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Abstract

Any Banach space can be realized as a direct summand of a uniform algebra, and one does not expect an arbitrary uniform algebra to have an abundance of properties not common to all Banach spaces. One general result concerning arbitrary uniform algebras is that no proper uniform algebra is linearly homeomorphic to a $C(K)$-space. Nevertheless many specific uniform algebras arising in complex analysis share (or are suspected to share) certain Banach space properties of $C(K)$. We discuss the family of tight algebras, which includes algebras of analytic functions on strictly pseudoconvex domains and algebras associated with rational approximation theory in the plane. Tight algebras are in some sense close to $C(K)$-spaces, and along with $C(K)$-spaces they have the Pełczyński and the Dunford–Pettis properties. We also focus on certain properties of $C(K)$-spaces that are inherited by the disk algebra. This includes a discussion of interpolation between H^p-spaces and Bourgain's extension of Grothendieck's theorem to the disk algebra. We conclude with a brief description of linear deformations of uniform algebras and a brief survey of the known classification results.

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Contents

1 Uniform Algebras

A *uniform algebra* is a closed subalgebra A of the complex algebra $C(K)$ that contains the constants and separates points. Here K is a compact Hausdorff space, and A is endowed with the supremum norm inherited from $C(K)$. The algebra A is said to be *proper* if $A \neq C(K)$. Uniform algebras arise naturally in connection with problems in approximation theory. The main examples of proper uniform algebras come from complex analysis. The prototypical proper uniform algebra is the *disk algebra*, which we denote simply by C_A, consisting of the analytic functions on the open unit disk in the complex plane that extend continuously to the boundary Γ. More generally, if K is a compact subset of C^n, we denote by $A(K)$, or by $C_A(K)$, the algebra of functions continuous on K and analytic on the interior of K. Also, we may consider the uniform closure of the restriction to K of some algebras of "elementary" holomorphic functions, such as analytic polynomials, or rational functions with singularities off K. The uniform closure of the rational functions with singularities off K is denoted by $R(K)$. If K is a compact subset of the complex plane, Runge's approximation theorem asserts that $R(K)$ includes the functions that are analytic in a neighborhood of K. It may occur that $R(K)$ is a proper subalgebra of $C(K)$ even when K has empty interior. Other uniform algebras, associated with a domain D in C^n, are the algebra $A(D)$ of analytic functions on D that extend continuously to the closure of D, and the algebra $H^\infty(D)$ consisting of all bounded analytic functions on D. Endowed with the supremum norm on D, the algebra $H^\infty(D)$ becomes a uniform algebra on the smallest compactification of D to which the functions extend continuously. In the case of the open unit disk $\Delta = \{|z| < 1\}$, we may identify $H^\infty(\Delta)$ with a closed subalgebra $H^\infty(d\theta)$ of $L^\infty(d\theta)$ via nontangential boundary values, where $d\theta$ is the arc-length measure on the unit circle.

One of the goals in studying uniform algebras is to use the tools of functional analysis and the Gelfand theory in order to prove approximation theorems or to understand why approximation fails. Mergelyan's theorem, that $R(K) = A(K)$ whenever K is a compact subset of the complex plane whose complement has a finite number of components, was eventually given a proof, by Glicksberg and Wermer, that depends on uniform-algebra techniques and a less difficult theorem of Walsh on approximation by harmonic functions. We quote two other approximation theorems, whose proofs depend on the algebra structure and on techniques of functional analysis.

Theorem (Wermer [We]). *Let A be a (not necessarily closed) algebra of analytic functions on the unit circle Γ in the complex plane. Suppose that A separates the points of Γ, and that each function in A extends to be analytic in a neighborhood of Γ. Then either A*

is dense in $C(\Gamma)$*; or else there is a finite bordered Riemann surface with border* Γ *such that the functions in* A *extend to be analytic on the surface.*

Theorem (Davie [Da]). *Let* λ_K *be the area measure on a compact subset* K *of the complex plane, and let* $H^\infty(\lambda_K)$ *be the weak-star closure of* $R(K)$ *in* $L^\infty(\lambda_K)$*. Then each* $f \in H^\infty(\lambda_K)$ *is approximable pointwise a.e. on* K *by a sequence of functions* $f_n \in R(K)$ *such that* $||f_n|| \leq ||f||$*.*

In some sense, uniform algebra theory can be regarded as an abstract study of the maximum principle for algebras. For an incisive account of this aspect of the theory, see [AWe].

Uniform algebra theory has also served as a source of interesting problems. One such problem, originally raised by S. Kakutani, asked whether the open unit disk Δ is dense in the spectrum of the algebra $H^\infty(\Delta)$. This problem became known as the "corona problem." It was answered affirmatively by L. Carleson (see [Gar]). While the corona theorem per se has not played a really significant role in analysis, the techniques that were devised to solve the problem have played an important role in function theory.

The question arises as to the extent that properties of various uniform algebras depend only on their linear structure. From the point of view of Banach spaces, how special are uniform algebrasΓ We will see in Section 2 that generic uniform algebras are as bad as generic Banach spaces, in the sense that any Banach space is isometric to a complemented subspace of a uniform algebra. On the other hand, we will show in Section 3 that a proper uniform algebra is distinct as a Banach space from $C(K)$. In fact, no proper uniform algebra is linearly isomorphic to a complemented subspace of a $C(K)$-space, or even to a quotient of a $C(K)$-space.

It is of interest to know which properties of $C(K)$ are inherited by uniform algebras. Towards answering this question, a great deal of effort has gone into determining which properties of $C(\Gamma)$ are passed down to the disk algebra C_A. Some of the known results are summarized in the table below.

At present there is no unified theory but rather only fragmented results on uniform algebras as Banach spaces. Our aim in this article is to present a selection of results in order to give an idea of what has been studied and what problems are currently open. Sometimes proofs or indications of proofs are also given, to convey the flavor of the techniques that are employed.

Section 2 includes a brief introduction to algebras of analytic functions on domains in

Banach spaces. In Section 3 we show how some basic facts about p-summing and p-integral operators lead to several characterizations of proper uniform algebras. Sections 4 and 5 are devoted to certain properties that involve weak compactness. The properties are well-known for $C(K)$-spaces, and they are inherited by algebras generated by analytic functions on planar sets and on strictly pseudoconvex domains, where the key ingredient is the solvability of a $\bar{\partial}$-problem.

Property of $X = C(K)$ or $X = C_A$	$C(K)$	C_A	Reference
X is a (linear) quotient of $C(S)$	Yes	No	§3
X^{**} is complemented in a Banach lattice	Yes	No	§3
X^* has the Dunford-Pettis property	Yes	Yes	§5
X has the Pełczyński property	Yes	Yes	§5
X^* is weakly sequentially complete	Yes	Yes	§5
X has a basis	Yes if K is metric	Yes	[Woj]
X verifies Grothendieck's theorem	Yes	Yes	§6
X has a complemented copy of $C(\Gamma)$	Yes if K is metric and uncountable	Yes	[Woj]

In Sections 6 through 8 we talk of properties that are more specific to the disk algebra C_A. We focus on Bourgain's extension of Grothendieck's theorem to the disk algebra obtained in [B1, B2]. This subject was treated in detail in the survey [K2]. The exposition here will follow the ideas of [K2] with slight simplifications at some points, and with emphasis on the interpolatory nature of the proofs. The extension of Grothendieck's theorem is discussed in Section 6, and the results on interpolation used in the proofs are dealt with in Section 7. Section 8 includes a brief account of Bourgain projections, the main technical tool used by Bourgain to transfer results from continuous to analytic functions.

A good deal of the material presented here has already been discussed in various monographs and expository papers. We mention particularly the early lecture notes of Pełczyński [Pe], and the research monographs of Wojtaszczyk [Woj] and of Diestel, Jarchow, and Tonge [DJT]. For background on uniform algebra theory, see [Gam1], [Sto], and [AWe]. For Davie's theorem, see also [Gam2]. The basic Hardy space theory that we refer to is covered in [Du], [Gar], and [Hof]. One reference for several complex variables and pseudoconvexity is [Ra]. The expository papers [K2] and [K4] also cover in part the material of the present article.

We collect here some standard notation and conventions. We denote by Γ the unit circle $\{|z| = 1\}$ in the complex plane, and by $dm = d\theta/2\pi$ the normalized arc-length measure on Γ. The Hilbert transform on $L^1(dm)$ is denoted by $\mathcal{H}$. We denote by C_A the disk algebra, which is the uniform closure of the analytic polynomials in $C(\Gamma)$. The algebra of bounded analytic functions on a domain D is denoted by $H^\infty(D)$. If A is a linear space of functions on K, and σ is a measure on K, then $H^\infty(A,\sigma)$ will denote the weak-star closure of A in $L^\infty(\sigma)$. The generic Banach space, or quasi-Banach space, is denoted by X. The closed unit ball of a Banach space X is denoted by B_X, and the open unit ball by B_X°. The image of a Banach space X in its bidual X^{**} under the canonical embedding is denoted by $\hat{X}$.

2 Analytic Functions on Banach Spaces

We wish to develop some examples of algebras of analytic functions defined on domains in a Banach space.

A complex-valued function on an open subset of a Banach space X is *analytic* if it is locally bounded and its restriction to every complex one-dimensional affine subspace of X is analytic. In other words, f is analytic on D if f is locally bounded, and if for every $x_0 \in D$ and direction $x \in X$, the function $\lambda \mapsto f(x_0 + \lambda x)$ depends analytically on λ. Sums, products, and uniform limits of analytic functions are analytic.

A locally bounded function f on D is analytic just as soon as its restriction to $D \cap Y$ is analytic for every finite-dimensional subspace Y of X. Thus any statement about analytic functions that involves only a finite number of points of X will hold in general once it holds for analytic functions of several complex variables.

Let f be analytic on a domain D in a Banach space X. Suppose $0 \in D$, and suppose $|f(x)| \leq C$ for $\|x\| < r$. For fixed $x \in X$, the function $\zeta \mapsto f(\zeta x)$ has a Taylor series expansion

$$f(\zeta x) = \sum_{m=0}^{\infty} A_m(x)\zeta^m, \qquad |\zeta| < r/\|x\|.$$

One checks easily that each $A_m(x)$ is m-homogeneous, that is, $A_m(\lambda x) = \lambda^m A_m(x)$. From the Cauchy estimates we have $|A_m(x)| \leq C\|x\|^m/r^m$, and consequently $A_m(x)$ is locally bounded. If we restrict the expansion $f(x) = \sum A_m(x)$ to a finite dimensional subspace of X, we obtain the usual expansion of an analytic function as a series of m-homogeneous polynomials. In particular, $A_m(x)$ depends analytically on x in any finite-dimensional subspace of X, and since it is locally bounded, it is analytic.

Let $P_m = P_m(X)$ denote the space of analytic functions on X that are m-homogeneous. We endow P_m with the supremum norm over the unit ball B_X of X, and then P_m becomes a Banach space. The Cauchy estimates show that the correspondence $f \mapsto A_m$ is a norm-one projection from the space $H^\infty(B_X^\circ)$ of bounded analytic functions on the open unit ball of X to P_m.

The first Taylor coefficient in the expansion of an analytic function f at 0 coincides with the usual Fréchet derivative $f'(0)$ of f at 0, whose defining property is that $f(x) = f(0) + f'(0)(x) + o(\|x\|)$ as $x \to 0$. The Fréchet derivative $f'(0)$ is a continuous linear functional on X, and the space P_1 coincides with the dual space X^* of X.

Theorem 2.1 (Milne [Mi]). *Any Banach space is isometric to a complemented subspace of a uniform algebra.*

To prove the theorem, we let $K = B_{X^*}$ be the closed unit ball of the dual space X^* of X, endowed with the weak-star topology. Recall that $\hat{X}$ denotes the canonical image of X in X^{**}. The restriction of $\hat{X}$ to K is a closed subspace of $C(K)$. Let A be the uniform algebra on K generated by $\hat{X}$. The functions in A are analytic on the open unit ball of X^*, hence have Taylor expansions $f(x^*) = \sum A_m(x^*)$. The norm-one projection $f \mapsto A_1$ into $P_1(X^*)$ is the identity on $\hat{X}$, and it projects any m-homogeneous polynomial in elements of $\hat{X}$ to 0 if $m \neq 1$. Thus if we pass to uniform limits of sums of polynomials in elements of $\hat{X}$, we obtain a norm-one projection of A onto $\hat{X} \cong X$.

We could as well obtain the same result by considering the algebra $A(B_{X^*})$ of weak-star continuous functions on B_{X^*} that are analytic on the open unit ball of X^*. It is not difficult to check that the projection of $A(B_{X^*})$ onto $P_1(X^*) \cong X^{**}$ maps $A(B_{X^*})$ onto linear functionals that are weak-star continuous on B_{X^*}, thus onto X. However, it is shown in [ACG] that the algebra $A(B_{X^*})$ need not coincide with the algebra generated by the weak-star continuous linear functionals. Along these lines, it is not even known whether the maximal ideal space of $A(B_{X^*})$ coincides with B_{X^*}.

