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Vienna, Preprint ESI 642 (1998)

December 15, 1998

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at

On the embedding of von Neumann subalgebras

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Abstract:
For a von Neumann algebra with a cyclic and separating vector it will be shown that the von Neumann subalgebras with the same cyclic vector can uniquely be characterized by one–parametric operator–valued functions obeying a set of conditions. Since the properties contain no reference to the subalgebra these operator–valued functions will be called characteristic functions. On the set of characteristic functions there exists a natural topology under which this set is complete.

1. Introduction

Quantum field theory of local observables in the sense of Araki, Haag and Kastler [Ha92] is concerned with von Neumann algebras $\mathcal{M}(\mathcal{O})$ associated with bounded open regions $\mathcal{O}$. In addition one needs states fulfilling some requirements in order to describe physical situations. Here we are dealing with the vacuum state. If in the vacuum representation the theory fulfils the nuclearity condition of Buchholz and Wichmann [BW86] then, by a result of Buchholz, D'Antoni and Fredenhagen [BDF87], all the local algebras $\mathcal{M}(\mathcal{O})$ are of the same von Neumann–Connes type III_1. Hence one local algebra alone does not contain any information about the structure of the theory except one is dealing with local quantum physics. Therefore, one has to look at several local algebras at the same time in order to obtain informations about the underlying physical structure of the theory. The simplest case is the situation of two local algebras. Here we want to treat the algebras of two regions where one is a subset of the other. The isotony requirement of the theory implies that we are dealing with two von Neumann algebras $\mathcal{N} \subset \mathcal{M}$. In addition by the Reeh–Schlieder theorem [RSch61] both algebras have many common cyclic and separating vectors and we will choose one of them and call it Ω.

It turns out that every subalgebra $\mathcal{N} \subset \mathcal{M}$ with the same cyclic and separating vector can uniquely be characterized by an operator–valued function $D(t)$ with certain properties. This will be shown in section 2. In the characterization of these functions there is no reference to the von Neumann subalgebra $\mathcal{N}$. Therefore, these functions will be called characteristic functions. The set of these functions is in one to one correspondence with the subalgebras $\mathcal{N} \subset \mathcal{M}$ which have Ω as cyclic vector. The properties of the characteristic functions are such that there exists a natural topology on this set. It will be shown in section 3 that the set of characteristic functions is complete in this topology. In section 4 two examples will be given in order to get a better understanding of the topology introduced on the set of characteristic functions. Some problems connected with the concepts develloped in the second section will be listed at the end. The investigations of this note are based on the Tomita–Takesaki theory [To67],[Ta70]. (See also [BR79] or [KR86].)

2. Characteristic functions

In this and the following sections we deal with a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector $\Omega \in \mathcal{H}$. We are interested in von Neumann subalgebras $\mathcal{N} \subset \mathcal{M}$ for which the vector Ω is also cyclic. The modular operator and the modular conjugation of the pairs $\{\mathcal{M}, \Omega\}$ and $\{\mathcal{N}, \Omega\}$ will be denoted by $\{\Delta_{\mathcal{M}}, J_{\mathcal{M}}\}$ and $\{\Delta_{\mathcal{N}}, J_{\mathcal{N}}\}$ respectively. Most of the objects we are dealing with depend also on the vector Ω. But since we keep the Hilbert space $\mathcal{H}$ and the vector Ω fixed we will suppress the index Ω in all our notations.

First we introduce some quantities which are needed for the investigation.

2.1 Definition:

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with a cyclic and separating vector Ω.
1. By $\mathcal{S}ub(\mathcal{M})$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ which have Ω as cyclic vector.
2. For $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$ we denote by $D_{\mathcal{M},\mathcal{N}}(t)$ the function $t \to \mathcal{B}(\mathcal{H})$ defined by

$$D_{\mathcal{M},\mathcal{N}}(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t} \Delta_{\mathcal{N}}^{\mathrm{i}t}. \tag{2.1}$$

3. We define the strip $S(a,b),\ a < b$ as

$$S(a,b) = \{\tau \in \mathbb{C}; a < \Im m\, \tau < b\}.$$

4. The action of the modular group $\mathrm{Ad}\,\Delta^{\mathrm{i}t}$ will be denoted by σ^t.

First we derive for $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$ some properties of the function $D_{\mathcal{M},\mathcal{N}}(t)$ which will be crucial for the coming investigation.

