On the Entropy Theory of Finitely Generated Nilpotent Group Actions

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Abstract

Let $G$ be a finitely generated nilpotent torsionfree group. The entropy theory for $G$-actions is investigated. The Pinsker algebra of such actions is described explicitly. The completely positive systems are shown to have a sort of ‘asymptotic independence property’, just as in the case of $\mathbb{Z}^d$-actions. A notion of K-system for $G$-actions is considered. The properties of invariant partitions are used to prove that the property of complete positivity is equivalent to the properties of K-system and K-mixing. The relationship between the K-systems and some spectral and mixing properties of $G$-actions is clarified.

Introduction

Entropy is an important notion in statistical physics and information theory. Initially, A. Kolmogorov introduced the notion of entropy for automorphisms in ergodic theory [5]. A more general approach to entropy for automorphisms was suggested by Ya. Sinai [15]; in this setting the new invariant found a large number of applications in studying the dynamical systems. Also, A. Kolmogorov distinguished a class of dynamical systems which possess a partition with several good properties of an algebraic nature [5]. Those systems were later called K-systems after A. Kolmogorov by V. Rokhlin and Ya. Sinai [11]. They considered, in particular, the dynamical systems with trivial Pinsker algebra (completely positive systems), and proved that this property is equivalent to the property of K-system. In order to do that, a techniques related to a class of invariant partitions were developed.

The K-systems possess a rather strong mixing property of (K-mixing [3]), which is equivalent to the complete positivity. It turns out that such systems are frequently used in applications, in particular, in the models of mathematical physics [3].

A deep result was obtained by D. Ornstein who proved the isomorphism of Bernoulli shifts with the same entropy [7]. This inspired extensive activities on generalizing this result onto a larger class of groups (specifically, amenable groups). A more advanced result in this sphere was obtained by D. Ornstein and B. Weiss [8]. A study of entropy of $\mathbb{Z}^d$-actions of an algebraic origin was made by K. Schmidt [14].

A description of Pinsker algebra for actions of $\mathbb{Z}^d$, $d < \infty$ was made by J. Conze [1], who also considered for the above group actions the completely positive systems and the notion of K-system. The equivalence of the two latter properties was proved.
by B. Kamiński [4], who developed the Rokhlin-Sinai’s ([11]) theory of invariant partitions in the context of $\mathbb{Z}^d$. Recently a new rise of activities coming up in this sphere. In particular, the work of D. Rudolf and B. Weiss [12] contains a refined study of strong mixing properties of completely positive actions of countable amenable groups. E. Glasner, J. P. Thouvenot, and B. Weiss [2] investigated some problems related to Pinsker algebras and completely positive systems. This work contains, in particular, some strong results on relative disjointness and quasi-factors.

On the other hand, there is still a problem of describing a structure of Pinsker algebra for actions of amenable groups, the properties of completely positive systems and their relationship to K-systems.

In this work, we approach this problem with considering the case of finitely generated nilpotent torsionfree groups and their subgroups. In particular, this class of groups includes upper triangular unipotent matricial groups with integer entries. We also consider some spectral and mixing properties for actions of such groups related to the entropy. We prove an analogue of the well known Pinsker formula (lemma 2.9) together with the associated asymptotic relations, which allow then to describe explicitly the Pinsker algebra (theorem 3.1). The next step is to consider the systems with trivial Pinsker algebra (completely positive systems, see definition A in section 3) and to establish that they are exactly the K-systems (definition C, section 5); the latter being introduced in a similar way as in [1, 4]. For doing this, we develop the approach of V. Rokhlin, Ya. Sinai and B. Kamiński [11, 4] in applying the techniques related to invariant partitions in our setting. We also define the notion of K-mixing for actions of a nilpotent group and show that it is equivalent to complete positivity (see theorem 6.5). On the other hand, the completely positive systems also possess a property of 'asymptotic independence' (see proposition 3.2), just as in the case of $\mathbb{Z}^d$.

To solve these problems, one has to overcome some difficulties. Unlike the case of $\mathbb{Z}^d$-actions, this requires a specific choice of a sequence of generators in the group with some good properties (see the appendix), together with an order relation on those. The presence of such sequence of generators, if proved, already implies the existence of an information past in the sense of B. Pitskel and A. Safonov [9, 13]. Another principal obstacle is that, while working with Følner sequences, a translate of a rectangle in a nilpotent group is no longer a rectangle; so, the problems of pavement arise in many arguments, which requires some additional estimational techniques.

To simplify our exposition, we consider the special case of Heisenberg group in all details, and then show how to transfer our techniques onto the general case.

The outline of this paper is as follows. Section 1 contains a brief review of definitions and basic properties related to the entropy of partitions and measure space transformations. Section 2 introduces the entropy for the Heisenberg group actions and discusses its standard properties. Section 3 presents a description of the Pinsker algebra for actions of the Heisenberg group. Section 4 shows how to transfer the results concerning the Pinsker algebra onto the case of general nilpotent group actions. Section 5 is devoted to studying the invariant and perfect partitions, complete positivity and K-systems. Section 6 clarifies the relationship between the K-systems and some spectral and mixing properties of the Heisenberg group actions. Section 7 exposes a generalization of the results of sections 5 and 6 onto actions of general nilpotent groups. Finally, appendix is devoted to forming a special sequence of generators in a general torsionfree nilpotent group.
1 Preliminaries

We start with recalling the basic definitions. For a partition $\alpha$ with at most countable number of elements on a Lebesgue space $$(X, \mathcal{A}, \mu)$$ define the entropy of $\alpha$ by

$$H(\alpha) = -\sum \mu(A_i) \log \mu(A_i).$$

Let $\mathcal{L}$ denote the collection of partitions $\alpha$ as above with $H(\alpha) < \infty$. Given any measurable partition $\beta$, the conditional entropy $H(\alpha | \beta)$ is defined in a standard way. In particular, $H(\alpha | \beta) = 0 \iff \alpha \subseteq \beta$, and $H(\alpha | \beta)$ is increasing in $\alpha$ and decreasing in $\beta$. The distance $d$ on $\mathcal{L}$ is given by

$$d(\alpha, \beta) = H(\alpha | \beta) + H(\beta | \alpha).$$

With $d$, $\mathcal{L}$ becomes a complete metric space. The basic property of the entropy for partitions is

$$\forall \alpha, \beta \in \mathcal{L}, \forall \gamma, \quad H(\alpha \beta | \gamma) = H(\alpha | \gamma) + H(\beta | \alpha \gamma),$$

and hence the subadditivity of $H$:

$$\forall \alpha, \beta \in \mathcal{L}, \forall \gamma, \quad H(\alpha \beta | \gamma) \leq H(\alpha | \gamma) + H(\beta | \gamma).$$

If $\{\gamma_n\}$ is an increasing sequence of partitions with $\gamma = \bigvee \gamma_n$, then for all $\alpha \in \mathcal{L}$ one has

$$\lim \frac{H(\alpha | \gamma_n)}{n} = H(\alpha | \gamma).$$

In a similar way, for a decreasing sequence of partitions $\{\gamma_n\}$ with $\gamma = \bigwedge \gamma_n$ one has

$$\lim \frac{H(\alpha | \gamma_n)}{n} = H(\alpha | \gamma).$$

Let $T$ be an automorphism of $(X, \mathcal{A}, \mu)$. For any partition $\alpha$, we use the notation

$$\alpha^n_T = \bigvee_{0}^{n-1} T^k \alpha, \quad \alpha^{-n}_{-T} = \bigvee_{1}^{\infty} T^{-k} \alpha, \quad \alpha_T = \bigvee_{-\infty}^{\infty} T^k \alpha.$$

The sequence $\frac{1}{n}H(\alpha^n_T)$ has a limit which is denoted by $h(\alpha, T)$; it can be also represented as a conditional entropy $H(\alpha | \alpha^{-n}_{-T})$. More generally, if $\gamma$ is a $T$-invariant measurable partition, the sequence $\frac{1}{n}H(\alpha^n_T | \gamma)$ has a limit equal to $H(\alpha | \alpha^{-n}_{-T} \gamma)$, which is denoted by $h(\alpha, T, \gamma)$. In particular, $h(\alpha, T) = h(\alpha, T, \nu)$, with $\nu$ being the trivial partition.

The entropy of the automorphism $T$ is defined by

$$h(T) = \sup_{\alpha \in \mathcal{L}} h(\alpha, T).$$

We remind also a very important Pinsker formula:

$$h(\alpha \beta, T) = h(\alpha, T) + H(\beta | \alpha_T \cup \beta_T),$$

and, in a more general setting,

$$h(\alpha \beta, T, \gamma) = h(\alpha, T, \gamma) + H(\beta | \alpha_T \cup \beta_T \cup \gamma)$$

for any $T$-invariant measurable partition $\gamma$. 

3
2 The entropy for actions of the Heisenberg group and its properties

Consider the two step nilpotent countable matrix group

\[ G = \left\{ \begin{pmatrix} 1 & n_3 & n_1 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} : n_i \in \mathbb{Z} \right\}. \]

We fix the generators

\[ T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad T_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Suppose \( G \) acts on a Lebesgue space \((X, A, \mu)\) with finite invariant measure \( \mu \). For any finite subset \( E \subset G \) and a measurable partition \( \alpha \), let \( S(E) \) stand for the number of elements of \( E \), and use the notation \( \alpha_E = \bigvee g \in E \alpha \). Among the finite subsets, we distinguish the rectangles \( \{ T_3^{i_3} T_2^{i_2} T_1^{i_1} : m_1 \leq i_1 \leq M_1, m_2 \leq i_2 \leq M_2, m_3 \leq i_3 \leq M_3 \} \).

Given a sequence of such rectangles \( \{ \rho_n \} \), we say that the modulus of \( \rho_n \) tends to infinity if \( \min_{k=1,2,3} (M_k(n) - m_k(n)) \to \infty \) as \( n \to \infty \),

\[ \frac{M_1(n) - m_1(n)}{M_2(n) - m_2(n)} \to \infty, \quad \frac{M_1(n) - m_1(n)}{M_3(n) - m_3(n)} \to \infty \text{ as } n \to \infty. \]

Proposition 2.1 Let \( \{ \rho_n \} \) be a sequence of rectangles in \( G \) whose modulus tends to infinity. Then \( \{ \rho_n \} \) is a Følner sequence of sets in \( G \).

Proof. It follows from the commutation relation

\[ (T_3^{i_3} T_2^{i_2} T_1^{i_1})^{-1} T_3^{k_3} T_2^{k_2} T_1^{k_1} = T_3^{k_3-i_3} T_2^{k_2-i_2} T_1^{k_1+i_2(k_3-i_3)-i_1}, \]

that for any \( g = (T_3^{i_3} T_2^{i_2} T_1^{i_1})^{-1} \) and \( \rho \) given by \( 0 \leq k_j \leq p_j, j = 1,2,3 \),

\[ S(g \rho_n \Delta \rho_n) \leq 2|i_3| p_1 p_2 + 2|i_2| p_1 p_3 + 2(|i_1| + |i_2| p_3 + |i_1| p_2) p_2 p_3, \]

and so

\[ \frac{S(g \rho_n \Delta \rho_n)}{S(\rho_n)} \leq \frac{2|i_3|}{p_3} + \frac{2|i_2|}{p_2} + \frac{2(|i_1| + i_2| i_3 + |i_1| p_3)}{p_1} \to 0 \]

as \( n \to \infty \). \( \blacksquare \)

It follows from proposition 2.1 and [9, theorem 1] that for any sequence \( \{ \rho_n \} \) of rectangles in \( G \) whose modulus tends to infinity and any \( \alpha \in \mathcal{L} \), there exists a limit \( \frac{1}{S(\rho_n)} H(\alpha, \rho_n) \). This limit is independent of the choice of \( \rho_n \). It is called the entropy of \( \alpha \) with respect to \( G \), and is denoted by \( h(\alpha, G) \). However, it admits different expressions.

