\odot = \min$. Obviously, it can be obtained as a result of a 'second dequantization' of \mathbb{R} or \mathbb{R}_+ .

The algebras presented in this section are the most important examples of idempotent semirings (for the general definition see section 3), the central algebraic structure of Idempotent Analysis. The basic object of the traditional calculus is a *function* defined on some set X and taking its values in the field \mathbb{R} (or \mathbb{C}); its idempotent analog is a map $X \to S$, where X is some set and $S = \mathbb{R}_{\min}$, \mathbb{R}_{\max} , or another idempotent *semiring*. Let us show that redefinition of basic constructions of traditional calculus in terms of Idempotent Mathematics can yield interesting and nontrivial results (see, e.g., [1] – [6] for details and generalizations).

EXAMPLE 2.1. INTEGRATION AND MEASURES. To define an idempotent analog of the Riemann integral, consider a Riemann sum for a function $\varphi(x)$, $x \in X = [a, b]$, and substitute semiring operations \oplus and \odot for operations + and \cdot in its expression (for the sake of being definite, consider the semiring \mathbb{R}_{\max}):

$$\sum_{i=1}^{N} \varphi(x_i) \cdot \Delta_i \quad \mapsto \quad \bigoplus_{i=1}^{N} \varphi(x_i) \odot \Delta_i = \max_{i=1,\dots,N} \left(\varphi(x_i) + \Delta_i \right),$$

where $a = x_0 < x_1 < \ldots < x_N = b$, $\Delta_i = x_i - x_{i-1}$, $i = 1, \ldots, N$. As $\max_i \Delta_i \to 0$, the integral sum tends to

$$\int_X^{\oplus} \varphi(x) \, dx = \sup_{x \in X} \varphi(x)$$

for any function $\varphi \colon X \to \mathbb{R}_{\max}$ that is bounded. In general, the set function

$$m_{\varphi}(B) = \sup_{x \in B} \varphi(x), \quad B \subset X,$$

is called an \mathbb{R}_{\max} -measure on X; since $m_{\varphi}(\bigcup_{\alpha} B_{\alpha}) = \sup_{\alpha} m_{\varphi}(B_{\alpha})$, this measure is completely additive. An idempotent integral with respect to this measure is defined as

$$\int_X^{\oplus} \psi(x) \, dm_{\varphi} = \int_X^{\oplus} \psi(x) \odot \varphi(x) \, dx = \sup_{x \in X} \left(\psi(x) + \varphi(x) \right).$$

EXAMPLE 2.2. FOURIER-LEGENDRE TRANSFORM. Consider the topological group $G = \mathbb{R}^n$. The usual Fourier-Laplace transform is defined as

$$\varphi(x) \mapsto \widetilde{\varphi}(\xi) = \int_G e^{i\xi \cdot x} \varphi(x) \, dx$$

where $\exp(i\xi \cdot x)$ is a *character* of the group G, i.e., a solution of the following functional equation:

$$f(x+y) = f(x)f(y).$$

The idempotent analog of this equation is

$$f(x+y) = f(x) \odot f(y) = f(x) + f(y).$$

Hence 'idempotent characters' of the group G are linear functions of the form $x \mapsto \xi \cdot x = \xi_1 x_1 + \ldots + \xi_n x_n$. Thus the Fourier-Laplace transform turns into

$$arphi(x)\mapsto \widetilde{arphi}(\xi) = \int_G^\oplus \xi\cdot x\odot arphi(x)\,dx = \sup_{x\in G}\left(\xi\cdot x + arphi(x)
ight).$$

This equation differs from the well-known Legendre-Fenchel transform (see, e.g., [12]) in insignificant details.

These examples suggest the following formulation of the idempotent correspondence principle [4]:

There exists a heuristic correspondence between interesting, useful, and important constructions and results over the field of real (or complex) numbers and similar constructions and results over idempotent semirings in the spirit of N. Bohr's correspondence principle in Quantum Mechanics.

So Idempotent Mathematics can be treated as a "classical shadow (or counterpart)" of the traditional Mathematics over fields.

3 Idempotent semirings

Now we are prepared to give a number of general definitions. Consider a set S equipped with two algebraic operations: *addition* \oplus and *multiplication* \odot . The triple $\{S, \oplus, \odot\}$ is an *idempotent semiring* (i.s.r. for short) if it satisfies the following conditions (here and below, the symbol \star denotes any of the two operations \oplus, \odot):

- the addition \oplus and the multiplication \odot are associative: $x \star (y \star z) = (x \star y) \star z$ for all $x, y, z \in S$;
- the addition \oplus is *commutative*: $x \oplus y = y \oplus x$ for all $x, y \in S$;
- the addition \oplus is *idempotent*: $x \oplus x = x$ for all $x \in S$;
- the multiplication \odot is *distributive* with respect to the addition \oplus : $x \odot$ $(y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ for all $x, y, z \in S$.

A unity of an i.s.r. S is an element $\mathbf{1} \in S$ such that for all $x \in S$

$$\mathbf{1} \odot x = x \odot \mathbf{1} = x.$$

A zero of an i.s.r. S is an element $0 \in S$ such that $0 \neq 1$ and for all $x \in S$

$$\mathbf{0} \oplus x = x, \qquad \mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}.$$

It is readily seen that if an i.s.r. S contains a unity (a zero), then this unity (zero) is determined uniquely.

Note that different versions of this axiomatics are used, see, e.g., [2] - [7] and some literature indicated in these books and papers.

The addition \oplus defines on an i.s.r. *S* the partial order: $x \leq y$ iff $x \oplus y = y$. We use the notation $x \prec y$ if $x \leq y$ and $x \neq y$. If an i.s.r. *S* contains a zero **0**, then **0** is its least element with respect to the order \leq . The operations \oplus and \odot are consistent with the order \leq in the following sense: if $x \leq y$, then $x \star z \leq y \star z$ and $z \star x \leq z \star y$ for all $x, y, z \in S$.