2.2 Lemma:

Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω and let $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$. Then the function $D(t) := D_{\mathcal{M},\mathcal{N}}(t)$ defined in Eq. (2.1) has the following properties:

(1) *$D(t)$ is unitary and strongly continuous in t. Moreover $D(0) = \mathbb{1}$.*

(2) *$D(t)\Omega = \Omega$, for all $t \in \mathbb{R}$.*
(3) *$D(t)$ has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$ and has strongly continuous boundary values at $\Im m\, t = 0$ and $\Im m\, t = \frac{1}{2}$.*
(4) *$D(t + \frac{\mathrm{i}}{2})$ is unitary and strongly continuous in t.*
(5) *$D(t)$ fulfils the following cocycle relation:*

$$D(s + t) = \sigma_{\mathcal{M}}^{-t}(D(s))D(t). \tag{2.2}$$

(6) *For complex values of the arguments one finds*

$$D(t + \frac{\mathrm{i}}{2})^* J_{\mathcal{M}} D(t) = D(t)^* J_{\mathcal{M}} D(t + \frac{\mathrm{i}}{2})$$

is independent of t.
(7) $\mathrm{Ad}\,\{D(t)D(\frac{\mathrm{i}}{2})^*\}\mathcal{M} \subset \mathcal{M}$ *holds for all $t \in \mathbb{R}$.*

Proof: (1) and (2) follow immediately from the definition of $D(t)$. The statements (3) and (4) are nothing else than Thm. A in [Bch95]. (5) From the definition of $D(t)$ we obtain $\sigma_{\mathcal{M}}^{-t}(D(s))D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{-\mathrm{i}s}\Delta_{\mathcal{N}}^{\mathrm{i}s}\Delta_{\mathcal{M}}^{\mathrm{i}t}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t} = \Delta_{\mathcal{M}}^{-\mathrm{i}(s+t)}\Delta_{\mathcal{N}}^{\mathrm{i}(s+t)} = D(s+t)$.
(6) From Thm. A in [Bch95] we know

$$D(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) J_{\mathcal{N}}. \tag{2.3}$$

This implies

$$D(t + \frac{\mathrm{i}}{2})^* J_{\mathcal{M}} D(t) = J_{\mathcal{N}} D(t)^* J_{\mathcal{M}} J_{\mathcal{M}} D(t) = J_{\mathcal{N}},$$

and

$$D(t)^* J_{\mathcal{M}} D(t + \frac{\mathrm{i}}{2}) = D(t)^* J_{\mathcal{M}} J_{\mathcal{M}} D(t) J_{\mathcal{N}} = J_{\mathcal{N}}.$$

This shows (6). For proving (7) we use Eqs. (2.1) and (2.3) and get

$$\mathrm{Ad}\,\{D(t)D(\frac{\mathrm{i}}{2})^*\}\mathcal{M} = \mathrm{Ad}\,\{\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t} J_{\mathcal{N}} J_{\mathcal{M}}\}\mathcal{M}.$$

Because of $\mathcal{N} \subset \mathcal{M}$ we know $\mathrm{Ad}\, J_{\mathcal{M}}\mathcal{M} = \mathcal{M}' \subset \mathcal{N}'$. Hence $\mathrm{Ad}\,\{J_{\mathcal{N}} J_{\mathcal{M}}\}\mathcal{M} \subset \mathcal{N}$ which implies $\mathrm{Ad}\,\{\Delta_{\mathcal{N}}^{\mathrm{i}t} J_{\mathcal{N}} J_{\mathcal{M}}\}\mathcal{M} \subset \mathcal{N}$. Since $\mathcal{N} \subset \mathcal{M}$ statement (7) is proved. □

2.3 Remarks:

1. *The functions $D_{\mathcal{M},\mathcal{N}}(t)$ fulfil the following chain rule: If $\mathcal{P} \subset \mathcal{N} \subset \mathcal{M}$ then*

$$D_{\mathcal{M},\mathcal{P}}(t) = D_{\mathcal{M},\mathcal{N}}(t) D_{\mathcal{N},\mathcal{P}}(t). \tag{2.4}$$

2. *With $\mathcal{N}' \supset \mathcal{M}'$ one obtains*

$$D_{\mathcal{N}',\mathcal{M}'}(t) = D_{\mathcal{M},\mathcal{N}}(-t)^*. \tag{2.5}$$

The function $D_{\mathcal{M},\mathcal{N}}(\bar{z})^*$ *is analytic in the conjugate complex domain, i.e. in* $S(-\frac{1}{2},0)$*. Therefore,* (2.5) *reads in the complex*

$$D_{\mathcal{N}',\mathcal{M}'}(z) = D_{\mathcal{M},\mathcal{N}}(-\bar{z})^*. \tag{2.5'}$$

Notice that the properties of $D(t)$ described in Lemma 2.1 do not contain any reference to the algebra $\mathcal{N}$. Therefore, we introduce the following notation:

2.4 Definition:

An operator–valued function $D(t)$ which fulfils the properties (1)–(7) of Lemma 2.2 will be called a characteristic function of $\mathcal{M}$. The set of charasteristic functions belonging to $\mathcal{M}$ will be denoted by $\mathcal{C}har(\mathcal{M})$.

2.5 Theorem:

Let $\mathcal{M}$ *be a von Neumann algebra with a cyclic and separating vector* Ω*. Then to every characteristic function* $D(t)$ *of* $\mathcal{M}$ *exists a von Neumann subalgebra* $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$ *such that* $D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}$*. The correspondence*

$$\mathcal{S}ub(\mathcal{M}) \Longleftrightarrow \mathcal{C}har(\mathcal{M})$$

is one to one.