Now introduce the concept of the past for a partition \( \alpha \) with respect to the \( G \)-action. Specifically, we set up \( \alpha_G^{-} = \bigvee T_3^{k_3} T_2^{k_2} T_1^{k_1} \alpha \), where the join is taken over all triples \((k_3, k_2, k_1) \in \mathbb{Z}^3\) which are lexicographically less than \((0,0,0)\). In more details, \( \alpha_G^{-} = \alpha^{-}_{T_1}(\alpha T_1 T_2)^{-}_{T_2}. \)
Theorem 2.2 ([9, 13]) For any partition \( \alpha \in \mathcal{L} \), \( h(\alpha, G) = H(\alpha | \alpha^{-} G) \).

We list here some properties of \( h(\alpha, G) \).

Proposition 2.3 If \( \alpha \in \mathcal{L} \),
\[
h(\alpha, G) \leq H(\alpha).
\]

**Proof.** This is a straightforward conclusion from theorem 2.2. \( \blacksquare \)

Proposition 2.4 If \( \alpha, \beta \in \mathcal{L} \),
\[
h(\alpha \lor \beta, G) \leq h(\alpha, G) + h(\beta, G).
\]

**Proof.** This follows from the sub-additivity of the functional \( H(\xi | \eta) \) in \( \xi \). \( \blacksquare \)

Proposition 2.5 If \( \alpha, \beta \in \mathcal{L} \),
\[
|h(\alpha, G) - h(\beta, G)| \leq d(\alpha, \beta).
\]

**Proof.** See [1] for a proof. \( \blacksquare \)

Let \( G_p \) be a subgroup of \( G \) of index \( p \). We consider only the subgroups generated by \( T_i^{p_1}, T_j^{p_2}, T_k^{p_3} \) with \( \frac{p_2}{p_1}, \frac{p_3}{p_1} \in \mathbb{Z} \). Consider a ‘fundamental domain’ \( \delta_p \) in \( G \) with respect to \( G_p \) which contains zero of \( G \). Given a partition \( \alpha \), we denote for simplicity’s sake by \( \alpha^p \) the partition \( \alpha^p = \bigcup_{g \in \delta_p} g \cdot \alpha \).

Lemma 2.6 For a subgroup \( G_p \subset G \) of index \( p \),
\[
h(\alpha^p, G_p) = ph(\alpha, G).
\]

**Proof.** Note that the ‘fundamental domain’ \( \delta_p \) can be chosen to be a rectangle; let it be \( \{ T_i^{p_1} T_j^{p_2} T_k^{p_3} : 0 \leq i_1 < p_1, 0 \leq i_2 < p_2, 0 \leq i_3 < p_3 \} \). In this case the index of \( G_p \) is \( p = p_2 p_3 \). Let also \( \rho = \{ T_i^{p_1} T_j^{p_2} T_k^{p_3} : 0 \leq k_1 \leq m_1, 0 \leq k_2 \leq m_2, 0 \leq k_3 \leq m_3 \} \) be a rectangle in \( G_p \). It follows from the commutation relation \( T_3^{i_1} T_2^{i_2} T_1^{i_3} T_3^{k_1} T_2^{k_2} T_1^{k_3} = T_3^{i_1+k_1} T_2^{i_2+k_2} T_1^{i_3+k_3} \) that the set \( \rho = \{ f g, f \in \delta_p, g \in \rho \} \) is inside the rectangle \( \overline{\rho} = \{ T_i^{j_1} T_j^{j_2} T_k^{j_3} : -p_2 m_3 p_3 \leq j_1 < m_1 p_1, 0 \leq j_2 < m_2 p_2, 0 \leq j_3 < m_3 p_3 \} \). Observe that \( S(\overline{\rho}) = S(\rho) S(\delta_p) = m_1 m_2 m_3 p_1 p_2 p_3, S(\overline{\rho}) = (m_1 p_1 + p_2 m_3 p_3) m_2 m_3 p_2 p_3, \) and hence \( \frac{S(\overline{\rho})}{S(\rho)} = 1 + \frac{p_2 p_3}{p_1}, \frac{m_3}{m_1} \).

Let us assume that the modulus of \( \rho \) tends to infinity. This means that \( \frac{m_3}{m_1} \to 0 \). We impose also the additional assumption that \( \frac{m_3}{m_1} \to 0 \). Note that under the latter assumption the modulus of \( \rho \) tends to infinity if and only if the modulus of \( \overline{\rho} \) tends to infinity, and \( \frac{S(\overline{\rho})}{S(\rho)} \to 1 \). In view of the above observation we have
\[
\lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) = \lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) \leq \lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) + \lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) \leq
\]

\[
\lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) = \lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) \leq \lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) + \lim_{S(\overline{\rho})} \frac{1}{S(\overline{\rho})} H(\alpha_{\overline{\rho}}) \leq
\]
and since the latter limit is zero, we deduce that

$$\lim \frac{1}{S(\rho')} H(\alpha_{\tau'}) \leq \lim \frac{1}{S(\rho)} H(\alpha_{\tau}).$$

What remains is to observe that the converse inequality is obvious, so finally we have

$$h(\alpha,G) = \lim \frac{1}{S(\rho')} H(\alpha_{\tau'}) = \lim \frac{1}{S(\rho)} H(\alpha_{\tau}) = \lim \frac{1}{pS(\rho)} H((\alpha_{\tau'})_{\rho}) = \frac{1}{p} h(\alpha^p,G_p).$$

**Proposition 2.7** For any transformation $T \in G$,

$$h(\alpha,G) \leq h(\alpha,T).$$

**Proof.** This is an easy consequence of theorem 2.2. \hfill \blacksquare

**Proposition 2.8** If $\alpha, \beta \in \mathcal{L}$,

$$h(\beta,G) \leq h(\alpha,G) + H(\beta | \alpha_G). \quad (2.2)$$

**Proof.** Consider the rectangles $\rho_n = \{T_3^i T_2^j T_1^k : 0 \leq i_1 < n^2; 0 \leq i, j < n\}$, $\rho_p = \{T_3^{k_2} T_2^{k_3} T_1^{k_1} : 0 \leq k_1 < p^2; 0 \leq k_2, k_3 < p\}$. It follows from the commutation relation $T_3^i T_2^j T_1^k = T_3^{i+k_3} T_2^{j+k_2} T_1^{i+k_1-k_2}$ that the set $\{T_3^i T_2^j T_1^k = fg : f \in \rho_n, g \in \rho_p\}$ is inside the rectangle $\rho_{n,p} = \{T_3^{j_3} T_2^{j_2} T_1^{j_1} : -(n-1)(p-1) \leq j_1 \leq p^2 + n^2 - 2, 0 \leq j_2, j_3 < n + p - 2\}$. Thus we obtain:

$$H(\beta_{\rho_n}) \leq H(\beta_{\rho_n} | \alpha_{\rho_{n,p}}) = H(\alpha_{\rho_{n,p}}) + H(\beta_{\rho_n} | \alpha_{\rho_{n,p}}) \leq$$

$$\leq H(\alpha_{\rho_{n,p}}) + \sum_{f \in \rho_n} H(f | \alpha_{\rho_{n,p}}) \leq H(\alpha_{\rho_{n,p}}) + \sum_{f \in \rho_n} H(f | \alpha_{f_{\rho_n}}) =$$

$$= H(\alpha_{\rho_{n,p}}) + S(\rho_n) H(\beta | \alpha_{\rho_p}).$$

After dividing out by $S(\rho_n) = n^4$ and observing via a routine calculation that $rac{S(\rho_{n,p})}{S(\rho_n)} \to 1$ as $n \to \infty$, we get for any fixed $p$

$$h(\beta,G) \leq h(\alpha,G) + H(\beta | \alpha_{\rho_p}).$$

Now what remains is to send $p$ to infinity and to observe that $H(\beta | \alpha_{\rho_p}) \to H(\beta | \alpha_G)$ in order to get the desired relation. \hfill \blacksquare

**Lemma 2.9** (Pinsker formula). If $\alpha, \beta \in \mathcal{L}$,

$$h(\alpha,\beta,G) = h(\alpha,G) + H(\beta | \alpha_G). \quad (2.3)$$
Proof. The proof of this formula is just the same as that one can find in [1]. It is based on (2.1) and (2.2).

In a similar way one can prove a relativized version of the above Pinsker formula. Specifically, if \( \alpha, \beta \in \mathcal{L} \) and \( \sigma \) is any \( G \)-invariant measurable partition, then
\[
h(\alpha \beta, G, \sigma) = h(\alpha, G, \sigma) + H(\beta|_G \alpha G \sigma).
\] (2.4)

Lemma 2.10 If \( \alpha, \beta \in \mathcal{L} \),
\[
\lim_{n \to \infty} H(\alpha|T^{-n}_3(\beta_G^-) \alpha_G^-) = H(\alpha|\alpha_G^-).
\] (2.5)

Proof. Apply the Pinsker formula (2.3) to \( \alpha \) and \( T_3^{-n}\beta \):
\[
h(\alpha \lor T_3^{-n}\beta, G) = h(\alpha, G) + H(T_3^{-n}\beta|T_3^{-n}\beta_G^- \lor \alpha_G) = h(\alpha, G) + H(\beta|_G \lor \alpha_G),
\]
with the latter equality being due to the obvious relation \( (T_3^{-n}\beta)_G^- = T_3^{-n}(\beta_G^-) \).

On the other hand, an application of (1.1) yields
\[
h(\alpha \lor T_3^{-n}\beta, G) = H(\alpha \lor T_3^{-n}\beta|_G \lor (T_3^{-n}\beta)_G^-) =
\]
\[
= H(\alpha|\alpha_G^- \lor T_3^{-n}(\beta_G^-)) + H(T_3^{-n}\beta^|_G \lor \alpha_G \lor T_3^{-n}(\beta_G^-)) =
\]
\[
H(\alpha|\alpha_G^- \lor T_3^{-n}(\beta_G^-)) + H(\beta|_G \lor \alpha_G \lor \beta_G^-)
\]
It is easy to see that \( T_3^n(\alpha \lor \alpha_G^-) \) is an increasing sequence of partitions whose join is \( \alpha_G \), and hence \( H(\beta|T_3^n(\alpha \lor \alpha_G^-) \lor \beta_G^-) \to H(\beta|_G \lor \beta_G^-) \). Now the comparison of the two expressions for \( h(\alpha \lor T_3^{-n}\beta, G) \) proves our statement. \( \blacksquare \)

There is also a relativized version of (2.5). Let \( \alpha, \beta \in \mathcal{L} \) and \( \sigma \) is a \( G \)-invariant measurable partition, then
\[
\lim_{n \to \infty} H(\alpha|T^{-n}_3(\beta_G^-) \alpha_G^- \sigma) = H(\alpha|\alpha_G^- \sigma).
\] (2.6)

The proof is similar to that of (2.5).

3 A description of Pinsker algebra for actions of the Heisenberg group

Associate to any measurable partition \( \alpha \) the partition \( \pi(\alpha) \) which is defined to be the least upper bound of all those partitions \( \beta \in \mathcal{L} \) that \( \beta \leq \alpha_G \) and \( h(\beta, G) = 0 \). In particular, with \( \alpha = \varepsilon \), one has \( \pi(\varepsilon) = \pi(G) \), the Pinsker partition of the dynamical system \((X, G)\). We are going to describe \( \pi(\alpha) \) more explicitly.