An i.s.r. S is said to be *a*-complete if for any subset $\{x_{\alpha}\} \subset S$, including \emptyset , there exists a sum $\bigoplus \{x_{\alpha}\} = \bigoplus_{\alpha} x_{\alpha}$ such that $(\bigoplus_{\alpha} x_{\alpha}) \odot y = \bigoplus_{\alpha} (x_{\alpha} \odot y)$ and $y \odot (\bigoplus_{\alpha} x_{\alpha}) = \bigoplus_{\alpha} (y \odot x_{\alpha})$ for any $y \in S$. An i.s.r. S containing a zero **0** is said to be *b*-complete if the conditions of *a*-completeness are satisfied for any nonempty subset $\{x_{\alpha}\} \subset S$ that is bounded from above. See [5] for details.

Note that $\bigoplus_{\alpha} x_{\alpha} = \sup\{x_{\alpha}\}$; in particular, an *a*-complete i.s.r. always contains a zero $\mathbf{0} = \bigoplus \emptyset$.

An i.s.r. S does not contain zero divisors if $x \odot y = 0$ implies that x = 0 or y = 0 for all $x, y \in S$. An i.s.r. S is said to satisfy the cancellation condition if for all $x, y, z \in S$ such that $x \neq 0$ it follows from $x \odot y = x \odot z$ or $y \odot x = z \odot x$ that y = z. Obviously, an i.s.r. satisfying the cancellation property contains no zero divisors. In i.s.r. S is said to be *idempotent semifield* if every nonzero element of S is invertible; in this case the cancellation condition is valid.

An i.s.r. S is said to be algebraically closed if the equation $x^n = y$, where $x^n = x \odot x \odot \ldots \odot x$ (n times), has a solution for all $y \in S$ and $n \in \mathbb{N}$; an i.s.r. S with a unity **1** satisfies the stabilization condition if the sequence $x^n \oplus y$ stabilizes whenever $x \leq \mathbf{1}$ and $y \neq \mathbf{0}$ [13], [14].

The most important examples of i.s.r. are those considered in section 2. We see that \mathbb{R}_{\max} is a *b*-complete algebraically closed idempotent semifield (satisfying stabilization and cancellation conditions; it contains no zero divisors). The i.s.r. \mathbb{R}_{\min} is isomorphic to \mathbb{R}_{\max} . Note that both \mathbb{R}_{\max} and \mathbb{R}_{\min} are linearly ordered with respect to the corresponding addition operations; the order \preccurlyeq in \mathbb{R}_{\max} coincides with the usual linear order \leqslant and is opposite to the order \preccurlyeq in \mathbb{R}_{\min} .

Consider the set $\mathbb{R}_{\max} = \mathbb{R}_{\max} \cup \{\infty\}$ with operations \oplus , \odot extended by $\infty \oplus x = \infty$ for all $x \in \mathbb{R}_{\max}$, $\infty \odot x = \infty$ if $x \neq 0$ and $\infty \odot 0 = 0$. It is easily shown that this set is an *a*-complete i.s.r. and ∞ is its maximal element. In general, if an i.s.r. *S* is *b*-complete but not *a*-complete, then the i.s.r. obtained by adding to *S* a maximal element ∞ is *a*-complete.

Let $\{S_1, S_2, \ldots\}$ be a collection if i.s.r. There are several ways to construct a new i.s.r. derived from the semirings of this collection.

EXAMPLE 3.1. Suppose S is an i.s.r., X is an arbitrary set. The set $\mathcal{M}(X;S)$ of all functions $X \to S$ is an i.s.r. with respect to the following operations:

$$(f \oplus g)(x) = f(x) \oplus g(x), \quad (f \odot g)(x) = f(x) \odot g(x), \quad x \in X.$$

If S contains a zero 0 and/or a unity 1, then the functions o(x) = 0 for all $x \in X$, e(x) = 1 for all $x \in X$ are zero and unity of the i.s.r. $\mathcal{M}(X;S)$. It is

also possible to consider various subsemirings of the i.s.r. $\mathcal{M}(X;S)$.

EXAMPLE 3.2. Let S_i be semirings with operations \oplus_i , \odot_i and zeros $\mathbf{0}_i$, $i = 1, \ldots, n$. The set $S = (S_1 \setminus \{\mathbf{0}_1\}) \times \ldots \times (S_n \setminus \{\mathbf{0}_n\}) \cup \mathbf{0}$ is an i.s.r. with respect to the following operations: $x \star y = (x_1, \ldots, x_n) \star (y_1, \ldots, y_n) = (x_1 \star_1 y_1, \ldots, x_n \star_n y_n)$; the element $\mathbf{0}$ is a zero of this i.s.r.

Note, that the set $S = S_1 \times \cdots \times S_n$ is also an i.s.r. with respect to the same operations; in this case the element $(\mathbf{0}_1, \cdots, \mathbf{0}_n)$ is a zero of this i.s.r.

Notice that even if primitive semirings in examples 3.1 and 3.2 are linearly ordered sets with respect to the orders induced by the correspondent addition operations, the derived semirings are only partially ordered.

EXAMPLE 3.3. Let S be an i.s.r., $\operatorname{Mat}_{mn}(S)$ be a set of all S-valued matrices. Define the sum \oplus of matrices $A = ||a_{ij}||$, $B = ||b_{ij}|| \in \operatorname{Mat}_{mn}(S)$ as $A \oplus B = ||a_{ij} \oplus b_{ij}|| \in \operatorname{Mat}_{mn}(S)$, and let \preccurlyeq be the corresponding order on $\operatorname{Mat}_{mn}(S)$. The product of two matrices $A \in \operatorname{Mat}_{lm}(S)$ and $B \in \operatorname{Mat}_{mn}(S)$ is a matrix $AB \in \operatorname{Mat}_{ln}(S)$ such that $AB = || \bigoplus_{k=1}^{m} a_{ik} \odot b_{kj} ||$. Thus the set $\operatorname{Mat}_{nn}(S)$ of square matrices is an i.s.r. with respect to these operations. If **0** is the zero of S, then the matrix $O = ||o_{ij}||$, where $o_{ij} = \mathbf{0}$, is the zero of the i.s.r. $\operatorname{Mat}_{nn}(S)$; if **1** is the unity of S, then the matrix $E = ||\delta_{ij}||$, where $\delta_{ij} = \mathbf{1}$ if i = j and $\delta_{ij} = \mathbf{0}$ otherwise, is the unity of the i.s.r. $\operatorname{Mat}_{nn}(S)$.