There is an expanding literature about polynomials on Banach spaces and about uniform algebras associated to Banach spaces (see [Din], [GJL]). The study of polynomials focuses on the spaces P_m, which can be viewed as spaces of multilinear functionals on X. Every continuous m-homogeneous polynomial f on X is the restriction to the diagonal of a unique continuous symmetric m-linear functional F on $X \times \cdots \times X$. This F is given by the polarization formula

$$F(x_1, \ldots, x_m) = \frac{1}{m!2^m} \sum \varepsilon_1 \cdots \varepsilon_m f(\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m),$$

the summation being extended over the 2^m independent choices of $\varepsilon_j = \pm 1$ (exercise). The same formula shows that the norm in P_m is equivalent to the multilinear functional norm, though the spaces are not isometric in general. The space P_m can also be viewed as the dual space of the m-fold symmetric projective tensor product of X with itself. For background, see [Mu], [Gam3].

3 Characterization of Proper Subalgebras

In this section we shall prove that if a uniform algebra is proper, then it differs as a Banach space from any space $C(K)$. The crux of the proof is that every absolutely summing operator from $C(K)$ to a reflexive space is compact, while we construct on any proper uniform algebra an operator to ℓ^2 that is absolutely summing but not compact.

Let X and Y be Banach spaces, and let $0 < p \leq q < \infty$. An operator $T : X \to Y$ is said to be (q,p)-*summing* if

$$\left(\sum \|Tx_j\|^q\right)^{1/q} \leq C \sup\left\{\left(\sum |x^*(x_j)|^p\right)^{1/p} : x^* \in X^*, \|x^*\| \leq 1\right\} \tag{3.1}$$

for every finite collection $\{x_j\}$ of elements of X. The best constant C is denoted by $\pi_{q,p}(T)$. The class of such operators forms an operator ideal, in the sense that precompositions and postcompositions with bounded operators remain within the class, and further the usual estimates for norms hold.

If $p = q$, (3.1) coincides with the definition of a p-*summing* operator (see Basic Concepts); in this case the notation $\pi_p(T)$ is used. By *absolutely summing* we mean 1-summing. We remind the reader that the p-summing operators are characterized as those that factor through a part of the inclusion $L^\infty(\mu) \hookrightarrow L^p(\mu)$, in the sense that T can be represented as a composition

$$T : X \xrightarrow{U} M \hookrightarrow M_p \xrightarrow{V} Y, \tag{3.2}$$

where μ is a probability measure, M is a subspace of $L^\infty(\mu)$, and M_p is a subspace of $L^p(\mu)$ containing M, or alternatively M_p is the closure of M in $L^p(\mu)$. For $p \geq 1$, T is said to be *strictly p-integral* if it factors through the entire inclusion $L^\infty(\mu) \hookrightarrow L^p(\mu)$, that is, we can take $M = L^\infty(\mu)$ and $M_p = L^p(\mu)$ in (3.2). This notion differs slightly from that of a p-integral operator, as defined in Basic Concepts. However, the two notions coincide if, say, Y is reflexive. As explained in Basic Concepts, for $p \geq 1$ every p-summing operator on $C(K)$ is strictly p-integral, so that the p-summing and the strictly p-integral operators on $C(K)$ coincide.

We recall that any absolutely summing operator is weakly compact. For 1-integral operators, we can use the Dunford-Pettis property of $L^1(\mu)$ to say more.

Lemma 3.1. *A (strictly) 1-integral operator from X to a reflexive Banach space is compact. In particular, an absolutely summing operator from $C(K)$ to a reflexive Banach space is compact.*

To see this, consider the factorization (3.2) above, with $p = 1$, $M = L^\infty(\mu)$, $M_p = L^p(\mu)$. Since Y is reflexive, V is weakly compact. By the Dunford-Pettis theorem, V maps weakly compact subsets of $L^1(\mu)$ to norm-compact subsets of Y. Since bounded subsets of $L^\infty(\mu)$ are weakly precompact in $L^1(\mu)$, the composed operator T maps bounded subsets of X into norm-compact subsets of Y, and T is compact.

The prototype for an absolutely summing operator that is not 1-integral is the Paley operator P on the disk algebra C_A. The Paley operator assigns to a function f on the unit circle Γ the sequence of 2^kth Fourier coefficients $\{\hat{f}(2^k)\}_{k=1}^\infty$. Paley's inequality is

$$\left(\sum_{k=1}^\infty |\hat{f}(2^k)|^2\right)^{1/2} \leq c_P\|f\|_1, \qquad f \in H^1(m),$$

for some constant $c_P > 0$. In other words, the restriction of the Paley operator P to $H^1(m)$ is a bounded operator from $H^1(m)$ to ℓ^2. For the proof, see [Hof], [Z].

Let M be the closed linear span in $L^2(m)$ of the exponential functions $\exp(i2^k\theta)$, $k \geq 1$. It is a classical fact (see [Z]) that the L^p-norms on M are equivalent, for $0 < p < \infty$. Further, for $1 < p < \infty$ there is a continuous projection Q_p of $L^p(m)$ onto M. Thus for $1 < p < \infty$, we can factor the Paley operator on $C(\Gamma)$ through the inclusion $L^\infty(\mu) \hookrightarrow L^p(\mu)$,

$$P : C(\Gamma) \hookrightarrow L^\infty(m) \hookrightarrow L^p(m) \xrightarrow{Q_p} M \longrightarrow \ell^2,$$

and P operating on $C(\Gamma)$ is p-integral for $1 < p < \infty$. On the other hand, P is not compact, so P is not absolutely summing on $C(\Gamma)$.

The story is different if we restrict P to C_A. Paley's inequality yields the factorization

$$P : A \hookrightarrow H^1(m) \xrightarrow{V} \ell^2,$$

where V is the Paley operator on $H^1(m)$. Thus P is absolutely summing on C_A. On the other hand, P maps the exponential functions $\exp(i2^k\theta)$, $k \geq 1$ to the standard basis vectors of ℓ^2, so that P is not compact, and P is not 1-integral. This shows incidentally that C_A is not complemented in $C(\Gamma)$, nor even isomorphic to any quotient space of a $C(K)$-space, or else the composition of the projection and P would produce an absolutely summing operator on $C(K)$ that is not compact.

Our aim is to transfer the Paley operator and this final observation to an arbitrary uniform algebra. Paley's inequality transfers directly, as follows.

Lemma 3.2. *Let A be a uniform algebra on K. Let $f \in A$ satisfy $\|f\| \leq 1$, and let*

τ be any measure on K orthogonal to A. Then the sequence

$$P(\tau) = \left\{ \int \overline{f}^{2^k} d\tau \right\}_{k=1}^{\infty}$$

belongs to ℓ^2, and $\|P(\tau)\|_2 \le c_P \|\tau\|$, where c_P is the best constant for Paley's inequality. The estimate persists for $f \in H^\infty(|\tau|)$, the weak-star closure of A in $L^\infty(|\tau|)$.

Proof. Define $U : C(\Gamma) \to C(K)$ by $(Ug)(s) = \tilde{g}(f(s))$ for $g \in C(\Gamma)$ and $s \in K$, where $\tilde{g}$ is the Poisson integral of g. Then $\|U\| \le 1$, $U(z^n) = f^n$, and $U(\bar{z}^n) = \bar{f}^n$. Now $U^* : M(K) \to M(\Gamma)$ sends $A^\perp$ to $C_A^\perp$, which by the F. and M. Riesz theorem is identified with $H_0^1(m)$. Thus $U^*(\tau) = h\, dm$ for some $h \in H^1(m)$, and

$$\int_K \overline{f}^{2^k} d\tau = \int_\Gamma \overline{z}^{2^k} h(z) dm, \qquad k \ge 1.$$

Paley's inequality for h yields $\|P(\tau)\|_2 \le c_P \|h\|_1 = c_P \|U^*(\tau)\| \le c_P \|\tau\|$. This proves the first statement of Lemma 3.2, and the second is obtained by applying the first to the uniform algebra $H^\infty(|\tau|)$ and noting that τ generates a functional on $L^\infty(|\tau|)$ orthogonal to $H^\infty(|\tau|)$.

Theorem 3.3 (Kislyakov [K1]). *If A is a proper uniform subalgebra of $C(K)$, then there is an absolutely summing operator from A to ℓ^2 that is not compact, hence not* 1-*integral.*

The proof depends on the Paley operator associated with an extremal function F for a certain dual extremal problem. Since A is proper, there is a measure μ on K such that $\mu - A$ but the complex conjugate $\bar{\mu}$ of μ is not orthogonal to A. We assume that the functional $f \mapsto \int f\, d\bar{\mu}$ on A has unit norm. By the Hahn-Banach and Riesz representation theorems, there is a measure λ on K such that $\|\lambda\| = 1$ and $\lambda - \bar{\mu} - A$. Let $\{f_n\}$ be a sequence of functions in A such that $\|f_n\| \le 1$ and $\int f_n d\bar{\mu} \to 1$, and let $F \in H^\infty(|\mu| + |\lambda|)$ be a weak-star limit point of the sequence $\{f_n\}$. Then $|F| \le 1$, and $\int F\, d\lambda = 1$, from which it follows that $|F| = 1$ a.e. $d\lambda$. Now $f(\lambda - \bar{\mu}) - A$ for all $f \in A$, hence for all $f \in H^\infty(|\mu| + |\lambda|)$. In particular, $F^2(\lambda - \bar{\mu}) - A$, and $\tau = \mu + F^2(\lambda - \bar{\mu}) - A$. We define

$$T(g) = \left\{ \int \overline{F}^{2^k} g\, d\tau \right\}_{k=1}^{\infty}, \qquad g \in A. \tag{3.3}$$

By Lemma 3.2, applied to $F \in H^\infty(|\mu| + |\lambda|)$ and the orthogonal measure $g\tau$, the sequence $T(g)$ is square summable and $\|T(g)\|_2 \le c\|g\tau\| = c \int |g| d|\tau|$. Thus T can be factored through the closure $H^1(|\tau|)$ of A in $L^1(|\tau|)$,

$$T : A \hookrightarrow H^1(|\tau|) \longrightarrow \ell^2,$$

and T is absolutely summing. To see that T is not compact, we compute the kth component of $T(F^{2^k \perp 1})$ (to be rigorous, rather we must consider $\lim_{n\to\infty} T(f_n^{2^k\perp 1})$):

$$T(F^{2^k\perp 1})_k = \int F^{2^k\perp 1}\overline{F}^{2^k}\, d\tau = \int F^{2^k+1}\overline{F}^{2^k}\, d\lambda + \int F^{2^k\perp 1}\overline{F}^{2^k}\, d\mu - \int F^{2^k+1}\overline{F}^{2^k}\, d\bar{\mu}. \quad (3.4)$$

Let E be the set on which $|F| = 1$. Since λ is carried by E, the first integral on the right is $\int F\, d\lambda = 1$. Since $|F|^n \to 0$ off E as $n \to \infty$, (3.4) tends to

$$1 + \int_E \overline{F}\, d\mu - \int_E F\, d\bar{\mu} = 1 + 2i\,\mathrm{Im}\left(\int_E \overline{F}\, d\mu\right), \quad (3.5)$$

which is not zero. Since the kth components of the vectors $T(F^n)$, $n \geq 1$, do not tend to zero uniformly in n, the vectors $T(F^n)$ do not lie in a compact subset of ℓ^2, and then neither does the image of the unit ball of A under T. Thus T is not compact, and by Lemma 3.2, T is not 1-integral on A.

If we analyze the proof of Theorem 3.4, we find that it extends to any closed subspace B of a uniform algebra A providing there is a function $f \in B$ such that $fA \subseteq B$ while $\bar{f} \notin A$. Indeed, let I be the set of all f satisfying $fA \subseteq B$. This is a closed ideal in A. We choose $\mu - A$ such that $\bar{\mu}$ generates a norm-one functional on I, and we proceed as before.

For some time it was an open problem, known as the *Glicksberg problem*, as to whether a proper uniform algebra on a compact space K can be complemented in $C(K)$. Theorem 3.4 settles the Glicksberg problem, and it does even more.

Theorem 3.4. *If A is a proper uniform subalgebra of $C(K)$, then A is not isomorphic to a quotient of a $C(J)$ space. In particular, A is not complemented in $C(K)$.*

Indeed, suppose A is the quotient of $C(J)$. If we compose the operator T from Theorem 3.3 with the quotient map, we obtain an operator

$$C(J) \longrightarrow C(J)/Z \cong A \xrightarrow{T} \ell^2$$

that is absolutely summing, hence compact, by Lemma 3.1. Since the projection is an open mapping, the operator T must be compact, and this contradicts Theorem 3.3.

We mention some further conclusions that can be drawn from this circle of ideas. Recall (see Basic Concepts) that a Banach space X has *Gordon-Lewis local unconditional structure* (GL l.u.st.) if, roughly speaking, its finite dimensional subspaces are well embeddable in spaces with unconditional basis. This occurs if and only if X^{**} is a complemented subspace of a Banach lattice. Thus $C(K)$ has GL l.u.st., as do all L^p-spaces, $1 \leq p \leq \infty$.

