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Abstract

We study the convergence rates of solutions to drift-diffusion systems (arising from plasma, semiconductors and electrolytes theories) to their self-similar or steady states. This analysis involves entropy-type Lyapunov functionals and logarithmic Sobolev inequalities.

Key words and phrases: drift-diffusion systems, asymptotic behavior of solutions, logarithmic Sobolev inequalities

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1 Introduction

We consider the long time asymptotics of solutions to drift-diffusion systems

$$u_t = \nabla \cdot (\nabla u + u \nabla \phi) \,, \tag{1.1}$$

$$v_t = \nabla \cdot (\nabla v - v \nabla \phi) \,, \tag{1.2}$$

$$\Delta \phi = v - u \,, \tag{1.3}$$

where u, v denote densities of negatively, respectively positively, charged particles. The Poisson equation (1.3) defines the electric potential ϕ coupling the equations (1.1)-(1.2) for the temporal evolution of charge distributions. The system (1.1)-(1.3) was formulated by W. Nernst and M. Planck at the end of the nineteenth century as a basic model for electrodiffusion of ions in electrolytes filling the whole space $\mathbb{R}^3$. Note that the case of multicharged particles is also covered by (1.1)-(1.3) since u and v denote the charge densities.

Supplemented with the no-flux boundary conditions

$$\frac{\partial u}{\partial \nu} + u \frac{\partial \phi}{\partial \nu} = 0 \,, \tag{1.4}$$

$$\frac{\partial v}{\partial \nu} - v \frac{\partial \phi}{\partial \nu} = 0 \tag{1.5}$$

on the boundary of a bounded domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, and either

$$\phi = 0 \quad \text{on } \partial\Omega \,, \tag{1.6}$$

or

$$\phi = E_d * (v - u) \,, \tag{1.7}$$

where E_d is the fundamental solution of the Laplacian in $\mathbb{R}^d$, the system (1.1)-(1.3) was also studied by P. Debye and E. Hückel in the 1920's. (1.6) signifies a conducting boundary of the container, while in the case of a bounded domain the "free" boundary condition (1.7) corresponds to a container immersed in a medium with the same dielectric constant as the solute.

These equations, together with their generalizations including *e.g.* an exterior potential, known as drift-diffusion Poisson systems, also appear in plasma physics and (supplemented with some mixed linear boundary conditions instead of (1.4)-(1.5)) in semiconductor device modelling.

To determine completely the evolution, the initial conditions

$$u(x,0)=u_0(x)\,,\quad v(x,0)=v_0(x) \tag{1.8}$$

are added. Obviously, positivity of $u_0\geq 0$, $v_0\geq 0$ is conserved: $u(x,t)\geq 0$, $v(x,t)\geq 0$, as well as the total charges

$$M_u=\int u_0(x)\,dx=\int u(x,t)\,dx\,,\quad M_v=\int v_0(x)\,dx=\int v(x,t)\,dx\,. \tag{1.9}$$

Here M_u, M_v are not necessarily the same, *i.e.* the electroneutrality condition

$$M_u=M_v \tag{1.10}$$

is not, in general, required. Condition (1.10) must be satisfied in the case of the homogeneous Neumann boundary conditions $\frac{\partial\phi}{\partial\nu}=0$ (*i.e.* an isolated wall of the container) leading together with (1.4)-(1.5) to

$$\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=\frac{\partial\phi}{\partial\nu}=0\,.$$

Our results (Theorem 1.2 below) are valid in that case, with even a simpler proof.

The asymptotic properties of solutions to (1.1)-(1.3), (1.7) have been studied recently in [1]. The authors proved that (for $d\geq 3$, $M_u=M_v=1$ and u_0 and v_0 regular enough) u, v tend to their self-similar asymptotic states at an algebraic rate. We improve these results by relaxing assumptions on the initial data and showing a stronger (still algebraic) decay rate, which we expect to be optimal (see Theorem 1.1 below).