Many additional examples can be found, e.g., in [2] - [7].

4 Idempotency and linearity

Now we discuss an idempotent analog of a linear space. A set V is called a *semimodule* over an i.s.r. S (or an S-semimodule) if it is equipped with an idempotent commutative associative addition operation \oplus_V and a multiplication $\odot_V : S \times V \to V$ satisfying the following conditions: for all $\lambda, \mu \in S, v, w \in V$

- $(\lambda \odot \mu) \odot_V v = \lambda \odot_V (\mu \odot_V v);$
- $\lambda \odot_V (v \oplus_V w) = (\lambda \odot_V v) \oplus_V (\lambda \odot_V w);$
- $(\lambda \oplus \mu) \odot_V v = (\lambda \odot_V v) \oplus_V (\mu \odot_V v).$

An S-semimodule V is called a semimodule with zero if S is an i.s.r. with zero $\mathbf{0} \in S$ and there exists a zero element $\mathbf{0}_V \in V$ such that for all $v \in V$, $\lambda \in S$

- $\mathbf{0} \oplus_V v = v;$
- $\lambda \odot_V \mathbf{0}_V = \mathbf{0} \odot_V v = \mathbf{0}_V$.

EXAMPLE 4.1. FINITELY GENERATED FREE SEMIMODULE. The simplest S-semimodule is the direct product $S^n = \{(a_1, \ldots, a_n) | a_j \in S, j = 1, \ldots, n\}$. The set of all endomorphisms $S^n \to S^n$ coincides with the semiring $\operatorname{Mat}_n(S)$ of all S-valued matrices (see example 3.3).

EXAMPLE 4.2. MATRIX SEMIMODULE. Take some $c \in S$, $A \in \operatorname{Mat}_{mn}(S)$. The product $c \odot A$ is defined as the matrix $||c \odot a_{ij}|| \in \operatorname{Mat}_{mn}(S)$. Then the set of all S-valued matrices of a given order $\operatorname{Mat}_{mn}(S)$ forms a semimodule under addition \oplus and multiplication by elements of S.

The theory of S-valued matrices is similar to the well-known Perron-Frobenius theory of nonnegative matrices (see, e.g., [3]). In particular, let S be an algebraically closed i.s.r. satisfying cancellation and stabilization conditions; then for any $A \in \operatorname{Mat}_{nn}(S)$ there exists a nontrivial subsemimodule $V \subset S^n$, or an *eigenspace*, and an element $\lambda \in S$, or an *eigenvalue*, such that $Av = \lambda \odot v$ for all $v \in V$; the eigenvalue is determined uniquely if A is irreducible [13], [14]. Similar results hold for semimodules of bounded or continuous functions [3].

EXAMPLE 4.3. FUNCTION SPACES. An idempotent function space $\mathcal{F}(X;S)$ is a subset of the set of all maps $X \to S$ such that if f(x), $g(x) \in \mathcal{F}(X;S)$ and $c \in S$, then $(f \oplus_{\mathcal{F}(X;S)} g)(x) = f(x) \oplus g(x) \in \mathcal{F}(X;S)$ and $(c \odot_{\mathcal{F}(X;S)} f)(x) =$ $c \odot f(x) \in \mathcal{F}(X;S)$; in other words, an idempotent function space is another example of an S-semimodule. If the semiring S contains a zero element **0** and $\mathcal{F}(X;S)$ contains the zero constant function $o(x) = \mathbf{0}$, then the function space $\mathcal{F}(X;S)$ has the structure of a semimodule with zero o(x) over the semiring S.

Recall that the idempotent addition defines a partial order in an i.s.r. S. An important example of an idempotent functional space is the space $\mathcal{B}(X;S)$ of all functions $X \to S$ bounded from above with respect to the partial order \preccurlyeq in S. There are many interesting spaces of this type including $\mathcal{C}(X;S)$ (a space of continuous functions defined on a topological space X), analogs of the Sobolev spaces, etc. (see, e.g., [2] – [5] for details).

According to the correspondence principle, many important concepts, ideas and results can be converted from usual functional analysis to Idempotent Analysis. For example, an idempotent scalar product in $\mathcal{B}(X;S)$ can be defined by the formula

$$\langle arphi,\psi
angle = \int_X^\oplus arphi(x)\odot\psi(x)\,dx,$$

where the integral is defined as the sup operation as in the example 2.1. Notice, however, that in the general case the ordering \preccurlyeq in S is not linear.

EXAMPLE 4.4. INTEGRAL OPERATORS. It is natural to construct idempotent analogs of *integral operators* of the form

$$K: \varphi(y) \mapsto (K\varphi)(x) = \int_Y^{\oplus} K(x,y) \odot \varphi(y) \, dy,$$

where $\varphi(y)$ is an element of a functional space $\mathcal{F}_1(Y;S)$, $(K\varphi)(x)$ belongs to a space $\mathcal{F}_2(X;S)$ and K(x,y) is a function $X \times Y \to S$. Such operators are gomomorphisms of the corresponding functional semimodules. If $S = \mathbb{R}_{\max}$, then this definition turns into the formula

$$(K\varphi)(x) = \sup_{y \in Y} (K(x, y) + \varphi(y)).$$

Formulas of this type are standard for optimization problems (see, e.g., [15]).

5 Superposition principle

In Quantum Mechanics the superposition principle means that the Schrödinger equation (which is basic for the theory) is linear. Similarly in Idempotent Mathematics the (idempotent) superposition principle means that some important and basic problem and equations (e.g. optimization problems, the Bellman equation and its versions and generalizations, the Hamilton-Jacobi equation) nonlinear in the usual sense can be treated as linear over appropriate idempotent semirings, see [1] - [4].