Theorem 3.5. *If A is a proper uniform subalgebra of $C(K)$, then A does not have GL l.u.st.*

The idea of the proof is to look for a factorization of the Paley operator $P : C_A \to \ell^2$ through A^{**} with the help of the second adjoint of the operator T defined in the proof of Theorem 3.3,

$$P : C_A \xrightarrow{U} A^{**} \xrightarrow{T^{**}} \ell^2.$$

However, in this way we only obtain an operator quite similar to P (but not P itself). We redefine F to be a weak-star limit point of the sequence $\{f_n\}$ in A^{**}, and we use the realization of A^{**} as a weak-star closed subspace of C^{**} described in the next section. The operator T^{**} may still be defined by (3.3), where now g belongs to A^{**}, and the functions being integrated in (3.3) are the projections of the functions in C^{**} into $L^\infty(|\tau|)$. The correspondence $p(Z) \mapsto p(F)$ mapping a polynomial in the coordinate function Z to $p(F)$ is of norm at most 1, hence extends to a bounded operator U from C_A into A^{**}. Now the hypothesis of GL l.u.st. implies by the Gordon-Lewis theorem (see [DJT, Theorem 17.7]) that the absolutely summing operator T^{**} factors through an L^1-space, and we obtain

$$T^{**}U : C_A \xrightarrow{U} A^{**} \xrightarrow{V} L^1(\nu) \xrightarrow{W} \ell^2.$$

By Bourgain's extension of the Grothendieck theorem (Theorem 6.5), the composition VU mapping C_A into an L^1-space is 2-summing, hence weakly compact. By the Dunford-Pettis theorem, the weakly compact operator W maps weakly compact subsets of $L^1(\nu)$ to norm-compact subsets of ℓ^2. Thus the composition WVU is compact, contradicting the noncompactness of $T^{**}U$ (see (3.4–3.5)).

Along similar lines, it can also be proved that if a proper uniform algebra A is a quotient of a Banach space X having GL l.u.st., then X contains a complemented copy of l^1. The crucial observation here is that if X fails to have a complemented copy of l^1, then every operator from X to $L^1(\nu)$ is weakly compact. To see this, combine the Pełczyński property of $L^\infty(\nu)$ with [LT, Proposition 2.e.8].

In another direction, Garling [Ga] showed that the dual A^* of a proper uniform algebra is not a subspace of the dual of a C^*-algebra. The proof is modeled on an earlier argument for C_A and uses the basic objects (λ, μ, and F) appearing in the proof of Theorem 3.4.

4 Tight Subspaces and Subalgebras of $C(K)$

The classical Hankel operator corresponding to a function g on the unit circle operates from $H^2(d\theta)$ to $H^2(d\theta)^\perp$, sending f to $gf - P(gf)$, where P is the orthogonal projection from $L^2(d\theta)$ onto $H^2(d\theta)$. The Hankel operator is equivalent to the operator $f \to gf + H^2$ from H^2 to the quotient space L^2/H^2. The analogue of these operators, acting on subspaces of $C(K)$, has proved a key to understanding uniform algebras.

Let A be a closed subspace of $C(K)$. To each $g \in C(K)$ we associate a *generalized Hankel operator* S_g from A to the quotient Banach space $C(K)/A$ by

$$S_g f = gf + A, \qquad f \in A.$$

We say that A is a *tight subspace* of $C(K)$ if the operators S_g are weakly compact for all $g \in C(K)$. We say that A is a *compactly tight subspace* of $C(K)$ if S_g is compact for all $g \in C(K)$. Tightness was introduced in [CG]. Our discussion is based on that paper, and on [Sac1], [Sac2].

We will use the representation of the bidual C^{**} of $C = C(K)$ as a uniform algebra. This representation is realized as follows. The dual space of C is the space $M(K)$ of finite (regular Borel) measures on K, with the total variation norm, and this can be regarded as the direct limit of the spaces $L^1(\mu)$, $\mu \in M(K)$. The bidual C^{**} is then represented as the inverse limit of their dual spaces $L^\infty(\mu)$, $\mu \in M(K)$. A "simple-minded" way to express this is to say that each element $F \in C^{**}$ determines for each $\mu \in M(K)$ a function $F_\mu \in L^\infty(\mu)$, and these satisfy the compatibility condition that $F_\nu = F_\mu$ almost everywhere with respect to ν whenever $\nu << \mu$. Conversely, each uniformly bounded compatible family $\{F_\mu\}$, $F_\mu \in L^\infty(\mu)$, determines an element of C^{**}. The norm of F in C^{**} is the supremum of the norms of F_μ in $L^\infty(\mu)$. The multiplication in the spaces $L^\infty(\mu)$ determines an obvious multiplication in C^{**}.

Let A be a subspace of C. Recall that $H^\infty(A, \mu)$ denotes the weak-star closure of A in $L^\infty(\mu)$. The bidual A^{**} can then be identified with the weak-star closed subspace of C^{**} consisting of $F \in C^{**}$ such that $F_\mu \in H^\infty(A, \mu)$ for all $\mu \in M(K)$. The bidual of the quotient space C/A is isometric to C^{**}/A^{**}, and the canonical embedding maps C/A isometrically onto C/A^{**}. In particular, C/A^{**} is a closed subspace of C^{**}/A^{**}, and consequently $A^{**} + C$ is a closed subspace of C^{**}.

Now the operator S_g is weakly compact if and only if the image of A^{**} under S_g^{**} is contained in the canonical image of C/A. Identifying C with its canonical image in C^{**}, we see that for $g \in C$,

$$S_g \text{ is weakly compact} \iff gA^{**} \subseteq A^{**} + C. \tag{4.1}$$

From this it follows that the g's for which S_g is weakly compact form a closed subalgebra of $C(K)$. Thus A is tight just as soon as S_g is weakly compact for any collection of g's that generates $C(K)$ as a uniform algebra.

If A is a subalgebra of C, then each space $H^\infty(A,\mu)$ is an algebra, and A^{**} is a subalgebra of C^{**}. In this case we obtain from (4.1) the following.

Theorem 4.1. *A uniform algebra A on a compact space K is a tight subalgebra of $C(K)$ if and only if $A^{**} + C(K)$ is a closed subalgebra of $C(K)^{**}$.*

For algebras of analytic functions, tightness is related to solving a $\overline{\partial}$-problem. Roughly speaking, the connection is as follows. Functions that belong to an algebra of analytic functions A are characterized as the functions f satisfying $\bar{\partial} f = 0$. Suppose that $\bar{\partial}^{\perp 1}$ is a solution operator for the $\bar{\partial}$-problem, which need not be linear. Let g be a smooth function. If $f \in A$, then from the Leibnitz rule we obtain $\bar{\partial}(fg) = f\bar{\partial}(g) + g\bar{\partial}(f) = f\bar{\partial}(g)$. This shows that the one-form $f\bar{\partial}(g)$ is $\bar{\partial}$-closed. We apply the solution operator and obtain a function $h = \bar{\partial}^{\perp 1}(f\bar{\partial}g)$ satisfying $\bar{\partial}h = \bar{\partial}(fg)$, so that $h - fg \in A$, and $S_g f = h + A$. Thus the action of S_g on f amounts to multiplying f by $\bar{\partial}g$, applying $\bar{\partial}^{\perp 1}$, and projecting into the quotient space C/A. It follows that if there is a weakly compact solution operator for the $\bar{\partial}$-problem, then each S_g is weakly compact, and A is tight. By the same token, if there is a compact solution operator for the $\bar{\partial}$-problem, then A is compactly tight.

If D is a bounded strictly pseudoconvex domain in complex n-space with smooth boundary, the $\bar{\partial}$-problem can be solved by means of integral operators, with Hölder estimates on the solutions, so that there are compact solution operators; see [Ra]. The argument outlined above can be made precise. It shows that the algebra $A(D)$ associated with any such domain is compactly tight. The connection between tightness and solving the $\bar{\partial}$-problem is not completely understood, but our line of reasoning does establish the following.

Theorem 4.2. *Let D be a bounded domain in complex n-space. Suppose there is a weakly compact subset E of $C(\bar{D})$ such that the equation $\bar{\partial}h = \omega$ on D has a solution $h \in E$ for every $\bar{\partial}$-closed smooth $(0,1)$-form ω on D that extends continuously to $\bar{D}$ and satisfies $\|\omega\|_\infty \leq 1$. Then $A(D)$ is a tight subalgebra of $C(\overline{D})$. If E is compact, then $A(D)$ is compactly tight.*

If D is strictly pseudoconvex with smooth boundary, the $\bar{\partial}$-problem solution techniques can be used to show that any $f \in H^\infty(D)$ can be approximated pointwise on D by a bounded sequence of functions in $A(D)$ that extend analytically across ∂D, with uniform convergence on $\bar{D}$ if $f \in A(D)$. In the strictly pseudoconvex case, every point p in ∂D is a peak point for $A(D)$, that is, there is $f \in A(D)$ satisfying $f(p) = 1$ and $|f| < 1$ on $\bar{D} \backslash \{p\}$. Thus the following theorem applies to strictly pseudoconvex domains with smooth boundaries.

Theorem 4.3. *Let D be a bounded domain in complex n-space for which the $\bar{\partial}$-problem is solvable as in Theorem 4.2, and let σ be the volume measure on D. Suppose that the functions in $A(D)$ that extend analytically across ∂D are pointwise boundedly dense in $H^\infty(D)$. Then $A(D)^*$ is the direct sum of $L^1(\sigma)/A(D)^\perp$ and an L^1-space, and the bidual $A(D)^{**}$ is isometrically isomorphic to the direct sum of $H^\infty(D)$ and an L^∞-space. Further, if every point of ∂D is a peak point for $A(D)$, then $H^\infty(D) + C(\bar{D})$ is a closed subalgebra of $L^\infty(\sigma)$, and $A(D)^{**} + C(\bar{D})$ is isometrically isomorphic to the direct sum of $H^\infty(D) + C(\bar{D})$ and an L^∞-space.*

The idea of the proof is as follows. Let B_s be the band of measures on $\bar{D}$ that are singular to every measure in $A(D)^\perp$, and let B_a be the band of measures generated by $A(D)^\perp$. There is a direct sum decomposition $M(\bar{D}) = B_a \oplus B_s$, with a corresponding decomposition $A(D)^{**} = H^\infty(B_a) \oplus L^\infty(B_s)$. We claim that the summand $H^\infty(B_a)$ is isometrically isomorphic to $H^\infty(D)$. We regard $H^\infty(D)$ as a subalgebra of $L^\infty(\sigma)$. If $F \in A(D)^{**}$, there is a bounded net $\{f_\alpha\}$ in $A(D)$ that converges weak-star in $A(D)^{**}$ to F. Then $\{f_\alpha\}$ converges weak-star to F_σ in $L^\infty(\sigma)$. Since the f_α's are uniformly bounded, they are equicontinuous at each point of D, and consequently any limit function is analytic on D. Thus $F_\sigma \in H^\infty(D)$. The hypothesis of pointwise bounded density implies that the projection $F \mapsto F_\sigma$ maps $A(D)^{**}$ onto $H^\infty(D)$. Suppose $F \in A(D)^{**}$ satisfies $F_\sigma = 0$. Let $\mu \in A(D)^\perp$. Let g be a smooth function, and let $\{f_\alpha\}$ be a bounded net in $A(D)$ that converges weak-star to F. Then $f_\alpha \to 0$ on D. Choose $h_\alpha \in E$ such that $\bar{\partial}(f_\alpha g) = \bar{\partial}(h_\alpha)$ on D. Passing to a subnet, we may assume that $h_\alpha \to h$ weakly, where $h \in C(\bar{D})$. Then $gf_\alpha - h_\alpha$ is analytic on D, and in the limit, h is analytic on D. Now $gf_\alpha + h - h_\alpha \to gF_\mu$ weak-star in $L^\infty(\mu)$. Thus $\mu - gF_\mu$, this for all smooth functions g, so that $F_\mu = 0$. It follows that $F_\nu = 0$ for all measures $\nu \in B_a$, and consequently the projection of F in $H^\infty(B_a)$ is 0. Thus the algebra homomorphism $H^\infty(B_a) \to H^\infty(D)$ is one-to-one and onto. Since any homomorphism of uniform algebras that is one-to-one and onto is an isometry, $H^\infty(B_a)$ is isometric to $H^\infty(D)$. The final statement of the theorem, that $H^\infty(D) + C(\bar{D})$ is isometric to a direct summand of $A(D)^{**} + C(\bar{D})$, is equivalent to the statement that $H^\infty(B_a) \to H^\infty(D)$ is a "local" isometry, in the sense that it is an isometry at every point of ∂D. An easy way to guarantee this is to assume that every point of ∂D is a peak point for $A(D)$.

Another class of examples of tight algebras are the algebras $R(K)$ and $A(K)$ associated with a compact subset K of the complex plane. For these, the solution operator for the $\bar{\partial}$-problem is the Cauchy transform operator

$$(\bar{\partial}^{-1}h)(\zeta) = -\frac{1}{\pi}\int\int \frac{h(z)}{z-\zeta}dxdy, \qquad \zeta \in K,$$

which is a compact operator on $C(K)$. Again the line of reasoning outlined above, together with a few technical details, establishes the following.

Theorem 4.4. *Let K be a compact subset of the complex plane. Then the algebras $R(K)$ and $A(K)$ are compactly tight. If σ is the area measure on K, and A is either of these algebras, then A^* is isometric to the direct sum of $L^1(\sigma)/A^{\perp}$ and an L^1-space. The bidual A^{**} is isometrically ismorphic to the direct sum of $H^\infty(\sigma)$ and an L^∞-space. Finally, $H^\infty(\sigma) + C(K)$ is a closed subalgebra of $L^\infty(\sigma)$, and $A^{**} + C(K)$ is isometrically ismorphic to the direct sum of $H^\infty(\sigma) + C(K)$ and an L^∞-space.*

Here $H^\infty(\sigma)$ is the weak-star closure of A in $L^\infty(\sigma)$. In the case of $A(K)$, the measure σ can be taken to be the area measure on the interior of K, or the harmonic measure on the boundary of the interior of K. In the case of $R(K)$, σ can be taken to be the area measure on the set of nonpeak points of $R(K)$, which serves in some sense as an interior for K with respect to $R(K)$. The proof of Theorem 4.4 is similar to that of Theorem 4.3, except that Davie's theorem is used to obtain the isometric isomorphism of $H^\infty(B_a)$ and $H^\infty(\sigma)$. The proof of the final statement depends upon estimating solutions of the $\bar{\partial}$-problem for some specific bump functions.