In the case of a bounded domain, the convergence (with no specific speed) in the L^1-norm of u and v solving (1.1)-(1.5) to their corresponding steady states has been proved in [5] (as well as the L^∞-convergence for more regular u_0, v_0). Here we prove the exponential convergence towards the steady states with a decay rate depending on $d\geq 2$, M_u, M_v, and the initial value of the entropy functional only (see Theorem 1.2 below).

Notation. The L^p-norm in $\mathbb{R}^d$ or $\Omega\subset\subset\mathbb{R}^d$ is denoted by $|\cdot|_p$, and inessential constants (which may vary from line to line) are denoted generically by C.

Define the asymptotic states in $\mathbb{R}^d$ by

$$u_{as}(x,t) = \frac{M_u}{(2\pi(2t+1))^{d/2}} \exp\left(-\frac{|x|^2}{2(2t+1)}\right), \tag{1.11}$$

$$v_{as}(x,t) = \frac{M_v}{(2\pi(2t+1))^{d/2}} \exp\left(-\frac{|x|^2}{2(2t+1)}\right), \tag{1.12}$$

where the charges of the solution $\langle u,v\rangle$ of (1.1)-(1.2) are given by (1.9), and the entropy functional by

$$L(t) = \int u(x,t)\log\left(\frac{u(x,t)}{u_{as}(x,t)}\right) dx + \int v(x,t)\log\left(\frac{v(x,t)}{v_{as}(x,t)}\right) dx + \frac{1}{2}|\nabla\phi(t)|_2^2. \tag{1.13}$$

Theorem 1.1 *There exists a constant $C = C(d, M_u, M_v, L_0)$ such that for each solution $\langle u,v\rangle$ of (1.1)-(1.3), (1.7)-(1.8) in $\mathbb{R}^d$, $d \geq 3$, if $L(0) = L_0$, then for all $t \geq 0$,*

$$L(t) \leq C\, H(t) \tag{1.14}$$

and

$$|u(t) - u_{as}(t)|_1^2 + |v(t) - v_{as}(t)|_1^2 + |\nabla\phi(t)|_2^2 \leq C\, H(t), \tag{1.15}$$

where

$$H(t) = \begin{cases} (2t+1)^{-1/2}, & d = 3, \\ (2t+1)^{-1}\Big(\log(2t+1)+1\Big), & d = 4, \\ (2t+1)^{-1}, & d > 4. \end{cases}$$

Moreover if $M_u = M_v$, then $H(t) = (2t+1)^{-1}$ for any $d \geq 3$.

In the case of a bounded domain, define the entropy functional

$$\begin{aligned} W(t) \ &= \int u(x,t)\log\left(\frac{u(x,t)}{U(x)}\right) dx + \int v(x,t)\log\left(\frac{v(x,t)}{V(x)}\right) dx \\ &+ \frac{1}{2}\int (\phi - \Phi)(u - U - v + V)\, dx, \end{aligned} \tag{1.16}$$

for the solution $\langle u,v,\phi\rangle$ of (1.1)-(1.5), (1.6) or (1.7), (1.8) and the unique steady state $\langle U,V,\Phi\rangle$ of the Debye-Hückel system with

$$M_u = \int U(x)\, dx, \quad M_v = \int V(x)\, dx. \tag{1.17}$$

Note that for the condition (1.6) the third term in $W(t)$ takes the form $\frac{1}{2}|\nabla(\phi-\Phi)|_2^2$.

Theorem 1.2 *If $d=2$, then there exist two constants $\lambda=\lambda(d,M_u,M_v,W_0)$ and $C=C(M_u,M_v,W_0)$ such that for each solution $\langle u,v,\phi\rangle$ of (1.1)-(1.6), (1.8) in a bounded domain Ω, if $W(0)=W_0$, then for all $t\geq 0$,*

$$W(t)\leq W(0)\,e^{-\lambda t}, \tag{1.18}$$

and

$$|u(t)-U|_1^2+|v(t)-V|_1^2+|\nabla(\phi-\Phi)|_2^2\leq C\,e^{-\lambda t}. \tag{1.19}$$

If $d\geq 3$, the same results hold except that λ and C may also depend on $\sigma=\sup\limits_{t\geq 0}|\phi(t)|_\infty$, which is supposed to be finite.