The linearity of the Hamilton-Jacobi equation over \mathbb{R}_{\min} and \mathbb{R}_{\max} can be deduced from the usual linearity (over \mathbb{C}) of the corresponding Schrödinger equation by means of the dequantization procedure described above (in Section 2). In this case the parameter h of this dequantization coincides with $i\hbar$, where \hbar is the Plank constant; so in this case \hbar must take imaginary values (because h > 0; see [5] for details). Of course, this is closely related to variational principles of mechanics.

The situation is similar for the differential Bellman equation, see [1], [3].

It is well-known that discrete versions of the Bellman equations can be treated as linear over appropriate idempotent semirings. The so-called *generalized stationary* (finite dimensional) *Bellman equation* has the form

$$X = A \odot X \oplus B,$$

where X, A, B are matrices with elements from an idempotent semiring and the corresponding matrix operations are described in Example 3.3 above; the matrices A and B are given (specified) and it is necessary to determine X from the equation.

B.A. Carre [16] used the idempotent linear algebra to show that different optimization problems for finite graphs can be formulated in a unified manner and reduced to solving these Bellman equations, i.e. systems of linear algebraic equations over idempotent semirings. For example, Bellman's method of solving shortest path problems corresponds to a version of the Jacobi method for solving systems of linear equations, whereas Ford's algorithm corresponds to a version of the Gauss-Seidel method.

6 Correspondence principle for computations

Of course, the (idempotent) correspondence principle is valid for algorithms as well as for their software and hardware implementations [4], [8]. Thus:

If we have an important and interesting numerical algorithm, then there is a good chance that its semiring analogs are important and interesting as well.

In particular, according to the superposition principle, analogs of linear algebra algorithms are especially important. Note that numerical algorithms for standard infinite-dimensional linear problems over i.s.r. (i.e. for problems related to idempotent integration, integral operators and transformations, the Hamilton-Jacobi and generalized Bellman equations) deal with the corresponding finite-dimensional (or finite) "linear approximations". Nonlinear algorithms often can be approximated by linear ones (and in this case idempotent linear algebra serves to solve problems nonlinear in the usual sense). Thus idempotent linear algebra is the basis of the idempotent numerical analysis.

Moreover, it is well-known that algorithms of linear algebra are convenient for parallel computations; their idempotent analogs admit parallelization as well. Thus we obtain a systematic way of applying parallel computation to optimization problems.

Basic algorithms of linear algebra (such as inner product of two vectors, matrix addition and multiplication, etc.) often do not depend on concrete semirings, as well as the nature of domains containing the elements of vectors and matrices. Thus it seems reasonable to develop *universal algorithms* that can deal equally well with initial data of different domains sharing the same basic structure [4], [8]; an example of such universal Gauss-Jordan elimination algorithm is found in [17].

Numerical algorithms are combinations of basic operations with 'numbers', which are elements of some numerical *domains* (e.g., real numbers, integers, etc). But every computer uses some finite *models* or finite representations of these domains. Discrepancies between 'ideal' numbers and their 'real' representations lead to calculation errors. This is another reason to deal with universal algorithms that allow to choose a concrete semiring and take into account the effects of its concrete finite representation in a systematic way; see [4], [8] for details and applications of the correspondence principle to hardware and software design.

7 Set-valued idempotent arithmetics

Due to imprecision of sources of input data in real-world problems, the data usually come in a form of intervals or other number sets rather than exact quantities. To deal with this problem in the context of optimization theory and Idempotent Analysis we develop a set-valued extension of idempotent arithmetics. In this sections we discuss several natural assumptions that this arithmetics is expected to satisfy.

Suppose S is an i.s.r. and S is a system of its subsets. We shall denote the elements of S by $\mathbf{x}, \mathbf{y}, \ldots$

Proposition 1 Suppose S satisfies the following conditions

- the system S contains all one-element subsets of S, i.e., the "exact values" of the elements of this i.s.r.;
- if $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ and \star is an algebraic operation in S, then there exists $\mathbf{z} \in \mathcal{S}$ such that $\mathbf{z} \supset \mathbf{x} \star \mathbf{y} = \{t \in S | (\exists x \in \mathbf{x}) (\exists y \in \mathbf{y}) t = x \star y\};$

if {x_α} is a subset of S such that ∩_α x_α ≠ Ø, then there exists the infimum of {x_α} in S with respect to the ordering ⊂, i.e., the set y ∈ S such that y ⊂ ∩_α x_α and z ⊂ y for any z ∈ S such that z ⊂ ∩_α x_α.

Let the algebraic operations $\overline{\oplus}$, $\overline{\odot}$ in S be defined as follows: if $\mathbf{x}, \mathbf{y} \in S$, then $\mathbf{x} \neq \mathbf{y}$ is the infimum of the set of all elements $\mathbf{z} \in S$ such that $\mathbf{z} \supset \mathbf{x} \star \mathbf{y}$. Then S is closed with respect to these operations and the element $\mathbf{x} \neq \mathbf{y}$ is optimal in the following sense: suppose the exact values of input variables \mathbf{x} and \mathbf{y} lie in sets \mathbf{x} and \mathbf{y} , respectively; then the result of an algebraic operation $\mathbf{x} \neq \mathbf{y}$ contains the quantity $\mathbf{x} \star \mathbf{y}$ and is the least subset of S in S with this property. In addition, the i.s.r. $\{S, \oplus, \odot\}$ is isomorphic to a subset of the algebra $\{S, \overline{\oplus}, \overline{\odot}\}$.

The proof is straightforward.

In general, not much can be said about the algebra $\{\mathcal{S}, \overline{\oplus}, \overline{\odot}\}$, as the following example shows.