Summarizing, we can say that very many standard uniform algebras of analytic functions are tight. We turn to an example of a tight subspace that is not an algebra and that has a different flavor. Let U_C be the space of continuous functions $f(e^{i\theta})$ on the unit circle Γ for which the symmetric partial sums $T_nf = \sum_{-n}^{n} \hat{f}(k)e^{ik\theta}$ of the Fourier series of f converge uniformly. Normed by $|||f||| = \sup ||T_nf||_\infty$, the space U_C becomes a Banach space. We may regard U_C as a subspace of a $C(K)$-space as follows. For each n, $0 \le n \le \infty$, let Γ_n be a copy of the unit circle Γ, and let K be the disjoint union of the Γ_n's, with the natural topology determined by declaring that $\Gamma_n \to \Gamma_\infty$ as $n \to \infty$. Each $f \in U_C$ determines $F \in C(K)$ by setting $F = T_nf$ on Γ_n, $0 \le n < \infty$, and $F = f$ on Γ_∞. Then U_C is isometric to a closed subspace of $C(K)$.

Theorem 4.5. *Let U_C be the Banach space of functions on Γ with uniformly convergent Fourier series, regarded as a closed subspace of $C(K)$ as above. Then U_C is a tight subspace of $C(K)$, though U_C is not compactly tight. Further, the weak-star closure*

$H^\infty(U_C, d\theta)$ of U_C in $L^\infty(d\theta)$, where $d\theta$ is the arc length measure on Γ_∞, coincides with the space of functions $f \in L^\infty(d\theta)$ such that the symmetric partial sums of the Fourier series of f are uniformly bounded. The bidual U_C^{**} is isometric to the direct sum of $H^\infty(U_C, d\theta)$ and an L^∞-space.

If $g \in C(K)$ is supported on one of the circles Γ_n for n finite, then the operator S_g is finite dimensional. Thus to check that U_C is tight, it suffices to show that the operators S_z and $S_{\bar{z}}$ are weakly compact, where $z = e^{i\theta}$ on each circle Γ_n. Let $f \in U_C$ have Fourier series $\sum a_k e^{ik\theta}$, and denote the corresponding function in $C(K)$ by $\Phi(f) = [T_0 f, T_1 f, \dots, f]$. With this notation,

$$z\Phi(f) - \Phi(zf) = [a_0 e^{i\theta} - a_{\perp 1}, \dots, a_k e^{i(k+1)\theta} - a_{\perp k \perp 1} e^{\perp ik\theta}, \dots, 0].$$

Since this expression is in $C(K)$ for all two-tailed ℓ^2-sequences $\{a_k\}$, and since $\sum |a_k|^2 < \infty$ for $f \in U_C$, the operator S_z factors through ℓ^2. Thus S_z is weakly compact, as is $S_{\bar{z}}$, and U_C is tight. To see that S_z is not compact, apply S_z to the sequence of exponential functions $\{e^{ik\theta}\}$.

There is another way to see that S_z is weakly compact. Since only two Fourier coefficients appear in each component above, we obtain

$$\|S_z f\| \leq \|z\Phi(f) - \Phi(zf)\|_\infty \leq 2\|f\|_{L^1(d\theta/2\pi)}.$$

This estimate shows that S_z is an absolutely summing operator, in fact, an integral operator. Since bounded subsets of L^∞ are weakly compact in L^1, S_z is weakly compact.

A similar theorem holds for the space U_A of analytic functions on the unit disk with uniformly convergent Taylor series. Again we may regard U_A as a subspace of $C(K)$ as above, and U_A is a tight subspace. In this case the weak-star closure $H^\infty(U_A, d\theta)$ coincides with the functions in $f \in H^\infty(d\theta)$ such that the partial sums of the power series of f are uniformly bounded. The bidual U_A^{**} is isometrically isomorphic to the direct sum of $H^\infty(U_A, d\theta)$ and an L^∞-space. The proof (see [Sac2]) depends on a generalization of the F. and M. Riesz theorem due to Oberlin [Ob], asserting that any measure on K orthogonal to U_A is absolutely continuous with respect to $d\theta$ on Γ_∞.

5 The Pełczyński and Dunford-Pettis Properties

As Banach spaces, tight subspaces of $C(K)$ share a number of properties of $C(K)$. We discuss the Pełczyński property, which is shared, and also the Dunford-Pettis property, which is partially shared.

A Banach space X has the *Pełczyński property* if whenever T is an operator from X to another Banach space that is not weakly compact, there is an embedding $c_0 \hookrightarrow X$ such that the restriction of T to c_0 is an isomorphism. The spaces $C(K)$ have the Pełczyński property (see [Woj, III.D.§ 33]). However, L^1-spaces do not have the Pełczyński property unless they are finite-dimensional. Reflexive Banach spaces have the Pełczyński property, by default.

Theorem 5.1 (Saccone). *Any tight subspace of $C(K)$ has the Pełczyński property.*

Before saying something about the proof, we discuss the Pełczyński property in more detail.

A series $\sum x_k$ in X is *weakly unconditionally convergent*, or *a wuc series*, if $\sum x^*(x_k)$ converges unconditionally for all $x^* \in X^*$. In this case, $\sum x^*(x_k)$ converges absolutely for each $x^* \in X^*$, and the closed graph theorem shows that the operator $x^* \mapsto \{x^*(x_k)\}$ is continuous from X^* to ℓ^1. In particular, there is $\beta > 0$ such that $\sum |x^*(x_k)| \le \beta \|x^*\|$ for all $x^* \in X^*$. The preadjoint operator $T : c_0 \to X$, defined on the standard basis vectors e_k of c_0 by $T(e_k) = x_k$, is then seen to be continuous and satisfy $\|T\| \le \beta$. Conversely, any (continuous) operator $T : c_0 \to X$ determines a wuc series $\sum T(e_k)$. Thus wuc series correspond to operators from c_0 into X.

Let E be a subset of X^*. If E is weakly precompact in X^*, and if $\sum x_k$ is a wuc series in X with corresponding operator T, then $T^*(E)$ is weakly precompact in ℓ^1. Consequently $T^*(E)(e_k)$ tends to 0 as $k \to \infty$, that is,

$$\sup_{x^* \in E} |x^*(x_k)| \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{5.1}$$

With a little more effort, it can be shown that if (5.1) fails, then there is an ℓ^1-basic sequence $\{x_k^*\}$ in E, that is, a sequence that is equivalent to an ℓ^1-basis. These statements characterize weak compactness precisely when X has the Pełczyński property. We state this result formally.

Theorem 5.2. *The following statements are equivalent, for a Banach space X.*
(i) *X has the Pełczyński property.*

(ii) *If E is a subset of X^* such that for any wuc series $\sum x_k$ in X we have $x^*(x_k) \to 0$ (as $k \to \infty$) uniformly for $x^* \in E$, then E is weakly precompact.*
(iii) *If E is a subset of X^* that is not weakly precompact, then there is an ℓ^1-basic sequence in E.*

We refer to [Woj] for the proof. A related result in this circle of ideas is that if X has the Pełczyński property, then its dual space X^* is weakly sequentially complete.

Now we return to Saccone's theorem, which is proved in [Sac1]. The crux of the matter is to find, for a given E that is not weakly compact, a wuc series in X for which (5.1) fails. To do this, Saccone begins with a characterization of weak compactness due to R.C.James, and eventually he throws the proof back on some difficult work of Bourgain [B3, B7], for which there is a clear treatment in [Woj, III.D. §§ 29–32].

Recall that an operator $T : X \to Y$ is *completely continuous* if T maps weakly convergent sequences in X to norm convergent sequences in Y. If X is reflexive, then the completely continuous operators coincide with the compact operators, while every operator on X is weakly compact. In contrast, for $X = C(K)$, the completely continuous operators coincide with the weakly compact operators.

An isomorphism of c_0 cannot be completely continuous, as the standard basis of c_0 converges weakly to 0. Thus if X has the Pełczyński property, then any completely continuous operator from X to another Banach space is weakly compact. As a corollary to Saccone's theorem, we then obtain the following.

Corollary 5.3. *If B is a tight subspace of $C(K)$, then any completely continuous operator $T : B \to Y$ is weakly compact.*

A Banach space X has the *Dunford-Pettis property* if every weakly compact operator from X to another Banach space is completely continuous. This occurs if and only if whenever the sequence $\{x_n\}$ in X converges weakly to 0, and the sequence $\{x_n^*\}$ in X^* converges weakly to 0, then $x_n^*(x_n) \to 0$. The spaces $C(K)$, and any L^1-space, have the Dunford-Pettis property. If a dual Banach space has the Dunford-Pettis property, then its predual does also. Reflexive Banach spaces do not have the Dunford-Pettis property unless they are finite-dimensional. See Basic Concepts.

Not every tight subspace of $C(K)$ has the Dunford-Pettis property. In fact, any infinite-dimensional reflexive subspace of $C(K)$ is tight but fails to have the Dunford-Pettis property. However, the Dunford-Pettis property *does* hold under hypotheses that are somewhat stronger than tightness. The following statement can be extracted from Bourgain's work in [B3].

Theorem 5.4 (Bourgain). *Let A be a subspace of $C(K)$. If S_g^{**} is completely continuous for all $g \in C$, then A^* and A have the Dunford-Pettis property.*

The collection of $g \in C$ such that S_g^{**} is completely continuous is a closed subalgebra of C. It is called the *Bourgain algebra* associated with A.

We wish to develop some criteria that guarantee that S_g^{**} is completely continuous. The following condition is a variant of the notion of a *rich subspace*, which stems from [Woj].

An operator T from A to another Banach space is *nearly dominated* if there is a probability measure μ on K such that if $\{f_m\}$ is a bounded (!) sequence in A that converges to 0 in $L^1(\mu)$, then $\|Tf_m\| \to 0$. Trivially, absolutely summing operators are nearly dominated. If $T_j : A \to X$ is nearly dominated by μ_j, and if $T_j \to T$ in operator norm, then T is nearly dominated by $\sum \mu_j/2^j$. It is straightforward to show that the collection of $g \in C$ such that S_g is nearly dominated forms a closed subalgebra of C. The pointwise bounded convergence theorem can be used to show that nearly dominated operators are completely continuous.

Theorem 5.5. *Let A be a subspace of $C(K)$. Each of the following conditions guarantees that S_g^{**} is completely continuous for all $g \in C$, hence that A^* and A have the Dunford-Pettis property.*
(i) *The subspace A is compactly tight.*
(ii) *The operators S_g are absolutely summing for a family of functions $g \in C$ that generates C as a uniform algebra.*
(iii) *The operators S_g are nearly dominated for a family of functions $g \in C$ that generates C as a uniform algebra.*

For (i), observe that if S_g is compact, then S_g^{**} is compact hence completely continuous. For (ii), we use the fact that if S_g is absolutely summing, then S_g^{**} is absolutely summing hence completely continuous. The proof under the condition (iii) is straightforward.

Note that either of the conditions (ii) or (iii) covers the subspaces U_A and U_C discussed earlier. Each of the three conditions covers the algebras $R(K)$ and $A(K)$ from rational approximation theory.

6 Absolutely Summing and Related Operators on the Disk Algebra

Now we consider some properties of the Banach space $C(\Gamma)$ of continuous functions on the unit circle Γ that are inherited by the disk algebra C_A. A prototypical theorem along these lines is the following (see [Pe]).

Theorem 6.1 (Mityagin-Pełczyński). *For $1 < p < \infty$, every p-summing operator from the disk algebra C_A to a Banach space Y extends to a p-summing operator from $C(\Gamma)$ to Y, hence is strictly p-integral.*

The Paley operator shows that the statement fails at the endpoint $p = 1$. The proof of the theorem depends on the boundedness of the Riesz projection R from $L^p(d\theta)$ onto $H^p(d\theta)$, $1 < p < \infty$, together with some Hardy space theory. Indeed, let T be a p-summing operator, and let μ be the measure on Γ for T given by the Pietsch theorem (see Basic Concepts), so that T extends to a continuous operator from the closure $H^p(\mu)$ of C_A in $L^p(\mu)$ to Y. Let $\mu = w d\theta + \mu_s$ be the Lebesgue decomposition of μ with respect to Lebesgue measure $d\theta$. The Hardy space theory gives $H^p(\mu) = H^p(w\, d\theta) \oplus L^p(\mu_s)$. Further, if $\log w \notin L^1(d\theta)$, then $H^p(\mu) = L^p(\mu)$, and T is p-integral. On the other hand, if $\log w \in L^1(d\theta)$, and $h = \exp(\log w + i\mathcal{H}(\log w))$ is the "outer" function in $H^1(d\theta)$ such that $|h| = w$, then $Q_p(f) = h^{-1/p} R(h^{1/p} f)$ projects $L^p(w d\theta)$ onto $H^p(w d\theta)$, and this projection allows us to factor T,

$$T : C_A \hookrightarrow L^\infty(\mu) \hookrightarrow L^p(\mu) \stackrel{Q_p \oplus I}{\longrightarrow} H^p(\mu) \longrightarrow Y,$$

again showing that T is strictly p-integral.