The proof of this theorem will be splitted into several steps. We start with

2.6 Lemma:

Define

$$U(t) = \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) \quad \text{and} \quad K = J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) \tag{2.6}$$

then there holds:

(1) $U(t)$ *is a strongly continuous unitary group.*
(2) K *is a conjugation i.e.* $K = K^* = K^{-1}$.
(3) K *commutes with* $U(t)$*, which implies that one can write* $U(t) = \Delta^{\mathrm{i}t}$ *with an invertible operator* Δ.
(4) *The function* $D(t)$ *can be reconstructed if we know* $U(t)$ *and* K.

$$D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t} U(t), \quad D(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) K. \tag{2.7}$$

Proof: Since $D(t)$ and $\Delta_{\mathcal{M}}^{\mathrm{i}t}$ are both unitary and strongly continuous it follows that $U(t)$ is unitary and weakly continuous. The unitarity implies that $U(t)$ is strongly continuous. From the cocycle relation (2.2) it follows that $U(t)$ is a unitary group. The relation $K = K^*$ is a consequence of property Lemma 1.2 (6). Using this again we find

$KK = D(\frac{\mathrm{i}}{2})^* J_{\mathcal{M}} J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) = \mathbb{1}$. For proving (3) we reformulate the cocycle relation (2.2). It reads $\Delta_{\mathcal{M}}^{-\mathrm{i}t} D(s) \Delta_{\mathcal{M}}^{\mathrm{i}t} = D(t+s)D(t)^*$. If we replace t by $-t$ and s by t we get

$$\Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) \Delta_{\mathcal{M}}^{-\mathrm{i}t} = D(-t)^*. \tag{2.8}$$

If we analytically continue the last but one equation in s then we find $\Delta_{\mathcal{M}}^{-\mathrm{i}t} D(\frac{\mathrm{i}}{2}) \Delta_{\mathcal{M}}^{\mathrm{i}t} = D(t+\frac{\mathrm{i}}{2})D(t)^*$. Using this equation and Lemma 2.2 (6) we obtain:

$$\begin{aligned} KU(t) &= J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) = J_{\mathcal{M}} \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t+\frac{\mathrm{i}}{2}) D(t)^* D(t) \\ &= \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) D(t)^* J_{\mathcal{M}} D(t+\frac{\mathrm{i}}{2}) = \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) = U(t)K. \end{aligned}$$

Finally the first relation of Lemma 2.6,(4) follows from the definition of $U(t)$. The second relation will be derived by using condition (6) of Lemma 2.2.

$$D(t+\frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) D(t)^* J_{\mathcal{M}} D(t+\frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) K.$$

This shows the lemma. □

Next we want to construct the von Neumann algebra $\mathcal{N}$ or better the algebra $\mathcal{N}'$ which we define

$$\mathcal{N}' = \bigvee_{t \in \mathbb{R}} \mathrm{Ad}\, U(t) \mathcal{M}'. \tag{2.9}$$

This algebra is invariant under $\mathrm{Ad}\, U(t)$. Now we show that Ω is separating for $\mathcal{N}'$. For this and the following calculation we set $\mathrm{Ad}\, U(t) = \sigma^t$.

2.7 Lemma:

The algebra $KJ_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}} K$ commutes with $\sigma^t(\mathcal{M}')$ and hence with $\mathcal{N}'$. Since $KJ_{\mathcal{M}}$ is unitary and maps Ω onto itself it follows that Ω is cyclic for $\mathcal{N}$.

Proof: Let $A \in \mathcal{M}$ and $B \in \mathcal{M}'$. By using Eq.(2.8) we obtain:

$$\begin{aligned} U(t)BU(t)^* KJ_{\mathcal{M}} A J_{\mathcal{M}} K &= \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) B D(t)^* \Delta_{\mathcal{M}}^{-\mathrm{i}t} D(\frac{\mathrm{i}}{2})^* J_{\mathcal{M}} J_{\mathcal{M}} A J_{\mathcal{M}} J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) \\ &= D(-t)^* \Delta_{\mathcal{M}}^{\mathrm{i}t} B \Delta_{\mathcal{M}}^{-\mathrm{i}t} D(-t) D(\frac{\mathrm{i}}{2})^* A D(\frac{\mathrm{i}}{2}) D(-t)^* D(-t). \end{aligned}$$

Property (7) of Lemma 2.2 and (2.8) leads to

$$\begin{aligned} &= D(-t)^* D(-t) D(\frac{\mathrm{i}}{2})^* A D(\frac{\mathrm{i}}{2}) D(-t)^* \Delta_{\mathcal{M}}^{\mathrm{i}t} B \Delta_{\mathcal{M}}^{-\mathrm{i}t} D(-t) \\ &= D(\frac{\mathrm{i}}{2})^* J_{\mathcal{M}} J_{\mathcal{M}} A J_{\mathcal{M}} J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) B D(t)^* \Delta_{\mathcal{M}}^{-\mathrm{i}t} \\ &= KJ_{\mathcal{M}} A J_{\mathcal{M}} K U(t) B U(-t). \end{aligned}$$