EXAMPLE 7.1. Let $S = 2^S$ and $\mathbf{x} \neq \mathbf{y} = \mathbf{x} \neq \mathbf{y}$. In general, the set S with these 'naïve' operations $\overline{\oplus}$, $\overline{\odot}$ satisfies assumptions of proposition 1 but is not an i.s.r. Indeed, let S be the i.s.r. $(\mathbb{R}_{\max} \setminus \{\mathbf{0}\})^2 \cup \{(\mathbf{0}, \mathbf{0})\}$ with operations \oplus , $\overline{\odot}$ defined as in example 3.2. Consider a set $\mathbf{x} = \{(0, 1), (1, 0)\} \in S$; we see that $\mathbf{x} \oplus \mathbf{x} = \{(0, 1), (1, 0), (1, 1)\} \neq \mathbf{x}$ and if $\mathbf{y} = \{(1, 0)\}, \mathbf{z} = \{(0, 1)\},$ then $\mathbf{x} \odot (\mathbf{y} \oplus \mathbf{z}) = \{(1, 2), (2, 1)\} \neq (\mathbf{x} \odot \mathbf{y}) \oplus (\mathbf{x} \odot \mathbf{z}) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Thus S with operations $\overline{\oplus}$, $\overline{\odot}$ does not satisfy axioms of idempotency and distributivity.

It follows that \mathcal{S} should satisfy some additional conditions in order to have the structure of an i.s.r. In the next section we consider the case when \mathcal{S} is a set of all closed intervals; this case is of particular importance since it represents an idempotent analog of the traditional Interval Analysis.

8 Interval extensions of idempotent semirings

Let S be an i.s.r. A (closed) interval in S is a set of the form $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{t \in S | \underline{\mathbf{x}} \preccurlyeq t \preccurlyeq \overline{\mathbf{x}}\}$, where $\underline{\mathbf{x}}, \overline{\mathbf{x}} \in S$ ($\underline{\mathbf{x}} \preccurlyeq \overline{\mathbf{x}}$) are said to be *lower* and *upper* bounds of the interval \mathbf{x} , respectively.

Note that if \mathbf{x} and \mathbf{y} are intervals in S, then $\mathbf{x} \subset \mathbf{y}$ iff $\underline{\mathbf{y}} \preccurlyeq \mathbf{x} \preccurlyeq \mathbf{\overline{x}} \preccurlyeq \mathbf{\overline{y}}$. In particular, $\mathbf{x} = \mathbf{y}$ iff $\mathbf{\underline{x}} = \mathbf{y}$ and $\mathbf{\overline{x}} = \mathbf{\overline{y}}$.

Let \mathbf{x} , \mathbf{y} be intervals in S. In general, the set $\mathbf{x} \star \mathbf{y}$ is not an interval in S. Indeed, consider a set $S = \{\mathbf{0}, a, b, c, d\}$ and let \oplus be defined by the following order relation: $\mathbf{0}$ is the least element, d is the greatest element, and a, b, and c are noncomparable with each other. If \odot is a zero multiplication, i.e., if $\mathbf{x} \odot \mathbf{y} = \mathbf{0}$ for all $x, y \in S$, then S is an i.s.r. without a unity. Let $\mathbf{x} = [\mathbf{0}, a]$ and $\mathbf{y} = [\mathbf{0}, b]$; thus $\mathbf{x} \star \mathbf{y} = \{\mathbf{0}, a, b, d\}$, which is not an interval because it does not contain c although $\mathbf{0} \leq c \leq d$.

Proposition 2 Suppose S is an i.s.r. and \mathbf{x} , \mathbf{y} are intervals in S; then the interval $[\mathbf{x} \star \mathbf{y}, \mathbf{\overline{x}} \star \mathbf{\overline{y}}]$ contains the set $\mathbf{x} \star \mathbf{y}$ and is contained in every other interval containing this set. If S is an a-complete (b-complete) i.s.r. and $\{\mathbf{x}_{\alpha}\}$

is a nonempty set of intervals in S (a nonempty set of intervals in S such that the set $\{\overline{\mathbf{x}}_{\alpha}\}$ is bounded), then the interval $[\bigoplus_{\alpha} \underline{\mathbf{x}}_{\alpha}, \bigoplus_{\alpha} \overline{\mathbf{x}}_{\alpha}]$ contains the set $N = \{t \in S | (\forall \alpha) (\exists x_{\alpha} \in \mathbf{x}_{\alpha}) t = \bigoplus_{\alpha} x_{\alpha}\}$ and is contained in every other interval containing N.

PROOF. Take $t \in \mathbf{x} \star \mathbf{y}$ and let $x \in \mathbf{x}$, $y \in \mathbf{y}$ be such that $t = x \star y$. By definition of interval, it follows that $\underline{\mathbf{x}} \preccurlyeq x \preccurlyeq \overline{\mathbf{x}}$ and $\underline{\mathbf{y}} \preccurlyeq y \preccurlyeq \overline{\mathbf{y}}$. Since the operation \star is consistent with the order \preccurlyeq , we see that $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \preccurlyeq x \star y \preccurlyeq \overline{\mathbf{x}} \star \overline{\mathbf{y}}$. In particular, $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \preccurlyeq \overline{\mathbf{x}} \star \overline{\mathbf{y}}$, i.e., the interval $[\underline{\mathbf{x}} \star \underline{\mathbf{y}}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}]$ is properly defined. It follows that $\mathbf{x} \star \mathbf{y} \subseteq [\underline{\mathbf{x}} \star \underline{\mathbf{y}}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}]$.

Now let an interval \mathbf{z} in S be such that $\mathbf{x} \star \mathbf{y} \subset \mathbf{z}$. We have $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \in \mathbf{x} \star \mathbf{y} \subset \mathbf{z}$ and $\overline{\mathbf{x}} \star \overline{\mathbf{y}} \in \mathbf{x} \star \mathbf{y} \subset \mathbf{z}$; hence $\underline{\mathbf{z}} \preccurlyeq \underline{\mathbf{x}} \star \underline{\mathbf{y}}$ and $\overline{\mathbf{x}} \star \overline{\mathbf{y}} \star \overline{\mathbf{z}}$. Since $\underline{\mathbf{x}} \star \underline{\mathbf{y}} \preccurlyeq \overline{\mathbf{x}} \star \overline{\mathbf{y}}$ by the above, it follows that $[\underline{\mathbf{x}} \star \underline{\mathbf{y}}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}] \subset \mathbf{z}$.