For \( \alpha \in \mathcal{L} \) we set up
\[
\hat{\alpha} = \bigwedge_n (T_1^{-n} \alpha_{T_1}^- \lor T_2^{-n}(\alpha_{T_1})^-_{T_2} \lor T_3^{-n}(\alpha_{T_1 T_2})^-_{T_3}).
\]
To make this partition invariant, set up \( \alpha_\infty = \bigvee_n \hat{\alpha}_n \) with \( \rho_n \) being a sequence of rectangles in \( G \) whose modulus tends to infinity.
Theorem 3.1 For any partition \( \alpha \in \mathcal{L} \), \( \pi(\alpha) = \alpha_\infty \).

Proof. First we suppose that \( \beta \in \mathcal{L} \), and \( \beta \leq \alpha_\infty \). Note that the set of partitions which are less than some \( \hat{\alpha}_p \) is dense in the set of partitions which are less than \( \alpha_\infty \). In this context, it suffices to show that \( h(\beta, G) = 0 \) for \( \beta \in \mathcal{L} \), \( \beta \leq \hat{\alpha}_p \). Actually, we consider here the case \( \beta \leq \hat{\alpha} \); the general case can be treated similarly. One has for all \( k_1, k_2, k_3 \)

\[
H(\beta | T_1^{-k_1} \alpha_{T_1}^{-} \lor T_2^{-k_2} (\alpha_{T_1})_{T_2}^{-} \lor T_3^{-k_3} (\alpha_{T_1,T_2})_{T_3}^{-} \lor (\beta_{T_1})_{T_2}^{-} \lor (\beta_{T_1,T_2})_{T_3}^{-} ) = 0.
\]

Let \( k_1 \to \infty \), then an application of an analogue of (2.6) for \( G = \mathbb{Z} [1] \) with a generator \( T_1 \) and \( \sigma = T_2^{-k_2} (\alpha_{T_1})_{T_2}^{-} \lor T_3^{-k_3} (\alpha_{T_1,T_2})_{T_3}^{-} \lor (\beta_{T_1})_{T_2}^{-} \lor (\beta_{T_1,T_2})_{T_3}^{-} \) yields

\[
H(\beta | T_2^{-k_2} (\alpha_{T_1})_{T_2}^{-} \lor T_3^{-k_3} (\alpha_{T_1,T_2})_{T_3}^{-} \lor (\beta_{T_1})_{T_2}^{-} \lor (\beta_{T_1,T_2})_{T_3}^{-} ) = 0,
\]

and hence

\[
H(\beta | T_2^{-k_2} (\alpha_{T_1})_{T_2}^{-} \lor (\alpha_{T_1,T_2})_{T_3}^{-} \lor (\beta_{T_1})_{T_2}^{-} \lor (\beta_{T_1,T_2})_{T_3}^{-} ) = 0.
\]

Apply again an analogue of (2.6) for \( G = \mathbb{Z}^2 [1] \) with \( \sigma = T_3^{-k_3} (\alpha_{T_1,T_2})_{T_3}^{-} \lor (\beta_{T_1,T_2})_{T_3}^{-} \) and the generators \( T_1, T_2 \) to get

\[
H(\beta | T_3^{-k_3} (\alpha_{T_1,T_2})_{T_3}^{-} \lor (\beta_{T_1,T_2})_{T_3}^{-} ) = 0,
\]

which also implies

\[
H(\beta | T_3^{-k_3} (\alpha_{T_1})_{T_2}^{-} \lor (\alpha_{T_1,T_2})_{T_3}^{-} \lor (\beta_{T_1})_{T_2}^{-} \lor (\beta_{T_1,T_2})_{T_3}^{-} ) = 0.
\]

Finally, an application of (2.6) assures \( H(\beta | T_3^{-k_3} \gamma_{G_p}^{-} ) = 0 \), which was to be proved.

Conversely, let \( \beta \in \mathcal{L} \), \( \beta \leq \alpha_G \), and \( h(\beta, G) = 0 \). Recall that for any subgroup \( G_p \) of a finite index \( p \) of the form described above, \( h(\beta, G_p) \leq h(\beta, G) = ph(\beta, G) = 0 \). Given any partition \( \gamma \in \mathcal{L} \), the double application of the Pinsker formula (2.5) yields

\[
H(\gamma | \gamma_{G_p}^{-} ) + H(\beta | \gamma_{G_p}^{-} \cdot \gamma_{G_p}^{-} ) = H(\beta | \gamma_{G_p}^{-} ) + H(\gamma | \gamma_{G_p}^{-} \cdot \beta_{G_p}^{-} ),
\]

and since \( H(\beta | \beta_{G_p}^{-} \cdot \gamma_{G_p}^{-} ) = H(\beta | \beta_{G_p}^{-} ) = 0 \), we obtain

\[
H(\gamma | \gamma_{G_p}^{-} ) = H(\gamma | \gamma_{G_p}^{-} \cdot \beta_{G_p}^{-} ),
\]

whence \( H(\gamma | \gamma_{G_p}^{-} ) = H(\gamma | \gamma_{G_p}^{-} \cdot \beta ) \).

Let \( G_p \) be generated by \( T_1^n, T_2^n, T_3^n \), then \( \gamma_{G_p}^{-} \leq T_1^{-n} \gamma_{T_1}^{-} \cdot T_2^{-n} (\gamma_{T_2}^{-} )_{T_3}^{-} \cdot T_3^{-n} (\gamma_{T_1,T_2}^{-} )_{T_3}^{-} \).

Thus one has

\[
H(\gamma | \beta ) \geq H(\gamma | \gamma_{G_p}^{-} \cdot \beta ) = H(\gamma | \gamma_{G_p}^{-} ) \geq H(\gamma | T_1^{-n} \gamma_{T_1}^{-} \cdot T_2^{-n} (\gamma_{T_2}^{-} )_{T_3}^{-} \cdot T_3^{-n} (\gamma_{T_1,T_2}^{-} )_{T_3}^{-} ) .
\]

Pass in this relation to a limit as \( n \to \infty \) to get \( H(\gamma | \beta ) \geq H(\gamma | \gamma ) \). In the case when \( \gamma \leq \alpha_p \) for some rectangle \( \rho \) we also deduce

\[
H(\gamma | \beta ) \geq H(\gamma | \hat{\alpha} ) \geq H(\gamma | \alpha_\infty ) .
\]

It follows from the condition \( \beta \leq \alpha_G \) that \( \beta \) can be represented as a limit in the metric \( d \) of some sequence of partitions \( \gamma_n \) with \( \gamma_n \leq \alpha_p \) for some sequence of rectangles.
\( \rho_n \) whose modulus tends to infinity. Thus a passage to a limit in the above inequality affords \( 0 = H(\beta|\beta) = H(\beta|\alpha_\infty) \), that is, \( \beta \leq \alpha_\infty \), and hence \( \pi(\alpha) \leq \alpha_\infty \). \( \blacksquare \)

**Definition A.** The dynamical system \((X, G)\) is said to have a completely positive entropy if \( \pi(G) = \pi(\varepsilon) = \nu \); equivalently, for any partition \( \beta \in \mathcal{L} \), \( h(\beta, G) = 0 \) implies \( \beta = \nu \).

**Proposition 3.2** A dynamical system \((X, G)\) has completely positive entropy if and only if for any partition \( \alpha \in \mathcal{L} \) one has \( H(\alpha) = \sup_{G_p} h(\alpha, G_p) \), where the sup is taken over some sequence of subgroups \( G_p \) of finite indices in \( G \).

**Proof.** If \((X, G)\) has a completely positive entropy, it was shown at the end of the proof of theorem 3.1 that \( \bigwedge_{G_p} \alpha^*_G \leq \pi(G) \), that is, under our assumptions, \( \alpha^*_G \rightarrow \nu \) for some sequence of finite index subgroups \( G_p \) whose index tends to infinity. This implies

\[
H(\alpha) \geq h(\alpha, G_p) = H(\alpha|\alpha^*_G) \rightarrow H(\alpha),
\]

which was to be proved.

Conversely, if \( h(\alpha, G) = 0 \), it follows from (2.1) that \( h(\alpha, G_p) = 0 \) for all finite index subgroups \( G_p \), that is, \( H(\alpha) = 0 \), and hence the complete positivity. \( \blacksquare \)

### 4 Entropic theory and Pinsker algebra: the general case

The results of sections 2, 3 admit a generalization onto a much broader class of transformation groups. Let \( G \) be a finitely generated nilpotent torsionfree group. It is shown in Appendix that there exists a sequence of generators of \( G \) submitted to its structure. Reverse the sequence of generators as in lemma A.3 and rewrite it in the form

\[
T_1, \ldots, T_{n_1}, T_{n_1+1}, \ldots, T_{n_2}, T_{n_2+1}, \ldots, T_{n_{N-1}}, T_{n_{N-1}+1}, \ldots, T_{n_N}
\]

with \( n_0 = 0 \), \( T_{n_i+1}, \ldots, T_{n_{i+1}} \in Z_i+i \), and \( T_i \notin Z_i \) for \( k > n_i \), \( j \in \mathbb{Z} \setminus \{0\} \), and the subgroups \( Z_i \) being chosen as in the Appendix.

Define the linear order relation on the above generators by setting \( T_1 < T_2 < \ldots < T_M \), \( (n_N = M) \), together with the associated lexicographic linear order relation on \( G \):

\[
T^{i_M}_M, \ldots, T^{i_1}_1 < T^{k_M}_M, \ldots, T^{k_1}_1 \text{ iff } (j_M, \ldots, j_1) \text{ is lexicographically less than } (k_M, \ldots, k_1).
\]

It is easy to verify that this order relation is invariant with respect to the left shifts on \( G \). So we can apply the results of [9, 13].

It follows from our construction that the multiplication law in \( G \) can be described by the relation as follows:

\[
T^{i_M}_M, \ldots, T^{i_1}_1, T^{k_M}_M, \ldots, T^{k_1}_1 = T^{i_M+k_M+f_M}_M, \ldots, T^{i_1+k_1+f_1}_1,
\]

where \( f_M = 0 \) and for \( j = 1, \ldots, M-1 \), the integer-valued (in fact, polynomial) function \( f_j = f_j(i_{j+1}, \ldots, i_M; k_{j+1}, \ldots, k_M) \) is independent of \( i_1, \ldots, i_j, k_1, \ldots, k_j \).

Given an increasing sequence of finite dimensional rectangles

\[
\rho_n = \{T^{k_M}_M, \ldots, T^{k_1}_1 : l_1(n) \leq k_1 \leq L_1(n), \ldots, l_M \leq k_M \leq L_M(n)\},
\]

with \( l = l(M) \) and \( \gamma = \gamma(M) \), the set of rectangles \( \rho_n \) satisfies

\[
\bigwedge_{G_p} \alpha^*_G \leq \pi(G).
\]

If \( G_p \) is a left shift with \( G_p \) replacing \( G \), then we have the following:

**Corollary 4.1** For any partition \( \alpha \in \mathcal{L} \) and any sequence of finite index subgroups \( G_p \),

\[
H(\alpha) \geq \sup_{G_p} h(\alpha, G_p) = \sup_{G_p} h(\alpha|\alpha^*_G),
\]

which implies

\[
\rho_n \leq \rho(n).
\]
we say that the modulus of \( \rho_n \) tends to infinity if \( \bigcup_n \rho_n = G \), \( (L_j(n) - l_j(n)) \to \infty \) for each \( j \) as \( n \to \infty \), and each \( j = 1, \ldots, M \), with

\[
F_j(n) = \max \{|f_j(i_{j+1}, \ldots, i_M; k_{j+1}, \ldots, k_M)| : -n \leq i_{j+1} \leq n, \ldots, -n \leq i_M \leq n; \\
l_{j+1}(n) \leq k_{j+1} \leq L_{j+1}(n), \ldots, l_M(n) \leq k_M \leq L_M(n)\},
\]

one has

\[
\frac{F_j(n)}{L_j(n) - l_j(n)} \to 0 \quad (4.2)
\]
as \( n \to \infty \).