This shows the lemma. □

¿From the invariance of $\mathcal{N}'$ and the last lemma we notice for later use

$$[\sigma^{t_1}(A_1'), K\sigma^{t_2}(A_2')K] = 0, \quad A_1', A_2' \in \mathcal{M}';\ t_1, t_2 \in \mathbb{R}. \tag{2.10}$$

This follows from $[K, \Delta^{\mathrm{i}t}] = 0$ and $\mathcal{M}' = J_{\mathcal{M}}\mathcal{M}J_{\mathcal{M}}$. Next we want to show that $U(t)$ is the modular group of $\mathcal{N}$. We start with the observation

2.8 Lemma:
With $U(t) = \Delta^{\mathrm{i}t}$ we obtain for $A' \in \mathcal{M}'$

$$\Delta^{-\frac{1}{2}}\sigma^t(A')\Omega = K\sigma^t(A'^*)\Omega.$$

Proof: Using Eqs. (2.6) and (2.8) we get $\Delta^{it}A'\Omega = \Delta^{it}_{\mathcal{M}}D(t)A'\Omega = D(-t)^*\Delta^{it}_{\mathcal{M}}A'\Omega$. This expression has an analytic continuation into the strip $S(0, \frac{1}{2})$ and we obtain with Eq. (2.7)

$$\begin{aligned}\Delta^{\mathrm{i}t-1/2}A'\Omega &= D(-t + \frac{\mathrm{i}}{2})^*\Delta_{\mathcal{M}}^{\mathrm{i}t-1/2}A'\Omega = KD(-t)^*J_{\mathcal{M}}\Delta^{\mathrm{i}t}_{\mathcal{M}}J_{\mathcal{M}}A'^*\Omega \\ &= KD(-t)^*\Delta^{\mathrm{i}t}_{\mathcal{M}}A'^*\Omega = K\Delta^{\mathrm{i}t}_{\mathcal{M}}D(t)A'^*\Omega = KU(t)A'^*\Omega,\end{aligned}$$

and the lemma is proved. □

Next we want to extend Lemma 2.8 to all of $\mathcal{N}'$. To this end we start with the following remark: Since $\sigma^t(A')$ is weakly continuous in t we can define for $f(t) \in \mathcal{L}^1(\mathbb{R})$ the weak integral

$$\sigma^f(A') = \int \sigma^t(A')f(t)\,dt.$$

If we take for $f(t)$ a function which is entire analytic in t then $\sigma^s(\sigma^f(A'))$ is entire analytic in s. Notice that every $\sigma^t(A')$ is the strong limit of elements of the form $\sigma^f(A')$. With this remark we obtain:

2.9 Lemma:
Let $C' \in \mathcal{N}'$ then we get

$$\Delta^{\mathrm{i}t-\frac{1}{2}}C'\Omega = K\Delta^{\mathrm{i}t}C'^*\Omega.$$

Proof: Choose $A_i \in \mathcal{M}'$ and $f_i \in \mathcal{L}^1(\mathbb{R})$ entire analytic, $i = 1, ..., n$. Then

$$\Delta^{\mathrm{i}t}\sigma^{f_1}(A_1)...\sigma^{f_n}(A_n)\Omega = \sigma^t\big(\sigma^{f_1}(A_1)\big)...\sigma^t\big(\sigma^{f_n}(A_n)\big)\Omega$$

can be analytically continued and we obtain with Lemma 2.8 and Eq. (2.10):

$$\begin{aligned}\Delta^{\mathrm{i}t-\frac{1}{2}}\sigma^{f_1}(A_1)...\sigma^{f_n}(A_n)\Omega &= \sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)...\sigma^{f_n}(A_n)\big)\Omega \\ &= \sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)\big)...\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_{n-1}}(A_{n-1})\big)\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_n}(A_n)\big)\Omega \\ &= \sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)\big)...\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_{n-1}}(A_{n-1})\big)K\sigma^t\big(\sigma^{f_n}(A_n^*)\big)K\Omega \\ &= K\sigma^t\big(\sigma^{f_n}(A_n^*)\big)K\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)\big)...\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_{n-1}}(A_{n-1})\big)\Omega.\end{aligned}$$

Repeating this manipulation we find

$$= K\sigma^t\big(\sigma^{f_n}(A_n^*)...\sigma^{f_1}(A_1^*)\big)\Omega.$$

Since the set $\{\sigma^{f_1}(A_1)...\sigma^{f_n}(A_n),\ n \in \mathbb{N},\ f \in \mathcal{L}^1(\mathbb{R})$ entire analytic$\}$ is weakly dense in $\mathcal{N}'$ and the $*$–operation is weakly continuous the lemma is proved. □