Our primary focus will be on Bourgain's extension of the Grothendieck theorem to the disk algebra, with emphasis on the interpolatory nature of the proofs. We will sketch the proofs modulo the interpolation theorems, which we defer to the next section.

The Grothendieck theorem (see Basic Concepts) asserts that any operator from an L^1-space to ℓ^2 is absolutely summing. A dual version of Grothendieck's theorem asserts that any operator from a $C(K)$-space to ℓ^1 is 2-summing. In fact, we can replace ℓ^1 in this statement by any space of cotype 2. In reading the following version of Grothendieck's theorem, recall that among the spaces L^p, precisely those with $1 \leq p \leq 2$ are of cotype 2.

Theorem 6.2. *Every operator from $C(K)$ to a space of cotype 2 is 2-summing.*

Theorem 6.2 is a simple consequence of the following two lemmas.

Lemma 6.3. *If Y is of cotype 2, and if $p \geq 2$, then any p-summing operator from a Banach space X to Y is 2-summing.*

Lemma 6.4. *For every Banach space Y and every finite rank operator $T : C(K) \to Y$ we have $\pi_p(T) \leq \pi_2(T)^\theta \|T\|^{1-\theta}$ for $2 < p < \infty$, where $\theta = 2/p$.*

If the lemmas are proved and Y is of cotype 2, we combine the lemmas to obtain $\pi_2(T) \leq c\pi_4(T) \leq c(\pi_2(T)\|T\|)^{1/2}$ for every $T : C(K) \to Y$ of finite rank, whence $\pi_2(T) \leq c^2\|T\|$. The finite rank assumption is easily lifted by approximation.

The first lemma is proved by an easy concatenation of inequalities, one of which is the Khinchin inequality (see [Woj, III.F § 36], and also Basic Concepts). It yields the estimate $\pi_2(T) \leq c_p C_q(Y)\pi_p(T)$, where $C_q(Y)$ is the cotype constant of Y, and c_p depends only on p. To prove the second lemma, we choose by the Pietsch theorem a probability measure μ on K such that the operator T acts from $L^2(\mu)$ to Y with norm $\pi_2(T)$. Also, T acts from $C(K)$ to Y with the norm $\|T\|$, and consequently T extends to $L^\infty(\mu)$ with (at most) the same norm,

$$T : L^2(\mu) \xrightarrow{\pi_2(T)} Y, \tag{6.1}$$

$$T : L^\infty(\mu) \xrightarrow{\|T\|} Y. \tag{6.2}$$

By interpolation, T acts from $L^p(\mu)$ to Y with norm not exceeding $\pi_2(T)^\theta\|T\|^{1-\theta}$, where θ is given by the convexity condition $1/p = \theta(1/2) + (1-\theta)(1/\infty) = \theta/2$. This proves the lemma and with it Grothendieck's theorem.

Now we turn to Bourgain's version of the theorem for the disk algebra.

Theorem 6.5 (Bourgain). *Every operator from the disk algebra C_A to a space of cotype 2 is 2-summing.*

As previously, the proof is an easy consequence of Lemma 6.3 and the following analog of Lemma 6.4.

Lemma 6.6. *For every Banach space Y and every finite rank operator $T : C_A \to Y$ we have $\pi_p(T) \leq c\pi_2(T)^\theta\|T\|^{1-\theta}$ for $2 < p < \infty$, where $\theta = 2/p$ and c is a universal constant.*

For the proof, we start as in the proof of Lemma 6.4 with a probability measure μ on the unit circle Γ such that T acts from the closure $H^2(\mu)$ of C_A in $L^2(\mu)$ to Y with norm $\pi_2(T)$. Our first problem is that the measure μ need not be absolutely continuous with respect to arc length $d\theta$. For this, we invoke the following.

Absolute Continuity Principle. *In problems like this, the singular parts of mea-*

sures can be disregarded.

One way to justify this is to refer to the decomposition $H^p(\mu) = H^p(wd\theta) \oplus L^p(\mu_s)$ used above and to the Hardy space theory underlying this decomposition. However, sometimes other arguments are also applicable. In the case under consideration, we may work with the operators $T_n f = T(K_n * f)$ in place of T, where K_n is the nth Fejér kernel. For them the above measure becomes absolutely continuous, and moreover the T_n's may be regarded directly as operators on $H^\infty(d\theta)$.

Thus we assume that $\mu = w\, d\theta$, where $w \geq 0$ is a weight, $\int wd\theta = 1$. We arrive at the following analogs of (6.1) and (6.2):

$$T : H^2(w\, d\theta) \xrightarrow{\pi_2(T)} Y, \tag{6.3}$$

$$T : H^\infty(w\, d\theta) \xrightarrow{\|T\|} Y. \tag{6.4}$$

The question now is whether we can interpolate between (6.3) and (6.4) as we did between (6.1) and (6.2). The answer is that we can replace w by a weight $v \geq w$, $\int v \leq C$, such that for this new weight the above interpolation is possible. This will follow from results in the next section. Indeed, since $L^1(d\theta)$ is BMO-regular (see Proposition 7.4), there is a majorant v for w such that $\log v$ belongs to BMO, and on account of Theorem 7.7 the desired interpolation holds for this majorant v. This proves Bourgain's theorem.

In a standard way, Bourgain's theorem implies that every operator from C_A^* (or from L^1/H_0^1) to l^2 is absolutely summing. Then from the relations

$$C_A \sim (C_A \oplus C_A \oplus \dots)_{c_0}, \quad L^1/H_0^1 \sim (L^1/H_0^1 \oplus L^1/H_0^1 \oplus \dots)_{l^1}$$

(see [Woj, III.E. § 12]), it is also standard to conclude that L^1/H_0^1 and C_A^* are of cotype 2. See [Woj, III.I, § 14] for more details.

We mention another approach to the Grothendieck theorem, due to Maurey. This method gives some information about operators $T : C(K) \to Y$, where Y is a space of arbitrary finite cotype. In particular, it applies to $Y = L^p$ for any $1 \leq p < \infty$. For the proof of the following theorem, see [DJT, Chapter 10] or [K2].

Theorem 6.7 (Maurey). *For $1 \leq p < q$, the class of (q,p)-summing operators defined on $C(K)$ does not depend on p and is contained in the class of $(q+\epsilon)$-summing operators for any $\epsilon > 0$.*

A related result, due to Pisier (see the above references), is that $T : C(K) \to Y$ is (q,p)-summing if and only if T factors through the inclusion $C(K) \hookrightarrow L_{q,1}(\mu)$ (the Lorentz space) for some probability measure μ on K.

If Y is of cotype q, the identity operator of Y is $(q,1)$-summing. Thus Maurey's theorem shows that every operator from $C(K)$ to Y is $(q+\epsilon)$-summing. We recover the Grothendieck theorem by setting $q=2$ and applying Lemma 6.3.

The following theorem allows us to transport these results to the disk algebra C_A.

Theorem 6.8 (Kislyakov). *For an arbitrary Banach space Y and $q > p \geq 1$, every (q,p)-summing operator $T : C_A \to Y$ extends to a (q,p)-summing operator from $C(\Gamma)$ to Y.*

According to the Mityagin-Pełczyński theorem, the theorem remains true if $q = p > 1$.

The remainder of this section is devoted to an outline of a proof of Theorem 6.8, with some simplifications compared to the exposition in [K2].

Lemma 6.9. *Under the conditions of Theorem* 6.8, *the operator T is $10q$-summing.*

We break the proof of Lemma 6.9 into four steps. First note that the family of (q,p)-summing operators grows as p decreases, so we may assume that $p=1$. We assume also that $\pi_{q,1}(T) = 1$. When convenient, we use $\langle \cdot, \cdot \rangle$ to denote the pairing between vectors and functionals.

Step 1. There is a probability measure λ on Γ such that

$$\|Tx\|^q \leq q\,|\langle 1-\varphi, \lambda\rangle| \quad \text{for all } x, \varphi \in C_A \text{ satisfying } |x| + |\varphi| \leq 1. \tag{6.5}$$

To see this, we use a trick invented by Pisier to prove his characterization of (q,p)-summing operators on $C(K)$ mentioned above. Let

$$C_n = \sup\left\{ \left(\sum \|Tx_j\|^q\right)^{1/q} : x_1, \dots, x_n \in C_A, \sum |x_j(t)| \leq 1 \right\}.$$

Clearly $C_n \nearrow \pi_{q,1}(T) = 1$. We choose $\delta_n \searrow 1$ and for every n find $x_1^{(n)}, \dots, x_n^{(n)} \in C_A$ such that $\sum_j \|Tx_j^{(n)}\|^q > 1$ and $\sum_j |x_j^{(n)}(t)| \leq \delta_n / C_n$. Then we choose $\xi_j^{(n)} \in Y^*$ such that $\sum_j \|\xi_j^{(n)}\|^{q'} \leq 1$ and $\sum_j \langle Tx_j^{(n)}, \xi_j^{(n)}\rangle = 1$, and define a functional λ_n on C_A by the formula $\lambda_n(\psi) = \sum_j \langle T(\psi x_j^{(n)}), \xi_j^{(n)}\rangle$. Then $\lambda_n(1) = 1$ and $\|\lambda_n\| \leq \delta_n$. We consider a weak-star limit point of the sequence $\{\lambda_n\}$. This is a functional on C_A. We extend it to $C(\Gamma)$ with preservation of norm, obtaining a measure λ on Γ. Since $\lambda(1) = 1$ and $\|\lambda\| \leq \lim \delta_n = 1$, λ is a probability measure.

Now let x, $\varphi \in C_A$ satisfy $|x| + |\varphi| \leq 1$. We define $y_1, \dots, y_{n+1} \in C_A$ by $y_j = \varphi x_j^{(n)}$ for $1 \leq j \leq n$ and $y_{n+1} = x$. Then

$$\left(\sum \|Ty_j\|^q\right)^{1/q} \leq C_{n+1} \sup_t \sum |y_j(t)| \leq C_{n+1}\delta_n / C_n,$$

which implies that $|\langle\varphi,\lambda_n\rangle|^q+\|Tx\|^q\le(C_{n+1}\delta_n/C_n)^q$ and, in the limit, $|\langle\varphi,\lambda\rangle|^q+\|Tx\|^q\le 1$. Finally $|\langle\varphi,\lambda\rangle|^q=|1-\langle 1-\varphi,\lambda\rangle|^q\ge 1-q|\langle 1-\varphi,\lambda\rangle|$, from which (6.5) follows.

Step 2. We apply the absolute continuity principle. It may be assumed that $\lambda=v d\theta$, $\int v d\theta=1$, and that (6.5) is valid for $x,\ \varphi\in H^\infty(d\theta)$. (Again, we may convolve with Fejér kernels to ensure this.)

Step 3. There is $a\ge v$, $\int a\le C$, such that

$$\|Tx\|\le C\|x\|_\infty^{1-1/8q}\|x\|_{L^1(a)}^{1/8q},\qquad x\in H^\infty.$$

To see this, we assume that $\|x\|_\infty\le 1/2$. It suffices to show that $\|Tx\|^q\le C\|x\|_{L^1(a)}^{1/8}$. We shall deduce this from (6.5) by a careful choice of φ. Denote by $\mathcal{H}$ the harmonic conjugation operator. As in the proof of Proposition 7.4, there exists $a\ge v$, $\int a\le C_0$, such that for $b=a^{1/2}$ we have $|\mathcal{H}(b)|\le C_0 b$, and $|b+i\mathcal{H}(b)|\le(1+C_0)b$. We put $\alpha=-\log(1-|x|)$, so that $\alpha\ge 0$, and also $\alpha\le C_1|x|$ since $|x|\le 1/2$. We define successively

$$\psi=\frac{\alpha^4 b+i\mathcal{H}(\alpha^4 b)}{b+i\mathcal{H}(b)},\qquad \Phi=\psi^{1/4},\qquad \varphi=\exp(-A\Phi),$$

where the constant $A>0$ will be chosen momentarily. Since ψ is the quotient of functions with values in the right half-plane, it omits the negative axis, and we choose the branch of Φ whose argument ranges between $\pm\pi/4$. Then Re $\Phi\ge|\Phi|/\sqrt{2}$ and $|1-\varphi|\le C_2|\Phi|$. Now $|\psi|\ge\alpha^4/(1+C_0)$, so Re $\Phi\ge C_3\alpha$. We set $A=1/C_3$, and then $|\varphi|=\exp(-A\ \mathrm{Re}\ \Phi)\le\exp(-AC_3\alpha)=1-|x|$. Thus we may apply (6.5) with this φ and $\lambda=v\,d\theta$:

$$\|Tx\|^q\le\int|1-\varphi|v\,d\theta\le C_2\int|\Phi|a\,d\theta\le C_4\left(\int|\Phi|^8 a\,d\theta\right)^{1/8}.$$

Now, $|\Phi|^8 a=|\psi|^2 b^2\le(\alpha^4 b)^2+\mathcal{H}(\alpha^4 b)^2$. Using the L^2-estimate for $\mathcal{H}$, we obtain

$$\int|\Phi|^8 a\,d\theta\le\int\left((\alpha^4 b)^2+\mathcal{H}(\alpha^4 b)^2\right)\,d\theta\le 2\int(\alpha^4 b)^2\,d\theta\le 2C_1^8\int|x|^8 a\,d\theta\le C_5\int|x|a\,d\theta.$$

Taking 8th roots, we obtain the required estimate for $\|Tx\|^q$.