Sufficient conditions for σ to be finite in the case of linear boundary conditions and explicit estimates of λ and C will be provided in the proof of this result.

2 Proof of Theorem 1.1

We begin with a rescaling of the system (1.1)-(1.3) which will lead to a system with a quadratic confinement potential, and therefore (eliminating the dispersion) to the expected exponential convergence to the steady states. This idea was applied in [8] and [7], as well as in [1], to a variety of problems ranging from kinetic equations to porous media equations.

Let $\bar{x}\in\mathbb{R}^d$, $\tau>0$, be the new variables defined by

$$\bar{x}=\frac{x}{R(t)}, \quad \tau=\log R(t), \quad R(t)=(2t+1)^{1/2}, \tag{2.1}$$

and consider the rescaled functions $\bar{u}$, $\bar{v}$, $\bar{\phi}$ such that

$$\begin{aligned} u(x,t)&=\frac{1}{R^d(t)}\bar{u}(\bar{x},\tau)\,, \\ v(x,t)&=\frac{1}{R^d(t)}\bar{v}(\bar{x},\tau)\,, \\ \phi(x,t)&=\bar{\phi}(\bar{x},\tau)\,. \end{aligned} \tag{2.2}$$

This whole section will deal with the rescaled system, so omitting the bars over x, u, v, ϕ will not lead to confusions with the original system, which now takes, after rescaling, the form

$$u_\tau = \nabla \cdot (\nabla u + ux + u\nabla\phi)\,, \tag{2.3}$$

$$v_\tau = \nabla \cdot (\nabla v + vx - v\nabla\phi)\,, \tag{2.4}$$

$$\Delta\phi = e^{-\tau(d-2)}(v-u)\,. \tag{2.5}$$

The scaling (2.2) preserves the L^1-norms, so the rescaled initial data u_0, v_0 still satisfy

$$M_u = \int u_0(x)\,dx = \int u(x,\tau)\,dx\,, \quad M_v = \int v_0(x)\,dx = \int v(x,\tau)\,dx\,. \tag{2.6}$$

Denote by $\langle u_\infty, v_\infty\rangle$ the steady state of (2.3)-(2.4), that is

$$u_\infty(x) = \frac{M_u}{(2\pi)^{d/2}} \exp\left(-\frac{|x|^2}{2}\right)\,, \tag{2.7}$$

$$v_\infty(x) = \frac{M_v}{(2\pi)^{d/2}} \exp\left(-\frac{|x|^2}{2}\right)\,. \tag{2.8}$$

Of course, going back to the original variables x, t, $\langle u_\infty, v_\infty\rangle$ corresponds to the asymptotic state $\langle u_{as}, v_{as}\rangle$ defined by (1.11)-(1.12). Writing $\phi = \beta\psi$ with $\beta = \beta(\tau) = e^{-\tau(d-2)} \to 0$ as $\tau \to +\infty$, we introduce the relative entropy

$$W(\tau) = \int u \log\left(\frac{u}{u_\infty}\right)\,dx + \int v \log\left(\frac{v}{v_\infty}\right)\,dx + \frac{\beta}{2}|\nabla\psi|_2^2 \tag{2.9}$$

corresponding to the original entropy functional L in (1.13). The evolution of W is given by

$$\frac{dW}{d\tau} = -\int u\left|\nabla\left(\log\frac{u}{U}\right)\right|^2 dx - \int v\left|\nabla\left(\log\frac{v}{V}\right)\right|^2 dx - \left(\frac{d}{2}-1\right)\beta|\nabla\psi|_2^2\,, \tag{2.10}$$