Just as in proposition 2.1, we claim that the sequence of rectangles \( \rho_n \) whose modulus tends to infinity, forms a Følner sequence of sets. Let \( \rho_n = \{T_M^{k_M} \cdots T_1^{k_1} : l_1(n) \leq k_1 \leq L_1(n), \ldots, l_M(n) \leq k_M \leq L_M(n)\} \) and \( g = T_M^{i_M} \cdots T_1^{i_1} \) be a fixed element of \( G \).

To verify as in the proof of proposition 2.1 that \( \frac{S(g \rho_n \Delta \rho_n)}{S(\rho_n)} \to \infty \), it suffices to observe that \( g \cdot T_M^{k_M} \cdots T_1^{k_1} = T_M^{k_M+i_M+\xi_M(k_1, \ldots, k_M)} \cdots T_1^{k_1+i_1+\xi_1(k_1, \ldots, k_M)} \), with \( \xi_j \) being derived from \( f_j \) as in (4.1) by fixing \( i_1, \ldots, i_M \) (in particular \( \xi_j \) is independent of \( k_1, \ldots, k_j \)).

Now our statement is an obvious consequence of the condition (4.2) as above.

In this setting, one can reproduce the results of sections 2 and 3, using in their proofs just the same ideas. We restrict ourselves to rewriting here only those definitions and statements whose analogues are not completely identical with what one can find above.

Let us demonstrate the possibility of transferring the results of the previous sections by sketching the proof of an analogue of lemma 2.6.

Let \( G_p \) be a subgroup of \( G \) of index \( p \). We consider only the subgroups generated by \( T_1^{p_1} \cdots T_M^{p_M} \) with the powers \( p_i \) being chosen so that \( T_1^{p_1} \cdots T_M^{p_M} \) never generate the same subgroup when \( p_i < p_i \) for at least one \( i \). Consider a ‘fundamental domain’ \( \delta_p \) in \( G \) with respect to \( G_p \) which contains zero of \( G \). This ‘fundamental domain’ \( \delta_p \) can be chosen to be a rectangle; let it be \( \{T_M^{i_M} \cdots T_1^{i_1} : 0 \leq i_1 < p_1, \ldots, 0 \leq i_M < p_M\} \).

In this case the index of \( G_p \) is \( p = p_1 \cdots p_M \). Given a partition \( \alpha \), we denote, just as in section 2, by \( \alpha^p \) the partition \( \alpha^p = \bigvee g^p, \quad g \in \delta_p \).

**Lemma 4.1** (analogue of lemma 2.6) For a subgroup \( G_p \) as above, the relation (2.1) is valid.

**Proof.** Let \( \rho = \{T_M^{k_M} \cdots T_1^{k_1} : 0 \leq k_1 < m_1, \ldots, 0 \leq k_M < m_M\} \) be a rectangle in \( G_p \). It is a routine verification that the set \( \overline{\rho} = \{T_M^{j_M} \cdots T_1^{j_1} = fg, \quad f \in \delta_p, \quad g \in \rho\} \)
is inside the rectangle \( \overline{\rho} = \{T_M^{j_M} \cdots T_1^{j_1} : m_1 j_1 + \Xi_1(m_1, \ldots, m_M) \leq m_M j_M + \Xi_M(m_1, \ldots, m_M)\} \), with \( \Xi_j = \max \{|f_j(i_{j+1}, \ldots, i_M; k_{j+1} + p_{j+1}, \ldots, k_M p_M)| : 0 \leq i_{j+1} < p_{j+1}, \ldots, 0 \leq i_M < p_M; \quad 0 \leq k_1 < m_1, \ldots, 0 \leq k_M < m_M\} \) and \( f_j \) being as in (4.1).

Observe that \( S(\overline{\rho}) = S(\rho) S(\delta_p) = m_1 \cdots m_M p_1 \cdots p_M \). Let us assume that the modulus of \( \rho \) tends to infinity as above. Then the modulus of \( \overline{\rho} \) tends to infinity, and it follows from our definitions that \( \frac{S(\overline{\rho})}{S(\rho)} \to 1 \). In view of the above observation we have

\[
\lim \frac{1}{S(\overline{\rho})} H(\alpha_\overline{\rho}) = \frac{1}{S(\overline{\rho})} H(\alpha_\overline{\rho} \bigvee \alpha_\overline{\rho} \Delta \rho) \leq \frac{1}{S(\overline{\rho})} H(\alpha_\overline{\rho}) + \frac{1}{S(\overline{\rho})} H(\alpha_\overline{\rho} \setminus \rho) \leq
\]
\[ \leq \lim \frac{1}{S(\overline{p})} H(\alpha) + \lim \frac{S(\overline{p} \setminus \overline{p})}{S(\overline{p})} H(\alpha), \]

and since the latter limit is zero, we deduce that

\[ \lim \frac{1}{S(\overline{p})} H(\alpha) \leq \lim \frac{1}{S(\overline{p})} H(\alpha). \]

What remains is to observe that the converse inequality is obvious, so finally we have

\[ h(\alpha, G) = \lim \frac{1}{S(\overline{p})} H(\alpha) = \lim \frac{1}{S(\overline{p})} H(\alpha) = \lim \frac{1}{p S(\rho)} H((\alpha_{\mathcal{E}})_p) = \frac{1}{p} h(\alpha^p, G_p). \]

Another example we demonstrate here is in applying very similar ideas to reproducing proposition 2.8 in the more general setting of this section.

**Proposition 4.2** (analogue of proposition 2.8) For \( \alpha, \beta \in \mathcal{L} \) and \( G \) a finitely generated nilpotent torsionfree group, (2.2) is valid.

**Proof.** Consider the rectangles \( \rho_n = \{ T_2^{i_1} \cdots T_1^{i_k} : 0 \leq k_1 < q_1(n), \ldots, 0 \leq k_M < q_M(n) \} \), \( \rho_p = \{ T_2^{j_1} \cdots T_1^{j_k} : 0 < j_1 < r_1(p), \ldots, 0 < j_M < r_M(p) \} \). One can verify that the set \( \{ T_2^{j_1} \cdots T_1^{j_k} = f g : f \in \rho_n, g \in \rho_p \} \) is inside the rectangle \( \rho_{n,p} = \{ T_2^{j_1} \cdots T_1^{j_k} : |j_1| < q_1(n) + q_M(n) + q_M(n) + \ldots, 0 \leq k_1 < q_1(n), \ldots, 0 \leq k_M < q_M(n) \} \), with

\[ \overline{\Xi}_j = \max\{|f_i(i_{j+1}, \ldots, i_M; k_{j+1}, \ldots, k_M)| : 0 \leq i_{j+1} < r_1(p), \ldots, 0 \leq i_M < r_M(p) ; 0 \leq k_1 < q_1(n), \ldots, 0 \leq k_M < q_M(n) \}. \]

Thus we obtain:

\[ H(\beta_{\rho_n}) \leq H(\beta_{\rho_n, \alpha_{\rho_n,p}}) = H(\alpha_{\rho_n,p}) + H(\beta_{\rho_n, \alpha_{\rho_n,p}}) \leq H(\alpha_{\rho_n,p}) + \sum_{f \in \rho_n} H(f | \beta | \alpha_{\rho_n,p}) \leq H(\alpha_{\rho_n,p}) + \sum_{f \in \rho_n} H(f | \beta | f | \lambda_{\rho_n}) = H(\alpha_{\rho_n,p}) + S(\rho_n) H(\beta | \alpha_{\rho_n,p}). \]

Assume that the modulus of \( \rho_n \) tends to infinity as \( n \to \infty \) as above. Then \( \rho_{n,p} \) tends to infinity too, and \( \frac{S(\rho_{n,p})}{S(\rho_n)} \to 1 \) as \( n \to \infty \). Thus after dividing out by \( S(\rho_n) \) we get for any fixed \( p \)

\[ h(\beta, G) \leq h(\alpha, G) + H(\beta | \alpha_{\rho_n,p}). \]

Now what remains is to send \( p \) to infinity and to observe that \( H(\beta | \alpha_{\rho_n,p}) \to H(\beta | \alpha_{\rho_n}) \) in order to get the desired relation. \( \square \)

**Lemma 4.3** (analogue of lemma 2.10) If \( \alpha, \beta \in \mathcal{L} \),

\[ \lim_{n \to \infty} H(\alpha | T_M^{-n}(\beta G) \alpha_G^-) = H(\alpha | \alpha_G^-). \]
5 K-systems and invariant partitions

This section is intended to reproduce in an appropriate form the theory of B. Kamiński [4] in the case of actions of the Heisenberg group as in section 2. Let \( G \) be such a group. We need a notion of \( \sigma \)-relative entropy for a \( G \)-invariant measurable partition \( \sigma \) which is defined by \( h_{\sigma}(G) = \sup_{\alpha \in \mathcal{L}} h(\alpha, G, \sigma) \). Also, by the Pinsker closure \( \overline{\sigma} \) of \( G \) with respect to \( \sigma \) we mean the join of all partitions \( \alpha \in \mathcal{L} \) such that \( h(\alpha, G, \sigma) = 0 \). Clearly \( \overline{\sigma} = \pi(G) \).

We list here some straightforward properties of \( \overline{\sigma} \):

1. \( \sigma \leq \overline{\sigma} \),
2. \( \overline{\sigma} \) is \( G \)-invariant,
3. \( \overline{\overline{\sigma}} = \overline{\sigma} \),
4. for any automorphism \( T \) commuting with \( G \) one has \( \overline{T \sigma} = \overline{\sigma} \).

There exists a \( \sigma \)-relative version of the Rokhlin-Sinai theorem concerning the existence of perfect partitions for a single automorphism \( S \) [11] to be generalized below. Specifically, we have:

**Lemma 5.1** There exists a measurable partition \( \zeta \) such that

(i) \( \sigma \leq S^{-1} \zeta \leq \zeta \),
(ii) \( \bigvee_{n=0}^{\infty} S^n \zeta = \epsilon \),
(iii) \( \bigwedge_{n=0}^{\infty} S^{-n} \zeta = \overline{\sigma} \),
(iv) \( h(\zeta, S, \sigma) = h(S, \sigma) \).

**Lemma 5.2** For every partition \( \alpha \in \mathcal{L} \) and \( G \)-invariant measurable partition \( \sigma_1, \sigma_2 \) one has \( h(\alpha, G, \overline{\sigma_1 \vee \sigma_2}) = h(\alpha, G, \sigma_1 \vee \sigma_2) \).

**Proof** is just the same as that of [4, lemma 2].

Define some special invariance properties related to the lexicographic order relation in \( G \). From now on we replace the ordinary notion of \( G \)-invariance \( (g \zeta = \zeta \text{ for all } g \in G) \) by the term total invariance.

A measurable partition \( \zeta \) is said to be \( G \)-invariant if \( T_1^{-1} \zeta \leq \zeta \), \( T_2^{-1} \zeta_{T_1} \leq \zeta \), and \( T_3^{-1} \zeta_{T_1T_2} \leq \zeta \). Note that \( \zeta \) is \( G \)-invariant iff \( T_3^{k_3}T_2^{k_2}T_1^{k_1} \zeta \leq \zeta \) for those triples \((k_1, k_2, k_3)\) which are lexicographically less than \( (0,0,0) \). Another important observation is that for \( \zeta \) \( G \)-invariant \( \zeta_G = T_1^{-1} \zeta \).