Step 4. The result of the preceding step shows that T acts from the interpolation space $(H^\infty,H^1(a))_{1/8q,1}$ of the real method to Y, with an appropriate norm estimate (see a calculation in the proof of Theorem 3.5.2(b) in [BL]). If we were dealing with the L^p-scale, the interpolation space would be readily identifiable as a Lorentz space, $(L^\infty,L^1(a))_{1/8q,1}=L^{8q,1}(a)$. Even in the case at hand, on account of the above choice of a we are able to interpolate similarly in the scale of weighted Hardy spaces, by Theorem 7.7. This leads to the estimate

$$\|Tx\|\le C'\|x\|_{L^{8q,1}(a)}\le C\|x\|_{L^{10q}(a)},\qquad x\in C_A\,.$$

Thus T is $10q$-summing, and Lemma 6.9 is established.

Next we need to introduce "vector coefficients." For a Banach space X, we consider the space $C_A(X)$ of all X-valued continuous functions f on the unit circle Γ that extend analytically to the unit disk, that is, that satisfy $\int f(z)z^n d\theta = 0$ for $n \geq 1$. An operator $T : C_A(X) \to Y$ is said to be (q, p, X)-summing if

$$\left(\sum \|Tx_j\|^q\right)^{1/q} \leq C \sup_{t\in\Gamma} \left(\sum \|x_j(t)\|_X^p\right)^{1/p} \tag{6.6}$$

for any finite collection $\{x_j\}$ in $C_A(X)$.

Lemma 6.10. *If $T : C_A(X) \to Y$ is (q, p, X)-summing, then there is a probability measure μ on Γ such that $\|Tx\| \leq C(\int \|x(t)\|_X^{10q} d\mu(t))^{1/10q}$.*

Indeed, it is routine to carry through the above proof of Lemma 6.9. For this, note that some functions will remain scalar-valued, as for instance the function φ in (6.5). The condition on x and φ in (6.5) becomes $\|x(t)\|_X + |\varphi(t)| \leq 1$ for $t \in \Gamma$.

For any operator $T : C_A \to Y$, we define the operator $\tilde{T} : \{x_j\} \mapsto \{Tx_j\}$ on sequences of functions in C_A. Evidently T is (q, p)-summing if and only if $\tilde{T}$ maps $C_A(\ell^p)$ to $\ell^q(Y)$. It is quite easy to see that even more is true.

Lemma 6.11. *If T is (q, p)-summing, then $\tilde{T}$ is (q, p, ℓ^p)-summing.*

Now we apply Lemma 6.10 to $\tilde{T}$ and proceed as in the proof of the Mityagin-Pełczyński theorem. As before, we may assume that the measure μ is absolutely continuous, and even that $\mu = a d\theta$ with $\log a \in L^1(d\theta)$. Then $\tilde{T}$ has the factorization

$$\tilde{T} : C_A(\ell^p) \hookrightarrow H^{10q}(\ell^p, a) \to \ell^q(Y), \tag{6.7}$$

where the first mapping is the identity embedding and the second is the extension of $\tilde{T}$ by continuity. To complete the proof, we need a projection.

Lemma 6.12. *If $1 < p, s < \infty$ and $\log a \in L^1(d\theta)$, then there is a projection $\tilde{Q}$ from $L^s(\ell^p, a)$ onto $H^s(\ell^p, a)$ having the form $\tilde{Q}(\{f_j\}) = \{Qf_j\}$ for a projection operator Q acting on scalar-valued functions.*

We take Q to be the projection Q_s in the proof of the Mityagin-Pełczyński theorem (Theorem 6.1). The boundedness of $\tilde{Q}$ follows from standard techniques.

It is now easy to establish Theorem 6.8 in the case $p > 1$. If $p > 1$ in (6.7), then $\tilde{T}$ extends to some operator $U : C(\ell^p) \to \ell^q(Y)$ of the form $U\{x_j\} = \{Sx_j\}$, where S acts from $C(\Gamma)$ to Y. The boundedness of U means that S is (q, p)-summing. Clearly S extends T.

It remains to treat the case where $p = 1$. The facts already proved and Maurey's Theorem 6.7 show that for $1 < r < q$ the class of (q, r)-summing operators from C_A to Y does not depend on r. It suffices to extend this statement to $r = 1$. For this, let $T : C_A \to Y$ be of finite rank, and let $1 < r < s < q$. Then we have

$$\begin{aligned} T &: C_A(\ell^s) \stackrel{\pi_{q,s}(T)}{\longrightarrow} \ell^q(Y), \\ T &: C_A(\ell^1) \stackrel{\pi_{q,1}(T)}{\longrightarrow} \ell^q(Y). \end{aligned} \tag{6.8}$$

By the remark after the proof of Lemma 7.6 (where H^∞-spaces are involved, but this does not matter too much), we can interpolate as if we had $C(\ell^s)$ and $C(\ell^1)$. This shows that $\pi_{q,r}(T) \le C\pi_{q,s}(T)^{1\perp\theta}\pi_{q,1}(T)^\theta$ for some $0 < \theta < 1$. Since the norms $\pi_{q,r}$ and $\pi_{q,s}$ are equivalent, we obtain the desired result.

7 Interpolation of Hardy-Type Subspaces

Several times in Section 6 we had to interpolate either between weighted Hardy spaces $H^p(a\,d\theta)$, or between Hardy spaces of vector-valued functions $H^p(\ell^r)$. To cover both cases, we consider the measure space $(\Gamma \times \Omega, m \times \mu)$, where $dm = d\theta/2\pi$ is normalized arc-length measure on the unit circle Γ, and (Ω, μ) is some fixed σ-finite measure space. Since we wish to use the full range $0 < p \leq +\infty$, we will refer to quasi-Banach spaces where appropriate. A *lattice of measurable functions* on $(\Gamma \times \Omega, m \times \mu)$ is a quasi-Banach space X of measurable functions such that if $f \in X$, g is measurable, and $|g| \leq |f|$, then $g \in X$ and $\|g\|_X \leq C\|f\|_X$. (Note that this is not the same as a Banach lattice, as defined in Basic Concepts, whose elements are measurable functions. Since we treat the term as an inseparable unit, lattice-of-measurable-functions, there should be no confusion.) The examples we have in mind are the spaces $L^p(w\,dmd\mu)$, and the spaces $L^p(dm, L^r(\mu))$ of measurable functions $x(t,\omega)$ such that $y(t) = \left(\int |x(t,\omega)|^r d\mu(\omega)\right)^{1/r}$ is in $L^p(dm)$.

Let N^+ be the Smirnov class of analytic functions on the unit disk (see [Du, Pr]), which we identify with their boundary value functions on the circle. (For our purposes the class N^+ could be replaced by $\bigcup_{p>0} H^p$.) We call a function on the circle *analytic* if it belongs to N^+. If X is a lattice of measurable functions on $(\Gamma \times \Omega, m \times \mu)$, we define its *analytic subspace* X_A to be the set of functions $f \in X$ such that $f(\cdot,\omega) \in N^+$ for almost all ω. In the case of functions of one variable, as when μ is a point mass, we have $L^p_A = H^p$.

In the sequel we also impose on X the following conditions:
(i) if $f \in X$, then $\int_\Gamma \log^+ |f(t,\omega)|\, dm(t) < \infty$ a.e. on Ω,
(ii) if $f_n \to 0$ in X, then $\int_\Gamma \log^+ |f_n(t,\omega)|\, dm(t) \to 0$ in μ-measure,
(iii) if $f \in X$, there exists $g \in X$ such that $|f| \leq |g|$, $\|g\|_X \leq C\|f\|_X$,
and $\log |g(\cdot,\omega)| \in L^1(m)$ for a.a. ω.

These conditions serve to exclude various degenerate possibilities. Under these conditions it is easy to prove, for instance, that X_A is closed in X.

Now let X and Y be lattices of measurable functions. The fundamental problem we consider is to determine when interpolation properties of the couple (X,Y) are inherited by the couple (X_A, Y_A). We shall deal with real interpolation only.

We remind the reader of the definition of the real interpolation spaces $(X_0, X_1)_{\theta,q}$ for a couple (X_0, X_1) of compatible quasi-Banach spaces. By compatible we mean that X_0 and X_1 are linear subspaces of some ambient space, so we may define the *K-functional*

$K(x, t; X_0, X_1)$ for $t > 0$ and $x \in X_0 + X_1$ by

$$K(x, t; X_0, X_1) = \inf\{\|x_0\|_0 + t\|x_1\|_1 : x_0 + x_1 = x,\ x_0 \in X_0,\ x_1 \in X_1\}.$$

For $0 < \theta < 1$ and $0 < q \le \infty$, we define the interpolation space $(X_0, X_1)_{\theta,q}$ to consist of $x \in X_0 + X_1$ such that $t^\theta K(x, t; X_0, X_1)$ belongs to $L^q(dt/t)$, and we define the norm of x in $(X_0, X_1)_{\theta,q}$ to be the norm of $t^\theta K(x, t; X_0, X_1)$ in $L^q(dt/t)$. Actually the specific expressions for the K-functional and the norm will not play a role for us.

Let $Y_0 \subset X_0$ and $Y_1 \subset X_1$ be closed subspaces. We say that the couple (Y_0, Y_1) is *K-closed in* (X_0, X_1) if there is $C > 0$ such that any decomposition $y = x_0 + x_1$ of an element $y \in Y_0 + Y_1$ with $x_i \in X_i$ can be modified to a decomposition $y = y_0 + y_1$ with $y_i \in Y_i$ and $\|y_i\|_i \le C\|x_i\|_i$, $i = 0, 1$. In this case we have

$$(Y_0, Y_1)_{\theta,q} = (Y_0 + Y_1) \cap (X_0, X_1)_{\theta,q}\,,$$

with equivalence of norms, and the interpolation properties of the couple (X_0, X_1) and its subcouple (Y_0, Y_1) are identical. Our basic problem can be formulated as follows.

Problem. *When is the couple (X_A, Y_A) K-closed in (X, Y)*Γ

We shall see that this happens fairly often.

We begin with a useful duality result. Assume that X_0 and X_1 are Banach spaces and that $X_0 \cap X_1$ is dense in both X_0 and X_1. Then, in a natural way, the spaces X_0^* and X_1^* are included in $(X_0 \cap X_1)^*$ and, consequently, form a compatible couple. If $Y_i \subset X_i$ (as above), we denote by $Y_i^\perp$ the annihilator of Y_i in X_i^*, that is, the set of $L \in X_i^*$ such that $L = 0$ on Y_i.

Lemma 7.1. *The couple (Y_0, Y_1) is K-closed in (X_0, X_1) if and only if the couple $(Y_0^\perp, Y_1^\perp)$ is K-closed in (X_0^*, X_1^*).*

The proof is left to the reader (see [Pi, K4]).

If X is a Banach lattice of measurable functions on $(\Gamma \times \Omega, m \times \mu)$, it often happens that under the duality $\langle f, g\rangle = \int\int fg\, dm d\mu$, X^* is also a lattice of measurable functions on the same measure space. We will assume that this is the case, and further that both X and X^* satisfy the conditions (i)-(iii) above. Then one easily sees that, as in the classical case of the H^p-spaces on the circle, we have $X_A^\perp = Z(X^*)_A$, where Z is the coordinate function on Γ. Thus Lemma 7.1 relates interpolation properties of the couples (X_A, Y_A) and $((X^*)_A, (Y^*)_A)$, X and Y being two lattices as above.

The class of BMO functions will play an important role in what follows. Recall that a function f on Γ is in BMO if $f = u + \mathcal{H}v$, where u, $v \in L^\infty$. As usual, we disregard the constant functions and define $\|f\|_{BMO}$ to be the infimum of $\|u\| + \|v\|$ over all such representations, where $\|\cdot\|$ is the norm in L^∞ modulo the constants.

Lemma 7.2. *Let $w > 0$ be a measurable function on Γ. Then $\log w \in BMO$ if and only if there exist constants $C > 1$, $0 < \rho < 1$, and a function $f > 0$ such that $w/C \le f \le Cw$ and $|\mathcal{H}(f^\rho)| \le Cf^\rho$. Moreover, C and ρ are controlled in terms of $\|\log w\|_{BMO}$, and vice versa.*

Proof. Suppose $\log w \in BMO$. Then $\log w = \alpha + u + \mathcal{H}v$, where α is a real constant, u and v are real functions, $\int v = 0$, and $\|u\|_\infty + \|v\|_\infty \le 2\|\log w\|_{BMO}$. We pick ρ so small that $\rho\|v\|_\infty < \pi/4$, and we set $F = \exp(-i\rho(v + i\mathcal{H}v)) = e^{\rho\mathcal{H}v}[\cos(\rho v) + i\sin(\rho v)]$ and $f = e^\alpha(\operatorname{Re} F)^{1/\rho} = we^{\perp u}[\cos(\rho v)]^{1/\rho}$. Since F is analytic in the unit disk and $F(0)$ is real, we have $\operatorname{Im} F = \mathcal{H}(\operatorname{Re} F)$. Since $|\sin\rho v| \le \cos\rho v$, we have $|\operatorname{Im} F| \le \operatorname{Re} F$, and consequently $|\mathcal{H}(f^\rho)| \le f^\rho$. The estimates $1 \ge \cos(\rho v) \ge 1/\sqrt{2}$ lead to $w/C \le f \le Cw$.