The statement concerning an *a*-complete i.s.r. is proved similarly. If *S* is a *b*-complete semiring and the set $\{\overline{\mathbf{x}}_{\alpha}\}$ is bounded from above, then the set $\{\underline{\mathbf{x}}_{\alpha}\}$ is also bounded from above, i.e., there exists $y \in S$ such that $\underline{\mathbf{x}}_{\alpha} \preccurlyeq \overline{\mathbf{x}}_{\alpha} \preccurlyeq y$ for all α . Thus there exist $\bigoplus_{\alpha} \underline{\mathbf{x}}_{\alpha} \in S$ and $\bigoplus_{\alpha} \overline{\mathbf{x}}_{\alpha} \in S$. Now the obvious adaptation of the above argument completes the proof.

We see that $\mathbf{x} \overline{\star} \mathbf{y} = [\mathbf{x} \star \mathbf{y}, \overline{\mathbf{x}} \star \overline{\mathbf{y}}]$.

Consider the set I(S) of all intervals in S that do not contain **0**.

Proposition 3 Suppose i.s.r. S either does not contain a zero or has a zero **0** but does not contain zero divisors; then the set I(S) is an i.s.r. with respect to the operations $\overline{\oplus}$ and $\overline{\odot}$ without a zero element. If S contains a unity **1**, then the interval $[\mathbf{1},\mathbf{1}]$ is the unity of i.s.r. I(S). If S is an a-complete i.s.r. (b-complete i.s.r.) and $\{\mathbf{x}_{\alpha}\}$ is a nonempty subset of I(S) (a nonempty subset of S such that the set $\{\overline{\mathbf{x}}_{\alpha}\}$ is bounded from above), then its sum $\overline{\bigoplus}_{\alpha} \mathbf{x}_{\alpha} \in I(S)$ and

$$\mathbf{y}\overline{\odot}\left(\overline{\bigoplus}_{\alpha}\mathbf{x}_{\alpha}\right) = \overline{\bigoplus}_{\alpha}\left(\mathbf{y}\overline{\odot}\mathbf{x}_{\alpha}\right), \quad \left(\overline{\bigoplus}_{\alpha}\mathbf{x}_{\alpha}\right)\overline{\odot}\mathbf{y} = \overline{\bigoplus}_{\alpha}\left(\mathbf{x}_{\alpha}\overline{\odot}\mathbf{y}\right)$$

for any $\mathbf{y} \in I(S)$.

PROOF. From proposition 2 it follows that if $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \in I(S)$ and $\mathbf{y} = [\underline{\mathbf{y}}, \overline{\mathbf{y}}] \in I(S)$, then $\mathbf{x} \cdot \overline{\mathbf{x}} = [\underline{\mathbf{x}} \cdot \mathbf{y}, \overline{\mathbf{x}} \cdot \overline{\mathbf{y}}]$. Let us check that $\mathbf{x} \cdot \overline{\mathbf{x}} \in I(S)$.

Indeed, this is obvious if S contains no zero element. Assume S has the zero **0**. First we shall show that $\mathbf{x} \oplus \mathbf{y} \in I(S)$. Clearly, an interval $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ in S belongs to I(S) iff $\underline{\mathbf{x}} \neq \mathbf{0}$; hence it is sufficient to check that if $\underline{\mathbf{x}} \neq \mathbf{0}$ and $\underline{\mathbf{y}} \neq \mathbf{0}$, then $\underline{\mathbf{x}} \oplus \underline{\mathbf{y}} \neq \mathbf{0}$. But $x \oplus y = \mathbf{0}$ iff $x = \mathbf{0}$ and $y = \mathbf{0}$; thus I(S) is closed with respect to the addition $\overline{\oplus}$.

Now let us check that $\mathbf{x} \overline{\odot} \mathbf{y} \in I(S)$; to do this, it is sufficient to show that if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{x} \odot \mathbf{y} \neq \mathbf{0}$. But this is so because S has no zero divisors. The reader will have no difficulty in showing that the operations $\overline{\oplus}$ and $\overline{\odot}$ are associative, the addition $\overline{\oplus}$ is commutative and each element of I(S) is

 \odot are associative, the addition \oplus is commutative and each element of I(S) is idempotent with respect to it, and the multiplication $\overline{\odot}$ is distributive. The proof is by direct calculation based on proposition 2.

Notice that the idempotent addition $\overline{\oplus}$ determines a partial order $\overline{\preccurlyeq}$ on I(S). Let us check that I(S) contains no zero element. The only nontrivial case is when the i.s.r. S has a least element l but l is not a zero. We see that I(S) has the least element [l, l] with respect to the order $\overline{\preccurlyeq}$. Since $l \neq \mathbf{0}$, there exists $x \in S$ such that $x \odot l \neq l$ or $l \odot x \neq l$. Hence $[x, x] \odot [l, l] \neq [l, l]$ or $[l, l] \odot [x, x] \neq [l, l]$. This means that [l, l] cannot be a zero element of I(S). Let us remember that if a zero exists, then it is the least element of the i.s.r.; thus I(S) has no zero element.

It follows easily from proposition 2 that if $1 \in S$, then the interval [1, 1] is the unity element of I(S).

The statements concerning *a*-complete and *b*-complete i.s.r.'s are proved by direct calculation based on proposition 2. \Box

Corollary-Theorem 1 Suppose an i.s.r. S has a zero element **0** but contains no zero divisors and denote the set $I(S) \cup [\mathbf{0}, \mathbf{0}]$ by $\mathbf{I}(S)$. Then the set $\mathbf{I}(S)$ is an i.s.r. with the zero element $[\mathbf{0}, \mathbf{0}]$ containing no zero divisors with respect to the operations $\overline{\oplus}$, $\overline{\odot}$. If S is an a-complete i.s.r. (b-complete i.s.r.), then $\mathbf{I}(S)$ is an a-complete i.s.r. (b-complete i.s.r.).

The proof is obvious.

Proposition 4 If an i.s.r. S without zero divisors satisfies the cancellation condition (the stabilization condition), then the i.s.r. I(S) satisfies the cancellation condition (the stabilization condition). If S is an i.s.r. with commutative multiplication $\overline{\odot}$.