A measurable partition \( \zeta \) is said to be strongly invariant iff \( T_1^{-1} \zeta \leq \zeta \), \( \bigwedge_{n=0}^{\infty} T_1^{-n} \zeta = T_2^{-1} \zeta_{T_1}, \text{ and } \bigwedge_{n=0}^{\infty} T_2^{-n} \zeta_{T_1} = T_3^{-1} \zeta_{T_1T_2} \). Of course this notion is stronger than invariance.

A measurable partition \( \zeta \) is said to be \( G \)-exhaustive if \( \zeta \) is strongly invariant and \( \zeta_G = \epsilon \).

**Lemma 5.3** Suppose that a measurable partition \( \zeta \) is strongly invariant, \( \alpha \in \mathcal{L}, \alpha \leq T_3^p \zeta_{T_1T_2} \) for some positive integer \( p \), and \( h(\alpha, G) = 0 \). Then \( \alpha \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2} \).
Proof. Let us first prove our statement under the assumption \( \alpha \in \mathcal{L} \), \( \alpha \leq T_3^p T_2^q \zeta_{T_1} \) for some integer \( q \). It follows from \( h(\alpha, G) = 0 \) and (2.1) that for the finite index subgroup \( G_k \) generated by \( T_1^k, T_2^k, T_3^k \) one has \( h(\alpha, G_k) = 0 \). An application of an analogue of the relativized Pinsker formula (2.4) for the case \( G = \mathbb{Z} \) generated by \( T_1^k \) [1] yields for any partition \( \gamma \in \mathcal{L} \)

\[
H(\alpha \lor \gamma | \gamma T_1^k \lor \gamma T_2^k \lor (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2}) =
\]

\[
= H(\alpha | \gamma T_1^k \lor (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2}) + H(\gamma | \gamma T_1^k \lor (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2})
\]

\[
= H(\gamma | (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2}) + H(\alpha | \gamma T_1^k \lor (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2})
\]

It follows from our assumptions on \( \alpha \) and the strong invariance of \( \zeta \) that \( (\alpha T_1 T_2) T_1 \leq T_3^{p-1} \zeta_{T_1 T_2} \), and so \( H(\alpha | (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2}) = H(\alpha | (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2}) \) for all \( k \). If we assume \( \gamma \leq T_3^{p} T_2^{q} T_1^{r} \zeta \) for some integer \( r \) we get

\[
H(\gamma | \alpha \lor T_3^{p-1} \zeta_{T_1 T_2}) \geq H(\gamma | (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2}) =
\]

\[
= H(\gamma | (\alpha T_1^k) T_2^k \lor (\gamma T_1^k)^p \lor T_3^{p-1} \zeta_{T_1 T_2}) \geq H(\gamma | T_3^{p} T_2^{q} T_1^{r-k} \zeta \lor T_3^{p} T_2^{q} T_1^{r-k} \zeta_{T_1 T_2} \lor T_3^{p-1} \zeta_{T_1 T_2}).
\]

If we let in this relation \( k \) go to infinity and apply the strong invariance of \( \zeta \), we obtain

\[
H(\gamma | \alpha \lor T_3^{p-1} \zeta_{T_1 T_2}) \geq H(\gamma | T_3^{p} T_2^{q} T_1^{r} \zeta \lor T_3^{p-1} \zeta_{T_1 T_2}) = H(\gamma | T_3^{p} T_2^{q} T_1^{r} \zeta_{T_1 T_2}).
\]

Since this is valid for a class of partitions \( \gamma \) which is \( d \)-dense in the class of partitions \( \alpha \in \mathcal{L} \), \( \alpha \leq T_3^{p} T_2^{q} \zeta_{T_1} \), we can replace \( \gamma \) with \( \alpha \) in the latter inequality to get

\[
H(\alpha | T_3^{p} T_2^{q} T_1^{r} \zeta_{T_1 T_2}) = 0,
\]

and thus we have deduced \( \alpha \leq T_3^{p} T_2^{q} T_1^{r} \zeta_{T_1 T_2} \) from \( \alpha \leq T_3^{p} T_2^{q} \zeta_{T_1 T_2} \). If we proceed this way infinitely many times, we obtain \( \alpha \leq \bigwedge T_3^{p} T_2^{q} \zeta_{T_1 T_2} = T_3^{p} T_2^{q} \zeta_{T_1 T_2} \). Now we can get rid of our more subtle assumption \( \alpha \leq T_3^{p} T_2^{q} \zeta_{T_1 T_2} \) as compared with that written in the statement of the lemma via an approximation argument since \( \bigvee T_3^{p} T_2^{q} \zeta_{T_1 T_2} = T_3^{p} \zeta_{T_1 T_2} \).

Thus we have deduced \( \alpha \leq T_3^{p-1} \zeta_{T_1 T_2} \) from \( \alpha \leq T_3^{p} \zeta_{T_1 T_2} \). Now proceed by induction in \( p \) to get the desired statement. \( \square \)

A relativised version of lemma 5.3 is also valid. Specifically we have

**Lemma 5.4** Let \( \sigma \) be a \( G \)-totally invariant partition. Suppose also that a measurable partition \( \zeta \) is strongly invariant, \( \zeta \geq \sigma \), \( \alpha \in \mathcal{L} \), \( \sigma \leq \alpha \leq T_3^{p} \zeta_{T_1 T_2} \) for some positive integer \( p \), and \( h(\alpha, G) = 0 \). Then \( \alpha \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1 T_2} \).
Proof. One can reproduce literally the proof of lemma 5.3 with functionals like $H(\cdot | \cdot)$ being replaced by $H(\cdot | \cdot \vee \sigma)$ and $\pi(G)$ by $\mathfrak{F}$.

Lemma 5.5 For every $G$-exhaustive partition $\zeta$, $\bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2} \geq \pi(G)$.

Proof. Let $\alpha \in \mathcal{L}$ and $\alpha \leq \pi(G)$, that is $h(\alpha, G) = 0$, and assume also $\beta \in \mathcal{L}$ with $\beta \leq T_3^p \zeta_{T_1T_2}$ for some positive integer $p$. Denote also by $G_k$ the finite index subgroup generated by $T_1^k, T_2^k, T_3^k$, then certainly $h(\alpha, G_k) = 0$. Since this subgroup has quite the same structure as $G$ itself, we can apply the Pinsker formula (2.3) as follows:

$$h(\alpha \vee \beta, G_k) = h(\alpha, G_k) + H(\beta|\beta G_k \vee \alpha G_k) = h(\beta, G_k) + H(\alpha|\alpha G_k \vee \beta G_k).$$

Since $h(\alpha, G_k) = H(\alpha|\alpha G_k \vee \beta G_k) = 0$, we deduce that $h(\beta, G_k) = H(\beta|\beta G_k \vee \alpha G_k)$ for all $k$, whence

$$H(\beta|\alpha) \geq H(\beta|\beta G_k \vee \alpha G_k) = h(\beta, G_k) \geq H(\beta|T_1^{-k+1}\beta, G_k \vee T_2^{-k+1}(\beta T_1) \vee T_3^{-k+1}(\beta T_1 T_2) T_3).$$

If we let here $k$ go to infinity, we get $H(\beta|\beta) = H(\beta|\hat{\beta})$ with $\hat{\beta}$ being as in the definition of $\pi(\beta)$. Now our purpose is to prove that $\hat{\beta} \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2}$.

Let $\gamma \in \mathcal{L}$ and $\gamma \leq \hat{\beta}$. It was shown in the proof of theorem 3.1 that $\hat{\beta} \leq \pi(G)$, that is $h(\gamma, G) = 0$. Since $\beta \leq T_3^p \zeta_{T_1T_2}$ and $\zeta$ is $G$-invariant we have $\hat{\beta} \leq T_3^p \zeta_{T_1T_2}$ and hence $\gamma \leq T_3^p \zeta_{T_1T_2}$. Now by lemma 5.3 we deduce that $\gamma \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2}$, and hence our statement.

This, together with our previous observations imply

$$H(\beta|\alpha) \geq H(\beta|\bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2}).$$

Since $\zeta_G = \varepsilon$, we observe that the latter inequality is valid for $\beta$ in a dense subset of $\mathcal{L}$, and hence it is also valid for $\alpha$. Thus we obtain

$$0 = H(\alpha|\alpha) \geq H(\alpha|\bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2}),$$

that is, $\alpha \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2}$, which was to be proved.

We also need a relativized version of lemma 5.5.

Lemma 5.6 Let $\sigma$ be a $G$-totally invariant partition. For every $G$-exhaustive partition $\zeta$, $\zeta \geq \sigma$, one has $\bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2} \geq \sigma$.

Proof is just the same as that of lemma 5.5, with some minor changes indicated in the proof of lemma 5.4.
Lemma 5.7 There exists a measurable partition \( \eta \) with the properties:

(a) \( T_1^{-1} \eta \leq \eta, T_2^{-1} \eta_{T_1} \leq \eta, T_3^{-1} \eta_{T_1 T_2} \leq \eta \);
(b) \( \eta_G = \varepsilon \),

(c) \( \bigwedge_{n=0}^{\infty} T_3^{-n} \eta_{T_1 T_2} \leq \pi(G) \),

(d) \( h(G) = H(\eta|\eta^c_G) = H(\eta|T_1^{-1} \eta) \).

**Proof.** Let \( \alpha_k \in \mathcal{L} \) be an increasing sequence whose join is \( \varepsilon \). We claim that there exists an increasing sequence of positive integers \( n_1 < n_2 < \ldots \) such that for \( \xi_p = \bigvee_{k=1}^{p} T_3^{-n_k} \alpha_k \) one has

\[
H(\xi_p | (\xi_p)_{G}^-) - H(\xi_p | (\xi_p)_{G}^c) \leq \frac{1}{p}, \quad s \geq 0. \tag{5.1}
\]

To form such sequence of \( n_k \), it suffices to assure the property

\[
H(\xi_p | (\xi_{p-1})_{G}^-) - H(\xi_p | (\xi_{p-1})_{G}^c) < \frac{1}{p^2}, \quad p < q. \tag{5.2}
\]

In fact, if we take a sum in (5.2), we get \( H(\xi_p | (\xi_p)_{G}^-) - H(\xi_p | (\xi_p)_{G}^c) \leq \frac{1}{p} \) as desired. In order to obtain (5.2), we have to proceed by induction. Specifically, suppose \( n_1, \ldots, n_{q-1} \) have already been chosen, then \( n_q \) should be selected to be so large that (5.2) is valid with \( p = 1, \ldots, q-1 \). This is possible by a virtue of (2.5).

Let \( \xi = \bigvee_{p=1}^{\infty} \xi_p \), and \( \eta = \xi \lor \xi_G^- \). It is easy to verify that (a) and (b) are satisfied for \( \eta \).

Let us prove that (c) holds for \( \eta \). Since \( (\xi_p)_{G}^- \) is an increasing sequence of partitions whose join is \( \xi_G^- = \eta_G^- \), we can pass to a limit in (5.1) as \( s \to \infty \) to get

\[
H(\xi_p | (\xi_p)_{G}^-) - H(\xi_p | \eta_G^-) \leq \frac{1}{p}, \quad p \geq 1. \tag{5.3}
\]

Now let \( \alpha \in \mathcal{L} \) and \( \alpha \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \eta_{T_1 T_2} \). Since the latter partition is totally invariant with respect to \( G \), we have \( \alpha_G \leq \bigwedge_{n=0}^{\infty} T_3^{-n} \eta_{T_1 T_2} = \bigwedge_{n=0}^{\infty} T_3^{-n} \eta_{T_1 T_2} \). Hence \( \alpha_G \leq T_3^{-n} \eta_{T_1 T_2} \) for all \( n \geq 1 \), and so \( \alpha_G \leq \eta_{T_1 T_2} \).