Conversely, given f as in the lemma, we put $G = f^\rho + i\mathcal{H}(f^\rho)$. Since $|\mathcal{H}(f^\rho)| \le Cf^\rho$, the values of G lie in a sector in the right half-plane, and the principal branch of $\log G$ is analytic. Writing $\log G = \log|G| + i\arg G$, we have $|\arg G| \le \tan^{\perp 1} C$ and $\mathcal{H}(\arg G) = -\log|G| + \log|G(0)|$, so $\log|G| \in BMO$. Since $f^\rho \le |G| \le \sqrt{1 + C^2}f^\rho$, we see that $\rho\log f - \log|G|$ is bounded, and $\log f \in BMO$. Finally, $\log f - \log w$ is bounded, so $\log w \in BMO$, with BMO-norm bounded in terms of C and ρ.

From the proof we see that we can always reduce ρ, at the expense of increasing C and changing f.

A *weight* is a function $w > 0$ on $\Gamma \times \Omega$ such that $\log w(\cdot, \omega) \in L^1(dm)$ for a.a. ω. For $0 < p < \infty$ we denote by $L^p(w)$ the usual space $L^p(w\,dmd\mu)$, with norm denoted by $\|f\|_{p,w}$, though we shall denote by $L^\infty(w)$ the space of functions f on $\Gamma \times \Omega$ such that f/w is bounded, with the norm

$$\|f\|_{\infty,\omega} = \operatorname{ess\,sup}\{|f(\zeta,\omega)|/w(\zeta,\omega) : (\zeta,\omega) \in \Gamma \times \Omega\}.$$

With this notation, $L^\infty(w) = L^1(w)^*$ under the non-weighted duality $\langle f, g\rangle = \iint fg\,dmd\mu$.

We denote $L^p_A(w)$ by $H^p(w)$, $0 < p \le \infty$.

For a weight w, we say that $\log w$ is *uniformly* (or *C-uniformly*) *in BMO* if the function $\log w(\cdot, \omega)$ is in BMO for almost all ω, with BMO-norm bounded by C. In this case, the analog of Lemma 7.2 holds, where the function f can be chosen to depend measurably on the parameter ω. We will use this extended version of Lemma 7.2.

A quasi-Banach lattice of measurable functions X on $\Gamma \times \Omega$ is said to be *BMO-regular* if for every $x \in X$, there exists $u \in X$ such that $|x| \le u$, $\|u\|_X \le C\|x\|_X$, and $\log u$ is C-uniformly in BMO, where C depends only on X. The function u will be referred to as a *BMO-majorant* of x.

As an easy consequence of (the extended version of) Lemma 7.2, we have the following.

Lemma 7.3. *A lattice X is BMO-regular if and only if there are $C, \rho > 0$ such that for every $x \in X$ there exists $u \in X$ with $|x| \le u$, $\|u\| \le C\|x\|$, and $|\mathcal{H}(u^\rho(\cdot,\omega))| \le Cu^\rho(\cdot,\omega)$ for a.a. $\omega \in \Omega$. At the expense of increasing C, we can take ρ to be arbitrarily small.*

For a quasi-Banach lattice of measurable functions X and $0 < \beta < \infty$, we define X^β to be the space of functions f such that $|f|^\beta \in X$, with quasi-norm

$$\|f\|_{X^\beta} = \|\,|f|^\beta\,\|_X^{1/\beta}.$$

Thus if $X = L^p$, then $X^\beta = L^{p\beta}$. Clearly X^β is BMO-regular if and only if X is. Our main examples of BMO-regular spaces will be based on the following proposition.

Proposition 7.4. *If the operator $\mathcal{H}$ (acting in the first variable) is bounded on X^β, then X is BMO-regular.*

Proof. It suffices to show that $Y = X^\beta$ is BMO-regular. We verify the conditions formulated in Lemma 7.3. Taking $y \in Y$, we put $y_0 = |y|$, $y_{n+1} = |\mathcal{H}(y_n)|$ for $n \ge 0$, and $v = \sum \delta^n y_n$, where $\delta > 0$ is a fixed small constant. Then $|y| \le v$, $\|v\|_Y \le C\|y\|_Y$, and $|\mathcal{H}v| \le v/\delta$.

Lemma 7.5. *If $\log w$ is uniformly in BMO, then $L^p(w)$ is BMO-regular for $0 < p \le \infty$.*

Proof. If $1 < p < \infty$ and $w = 1$, we may apply Proposition 7.4, with $\beta = 1$. The case $0 < p < \infty$ and w arbitrary then follows easily from the definitions. For $p = \infty$, the definition of the norm in $L^\infty(w)$ as $\sup(|x|/w)$ shows that $\|x\|_\infty w$ is a BMO-majorant of $x \in L^\infty(w)$.

Lemma 7.6. *The space $L^p(dm, L^r(\Omega))$ is BMO-regular for $0 < p < \infty$, $0 < r \le \infty$.*

Proof. The case where $r < \infty$ is a consequence of Proposition 7.4, and the case where $r = \infty$ of its proof. Indeed, given x, we construct a BMO-majorant for $y = \operatorname{ess\,sup}_\omega |x(\cdot,\omega)|$ in $L^p(dm)$, and then treat this majorant as a function of two variables.

It is easy to find other examples of BMO-regular spaces on the basis of the same ideas. A less trivial example is the space $L^\infty(dm, \ell^s)$, $0 < s < \infty$ (see [K4]). While we could have cited this example (and, of course, Theorem 7.7) when interpolating in (6.8), that proof can be based also on the duality $L^\infty(dm; \ell^s) = L^1(dm; \ell^{s'})^*$ for $s \geq 1$, where the latter space is BMO-regular by Lemma 7.6. Thus, to interpolate in (6.8) we can refer to Corollary 7.8.

Now we state our main interpolation result. We are assuming that X and Y are Banach lattices of measurable functions on $\Gamma \times \Omega$, and (when applicable) that X^* and Y^* are also lattices of measurable functions on $\Gamma \times \Omega$, all satisfying the conditions (i)-(iii) above. Also, the density of $X \cap Y$ in X and Y is assumed when needed.

Theorem 7.7. *If X and Y are BMO-regular, then (X_A, Y_A) is K-closed in (X, Y).*

There is some evidence (see [Ka]) in favor of the conjecture that BMO-regularity is a self-dual property. However, this has not yet been verified in the general case. Thus, we combine Lemma 7.1 with Theorem 7.7 to obtain more information.

Corollary 7.8. *In any of the following three cases, (X_A, Y_A) is K-closed in (X, Y), and $((X^*)_A, (Y^*)_A)$ is K-closed in (X^*, Y^*).*

a) *X and Y are BMO-regular,*

b) *X^* and Y^* are BMO-regular,*

c) *X and Y^* are BMO-regular.*

Cases (a) and (b) of the corollary are direct consequences of Theorem 7.7 and Lemma 7.1. Case (c) is not needed in Section 6, so we leave it as an exercise.

We pass to the proof of Theorem 7.7. Let w_0, w_1 be two weights whose logarithms are C-uniformly in BMO.

Lemma 7.9. *If $f(\cdot, \omega) \in N^+$ for a.a. ω and $f = g + h$ with $g \in L^\infty(w_0)$, $h \in L^\infty(w_1)$, then $f = \varphi + \psi$ with $\varphi \in H^\infty(w_0)$, $\psi \in H^\infty(w_1)$ and $\|\varphi\|_{\infty, w_0} \leq C' \|g\|_{\infty, w_0}$, $\|\psi\|_{\infty, w_1} \leq C' \|h\|_{\infty, w_1}$, where C' is determined by C.*

Clearly Theorem 7.7 is a consequence of this lemma. Given $f = x + y$ with $f \in X_A + Y_A$, $x \in X$, $y \in Y$, we find BMO-majorants for x and y in their respective spaces, and apply the lemma to these majorants as the weights.

To prove Lemma 7.9, we first assume that $f \in H^\infty(w_0) + H^\infty(w_1)$. Then the statement to be proved is precisely the K-closedness of the couple $(H^\infty(w_0), H^\infty(w_1))$ in

$(L^\infty(w_0), L^\infty(w_1))$. By duality (Lemma 7.1), it suffices to check that the couple $(H^1(w_0), H^1(w_1))$ is K-closed in $(L^1(w_0), L^1(w_1))$. So let $z = a + b \in H^1(w_0) + H^1(w_1)$, where $a \in L^1(w_0)$, $b \in L^1(w_1)$. We must replace a and b with functions roughly of the same size but analytic in the first variable.

Let v be a BMO-majorant for b in the BMO-regular space $L^1(w_1)$. Setting $w = w_0 v$, we apply Lemma 7.2 and the remark after its proof to find a function k and a constant $\rho < 1$ such that $w/C \le k \le Cw$ and $|\mathcal{H}k^\rho(\cdot,\omega)| \le Ck^\rho(\cdot,\omega)$. Fixing an integer $n > 1/\rho$, we define

$$\alpha = \max\left\{1, (|a|/v)^{1/n}\right\}, \quad F = \frac{k^\rho + i\mathcal{H}(k^\rho)}{k^\rho\alpha + i\mathcal{H}(k^\rho\alpha)}, \quad G = 1 - (1 - F^n)^n.$$

We claim that $z = (1-G)z + Gz$ is the required decomposition.

Indeed, the summands are analytic in the first variable, and it suffices to estimate the norms. Since $|\mathcal{H}(k^\rho)| \le Ck^\rho$, we see that $|F| \le C_1/\alpha \le C_1$ and $|G| \le C_2/\alpha^n \le C_2$, whence $|Gz| \le C_3|a|\alpha^{-n} + C_2|b| \le C_3 v + C_2|b|$. By the choice of v, we obtain the required inequality $\|Gz\|_{1,w_1} \le C_4\|b\|_{1,w_1}$.

We estimate the quantity $\|(1-G)z\|_{1,w_0} \le (1+C_2)\|a\|_{1,w_0} + \|(1-G)b\|_{1,w_0}$. We have

$$|1 - F| = \frac{|(\alpha-1)k^\rho + i\mathcal{H}((\alpha-1)k^\rho)|}{|\alpha k^\rho + i\mathcal{H}(\alpha k^\rho)|} \le (\alpha - 1) + \frac{|\mathcal{H}((\alpha-1)k^\rho)|}{k^\rho}.$$

Since also $|F| \le C_1$, $|b| \le v$, and $w \le Ck$, we see that

$$\begin{aligned}
\int |1-G||b|w_0 &\le C_5 \int |1-F|^n w \le C_6 \int |1-F|^{1/\rho} w \\
&\le C_7\left(\int (\alpha-1)^{1/\rho} w + \int |\mathcal{H}((\alpha-1)k^\rho)|^{1/\rho}\frac{w}{k}\right) \\
&\le C_8\left(\int (\alpha-1)^{1/\rho} w + \int |\mathcal{H}((\alpha-1)k^\rho)|^{1/\rho}\right)
\end{aligned}$$

Since $\mathcal{H}$ acts on $L^{1/\rho}(dm\,d\mu)$ and $k \le Cw$, it follows that the second integral in parentheses is dominated by the first. Now $\alpha - 1 = 0$ if $|\alpha| \le v$ and $\alpha - 1 \le (|a|/v)^{1/n} \le (|a|/v)^\rho$ otherwise. Therefore

$$\int (\alpha-1)^{1/\rho} w \le \int (|a|/v) w = \int |a| w_0,$$

and we are done.

It remains to get rid of the asumption $f \in H^\infty(w_0) + H^\infty(w_1)$. This is done by a standard approximation argument based on the Hardy space theory. Suppose only $f \in N^+$. For $u = \log|f|$ define $G(\cdot,\omega) = \exp(u(\cdot,\omega) + i\mathcal{H}u(\cdot,\omega))$ and $F = f/G$, so that $|F| = 1$

a.e. Thus $f = FG$ is the "inner-outer" factorization of f. Set $u_j = |f| \wedge (jw_0)$ and $G_j(\cdot, \omega) = \exp(u_j(\cdot, \omega) + i\mathcal{H}u_j(\cdot, \omega))$. Then $|G_j| \leq |G|$, and $G_j \to G$ in measure. Set $f_j = FG_j \in H^\infty(w_0)$, then $f_j \to f$ in measure, and $|f_j| \leq |f| \leq |g| + |h|$. By the first part of the proof, $f_j = \varphi_j + \psi_j$, where $\varphi_j \in H^\infty(w_0)$, $\psi_j \in H^\infty(w_1)$, $\|\varphi_j\|_{\infty, w_0} \leq C'\|g\|_{\infty, w_0}$, and $\|\psi_j\|_{\infty, w_1} \leq C'\|h\|_{\infty, w_1}$. For some subnet of the integers we have $\varphi_j \to \varphi$ and $\psi_j \to \psi$ weak-star. Simultaneous convex combinations of the φ_j's and ψ_j's can be chosen to converge a.e. to φ and ψ, and this guarantees that $f = \varphi + \psi$ with the appropriate estimates for $|\varphi|$ and $|\psi|$.