with U, V denoting the local Maxwellians

$$U(x,\tau) = M_u\,\frac{\exp\left(-\frac{1}{2}|x|^2 - \phi(x,\tau)\right)}{\int \exp\left(-\frac{1}{2}|y|^2 - \phi(y,\tau)\right)\,dy}\,, \tag{2.11}$$

$$V(x,\tau) = M_v \frac{\exp\left(-\frac{1}{2}|x|^2 + \phi(x,\tau)\right)}{\int \exp\left(-\frac{1}{2}|y|^2 + \phi(y,\tau)\right)\, dy}, \tag{2.12}$$

so that $\nabla U/U = -(x+\nabla\phi)$, $\nabla V/V = -(x-\nabla\phi)$. Using the notation

$$J = \frac{1}{2}\int u\left|\frac{\nabla u}{u} + x\right|^2 dx + \frac{1}{2}\int v\left|\frac{\nabla v}{v} + x\right|^2 dx, \tag{2.13}$$

(2.10) can be rewritten as

$$\begin{aligned} \frac{dW}{d\tau} &= -2J - 2\int (\nabla u - \nabla v)\cdot\nabla\phi\, dx - 2\int (u-v)\, x\cdot\nabla\phi\, dx \\ &\qquad - \int (u+v)|\nabla\phi|^2\, dx - \left(\frac{d}{2} - 1\right)\beta|\nabla\psi|_2^2 \\ &= -2J - \beta^2\int (u+v)|\nabla\psi|^2\, dx - 2\beta|u-v|_2^2 + \left(\frac{d}{2}-1\right)\beta|\nabla\psi|_2^2. \end{aligned} \tag{2.14}$$

The quantity J in (2.13) can be estimated from below using the Gross logarithmic Sobolev inequality

$$\int f\log\left(\frac{f}{|f|_1}\right) dx + d\left(1 + \frac{1}{2}\log(2\pi a)\right)|f|_1 \le \frac{a}{2}\int \frac{|\nabla f|^2}{f}\, dx \tag{2.15}$$

valid for each $a>0$, see *e.g.* [11] or a thorough discussion of different versions of logarithmic Sobolev inequalities in [2]. (2.15) becomes an equality if and only if $f(x) = C\exp(-|x|^2/(4a))$ (up to a translation).

Taking $a=1$ in (2.15), the relation (2.14) leads to

$$-\left(\frac{dW}{d\tau} + 2W\right) \ge 2\beta|u-v|_2^2 - \beta\frac{d}{2}|\nabla\psi|_2^2 \ge -C\beta(M_u+M_v)^2 \tag{2.16}$$

with a constant $C = C(d) = \frac{2}{d}\left(\frac{d-2}{4}\right)^{(d-2)/2}\Sigma^{d/2}$, because by the Hardy-Littlewood-Sobolev inequality and an interpolation

$$|\nabla\psi|_2^2 \le \Sigma|u-v|_{2d/(d+2)}^2 \le \Sigma|u-v|_1^{4/d}|u-v|_2^{2-4/d} \le \frac{4}{d}|u-v|_2^2 + C|u-v|_1^2.$$

Clearly, (2.16) implies

$$\frac{d}{d\tau}\left(e^{2\tau}W(\tau)\right) \le C(M_u+M_v)^2 e^{\tau(4-d)}$$

and, after one integration, we obtain

$$W(\tau)\leq\Big(W(0)e^{-\tau}+C(M_u+M_v)^2\Big)e^{-\tau} \tag{2.17}$$

in the case $d=3$,

$$W(\tau)\leq\Big(W(0)+C(M_u+M_v)^2\tau\Big)e^{-2\tau} \tag{2.18}$$

if $d=4$, and finally for all $d>4$

$$W(\tau)\leq\Big(W(0)+C(M_u+M_v)^2\Big)e^{-2\tau}\,. \tag{2.19}$$

Since from the Csiszár-Kullback inequality (*cf.* (1.9) in [2], App. D in [7], [6] or [10]) $W(\tau)$ controls the L^1-norm of $u-u_\infty$ and $v-v_\infty$, we get the same decay rates as in (2.17)-(2.19) for