Proofs of commutativity of the multiplication $\overline{\odot}$ and of the cancellation condition follow easily from proposition 2. Suppose S satisfies the stabilization condition. By definition of $\mathbf{I}(S)$, $\mathbf{x} \neq [\mathbf{0}, \mathbf{0}]$ iff $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{\overline{x}} \neq \mathbf{0}$. Thus it follows from proposition 2 that the stabilization holds in S for both bounds of the involved intervals and hence for the whole intervals as elements of I(S).

Corollary 2 If the *i.s.r.* S has a zero element, then the above proposition is true for the *i.s.r.* I(S).

Proposition 5 If S is an i.s.r. without zero divisors that satisfies the conditions of cancellation and algebraic completeness, then the i.s.r. I(S) satisfies the same conditions.

PROOF. By proposition 4, it follows that the i.s.r. I(S) satisfies the cancellation condition. Suppose $\mathbf{x}^n = \mathbf{x} \odot \mathbf{x} \odot \ldots \odot \mathbf{x} = \mathbf{y}$. By proposition 2, we see that $\underline{\mathbf{x}}^n = \underline{\mathbf{y}}$ and $\overline{\mathbf{x}}^n = \overline{\mathbf{y}}$. Let $\underline{\mathbf{z}} \in S$ and $\overline{\mathbf{z}} \in S$ be the solutions of these two equations. We claim that $\underline{\mathbf{z}}$ and $\overline{\mathbf{z}}$ can be chosen such that $\underline{\mathbf{z}} \preccurlyeq \overline{\mathbf{z}}$, i.e., the interval $[\underline{\mathbf{z}}, \overline{\mathbf{z}}]$ is well defined.

Take $\overline{\mathbf{z}}' = \underline{\mathbf{z}} \oplus \overline{\mathbf{z}}$; hence $\underline{\mathbf{z}} \preccurlyeq \overline{\mathbf{z}}'$. Since S satisfies the cancellation property, it follows that $\overline{\mathbf{z}}'^n = (\underline{\mathbf{z}} \oplus \overline{\mathbf{z}})^n = \underline{\mathbf{z}}^n \oplus \overline{\mathbf{z}}^n$ (see, e.g., [14, assertion 2.1]). Thus $\overline{\mathbf{z}}'^n = \mathbf{y} \oplus \overline{\mathbf{y}} = \overline{\mathbf{y}}$. We see that $[\underline{\mathbf{z}}, \overline{\mathbf{z}}']^n = [\mathbf{y}, \overline{\mathbf{y}}] = \mathbf{y}$.

Corollary 3 Is the i.s.r. S contains a zero, then the same statement holds for the i.s.r. I(S).

Suppose an i.s.r. S has no zero element; then the map $\iota: S \to I(S)$ defined by $\iota(x) = [x, x]$ is an isomorphic imbedding of the i.s.r. S into the i.s.r. I(S). Suppose an i.s.r. S contains a zero **0** and the map $\iota_0: S \to \mathbf{I}(S)$ is such that $\iota_0(\mathbf{0}) = [\mathbf{0}, \mathbf{0}]$ and ι_0 coincides with $\iota: S \setminus \mathbf{0} \to I(S) \subset \mathbf{I}(S)$ on $S \setminus \mathbf{0}$; then the map ι_0 is an isomorphic imbedding of the i.s.r. S into the i.s.r. $\mathbf{I}(S)$. In the sequel, we will identify the i.s.r. S with the subsemiring $\iota(S) \subset I(S)$ or $\iota_0(S) \subset \mathbf{I}(S)$ and denote the operations in I(S) of $\mathbf{I}(S)$ by \oplus , \odot . If the i.s.r. S contains a unity **1**, then we denote the unity $[\mathbf{1}, \mathbf{1}] \in I(S)$ by **1**; also, denote $[\mathbf{0}, \mathbf{0}] \in \mathbf{I}(S)$ by **0**.

We stress that in the traditional Interval Analysis multiplication of intervals is not distributive with respect to interval addition. On the contrary, in idempotent interval mathematics most of algebraic properties of an i.s.r. are conserved in its interval extension. On the other hand, even if S is an idempotent semifield, then the set I(S) is only an i.s.r. satisfying the cancellation condition (but not a semifield).

Note also that it is monotonicity of operations \oplus , \odot and positivity of all elements of an i.s.r. with respect to the ordering \preccurlyeq that makes the idempotent interval arithmetics so simple. In general, idempotent interval analysis appears to be best suited for treating the problems with order-preserving transformations. An important instance of this kind of transformations is the linear operator in a semimodule over an i.s.r.

9 Application to linear algebra

Suppose S is an i.s.r. with zero **0** and unity **1** and I(S) is its interval extension. Thus $Mat_{nn}(I(S))$ is an i.s.r. We stress that that in the traditional Interval Analysis the set of all interval matrices is not even a semigroup with respect to matrix multiplication; indeed, the operation of matrix multiplication is not associative since the operation of interval multiplication is not distributive.

Recall that the matrix $\mathbf{A} = \|\mathbf{a}_{ij}\| \in \operatorname{Mat}_{nn}(\mathbf{I}(S))$ is said to be *irreducible* (see [13], [14]) if $\mathbf{b}_{ij} \neq \mathbf{0}$ for all i, j such that $1 \leq i \leq j \leq n$, where $\mathbf{B} = \|\mathbf{b}_{ij}\| = \bigoplus_{m=1}^{n} \mathbf{A}^{m}$.

If $\mathbf{A} = ||\mathbf{a}_{ij}|| \in \operatorname{Mat}_{mn}(\mathbf{I}(S))$, then the matrices $\underline{\mathbf{A}} = ||\mathbf{a}_{ij}||$ and $\overline{\mathbf{A}} = ||\overline{\mathbf{a}_{ij}}||$ are called *lower* and *upper matrices* of the *interval matrix* $\overline{\mathbf{A}}$. The unity of the i.s.r. $\operatorname{Mat}_{nn}(\mathbf{I}(S))$ is denoted by \mathbf{E} .