It follows from the Pinsker formula (2.3) that

\[
h(\alpha \lor \xi_p, G) = h(\xi_p, G) + H(\alpha | \alpha_G \lor \xi_G) = h(\alpha, G) + H(\xi_p | (\xi_p)_{G}^- \lor \alpha_G),
\]

and hence

\[
h(\alpha, G) = h(\xi_p, G) - H(\xi_p | (\xi_p)_{G}^- \lor \alpha_G) + H(\alpha | \alpha_G \lor \xi_G).
\]

(5.4)

It follows from lemma 5.2 that

\[
H(\xi_p | (\xi_p)_{G}^- \lor \alpha_G) \geq H(\xi_p | (\xi_p)_{G}^- \lor \eta_{T_1 T_2} \lor \eta_{T_1 T_2}^{-1} T_3) = H(\xi_p | (\xi_p)_{G}^- \lor (\eta_{T_1 T_2} \lor \eta_{T_1 T_2}^{-1} T_3)) \geq
\]

\[
H(\xi_p | (\xi_p)_{G}^- \lor \eta_G) = H(\xi_p | \eta_G^-),
\]

and hence
and so (5.4) implies

\[ h(\alpha,G) \leq h(\xi_0,G) - H(\xi_0 \eta_G^{-1}) + H(\alpha|\alpha_\eta^{-1} \vee (\xi_0)_G) \leq \frac{1}{p} + H(\alpha|\alpha_\eta^{-1} \vee (\xi_0)_G). \]

Since \( \bigvee_{p=1}^{\infty} (\xi_p)_G = \varepsilon \), we can pass to a limit in the latter inequality as \( p \to \infty \) to get \( h(\alpha,G) = 0 \), that is \( \alpha \leq \pi(G) \).

To prove (d), observe that since \( \alpha_k \) increases up to \( \varepsilon \), \( H(\xi_0|\xi_0)_G = h(\xi_0,G) = h(\alpha_\eta,G) \to h(G) \). On the other hand, since \( \xi_p \) is increasing up to \( \xi \), one has also \( H(\xi_0|\xi_0)_G \to H(\xi|\xi_0)_G \). Now it follows from (5.3) that \( H(\xi_0|\xi_0)_G \) and \( H(\xi_0|\xi_0)_G = H(\xi_0|\xi_0)_G \) have the same limit, and so \( h(G) = H(\xi|\xi_0)_G = H(\eta|\xi_0)_G = H(\eta|\xi_0)_G \).

**Definition B.** A measurable partition \( \zeta \) is said to be \( G \)-perfect if

(i) \( T_1^{-1} \zeta \leq \zeta, T_2^{-1} \zeta t_1 \leq \zeta, T_3^{-1} \zeta t_1 t_2 \leq \zeta \),

(ii) \( \zeta_G = \varepsilon \),

(iii) \( \bigcap_{n=0}^{\infty} T_1^{-n} \zeta = T_2^{-1} \zeta t_1, \bigcap_{n=0}^{\infty} T_2^{-n} \zeta t_1 = T_3^{-1} \zeta t_1 t_2 \),

(iv) \( \bigcap_{n=0}^{\infty} T_3^{-n} \zeta t_1 t_2 = \pi(G) \),

(v) \( h(G) = H(\zeta|\zeta_0)_G = H(\zeta|T_1^{-1} \zeta) \).

**Theorem 5.8** For any \( G \)-action there exists a \( G \)-perfect partition.

**Proof.** Let \( \eta \) be a measurable partition which satisfies the properties (a) - (d) of lemma 5.7. The property (a) implies \( T_3^{-1} \eta t_1 t_2 \leq \eta t_1 t_2 \). Now a relativized version of this theorem for the case of \( \mathbb{Z}^2 \) generated by \( T_1, T_2 \) on the quotient space \( X/\eta t_1 t_2 \) implies that there exists a measurable partition \( \zeta \) with the properties:

\[ T_3^{-1} \eta t_1 t_2 \leq \zeta \leq \eta t_1 t_2, \quad (5.5) \]

\[ T_1^{-1} \zeta \leq \zeta, \quad T_2^{-1} \zeta t_1 \leq \zeta, \quad (5.6) \]

\[ \zeta t_1 t_2 = \eta t_1 t_2, \quad (5.7) \]

\[ \bigcap_{n=0}^{\infty} T_1^{-n} \zeta = T_2^{-1} \zeta t_1, \quad \bigcap_{n=0}^{\infty} T_2^{-n} \zeta t_1 = T_3^{-1} \eta t_1 t_2, \quad (5.8) \]

\[ h_{\eta t_1 t_2}(\zeta, (T_1, T_2)) = h_{\eta t_1 t_2}(T_1, T_2) = H(\zeta|T_1^{-1} \zeta \vee T_3^{-1} \eta t_1 t_2). \quad (5.9) \]

To prove this, one has to repeat the proof of [4, theorem 4], with some slight changes being introduced like those in the proof of lemma 5.4 and \( \sigma = T_3^{-1} \eta t_1 t_2 \).

It follows from (5.5) that \( T_3^{-1} \zeta t_1 t_2 \leq \eta t_1 t_2 \leq \zeta \); this together with (5.6) implies (i). An application of (5.5), (5.7), and (b) yields \( \bigvee_{n=0}^{\infty} T_3^n \zeta t_1 t_2 \geq \bigvee_{n=0}^{\infty} T_3^n \eta t_1 t_2 \geq \bigvee_{n=0}^{\infty} T_3^n \eta t_1 t_2 = \varepsilon, \) i.e. (ii) is true. (5.7), (5.8) imply (iii). Thus we have checked that \( \zeta \) is \( G \)-exhaustive, and so lemma 5.5 implies \( \bigvee_{n=0}^{\infty} T_3^n \zeta t_1 t_2 \geq \pi(G) \). It also follows from
(c) that $\bigwedge_{n=0}^{\infty} T_3^{-n} \bar{\eta}_{T_1T_2} \leq \pi(G)$, so in view of (5.7) we get (iv). Now what remains is to verify (v).

Let $\alpha_k \in \mathcal{L}$ be an increasing sequence of partitions with $\bigvee_{k} \alpha_k = \zeta$. Hence $(\bigvee_{k} \alpha_k)_G = \zeta_G = T_1^{-1} \zeta$, and so

$$H(\alpha_k | T_1^{-1} \zeta) \leq H(\alpha_k | (\alpha_k)_G) = h(\alpha_k, G) \leq h(G), \quad k \geq 1.$$ 

Pass to a limit as $k \to \infty$ to get $H(\zeta | T_1^{-1} \zeta) \leq h(G)$. Rewrite (5.9) in the form

$$H(\zeta | \zeta_{(T_1T_2)} \vee T_3^{-1} \bar{\eta}_{T_1T_2}) = \sup_{\alpha \in \mathcal{L}} H(\alpha | \alpha^{-}_{(T_1T_2)} \vee T_3^{-1} \bar{\eta}_{T_1T_2}). \quad (5.10)$$

In view of (5.5) one has $T_3^{-1} \bar{\eta}_{T_1T_2} \leq \zeta^{-}_{(T_1T_2)} \leq \bar{\eta}_{T_1T_2}$, and so

$$H(\zeta | \zeta^{-}_{(T_1T_2)} \vee T_3^{-1} \bar{\eta}_{T_1T_2}) = H(\zeta | \zeta^{-}_{(T_1T_2)}) = H(\zeta | T_1^{-1} \zeta). \quad (5.11)$$

It follows from lemma 5.2 that $H(\alpha | \alpha^{-}_{(T_1T_2)} \vee T_3^{-1} \bar{\eta}_{T_1T_2}) = H(\alpha | \alpha^{-}_{(T_1T_2)} \vee T_3^{-1} \eta_{T_1T_2})$, and so

$$\sup_{\alpha \in \mathcal{L}} H(\alpha | \alpha^{-}_{(T_1T_2)} \vee T_3^{-1} \bar{\eta}_{T_1T_2}) \geq \sup_{\alpha \in \mathcal{L}} H(\alpha | \alpha^{-}_{(T_1T_2)} \vee T_3^{-1} \eta_{T_1T_2}) \geq \sup_{\alpha \in \mathcal{L}, \alpha \leq g} H(\alpha | T_1^{-1} \eta) = H(\eta | T_1^{-1} \eta).$$

A comparison of this result to (d), (5.10), (5.11) yields $h(G) = H(\eta | T_1^{-1} \eta) \leq H(\zeta | T_1^{-1} \zeta)$ which completes the proof. 

\textbf{Lemma 5.9} The following conditions are equivalent:

(a) $h(G) = 0$,

(b) the only $G$-exhaustive partition is $\varepsilon$,

(c) each $G$-strongly invariant partitions is totally invariant.

\textbf{Proof.} We start with demonstrating the equivalence of (a) and (b). Let $h(G) = 0$, and $\zeta$ a $G$-exhaustive partition. It follows from lemma 5.5 that $\bigwedge_{n=0}^{\infty} T_3^{-n} \zeta_{T_1T_2} \geq \pi(G)$.

Since $h(G) = 0$, we have $\pi(G) = \varepsilon$, and then $\zeta_{T_1T_2} = \varepsilon$. On the other hand zeta has the property $\bigwedge_{n=0}^{\infty} T_2^{-n} \zeta_{T_1T_2} = T_3^{-1} \zeta_{T_1T_2}$, and hence $\zeta_{T_1} = \varepsilon$. In a similar way, apply the property $\bigwedge_{n=0}^{\infty} T_1^{-n} \zeta = T_2^{-1} \zeta_{T_1}$ to get $\zeta = \varepsilon$, so (a) implies (b).

Now let us suppose that (b) holds, and $\zeta$ be a perfect partition. In particular, $\zeta$ is $G$-exhaustive, and hence $\zeta = \varepsilon$. On the other hand the property (iv) implies $\pi(G) = \varepsilon$, that is $h(G) = 0$. Thus we have the equivalence of (a) and (b).

Check the equivalence of (a) and (c). Let $h(G) = 0$ and $\zeta$ be a strongly $G$-invariant partition. We have $0 = H(\zeta | \zeta_G) = H(\zeta | T_1^{-1} \zeta)$, and so $\zeta \leq T_1^{-1} \zeta$. Thus the strong invariance of $\zeta$ implies $T_1^{-1} \zeta = \zeta$, hence $\zeta = \bigwedge_{n=0}^{\infty} T_1^{-n} \zeta = T_2^{-1} \zeta_{T_1}$. Another application of the strong invariance yields $\zeta = \bigwedge_{n=0}^{\infty} T_2^{-n} \zeta_{T_1} = T_3^{-1} \zeta_{T_1T_2}$, and hence $\zeta$ is totally invariant.
Conversely, let us suppose that (c) is valid, and \( \zeta \) be a perfect partition. In particular, 
\( \zeta \) is strongly invariant and \( h(G) = H(\zeta | T^{-1}_1 \zeta) \). It follows from (c) that \( T^{-1}_1 \zeta = \zeta \), and hence \( h(G) = 0 \), which completes the proof. \[ \square \]

**Definition C.** \( G \) is said to be a K-group if there exists a measurable partition \( \zeta \) such that 
(i) \( T^{-1}_1 \zeta \leq \zeta \), \( T^{-1}_2 \zeta_{T_1} \leq \zeta \), \( T^{-1}_3 \zeta_{T_1 T_2} \leq \zeta \),
(ii) \( \zeta_G = \varepsilon \),
(iii) \( \bigwedge_{n=0}^{\infty} T^{-n}_1 \zeta = T^{-1}_2 \zeta_{T_1}, \bigwedge_{n=0}^{\infty} T^{-n}_2 \zeta_{T_1} = T^{-1}_3 \zeta_{T_1 T_2} \),
(iv) \( \bigwedge_{n=0}^{\infty} T^{-n}_3 \zeta_{T_1 T_2} = \nu \).