Proof of the theorem: In order that $U(t)$ is the modular group of $\mathcal{N}$ we have to show that $U(-t)$ fulfils the KMS–condition for $\mathcal{N}'$. Let $C_1', C_2' \in \mathcal{N}'$ then by Lemma 2.9 $(\Omega, C_1'U(t)C_2'\Omega)$ has an analytic continuation into the strip $S(0, \frac{1}{2})$ and we obtain

$$\begin{aligned}&(\Omega, C_1'\Delta^{\mathrm{i}(t+\frac{\mathrm{i}}{2})}C_2'\Omega) = ({C'}_1^*\Omega, K\sigma^t({C'}_2^*)\Omega) = (\sigma^t({C'}_2^*)\Omega, K{C'}_1^*\Omega) = \\ &(\Omega, C_2'\Delta^{-\mathrm{i}t}K{C'}_1^*\Omega) = (\Omega, C_2'\Delta^{-\mathrm{i}t-\frac{1}{2}}C_1'\Omega).\end{aligned}$$

The last expression can again be analytically continued into $S(0, \frac{1}{2})$ and we obtain at the upper boundary $(\Omega, C_2'\Delta^{-\mathrm{i}t}C_1'\Omega)$. This shows the KMS–condition. It remains to show the uniqueness of the mapping. If $D_1(t)$ and $D_2(t)$ are different then follows from the construction used above that the algebras are different. Conversely assume $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{S}ub(\mathcal{M})$ and $D_1(t)$ and $D_2(t)$ coincide. Then $\Delta_1^{\mathrm{i}t}$ and $\Delta_2^{\mathrm{i}t}$ coincide and also J_1 and J_2 coincide by Eq. (2.6). This implies that $\mathcal{N}_1 \cap \mathcal{N}_2$ is invariant under $\Delta_1^{\mathrm{i}t} = \Delta_2^{\mathrm{i}t}$. Since $J_1\mathcal{M}'J_1$ is contained in the intersection it follows that Ω is cyclic for $\mathcal{N}_1 \cap \mathcal{N}_2$. Hence $\mathcal{N}_1$ and also $\mathcal{N}_2$ coincide with $\mathcal{N}_1 \cap \mathcal{N}_2$. (See [KR86] Thm. 9.2.36.) Hence the map $\mathcal{S}ub(\mathcal{M}) \Leftrightarrow \mathcal{C}har(\mathcal{M})$ is one to one. □

3. Topology for the set of characteristic functions

The set $\mathcal{C}har(\mathcal{M})$ can easily be furnished with a topology.

3.1 Definition:

Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. We introduce on $\mathcal{C}har(\mathcal{M})$ the topology τ of simultanious $*$–strong convergence of $D_\alpha(t)$ and $D_\alpha(t + \frac{\mathrm{i}}{2})$ and this uniformly on every compact K of the real line. The neighbourhoods of an element $D(t)$ are given by

$$\begin{aligned}U(\psi_1, ..., \psi_n, K, D(t)) = \{&D'(t) \in \mathcal{C}har(\mathcal{M});\ \|(D(t) - D'(t))\psi_i\| + \\ &\|(D(t)^* - D'(t)^*)\psi\| + \|(D(t + \frac{\mathrm{i}}{2}) - D'(t + \frac{\mathrm{i}}{2}))\psi\| + \\ &\|(D(t + \frac{\mathrm{i}}{2})^* - D'(t + \frac{\mathrm{i}}{2})^*)\psi\| \leq 1, \quad i = 1, ..., n\ t \in K\}.\end{aligned} \tag{3.1}$$

The importance of this definition is due to the observation that $\mathcal{C}har(\mathcal{M})$ is complete in this topology.

3.2 Theorem:

The space $Char(\mathcal{M})$ is τ complete.

Proof: Let $D_\alpha(t)$ be a Cauchy net in $Char(\mathcal{M})$ with limit $D(t)$. Since $D_\alpha(t)$ converges $*$–strong it follows that $D_\alpha(t)^* D_\alpha(t)$ converges weakly. This implies that $D(t)$ is unitary and the convergence of $D_\alpha(t)$ is strong. As the convergence is uniform on every compact it follows that $D(t)$ is continuous. $D_\alpha(0) = \mathbb{1}$ implies $D(0) = \mathbb{1}$ and $D_\alpha(t)\Omega = \Omega$ yields the same relation for $D(t)$. Before discussing the property (3) of Lemma 2.2 let us show the other properties.