$$|u(\tau)-u_\infty|_1^2+|v(\tau)-v_\infty|_1^2+\beta|\nabla\psi(\tau)|_2^2\leq 2\Big(\max(M_u,M_v)+1\Big)W(\tau)\,. \tag{2.20}$$

Returning to the original variables x, t, this implies, of course, the estimates (1.14)-(1.15) of Theorem 1.1 in the general case.

In the electroneutrality case (1.10): $M_u=M_v$, since $u_\infty=v_\infty$, so for $d=3$, $|u-v|_1^2=|u-u_\infty+v_\infty-v|_1^2\leq Ce^{-\tau}$. Next, a modification of (2.16) reads

$$\frac{d}{d\tau}\Big(e^{2\tau}W(\tau)\Big)\leq Ce^{2\tau}\beta|u-v|_1^2\leq C\,,$$

and this leads to $W(\tau)\leq C(1+\tau)e^{-2\tau}$. Inserting this into (2.20) and (2.16) once again implies

$$\frac{d}{d\tau}\Big(e^{2\tau}W(\tau)\Big)\leq C(1+\tau)e^{-\tau}\,,$$

so that $W(\tau)\leq Ce^{-2\tau}$. If $d=4$, the same reasoning once again applies providing also the same improved decay rate. □

Remark 2.1 *Note that the constant C in (1.15) depends on d, M_u, M_v and $L(0)$ only, and is independent of* e.g. *$|u_0|_r$, $|v_0|_r$ with some $r>d/2$ — as it was in fact in [1]. Conditions like $|u_0|_r+|v_0|_r<\infty$ are sufficient for (local in* *time) existence of solutions to the considered systems (cf. Theorem 2 in [5]), but they can be relaxed — as it was done for a related parabolic-elliptic system describing the gravitational interaction of particles in [4]. Thus, compared to [1], Theorem 1.1 gives not only an improvement of the exponents but also gets rid of the unnecessary dependence on quantities other than $L(0)$, M_u, M_v. We do not know if the exponents in Theorem 1.1 are optimal, but such a conjecture is supported by the calculations in the proof of the following*

Proposition 2.2 *There exists a constant $\lambda > 0$ depending only on d with $\lambda \geq \lambda(d) = (d-2)\left(\sqrt{(d-1)^2+3}-(d-1)\right)$, such that*

$$W(\tau) \quad \leq \quad W(0)\, e^{-\lambda\tau} \tag{2.21}$$

and hence

$$L(t) \quad \leq \quad L(0)\,(2t+1)^{-\lambda/2}$$

for each solution $\langle u,v\rangle$ to the Nernst-Planck system.

Remark 2.3 *The interest of this proposition is that the constants controlling the convergence of $W(t)$, $L(t)$, and hence $|u-u_{as}|_1$, $|v-v_{as}|_1$ in (1.15), depend on the initial values of $W(0)$, $L(0)$ only (and not on $|u|_1 = M_u$, $|v|_1 = M_v$, which are quantities not comparable with, say, $\int u \log u\, dx$, $\int v \log v\, dx$ in the whole $\mathbb{R}^d$ space case). However, the exponent λ — which is evaluated explicitly — is not as good as the one in Theorem 1.1.*