**Theorem 5.10** \( G \) is a K-group iff it has completely positive entropy.

**Proof.** Let \( G \) be a K-group and \( \zeta \) be a partition as in the definition above. In particular, \( \zeta \) is \( G \)-exhaustive, and thus by lemma 5.5 \( \pi(G) = \nu \), that is \( G \) has completely positive entropy. The converse is an easy consequence of theorem 5.8. \[ \square \]

The results of this section can be literally reformulated in the case of a general finitely generated nilpotent torsionfree group.

### 6 Relation between entropy, spectral and mixing properties for the Heisenberg group actions

Remind that a representation of \( G \) is said to have a countable Lebesgue spectrum if it is equivalent to a countable direct sum of the regular representation of \( G \).

**Lemma 6.1** ([10]). Let \( \zeta \) be a measurable partition such that \( T^{-1}_1 \zeta \leq \zeta \), and \((X/\zeta, \mu_\zeta)\) is nonatomic, then \( \dim(L^2(\zeta) \ominus L^2(T^{-1}_1 \zeta)) = \infty \).

**Theorem 6.2** If \( h(G) > 0 \) then \( G \) has a countable Lebesgue spectrum in \( L^2(X, \mu) \ominus L^2(\pi(G)) \).

**Proof.** Let \( \zeta \) be a \( G \)-perfect partition. It follows from \( \zeta_G = \varepsilon \) and \( \bigwedge_n T^n_3 \zeta(T_1 T_2) = \pi(G) \) that

\[ L^2(X, \mu) \ominus L^2(\pi(G)) = \bigoplus_{n=\infty}^{+\infty} L^2(T^n_3 \zeta(T_1 T_2)) \ominus L^2(T^{-n}_3 \zeta(T_1 T_2)) = \]

\[ = \bigoplus_{n=\infty}^{+\infty} U^n_{T_3}(L^2(\zeta(T_1 T_2)) \ominus L^2(T^{-n}_3 \zeta(T_1 T_2))). \]

In a similar way, the relation \( \bigwedge_n T^n_2 \zeta_{T_1} = T^{-n}_3 \zeta(T_1 T_2) \) implies

\[ L^2(\zeta(T_1 T_2)) \ominus L^2(T^{-n}_3 \zeta(T_1 T_2)) = \bigoplus_{n=\infty}^{+\infty} L^2(T^n_2 \zeta_{T_1}) \ominus L^2(T^{-n}_2 \zeta_{T_1}). \]
and, finally,

$$L^2(\zeta_{T_1}) \ast L^2(T_2^{-1}\zeta_{T_1}) = \bigoplus_{n=-\infty}^{+\infty} U_n^T(L^2(T_1^n \zeta) \ast L^2(T_1^{-n-1} \zeta)) = \bigoplus_{n=-\infty}^{+\infty} U_n^T(L^2(\zeta) \ast L^2(T_1^{-1} \zeta)).$$

Thus we obtain

$$L^2(X, \mu) \ast L^2(\pi(G)) = \bigoplus_{(k,m,n) \in \mathbb{Z}^3} U_k^T U_m^T U_n^T (L^2(\zeta) \ast L^2(T_1^{-1} \zeta)). \quad (6.1)$$

We claim that $T_1^{-1} \zeta \leq \zeta$. In fact, if one assumes the contrary $T_1^{-1} \zeta = \zeta$, one can deduce from the properties (ii), (iii), (iv) of a perfect partition that $\pi(G) = \varepsilon$, that is $h(G) = 0$, which contradicts to our assumptions on the $G$-action. Thus $T_1^{-1} \zeta \leq \zeta$. Now the properties (i) and (ii) imply that the space $(X/\zeta, \mu_\zeta)$ is nonatomic, and so by lemma 6.1 one has $\dim (L^2(\zeta) \ast L^2(T_1^{-1} \zeta)) = \infty$. Let $f_1, f_2, \ldots$ be a basis in $L^2(\zeta) \ast L^2(T_1^{-1} \zeta)$. It follows from (6.1) that $f_{g;i} = U_g f_i$, $1 \in \mathbb{N}$, $g \in G$, form an orthonormal basis in $L^2(X, \mu) \ast L^2(\pi(G))$, and $U_g f_{g;i} = f_{gh;i}$. This means that $G$ has a countable Lebesgue spectrum in $L^2(X, \mu) \ast L^2(\pi(G))$. \hfill \Box

As a consequence of theorems 5.10 and 6.2 we obtain

**Corollary 6.3** If $G$ is a $K$-group then it has a countable Lebesgue spectrum.

**Corollary 6.4** If $G$ has a singular spectrum or a spectrum with finite multiplicity then $h(G) = 0$. In particular, every group $G$ with discrete spectrum has zero entropy.

We describe here the relationship between the K-property and different mixing properties of $G$-actions. Given a finite collection $A_1, \ldots, A_r$ of sets of positive measure, we associate a finite partition $\pi$ corresponding to a (finite) $\sigma$-algebra of sets generated by the above sets. Let further $\rho_n$ be a sequence of rectangles in $G$ whose modulus tends to infinity, and $\mathcal{E}_{mn}(A_1, \ldots, A_r)$ be the sigma algebra of sets corresponding to the partition

$$\mathcal{E}_{mn} = T_1^{-m}(\mathcal{E}_{\rho_n}^T)_{T_1} \ast T_2^{-m}((\mathcal{E}_{\rho_n}^T)_{T_1})_{T_2} \ast T_3^{-m}((\mathcal{E}_{\rho_n}^{T_1 T_2})_{T_3}.$$  

**Definition**. A $G$-action is said to have the property of K-mixing (or briefly K-mixing) if for any sets $A_0, A_1, \ldots, A_r$, $r \geq 0$, and for each fixed $n$

$$\lim_{m \to \infty} \sup_{B \in \mathcal{E}_{mn}(A_1, \ldots, A_r)} |\mu(A_0 \cap B) - \mu(A_0) \mu(B)| = 0. \quad (6.2)$$

**Theorem 6.5** The following conditions are equivalent:

(i) the $G$-action is a $K$-system;

(ii) the $G$-action is K-mixing.
Proof. Suppose the $G$-action is a $K$-system, and $A_0, A_1, \ldots, A_r$, $r \geq 0$, be a finite collection of sets involved into the definition of the K-mixing property. Let $\pi_{mn}$ be a projection of $X$ onto the quotient space $X/\mathcal{Z}_{mn}$, and consider the associated decomposition of $\mu$ with respect to this projection: $\mu = \int \mu_{mn}^b d(\mu \circ \pi^{-1})(y)$. Given any $B \in \mathcal{E}_{mn}^\infty(A_1, \ldots, A_r)$, one has

$$|\mu(A_0 \cap B) - \mu(A_0)\mu(B)| = \left| \int_{\pi(B)} \mu_{mn}^b (A_0) d(\mu \circ \pi^{-1})(y) - \int_{\pi(B)} \mu(A_0) d(\mu \circ \pi^{-1})(y) \right| \leq \int_{\pi(B)} |\mu_{mn}^b(A_0) - \mu(A_0)| d(\mu \circ \pi^{-1})(y).$$

We know from theorems 3.1 and 5.10 that $\mathcal{X}_\infty = \pi(\mathcal{X}) = \nu$, and hence $\bigcap_m \mathcal{E}_{mn}^\infty(A_1, \ldots, A_r)$ may contain only sets of measure 0 or 1. Now an application of Doob’s theorem on convergence of conditional expectations shows that $|\mu_{mn}^b(A_0) - \mu(A_0)| \to 0$ as $m \to \infty$, and hence the limit relation (6.2).

Conversely, suppose the $G$-action is K-mixing, and let $\mathcal{X} = \{A_1, \ldots, A_r\}$ be a finite partition such that $h(\mathcal{X}, G) = 0$. Since clearly $\mathcal{X} \leq \mathcal{X}_G$ we can deduce from theorem 3.1 that $\mathcal{X} \leq \pi(\mathcal{X}) = \mathcal{X}_\infty$.

Let $A_i$ be any of the sets that constitute $\mathcal{X}$. It follows from $\mathcal{X} \leq \mathcal{X}_\infty = \bigvee \mathcal{Z}_{p,m}$ that one can select a sequence of sets $B_n \in \bigcap_m \mathcal{E}_{mn}^\infty(A_1, \ldots, A_r)$ such that $\mu(A_i \Delta B_n) \to 0$ as $n \to \infty$. Since one also has $B_n \in \mathcal{E}_{mn}^\infty(A_1, \ldots, A_r)$ for all $m$, we deduce from (6.2) that $|\mu(A_i \cap B_n) - \mu(A_i)| \mu(B_n)| = 0$ for all $n$, and after passage to a limit in $n$ one has $\mu(A_i) = \mu(A_i)\mu(A_i)$. Thus, $\mu(A_i) = 0$ or 1, which means that $\mathcal{X} = \nu$. Thus we have proved the complete positivity of the $G$-action, and theorem 5.10 implies that this is a K-system.

Definition E. A $G$-action is said to have the property of $r$-mixing if for all measurable sets $A_0, A_1, \ldots, A_r$

$$\lim_{g_1, \ldots, g_r \to -\infty} \mu(A_0 \cap g_1 A_1 \cap \ldots \cap g_r \cdot \ldots \cdot g_r A_r) = \mu(A_0) \cdot \ldots \cdot \mu(A_r).$$

Here $g_p = T_3^{l_p} T_2^{m_p} T_1^{n_p} \to -\infty$ means that for any $h = T_3^{k_3} T_2^{k_2} T_1^{k_1}$ there exists $N$ such that for all $p > N$ $(l_p, m_p, n_p)$ is lexicographically less than $(k_1, k_2, k_3)$.

Theorem 6.6 If a $G$-action is a $K$-system, then it is $r$-mixing for all $r \geq 1$.

Proof. Our argument is based on the property of K-mixing which is equivalent to the K-property by theorem 6.5. We proceed by induction. If $r = 1$, our statement is straightforward from the definition of K-mixing. Suppose the $G$-action is $r$-mixing and we are given sets $A_0, A_1, \ldots, A_{r+1}$ and an $\varepsilon > 0$. It follows from the K-mixing property that there exists some $h \in G$ such that for all $g_1, \ldots, g_{r+1}$ which are lexicographically less than $h$

$$\mu(A_0 \cap g_1 A_1 \cap \ldots \cap g_{r+1} A_{r+1}) - \mu(A_0)\mu(g_1 A_1 \cap \ldots \cap g_{r+1} \cdot \ldots \cdot g_{r+1} A_{r+1}) < \varepsilon. \quad (6.3)$$
Replace in (6.3) \( \mu(g_1 A_1 \cap \cdots \cap g_1 A_{r+1}) \) by \( \mu(A_1 \cap \cdots \cap g_1^{-1} g_{r+1} \cdots g_1 A_{r+1}) \). Observe that \( g_1^{-1} g_{r+1} \cdots g_1 \) coincides with \( g_{r+1} \cdots g_2 \) modulo center of \( G \) (which is of the lowest lexicographical order), and hence we may apply the induction hypothesis (r-mixing) to obtain the desired result

\[ |\mu(A_0 \cap g_1 A_1 \cap \cdots \cap g_1 \cdots g_{r+1} A_{r+1}) - \mu(A_0) \cdots \mu(A_{r+1})| < \varepsilon. \]

\[ \blacksquare \]

7 Perfect partitions and K-systems: the general case

Describe briefly how to extend the results of sections 5, 6 onto actions of finitely generated nilpotent torsionfree groups. The reader is referred to section 4 and Appendix for a notation. We again restrict ourselves to those statements which are reproduced not literally.

A measurable partition \( \zeta \) is said to be \( G \)-invariant if \( T_1^{-1} \zeta \leq \zeta, \ldots, T_M^{-1} \zeta_{T_1 \cdots T_{M-1}} \leq \zeta \).