Since $D_\alpha(t + \frac{\mathrm{i}}{2})$ converges $*$–strongly on every compact it follows that $D(t + \frac{\mathrm{i}}{2})$ is also unitary and continuous. Writing the cocycle relation in the form $\Delta_{\mathcal{M}}^{\mathrm{i}s} D_\alpha(s+t) = D_\alpha(s)\Delta_{\mathcal{M}}^{\mathrm{i}s} D_\alpha(t)$ we see that the righthand side converges weakly and since all operators involved are unitary the convergence is strong. Hence the limit function fulfils the cocycle relation. The same arguments are applicable for the property (6) of Lemma 2.2. To show (7) let $A \in \mathcal{M}$ then $D_\alpha(t)D_\alpha(\frac{\mathrm{i}}{2})^* A D_\alpha(\frac{\mathrm{i}}{2})D_\alpha(t)^* \in \mathcal{M}$. Since $D_\alpha(t)$ and $D_\alpha(\frac{\mathrm{i}}{2})$ converge $*$–strongly to unitary operators it follows that also the products $D_\alpha(t)D_\alpha(\frac{\mathrm{i}}{2})^*$ and $D_\alpha(\frac{\mathrm{i}}{2})D_\alpha(t)^*$ converge strongly to $D(t)D(\frac{\mathrm{i}}{2})^*$ and $D(\frac{\mathrm{i}}{2})D(t)^*$. Consequently $D_\alpha(t)D_\alpha(\frac{\mathrm{i}}{2})^* A D_\alpha(\frac{\mathrm{i}}{2})D_\alpha(t)^*$ converges weakly to $D(t)D(\frac{\mathrm{i}}{2})^* A D(\frac{\mathrm{i}}{2})D(t)^*$. As $\mathcal{M}$ is weakly closed the limit belongs to $\mathcal{M}$.

For the proof of property (3) of Lemma 2.2 exist two different procedures. One of them uses methods of bounded analytic functions and the other is based on operator theoretic technics. We will use the second method. Since $D_\alpha(t)$ converges $*$–strongly we conclude that also the unitary groups $\Delta_\alpha^{\mathrm{i}t} = \Delta_{\mathcal{M}}^{\mathrm{i}t} D_\alpha(t)$ converge to a unitary group $\Delta^{\mathrm{i}t}$. (We write the index α instead of $\mathcal{N}_\alpha$.) Since the convergence of $D_\alpha(t)$ is uniform on every compact and since $\Delta^{\mathrm{i}t}$ is strongly continuous in t it follows that also $\Delta^{\mathrm{i}t}$ is strongly continuous in t. On the other hand we know that $J_\alpha = J_{\mathcal{M}} D(\frac{\mathrm{i}}{2})$ converges strongly to a conjugation J. From $J_\alpha \Delta_\alpha^{\mathrm{i}t} = \Delta_\alpha^{\mathrm{i}t} J_\alpha$ we conclude $J\Delta^{\mathrm{i}t} = \Delta^{\mathrm{i}t} J$. This implies that Δ is an invertible operator.

¿From the uniform strong convergence of the unitary groups $\Delta_\alpha^{\mathrm{i}t}$ on every compact it follows that $\log\Delta_\alpha$ converges to $\log\Delta$ in the strong resolvent sense. (See, e.g. [RSi72] Thm. VIII.21.) Since $S_{\mathcal{M}}$ is an extension of S_α one has $(1+\Delta_{\mathcal{M}})^{-1} \geq (1+\Delta_\alpha)^{-1}$. (See also the beginning of section 4.) Since $(1+\Delta)^{-1}$ is a bounded function of $\log\Delta$ it follows that $(1+\Delta_\alpha)^{-1}$ converges strongly to $(1+\Delta)^{-1}$. (See [RSi72] Thm. VIII.20.) ¿From $(1+\Delta_{\mathcal{M}})^{-1} \geq (1+\Delta_\alpha)^{-1}$ follows $(1+\Delta_{\mathcal{M}})^{-1} \geq (1+\Delta)^{-1}$ and we conclude by standard arguments that $D(t)$ has an analytic extension into $S(0,\frac{1}{2})$ wich is norm–bounded by 1. [If X, Y are selfadjoint operators with $X^2 \leq Y^2$ then this implies $\mathcal{D}(Y) \subset \mathcal{D}(X)$ and for $\psi \in \mathcal{D}(Y)$ holds $\|X\psi\| \leq \|Y\psi\|$. If Y is an invertible operator then one has $\mathcal{D}(Y) = \operatorname{range} Y^{-1}$ and for $\varphi \in \mathcal{D}(Y^{-1})$ it follows $\|XY^{-1}\varphi\| \leq \|YY^{-1}\varphi\| = \|\varphi\|$. Hence XY^{-1} has an extension which is bounded in norm by 1. Since for $0 \leq \lambda \leq 1$ the function $x \to x^\lambda$ is operator–monoton on positive operators one easily shows that also $X^\lambda Y^{-\lambda}$ has an extension which is norm–bounded by 1.] This proves the theorem. □

4. Conclusions, examples

The two spaces $\mathcal{S}ub(\mathcal{M})$ and $\mathcal{C}har(\mathcal{M})$ are isomorphic. The first carries a semi–ordering (by inclusion) and the second a topology. Therefore, it is natural to try to make use of the semi–order also for convergence problems. For example the following problem is easier to answer with help of the τ–topology: Let $\mathcal{N}_n \subset \mathcal{S}ub(\mathcal{M})$ be a decreasing sequence of subalgebras and let Δ_n be their associated modular operators. Does this sequence converge and if the limit exists, is it the modular operator of a subalgebra?