Proof of Proposition 2.2. Using (2.9), (2.13), (2.14), we may write for any positive λ

$$\begin{aligned} -\left(\frac{dW}{d\tau}+\lambda W\right) &= \lambda\left(J - \int u \log\left(\frac{u}{u_\infty}\right) - \int v \log\left(\frac{v}{v_\infty}\right)\right) \\ &+(2-\lambda)J + B + 2E - \mu F\,, \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} &B = \beta^2 \int (u+v)|\nabla\psi|^2\, dx\,, \\ &E = \beta |u-v|_2^2, \\ &F = \left(\frac{d}{2}-1\right)\beta|\nabla\psi|_2^2, \\ &\mu = 1 + \frac{\lambda}{d-2}\,. \end{aligned}$$

Observe that if we define

$$G_1 = \int u\left(\frac{\nabla u}{u} + x\right)\cdot\nabla\phi\,dx\,,\quad G_2 = \int v\left(\frac{\nabla v}{v} + x\right)\cdot\nabla\phi\,dx\,,$$

then

$$G_1 - G_2 = \int \nabla(u-v)\cdot\nabla\phi\,dx + \int (u-v)\,(x\cdot\nabla\phi)\,dx = E - F.$$

Define now

$$f_1 = \sqrt{2-\lambda}\cdot\sqrt{u}\left(\frac{\nabla u}{u} + x\right)\,,\quad g_1 = \sqrt{u}\nabla\phi\,,$$

$$f_2 = \sqrt{2-\lambda}\cdot\sqrt{v}\left(\frac{\nabla v}{v} + x\right)\,,\quad g_2 = \sqrt{v}\nabla\phi\,,$$

$$a_1 = |f_1|_2\,,\quad b_1 = |g_1|_2\,,\quad a_2 = |f_2|_2\,,\quad b_2 = |g_2|_2\,.$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}(2-\lambda)^{1/2}|E-F| = &\ (2-\lambda)^{1/2}|G_1 - G_2| \\ = &\ \left|\int (f_1 g_1 - f_2 g_2)\,dx\right| \\ \leq &\ a_1 b_1 + a_2 b_2\,.\end{aligned}$$

But

$$0 \leq (a_1 b_2 - a_2 b_1)^2 = (a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1 b_1 + a_2 b_2)^2\,,$$

$$\begin{aligned}a_1 b_1 + a_2 b_2 &\leq \sqrt{2}\sqrt{(a_1^2 + a_2^2)/2}\sqrt{b_1^2 + b_2^2} \\ &\leq \frac{1}{\sqrt{2}}\left(\frac{1}{2}(a_1^2 + a_2^2) + (b_1^2 + b_2^2)\right) \\ &= \frac{1}{\sqrt{2}}\Big((2-\lambda)J + B\Big)\,,\end{aligned}$$

and thus

$$(2-\lambda)^{1/2}|E-F| \leq \frac{1}{\sqrt{2}}\Big((2-\lambda)J + B\Big)\,.$$

Using (2.22) we get

$$\begin{aligned}-\left(\frac{dW}{d\tau} + \lambda W\right) \geq &\ \sqrt{2(2-\lambda)}|E-F| + 2E - \mu F \\ &= F\cdot\left(\sqrt{2(2-\lambda)}|X-1| + 2X - \mu\right) \qquad (2.23)\end{aligned}$$

with $X = E/F \geq 0$. For either $d \geq 4$ and $\lambda \leq 2$, or $d=3$ and $\lambda \leq 1$, we have $\mu \leq 2$. The right hand side of (2.23) (positive for $X \geq \mu/2$) equals (for $X \leq \mu/2 \leq 1$)

$$\sqrt{2(2-\lambda)}(1-X)+2X-\mu=\left(2-\sqrt{2(2-\lambda)}\right)X+\sqrt{2(2-\lambda)}-\mu\,,$$

so that

$$\sqrt{2(2-\lambda)} \geq \mu \tag{2.24}$$

guarantees $\frac{dW}{d\tau}+\lambda W \leq 0$, which implies (2.21). The condition (2.24) is equivalent to $\lambda \leq \lambda(d)$. In particular, $\lambda(d)$ is an increasing function of d, $\lambda(3)=\sqrt{7}-2<1$, $\lambda(4)=4\sqrt{3}-6$ and $\lim_{d\to+\infty}\lambda(d)=\frac{3}{2}$. □