It follows easily from [14, theorem 6.2] that

Proposition 6 If an i.s.r. S with commutative multiplication \odot is algebraically closed and satisfies cancellation and stabilization conditions, then for any matrix $\mathbf{A} \in \operatorname{Mat}_{nn}(\mathbf{I}(S))$ there exist an 'eigenvector' $V \in \operatorname{Mat}_{n1}(\mathbf{I}(S))$ and an 'eigenvalue' $[\underline{\lambda}, \overline{\lambda}] \in \mathbf{I}(S)$ such that $\mathbf{AV} = [\underline{\lambda}, \overline{\lambda}] \odot \mathbf{V}$. If the matrix \mathbf{A} is irreducible, then the 'eigenvalue' $[\underline{\lambda}, \overline{\lambda}]$ is determined uniquely.

It follows from proposition 2 that addition and multiplication of interval matrices are reduced to separate addition and multiplication of their lower and upper matrices. Hence $\underline{\mathbf{A}} \ \underline{\mathbf{V}} = \underline{\lambda} \odot \underline{\mathbf{V}}$ and $\overline{\mathbf{A}} \ \overline{\mathbf{V}} = \overline{\lambda} \odot \overline{\mathbf{V}}$.

Consider the following *discrete stationary Bellman equation* (see Section 5 above), which plays an important part in optimization theory:

$$\mathbf{X} = \mathbf{A}\mathbf{X} \oplus \mathbf{B},\tag{1}$$

where $\mathbf{A} \in \operatorname{Mat}_{nn}(\mathbf{I}(S))$, \mathbf{B} , $\mathbf{X} \in \operatorname{Mat}_{nl}(\mathbf{I}(S))$. Consider the following iterative process:

$$\mathbf{X}_{k+1} = \mathbf{A}\mathbf{X}_k \oplus \mathbf{B} = \mathbf{A}^{k+1}\mathbf{X}_0 \oplus \left(\bigoplus_{l=0}^k \mathbf{A}^l\right)\mathbf{B},$$
 (2)

where $\mathbf{X}_k \in \operatorname{Mat}_{nl}(\mathbf{I}(S))$, $k = 0, 1, \dots$ If S is an a-complete i.s.r., then for any matrix **A** there exists a *closure matrix* $\mathbf{A}^* = \bigoplus_{l=1}^{\infty} \mathbf{A}^l$ and $\mathbf{X} = \mathbf{A}^* \mathbf{B}$ satisfies (1).

Proposition 7 If an a-complete i.s.r. S satisfies the assumptions of proposition 6 and $\mathbf{A} \in \operatorname{Mat}_{nn}(\mathbf{I}(S))$ is an irreducible matrix with the 'eigenvalue' $[\underline{\lambda}, \overline{\lambda}] \leq \mathbf{1}$, then the sequence $\{\mathbf{X}_k\}$ stabilizes to the solution $\mathbf{X} = \mathbf{A}^* \mathbf{B}$ of equation (1) for all k > n whenever $\mathbf{X}_0 \leq \mathbf{X}$.

PROOF. It follows from proposition 2 that it is sufficient to prove that sequences of lower and upper matrices of $\{\mathbf{X}_k\}$ stabilize separately.

Consider the sequence of upper matrices $\overline{\mathbf{X}}_k$. We have

$$\overline{\mathbf{X}}_{k+1} = \overline{\mathbf{A}}\overline{\mathbf{X}}_k \oplus \overline{\mathbf{B}} = \overline{\mathbf{A}}^{k+1}\overline{\mathbf{X}}_0 \oplus \left(\bigoplus_{l=0}^k \overline{\mathbf{A}}^l \right) \overline{\mathbf{B}},$$

where the eigenvalue of the matrix $\overline{\mathbf{A}}$ is $\overline{\lambda} \leq \mathbf{1}$ and $\overline{\mathbf{X}}_0 \leq \overline{\mathbf{A}}^* \overline{\mathbf{B}}$. Suppose that there exist $l \in \mathbb{N}$, $1 \leq l \leq n$, and a cycle $\{i_1, \ldots, i_l\} \subset \mathbb{N}$ such that $1 \leq i_1, \ldots, i_l \leq n, i_r \neq i_s$ if $r \neq s$, and $\mathbf{1} \prec \mathbf{1} \oplus P$, where the cycle invariant Pis defined as $\overline{\mathbf{a}_{i_1 i_2}} \odot \ldots \overline{\mathbf{a}_{i_{l-1} i_l}} \odot \overline{\mathbf{a}_{i_l i_1}}$. Using the formula (4) of [14] to express the greatest eigenvalue of the matrix $\overline{\mathbf{A}}$, we obtain $\mathbf{1} \prec \mathbf{1} \oplus P \preccurlyeq \mathbf{1} \oplus \overline{\lambda}^{\varphi(n)}$, where $\varphi(n)$ is the least common multiple of $1, 2, \ldots, n$. On the other hand, since $\overline{\lambda} \preccurlyeq \mathbf{1}$, we see that $\overline{\lambda}^{\varphi(n)} = \overline{\lambda}^{\varphi(n)-1} \odot \overline{\lambda} \preccurlyeq \overline{\lambda}^{\varphi(n)-1} \preccurlyeq \ldots \preccurlyeq \overline{\lambda} \preccurlyeq \mathbf{1}$, that is $\mathbf{1} \oplus \overline{\lambda}^{\varphi(n)} = \mathbf{1}$. This contradiction proves that for any cycle $P \preccurlyeq \mathbf{1}$. This means that the matrix $\overline{\mathbf{A}}$ is semi-definite in the sense of Carré [16]. Thus theorem 6.2 of [16] implies that for any $\overline{\mathbf{X}}_0$ such that $\overline{\mathbf{X}}_0 \preccurlyeq \overline{\mathbf{A}}^* \overline{\mathbf{B}}$ the sequence $\{\overline{\mathbf{X}}_k\}$ stabilizes after at most n iterations.

Continuing this line of reasoning, we see that $\{\underline{\mathbf{X}}_k\}$ also stabilizes after n steps since $\underline{\lambda} \leq \overline{\lambda} \leq \mathbf{1}$.

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