Since Tomita conjugations are a decreasing sequence (in the sense of the extension theory) of closed operators, it follows by standard arguments that $(1+\Delta_n)^{-1}$ is a decresing sequence of positive bounded operators. Hence exists a strong limit $(1+\Delta)^{-1}$. But is Δ the modular operator of some subalgebra of $\mathcal{M}$? Since the multiplication of a sequence $\Delta_n^{\mathrm{i}t}$ by $\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ does not change the convergence property the answer is positive if $D_n(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_n^{\mathrm{i}t}$ converges in the τ–topology. That this is true if the intersection of the $\mathcal{N}_n$ belongs also to $\mathcal{S}ub(\mathcal{M})$ coincides with a result of D'Antoni, Doplicher, Fredenhagen and Longo [DDFL87].

4.1 Corollary:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Let $\mathcal{N}_n \in \mathcal{S}ub(\mathcal{M}), n \in \mathbb{N}$ be a decreasing sequence of von Neumann subalgebras of $\mathcal{M}$. If Ω is still cyclic for $\mathcal{N} := \bigcap_n \mathcal{N}_n$, then $D_n(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_n^{\mathrm{i}t}$ converges to $D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}$ in the τ–topology.

Proof: From

$$(1+\Delta_{\mathcal{M}})^{-1} \geq (1+\Delta_n)^{-1} \geq (1+\Delta_{n+1})^{-1} \tag{4.1}$$

one obtains

$$\Delta_{\mathcal{M}} \leq \Delta_n \leq \Delta_{n+1}. \tag{4.1a}$$

¿From Eq. (4.1a) we conclude that the domain of $\Delta_n^{1/2}$ is contained in the domain of $\Delta_{\mathcal{M}}^{1/2}$. Since the domain of $\Delta_{\mathcal{N}}^{1/2}$ is the range of $\Delta_{\mathcal{N}}^{-1/2}$, the expression

$$\Delta_{\mathcal{N}}^{-1/2}\Delta_{\mathcal{M}}\Delta_{\mathcal{N}}^{-1/2}$$

is a densely defined bounded and hence a closable operator, and one gets

$$\text{closure}\ \Delta_{\mathcal{N}}^{-1/2}\Delta_{\mathcal{M}}\Delta_{\mathcal{N}}^{-1/2} \leq \mathbb{1}. \tag{4.2}$$

The map $A \to A^\alpha,\ 0 \leq \alpha \leq 1$ is an operator monotone function on positive operators (see e.g. G.K. Pedersen [Ped79] Prop. 1.3.8.). Hence we obtain from Eq. (4.2)

$$\Delta_n^\alpha \geq \Delta_{\mathcal{M}}^\alpha, \quad 0 \leq \alpha \leq 1$$

and consequently

$$\text{closure}\ \{\Delta_n^{-\alpha}\Delta_{\mathcal{M}}^{2\alpha}\Delta_n^{-\alpha}\} \leq \mathbb{1}, \quad 0 \leq \alpha \leq \frac{1}{2}.$$

This implies

$$\|\text{closure}\ \Delta_{\mathcal{M}}^{\alpha}\Delta_n^{-\alpha}\| \leq 1, \quad 0 \leq \alpha \leq \frac{1}{2}$$

$(1+\Delta_n)^{-1}$ is a decreasing sequence of positive bounded operators which converges strongly to $(1+D)^{-1}$. Since all these operators are bounded below by $(1+\Delta)^{-1}$ where Δ is the modular operator of the intersection, we conclude, that D is an invertible operator. As $(1+\Delta_n^\alpha)^{-1}$ are bounded functions of $\log \Delta_n$ it follows that $(1+\Delta_n^\alpha)^{-1}$ converges strongly to $(1+D\alpha)^{-1}$. (See e.g. [RSi72] Thm. VIII.21.) This implies that $\Delta_{\mathcal{M}}^\alpha \Delta_n^{-\alpha}$ converges strongly to $\Delta_{\mathcal{M}}^\alpha D^{-\alpha}$. The convergence of $(1+\Delta_n)^{-1}$ implies the convergence of the resolvent and hence the unitary groups $\Delta_n^{\mathrm{i}t}$ converge $*$–strong to $D^{\mathrm{i}t}$, and this uniformly on every compact of the real line. This implies $D_n(t+\mathrm{i}\alpha) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{\alpha}\Delta_n^{-\alpha}\Delta_n^{\mathrm{i}t}$ converges $*$–strongly to $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{\alpha} D^{-\alpha} D^{\mathrm{i}t}$. This shows that we have convergence in the τ–topology. Hence exists a von Neumann algebra $\mathcal{N}_l \in \mathcal{S}ub(\mathcal{M})$ which represents this limit. From $\mathcal{N}_n \supset \mathcal{N}_l \supset \bigcap_n \mathcal{N}_n$ we conclude that $\mathcal{N}_l$ coincides with the intersection. □

In order to get a better understanding of the τ–topology of $\mathcal{C}har(\mathcal{M})$ let us look at two examples of decreasing families of von Neumann subalgebras, wher Ω is no longer cyclic for the intersection.