Remark 2.4 *In the case of one species of particles, i.e.* $v \equiv 0$ *as was in [3] and [4], the result of Proposition 2.2 still holds.*

Finally, we remark that there is, in general, no hope to have $\lambda > 2$ *in non-trivial cases. This can be inferred from the formula (2.22), where for each* $\chi > 1$, $J-\chi\left(\int u \log\left(\frac{u}{u_\infty}\right)dx+\int v\log\left(\frac{v}{v_\infty}\right)dx\right)$ *could be negative and dominate the other terms (for instance, in the limit* $M_u,\ M_v \to 0^+$*).*

3 Proof of Theorem 1.2

First, we recall that steady states $\langle U,V,\Phi\rangle$ of (1.1)-(1.3) satisfy the relations

$$\nabla\cdot(e^{-\Phi}\nabla(e^{\Phi}U))=0\,,\quad \nabla\cdot(e^{\Phi}\nabla(e^{-\Phi}V))=0\,,$$

hence

$$U=M_u\frac{e^{-\Phi}}{\int e^{-\Phi}\,dx},\quad V=M_v\frac{e^{\Phi}}{\int e^{\Phi}\,dx}\,. \tag{3.1}$$

Together with (1.3) this leads to the Poisson-Boltzmann equation

$$\Delta\Phi=M_v\frac{e^{\Phi}}{\int e^{\Phi}\,dx}-M_u\frac{e^{-\Phi}}{\int e^{-\Phi}\,dx}\,. \tag{3.2}$$

This equation, supplemented with the Dirichlet boundary condition (1.6) or the free condition (1.7), for every $M_u, M_v \geq 0$, has a unique (weak) solution

Φ, see [9] or Proposition 2 in [5] (and this solution is classical whenever $\partial\Omega$ is of class $C^{1+\epsilon}$ for some $\epsilon > 0$).

The evolution of the Lyapunov functional defined by (1.16) in the case of the Dirichlet boundary condition (1.6) or in the case (1.7) is given by

$$\frac{dW}{dt} = -\int u|\nabla(\log u + \phi)|^2\,dx - \int v|\nabla(\log v - \phi)|^2\,dx\,, \tag{3.3}$$

cf. (35) in [5], where the above relation is obtained for weak solutions to the Debye-Hückel system.

Concerning the global in time existence of solutions to the Debye-Hückel system with nonlinear boundary conditions (1.4)-(1.5), we note that this was done for $d=2$ only in Theorem 3 of [5]. Thus, in higher dimensions $d \geq 3$, we assume that $\langle u(t), v(t)\rangle$ exists for all $t \geq 0$ and

$$\sigma = \sup_{t\geq 0} |\phi(t)|_\infty < \infty \tag{3.4}$$

holds. If equations (1.1)-(1.3) are supplemented with linear type boundary conditions (as it is the case in semiconductor modelling), the assumption u_0, $v_0 \in L^r(\Omega)$ with an exponent $r > d/2$ (*cf.* Theorem 2 (ii) in [5] and [1] for the case of the whole space $\mathbb{R}^d$) guarantees the existence of $\langle u(t), v(t)\rangle$ for all $t \geq 0$.