A measurable partition \( \zeta \) is said to be strongly invariant iff \( T_1^{-1} \zeta \leq \zeta, \bigwedge_{n=0}^{\infty} T_n^{-1} \zeta = T_2^{-1} \zeta_{T_1}, \ldots, \bigwedge_{n=0}^{\infty} T_{M-1}^{-n} \zeta_{T_1 \cdots T_{M-2}} = T_M^{-1} \zeta_{T_1 \cdots T_{M-1}}. \)

**Lemma 7.1** (analogue of lemma 5.3) Suppose that a measurable partition \( \zeta \) is strongly invariant, \( \alpha \in \mathcal{L}, \alpha \leq T_M^{p} \zeta_{T_1 T_2} \) for some positive integer \( p \), and \( h(\alpha, G) = 0 \). Then

\[ \alpha \leq \bigwedge_{n=0}^{\infty} T_M^{-n} \zeta_{T_1 \cdots T_{M-1}}. \]

**Lemma 7.2** (analogue of lemma 5.5) For every \( G \)-exhaustive partition \( \zeta \) one has

\[ \bigwedge_{n=0}^{\infty} T_M^{-n} \zeta_{T_1 \cdots T_{M-1}} \geq \pi(G). \]

**Lemma 7.3** (analogue of lemma 5.7) There exists a measurable partition \( \eta \) with the properties:

(a) \( T_1^{-1} \eta \leq \eta, \ldots, T_M^{-1} \eta_{T_1 \cdots T_{M-1}} \leq \eta \),
(b) \( \eta_G = \varepsilon \),
(c) \( \bigwedge_{n=0}^{\infty} T_M^{-n} \eta_{T_1 \cdots T_{M-1}} \leq \pi(G) \),
(d) \( h(G) = H(\eta|\eta_G) = H(\eta|T_1^{-1} \eta). \)

**Definition F** (analogue of Definition B) A measurable partition \( \zeta \) is said to be \( G \)-perfect if

(i) \( T_1^{-1} \zeta \leq \zeta, \ldots, T_M^{-1} \zeta_{T_1 \cdots T_{M-1}} \leq \zeta \),
(ii) \( \zeta_G = \varepsilon \),
(iii) \( \bigwedge_{n=0}^{\infty} T_1^{-n} \zeta = T_2^{-1} \zeta_{T_1}, \ldots, \bigwedge_{n=0}^{\infty} T_{M-1}^{-n} \zeta_{T_1 \cdots T_{M-2}} = T_M^{-1} \zeta_{T_1 \cdots T_{M-1}} \),
(iv) \( \bigwedge_{n=0}^{\infty} T_M^{-n} \zeta_{T_1 \cdots T_{M-1}} = \pi(G) \),
(v) \( h(G) = H(\zeta|\zeta_G) = H(\zeta|T_1^{-1} \zeta). \)
With the above background, the results of section 6 concerning the spectral and mixing properties can be literally reformulated in the case of a general finitely generated nilpotent torsionfree group.

Appendix

Let $G$ be a countable nilpotent finitely generated torsionfree group and

$$G = G^N \supset G^{N-1} \supset G^{N-2} \supset \ldots \supset G^2 \supset G^1 \supset G^0 = \{e\} \quad (A.1)$$

its lower central series. Specifically, $G^{N-1} = [G, G]$, the commutant subgroup, and $G^i = [G, G^{i+1}]$. Our first step is to verify that each $G^i$ is finitely generated. For that, we prove

**Lemma A.1.** Let $G$ be a nilpotent finitely generated group. There exists a finite (ordered) sequence of generators

$$S_1, \ldots, S_{n_1}, S_{n_1+1}, \ldots, S_{n_2}, S_{n_2+1}, \ldots, S_{n_{N-1}}, S_{n_{N-1}+1}, \ldots, S_{n_N} \in G$$

such that $n_0 = 0$, $S_{n_i+1}, \ldots, S_{n_N} \in G^{N-i}$, and $S_k \notin G^{N-i}$ for $k \leq n_i$, $j \in \mathbb{Z} \setminus \{0\}$.

**Proof.** We proceed by induction in the nilpotence order $N$. In the case $N = 1$ our statement is trivial. Now suppose that this statement is true for any finitely generated $N - 1$-step nilpotent group, and consider $N$-step nilpotent $G$.

Consider the quotient group $G/G^1$. We can apply the induction hypothesis to $G/G^1$ in order to choose in it a sequence of generators

$$\overline{S}_1, \ldots, \overline{S}_{n_1}, \overline{S}_{n_1+1}, \ldots, \overline{S}_{n_2}, \overline{S}_{n_2+1}, \ldots, \overline{S}_{n_{N-1}+1}, \ldots, \overline{S}_{n_N}$$

with the required properties. Let it be a projection of a (ordered) sequence

$$S_1, \ldots, S_{n_1}, S_{n_1+1}, \ldots, S_{n_2}, S_{n_2+1}, \ldots, S_{n_{N-1}+1}, \ldots, S_{n_N}$$

of elements of $G$. We are to complete it with a finite sequence of generators $S_{n_{N-1}+1}, \ldots, S_{n_N}$ of $G^1$. For that, we consider the finite subset of $G^1$

$$\{(S_i^+, S_j^-) : 1 \leq i \leq n_{N-1}; n_{N-2} + 1 \leq j \leq n_{N-1}\}. \quad (A.2)$$

We claim this set generates $G^1$. This is an easy consequence of the definition $G^1 = [G, G^2]$, together with the following observation, based on the fact that all the above commutators are central in $G$. Given $f_1, f_2 \in G$, $g \in G^2$, one has

$$[f_1 f_2, g] = f_1 (f_2 g f_2^{-1} g^{-1}) g f_2^{-1} g^{-1} = [f_1, g][f_2, g].$$
and in a similar way, for \( f \in G, g_1, g_2 \in G^2 \),
\[
[f, g_1g_2] = [f, g_1][f, g_2].
\]

What remains is to replace the set \((A.2)\) with a finite minimal subset which still generates \(G\). 

**Corollary A.2.** The commutant subgroup \([G, G]\) is finitely generated.

Now consider the upper central series of \(G\). Specifically, let \(Z_i\) be the center of \(G\). Define the subgroup \(Z_2\) in such a way that \(Z_2/Z_1 = \text{center}(G/Z_1)\). Continue this procedure by choosing at step \(i\) the subgroup \(Z_i\) so that \(Z_i/Z_{i-1} = \text{center}(G/Z_{i-1})\). It is easy to verify that after finitely many steps we get \(Z_N = G\), with \(N\) being exactly the same as in \((A.1)\) [6]. Thus we have a finite sequence of subgroups
\[
G = Z_N \supset Z_{N-1} \supset Z_{N-2} \supset \ldots \supset Z_2 \supset Z_1 \supset Z_0 = \{e\} \tag{A.3}
\]

We start with observing that the property of being torsionfree is inherited by the quotient groups in \((A.3)\).

**Proposition A.3.** For each \( i = 1, \ldots, N, n > 0, x \in Z_i, \) and \( x^n \in Z_{i-1} \) implies \( x \in Z_{i-1} \).

**Proof.** We proceed by an induction in \(i\). In the case \( i = 1 \) our statement reduces to observing that \( Z_0 = \{e\} \) and \( Z_1(\subseteq G) \) is torsionfree.

Now suppose that our proposition is valid for \(i - 1\), and prove it for \(i\). That is, we assume \( x \in Z_i \) and \( x^n \in Z_{i-1} \). For the sake of simplicity we stick to the case \( n = 2\); the general case can be treated in a similar way. Suppose the contrary: \( x \notin Z_{i-1} \). Since \(Z_{i-1}/Z_{i-2} = \text{center}(G/Z_{i-2})\), we deduce that there is some \( y \in G \) with \([x, y] \notin Z_{i-2}\). On the other hand, of course \([x, y] \in Z_i\), since \(Z_i/Z_{i-1} = \text{center}(G/Z_{i-1})\). In a similar way one can verify that \([x^2, y] \in Z_{i-2}\). Since also
\[
[x^2, y] = xx^i y^{-1} x^{-1} y^{-1} = x[x, y]y x^{-1} y^{-1} \in [x, y]^2 Z_{i-2},
\]
we deduce that \([x, y]^2 \in Z_{i-2}\). Now the induction hypothesis implies \([x, y] \in Z_{i-2}\), which contradicts to our choice of \(y\). This proves our statement.

**Corollary A.4.** Let \( x \in G \) be such that \( x^n \in Z_i \) for some \( n > 0 \). Then \( x \in Z_i \).

**Proof.** Choose \( i \) so that \( x \in Z_i \setminus Z_{i-1} \). If we assume \( i > 1 \), then \( Z_{i-1} \supset Z_1 \), and so \( x^n \in Z_{i-1} \). Now an application of proposition A.1 yields \( x \in Z_{i-1} \). This contradicts to our choice of \(i\). Thus \( i = 1 \), which is exactly our statement.

**Lemma A.5.** Let \( G \) be a nilpotent finitely generated torsionfree group. There exists a finite (ordered) sequence of generators
\[
T_1, \ldots, T_n, T_{n+1}, \ldots, T_{n_2}, T_{n_2+1}, \ldots, T_{n_N}, T_{n_N+1}, \ldots, T_{n_N} \in G
\]
such that \( n_0 = 0, T_{n+1}, \ldots, T_{n_N} \in Z_{N-i} \), and \( T_{k} \notin Z_{N-i} \) for \( k \leq n_i, j \in \mathbb{Z} \setminus \{0\} \).

**Proof.** We proceed by induction in \(N\). In the case \( N = 1 \) our statement is trivial. Now suppose that this statement is true for any nilpotent finitely generated torsionfree group with the length \( N - 1 \) series \((A.3)\), and consider \( G \) with \((A.3)\) of length \( N\).

Consider the quotient group \( G/Z_i \). By corollary A.4, it is torsionfree. We can apply the induction hypothesis to \( G/Z_i \) in order to choose in it a sequence of generators
\[
\overline{T_1}, \ldots, \overline{T_{n_1}}, \overline{T_{n_1+1}}, \ldots, \overline{T_{n_2}}, \overline{T_{n_2+1}}, \ldots, \overline{T_{n_N+1}}, \ldots, \overline{T_{n_N}}
\]
with the required properties. Let it be a projection of a (ordered) sequence

\[ T_1, \ldots, T_n, T_{n+1}, \ldots, T_{n_2}, T_{n_2+1}, \ldots, T_{n_{N-2}+1}, \ldots, T_{n_N-1} \]

of elements of \( G \). We are to complete it with a sequence of generators \( T_{n_{N-1}+1}, \ldots, T_{n_N} \) of \( Z_1 = \text{center}(G) \). For this sequence to be finite, we claim that \( Z_1 \) is always finitely generated. This is accessible by an induction in \( N \).

In the case \( N = 1 \) (\( G \) Abelian) our statement is trivial. Suppose it is valid for \( N-1 \)-step nilpotent finitely generated torsionfree groups, and let \( G \) be \( N \)-step nilpotent. Consider \( Z_c = [G, G] \cap Z_1 \). Since \([G, G]\) is a finitely generated by corollary \( A.2 \), it follows from the induction hypothesis that \( \text{center}([G, G]) \) is finitely generated, and hence so is its subgroup \( Z_c \). On the other hand, since \( Z_c \subset [G, G] \), the quotient group \( Z_1/Z_c \) is embeddable into an Abelian finitely generated group \( G/[G, G] \), and so one can form a finite sequence of generators for \( Z_1 \) via merging those for \( Z_c \) and \( Z_1/Z_c \), after raising the latter up to elements of \( Z_1 \).

\[ \blacksquare \]

References