4.2 Examples:

(1) Let W be a wedge and W_x be the shifted wedge. Then Δ_x, the modular operator of $\mathcal{M}(W_x)$, can be expressed as follows:

$$\Delta_x^{\mathrm{i}t} = T(x)\Delta_0^{\mathrm{i}t}T(-x)$$

where $T(x)$ is the representation of the translations. By a well known result [Bch92] we find

$$D_x(t) = \Delta_0^{-\mathrm{i}t}\Delta_x^{\mathrm{i}t} = T((\Lambda(-t)-1)x)$$

where $\Lambda(t)$ are the Lorentz boosts associated with W. If now x tends to spacelike infinity inside the wedge, then $\mathcal{M}(W_x)$ tends to the global center. But one sees that in this situation $D_x(t)$ does not converge on the real axis and hence not in the τ–topology.

(2) Let D_r, $r \to 0$ be a decreasing family of double cones. It is known that $\cap_r \mathcal{M}(D_r)$ is contained in the center of the global algebra. (See e.g. [Bch96] Thm. 4.6.) In order to be able to compute $D_r(t)$ we use a conformal field theory. In this case the modular group of the algebra of the double cone has been computed by Hislop and Longo [HL82]. The result is the following:
Let D be the double cone

$$D = \{x : |x_0| + \|\vec{x}\| < 1\}$$

and denote

$$x^\pm = x_0 \pm \|\vec{x}\|.$$

Then the modular group of the pair $(\mathcal{M}(D), \Omega)$ induces on D a geometric transformation given by the formula:

$$x^\pm(\lambda) = \frac{-(1-x^\pm) + \mathrm{e}^{-2\pi\lambda}(1+x^\pm)}{(1+x^\pm) + \mathrm{e}^{-2\pi\lambda}(1+x^\pm)}.$$

The formula for D_r is obtained if one replaces $x^\pm$ by $\frac{x^\pm}{r}$. Both modular groups are given by conformal transformations, hence $D_r(t) = \Delta_1^{-\mathrm{i}t}\Delta_r^{\mathrm{i}t}$, $0 < r < 1$ is given by the conformal transformation

$$x_1^\pm(-t) \circ x_r^\pm(t) = \frac{-r^2 - r(1-r)\sinh 2\pi t + x^\pm(1-(1-r)\cosh 2\pi t)}{r - r(1-r)\cosh 2\pi t + x^\pm(-r + (1-r)\sinh 2\pi t}.$$

For $r = 0$ we obtain

$$\frac{1-\cosh 2\pi t}{\sinh 2\pi t}.$$

This expression is well defined on the real axis also for $r \to 0$. If we replace t by $t + \mathrm{i}/2$ then the hyperbolic functions pick up a factor -1 which has a singularity at $t = 0$. Therefore, $D_r(t)$ does not converge in the τ–topology.

We end this paper with some remaks and questions.
Using the modular automorphisms of $\mathcal{M}$ one sees that $\mathcal{S}ub(\mathcal{M})$ contains a continuous family of different elements if it contains a non-trivial element. With help of the Longo endomorphism one can construct a decreasing family (by inclusion) of elements. (For $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$ the Longo endomorphism applied to $\mathcal{N}$ is $\mathrm{Ad}\,(J_\mathcal{N} J_\mathcal{M})\mathcal{N}$.)

If $\mathcal{N} \in \mathcal{S}ub(\mathcal{M})$, then there is a natural injection of $\mathcal{S}ub(\mathcal{N})$ into $\mathcal{S}ub(\mathcal{M})$. Hence if $\mathcal{S}ub(\mathcal{M})$ is non–trivial it must have a rich structure.

Problems: (α) If $U \in \mathcal{M}$ is unitary and $\psi = U\Omega$ then one has $\mathcal{S}ub_\psi(\mathcal{M}) = U\mathcal{S}ub_\Omega(\mathcal{M})U^*$. If one has $\psi = A\Omega$ with A and A^{-1} both in $\mathcal{M}$ then one obtains $\mathcal{S}ub_\psi(\mathcal{M}) = \mathcal{S}ub_\Omega(\mathcal{M})$. This implies that for a dense set of cyclic and separating vectors the set $\mathcal{S}ub_\psi(\mathcal{M})$ is homeomorphic to $\mathcal{S}ub_\Omega(\mathcal{M})$. Is this true for all ψ which are cyclic and separating for $\mathcal{M}$?
(β) If $\mathcal{M}$ is a finite algebra then $\mathcal{S}ub(\mathcal{M})$ consists only of one point, namely $\mathcal{M}$ itself. If (α) is true then for every infinite algebra the set $\mathcal{S}ub(\mathcal{M})$ contains non–trivial points.
(γ) Using the Longo endomorphisms one observes that at least every second element in $\mathcal{S}ub(\mathcal{M})$ is obtained by applying an endomorphism to $\mathcal{M}$. Is this true for every element in $\mathcal{S}ub(\mathcal{M})$?

Acknowledgements

I thank A. Uhlmann for helpful discussions. Hospitality of the University Leipzig and the Erwin Schrödinger Institute in Vienna is gratefully acknowledged.

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