For $d=2$ the uniform bound (3.4) for the potential ϕ follows from $W(0) < \infty$. Indeed, for either of conditions (1.6) or (1.7)

$$\phi(x,t) = \int_\Omega K(x,y)\,(v(y,t) - u(y,t))\,dy,$$

where the kernel K is either the Green function of the domain Ω or the fundamental solution E_2 of the Laplacian in the plane. Since $|G(x,y) - E_2(x,y)| \leq C = C(\Omega)$, we have, using the inequality

$$AB \leq A\log A + e^{B-1}$$

for all $A \geq 0$, $B \in \mathbb{R}$, with $A = u$, v and $B = \frac{1}{2\pi}|\log|x-y||$,

$$\begin{aligned} |\phi(x,t)| &\leq \int_\Omega (u+v)\Big(\frac{1}{2\pi}|\log|x-y|| + C\Big)\,dy \\ &\leq C\Big(\int_\Omega u\log u\,dy + \int_\Omega v\log v\,dy + 1\Big)\,, \end{aligned}$$

independently of x. Now, (3.3) gives

$$|\phi(t)|_\infty \le C\Big(W(0)+1\Big) = C\Big(W(0), M_u, M_v, \Omega\Big) \equiv \sigma < \infty$$

for all $t \ge 0$.

The calculations below are reminiscent of the proof of Theorem 2.1 in [1], but we do not use the Holley-Stroock perturbation lemma for logarithmic Sobolev inequalities as was in [1]. First, apply the inequality (2.15) with $a = 2e^{-\sigma}/\lambda$ and a small $\lambda > 0$ (to be determined later) to the functions $f = ue^{\phi}$ and $f = ve^{-\phi}$. Taking into account (3.3), (1.16) and (3.2) we arrive at

$$\begin{aligned}
-\left(\frac{dW}{dt}+\lambda W\right) \ \ge\ & \lambda\Big(\int ue^{\phi}\log u\,dx + \int ve^{-\phi}\log v\,dx\Big) + \lambda\int (ue^{\phi}-ve^{-\phi})\phi\,dx \\
& -\lambda|ue^{\phi}|_1\log|ue^{\phi}|_1 - \lambda|ve^{-\phi}|_1\log|ve^{-\phi}|_1 \\
& +\lambda d\left(1+\frac{1}{2}\log\left(\frac{4\pi}{\lambda e^{\sigma}}\right)\right)(|ue^{\phi}|_1+|ve^{-\phi}|_1) \\
& -\lambda\int (u\log u + v\log v)\,dx + \lambda(M_u\log M_u + M_v\log M_v) \\
& -\lambda\int (u-v)\Phi\,dx \\
& -\lambda M_u\log\Big(\int e^{-\Phi}\,dx\Big) - \lambda M_v\log\Big(\int e^{\Phi}\,dx\Big) \\
& -\frac{\lambda}{2}\int (\phi-\Phi)(u-U-v+V)\,dx\,.
\end{aligned}$$

Since the both exponential factors $e^{\pm\phi}$ are uniformly bounded from below by $e^{-\sigma}$, so $e^{\sigma}e^{\pm\phi}-1 \ge 0$, and since by the Jensen inequality, $\int_\Omega u\log(u/M_u)\,dx \ge -M_u\log|\Omega|$, the above inequality leads to

$$-\left(\frac{dW}{dt}+\lambda W\right) \ge C_1\lambda + C_2\lambda\log\left(\frac{1}{\lambda}\right)$$

for some $C_1 \in \mathbb{R}$, $C_2 > 0$ — depending on M_u, M_v, $W(0)$, σ and $|\Omega|$ only.

Now it is clear that for some small $\lambda = \lambda(d, M_u, M_v, W(0)) > 0$, $\frac{dW}{dt} + \lambda W \le 0$, *i.e.* $W(t)$ decays exponentially in t

$$W(t) \le W(0)\,e^{-\lambda t}\,. \tag{3.5}$$

By the Csiszár-Kullback inequality (as was in Section 2), $W(t)$ controls the L^1-convergence to the unique steady state, so the conclusion (1.19) of Theorem 1.2 follows from (3.5). This improves (34) in Theorem 6 of [5] in two ways. First, there is an exponential decay rate. Second, (34) is proved under the assumption (3.4), which is weaker than that in Theorem 6 of [5]. Evidently, this result is also valid for one species case (M_u or M_v equal to 0), so Theorem 2 in [3] is also improved. $\square$

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