8 Bourgain Projections

Let w be a weight on the unit circle. We say that w *admits an analytic projection* if there is an operator Q that projects $L^p(w)$ onto $H^p(w)$ for all $1 < p < \infty$ at once and, together with Q^*, is of weak type $(1,1)$ relative to w,

$$w\{|Qf| > \lambda\} \le \frac{c}{\lambda}\int |f|w, \qquad w\{|Q^*f| > \lambda\} \le \frac{c}{\lambda}\int |f|w, \qquad f \in L^1(w),$$

where we denote $w(e) = \int_e w$. Here the adjoint Q^* is calculated relative to the duality $\langle f, g\rangle = \int f\bar{g}w\,dm$.

Operators of this sort served as the main technical tool in Bourgain's work extending Grothendieck's theorem to the disk algebra. Bourgain [B2] proved that for every integrable weight u there exists a weight w admitting an analytic projection and satisfying $w \ge u$ and $\int w \le C \int u$, where C is a universal constant. The existence of such a projection does not imply the K-closedness in the scale $H^p(w)$ if the "extreme" exponents $p = 1$ and $p = \infty$ are involved. However, it still implies certain "nice" interpolation properties of this scale, sufficient for instance for proving Lemma 6.4.

Though here we have used different (simpler) techniques, Bourgain's projections remain interesting in themselves. We shall show that a weight w admits an analytic projection if and only if $\log w \in BMO$. Bourgain's majorization result quoted in the preceding paragraph then follows from the fact that $L^1(d\theta)$ is BMO-regular.

We need a technical notion. A two-tailed sequence of functions $\varphi_j \in H^\infty$, $-\infty < j < \infty$, is called an *analytic decomposition of unity subordinate to a weight* w if there exists a constant c such that

(i) $|\varphi_j|^{1/8}w \le c2^j, \qquad -\infty < j < \infty$,
(ii) $\sum |\varphi_j|^{1/8}2^j \le cw$,
(iii) $\sum |\varphi_j|^{1/8} \le c$,
(iv) $\sum \varphi_j = 1$.

Roughly speaking, the functions φ_j behave like the characteristic functions of the sets where $2^{j-1} \le w < 2^j$. The exponent $1/8$ is convenient technically but, in principle, may be replaced by any $\alpha \in (0,1]$; see [K3].

Theorem 8.1. *For a weight w, the following conditions are equivalent*:

1) $\log w \in BMO$,

2) *there is an analytic decomposition of unity subordinate to w*,

3) *w admits an analytic projection,*

4) *there is an operator Q projecting $L^p(w)$ onto $H^p(w)$ for two different values of p.*

We focus on the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3). The implication 3) $\Rightarrow$ 4) is trivial, and 4) $\Rightarrow$ 1) is proved in [KX, Corollary 2.2].

To prove that 1) $\Rightarrow$ 2), we need a lemma.

Lemma 8.2. *Suppose u is a weight such that $\log u \in BMO$. Then $\log(1 \wedge u) \in BMO$, and the BMO-norm of $\log(1 \wedge u)$ is controlled by the BMO-norm of $\log u$.*

Proof. By Lemma 6.2, there exist $C > 1$, $0 < \rho < 1$, and f such that $u/C \le f \le Cu$ and $|\mathcal{H}(f^\rho)| \le Cf^\rho$. We put $g = (1 + f^\rho)^{1/\rho}$, then $|\mathcal{H}(g^\rho)| \le Cg^\rho$ and $(u+1)/C_1 \le g \le C_1(u+1)$. Thus the BMO-norm of $\log(u+1)$ is controlled by the BMO-norm of $\log u$, and consequently we have similar control over the BMO-norm of $\log(u/(u+1))$. Finally note that $(1 \wedge u)/2 \le u/(u+1) \le 1 \wedge u$.

Now to show 1) $\Rightarrow$ 2), let $\log w \in BMO$. For any $\lambda > 0$, we introduce two weights: $u_0 = 1 \wedge (w/\lambda)^{16}$, $u_1 = 1 \wedge (\lambda/w)^8$. Then $\|\log u_0\|_{BMO}$, $\|\log u_1\|_{BMO} \le C = C(\|\log w\|_{BMO})$ by Lemma 8.2. Since $1 \le u_0 + u_1$, by Lemma 7.9 we find $g \in H^\infty(u_0)$, $h \in H^\infty(u_1)$ such that $1 = g + h$, $|g| \le Cu_0$, $\|h\| \le Cu_1$, where here and below all constants are determined by $\|\log w\|_{BMO}$.

We do this for each $\lambda = 2^n$, $n \in Z$, and denote the resulting functions by g_n and h_n. Next we put $\varphi_n = g_n - g_{n+1} = h_{n+1} - h_n$, then

$$|\varphi_n| \le c \min\{(2^{\perp n} w)^{16}, (2^n w^{\perp 1})^8\}. \tag{8.1}$$

We claim that $\{\varphi_n\}$ is the required analytic decomposition of unity. Indeed, (iv) is clear; the convergence of the series a.e. easily follows from (8.1), and the sum telescopes. Again by (8.1), $|\varphi_n|^{1/8} w \le C2^n$, which is (i). We verify (ii) (condition (iii) is proved similarly). Let $e_k = \{2^k \le w < 2^{k+1}\}$, then, again by (8.1),

$$\sum_{n \in Z} 2^n |\varphi_n|^{1/8} \le C \sum_{n \in Z} 2^n \Big(\sum_{k \le n} 2^{\perp 2n} 2^{2k} \chi_{e_k} + \sum_{k > n} 2^n 2^{\perp k} \chi_{e_k} \Big).$$

Changing the order of summation, we see that the latter expression is dominated by $\sum_k 2^k \chi_{e_k} \le Cw$.

Now we sketch the proof of the implication 2) $\Rightarrow$ 3). Let $\{\varphi_j\}$ be an analytic decomposition of unity for w. We write $\varphi_j = \theta_j \psi_j^8$ with θ_j inner and ψ_j outer, and put

$$Qf = \sum_{j \in Z} \theta_j \psi_j^4 R(f\psi_j^4), \tag{8.2}$$

where R is the Riesz projection. Then Q is the required operator. We only check the weak type $(1,1)$ property of Q; the weak type (1.1) for Q^* is similar, and the $L^p(w)$-boundedness of Q for $1 < p < \infty$ is simpler. Clearly the values of Q are analytic functions, and Q fixes analytic functions because $\sum \varphi_j = 1$.

Lemma 8.3. *An operator T acting from a subset of $L^1(\mu)$ to measurable functions is of weak type $(1,1)$ if and only if it satisfies the estimate*

$$\int |Tf|^{1/2}|g|d\mu \le C\|f\|_{L^1(\mu)}^{1/2}\|g\|_{L^1(\mu)}^{1/2}\|g\|_{L^\infty(\mu)}^{1/2}.$$

Proof hint. To prove the "if" part, take $g = \chi_E$, where $E \subset \{|Tf| > \lambda\}$ is an arbitrary set of finite measure.

Now we check the estimate of Lemma 8.3 for Q, using the fact that it is true for R:

$$\begin{aligned}\int |Qf|^{1/2}|g|w &\le \sum_j \int |\psi_j|^2 |R(f\psi_j^4)|^{1/2}|g|w \\ &\le C\sum_j 2^j \int |R(f\psi_j^4)|^{1/2}|\psi_j g| \le C\|g\|_\infty^{1/2}\sum_j 2^j \|f\psi_j^4\|_{L^1}^{1/2}\|\psi_j g\|_{L^1}^{1/2} \\ &\le C\|g\|_\infty^{1/2}(\sum_j 2^j\|f\psi_j\|_{L^1})^{1/2}(\sum_j 2^j\|g\psi_j\|_{L^1})^{1/2} \le C\|g\|_\infty^{1/2}\|f\|_{L^1(w)}^{1/2}\|g\|_{L^1(w)}^{1/2}.\end{aligned}$$

See [K3] for more information on Bourgain projections. To illustrate their usefulness, we mention an application to conformal mapping.

Let G be a Jordan domain with rectifiable boundary. The analogues for G of the classical H^p-spaces are the spaces $E^p(G)$ consisting of the analytic functions f on G for which $\sup \int_{\gamma_n} |f(z)|^p|dz|$ is finite, where $\{\gamma_n\}$ is a fixed sequence of rectifiable curves that tend to ∂G in a natural sense. The domain G is a *Smirnov domain* if the derivative g' of the conformal mapping g of the unit disk onto G is outer. In this case we may identify the scale $E^p(G)$ with the scale $H^p(|g'|)$ of weighted Hardy spaces on the disk.

For a Smirnov domain G with conformal map g, it is important to know when $\log|g'|$ is in BMO. See [Po] for a detailed discussion. The following theorem is an immediate consequence of Theorem 8.1.

Theorem 8.4. *The following statements are equivalent for the conformal mapping g of the unit disk onto a Smirnov domain G.*

1) $\log |g'| \in BMO$.

2) *There is an operator projecting $L^p(\partial G)$ onto $E^p(G)$ for all $p \in (1, \infty)$ and having weak type* $(1,1)$.

3) *There is an operator projecting $L^p(\partial G)$ onto $E^p(G)$ for two different values of p.*

Theorem 8.4 together with a theorem of G. David [D] yields a proof of the (known) fact that if the arc-length measure on the boundary of G satisfies a Carleson condition, then $\log |g'| \in BMO$. The Carleson condition is that the length of $\partial G \cap D_r$ is bounded by cr for any small disk D_r of radius r. By David's theorem, this implies that the Cauchy integral over ∂G is a bounded operator on $L^p(\partial G)$ for $1 < p < \infty$. In particular, condition (3) above holds, and from (1) we obtain $\log |g'| \in BMO$. Note that the Cauchy integral operator need not have weak type $(1,1)$, and the Bourgain projection in statement (2) has the form (8.2).

Finally, we note that the theory discussed in Section 7 can be carried over, nearly word for word, to the Hardy spaces related to a weak-star Dirichlet algebra, as can the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) in Theorem 8.1. See [SW] for background on weak-star Dirichlet algebras.

9 Perturbation of Uniform Algebras

Often a Banach function space remembers almost nothing about the set on which the functions are defined. For example, if p is fixed, the spaces $L^p(\Omega, \mu)$ for μ separable and atomless are all isometric. The reason is that, up to isomorphism in a proper sense, there are no separable atomless measures other than Lebesgue measure on $[0, 1]$. Something similar occurs in the context of the spaces $C(K)$. The celebrated Milyutin theorem (see [Woj, III.D, §19]) asserts that for any uncountable compact metric space K, $C(K)$ is linearly homeomorphic to $C[0, 1]$. On the other hand, if we do not change the norm of $C(K)$ by too much, we keep K in sight. The Amir-Cambern theorem (see [Ja1]) asserts that if T is a linear isomorphism of two $C(K)$-spaces such that $\|T\|\,\|T^{-1}\| < 2$, then the underlying compact spaces are homeomorphic.

This leads us to consider linear isomorphisms of uniform algebras that are not too far from being isometries. For an exposition of this area, see [Ja1]. The idea of nearness of two uniform algebras can be given several equivalent formulations, but we focus only on the Banach space notion of nearness to an isometry. We say that two uniform algebras are $(1+\varepsilon)$-*isomorphic* if there is a linear isomorphism T between them that satisfies $\|T\|\,\|T^{-1}\| < 1+\varepsilon$, that is, if the Banach-Mazur distance between them is less than $\log(1+\varepsilon)$. Algebras that are 1-isomorphic in this sense are isometrically isomorphic as Banach spaces, and for these we have the following.

Theorem (Nagasawa [Na]). *If two uniform algebras A and B are isometrically isomorphic as Banach spaces, then they are isometrically isomorphic as uniform algebras.*

In particular, their maximal ideal spaces are then homeomorphic, as are their Shilov boundaries. As a consequence, if K_1 and K_2 are compact subsets of the complex plane such that the algebras $A(K_1)$ and $A(K_2)$ are linearly isometrically isomorphic, then there is a homeomorphism of K_1 onto K_2 that maps the interior of K_1 conformally onto the interior of K_2. On the other hand, deformation of the compact set K leads to linear isomorphisms of $A(K)$ that are close to being isometries. For example, consider the scale of annuli $G_r = \{r < |z| < 1\}$, and the associated algebras $A(G_r)$. Since these annuli are conformally distinct, no two of the algebras $A(G_r)$ can be isometric. On the other hand, the linear operator T from A_r to A_s defined by

$$T\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{n=-1} a_n \left(\frac{r}{s}\right)^n z^n + \sum_{n=0}^{\infty} a_n z^n$$

is a linear isomorphism of A_r onto A_s, and further $\|T\|\,\|T^{-1}\| \to 1$ as $s \to r$.

We say that a uniform algebra A is *stable* if there is $\varepsilon > 0$ such that any uniform algebra B that is $(1+\varepsilon)$ isomorphic to A is actually isometrically isomorphic to A. Thus the algebras $C(K)$ are stable, while the annulus algebras are not. R. Rochberg [Ro1] proved in 1972 that the disk algebra C_A is stable, and he went on to study the perturbations of the algebras $A(K)$ for K a finitely connected subset of the complex plane with smooth boundary, or more generally for K a finite bordered Riemann surface. The flavor of his work is given by the following result.

Theorem (Rochberg). *Let K be a finite bordered Riemann surface. Then for $\varepsilon > 0$ small, any uniform algebra B that is $(1+\varepsilon)$-isomorphic to $A(K)$ has the form $A(J)$ for a compact bordered Riemann surface J that is a deformation of K by a quasi-conformal homeomorphism with dilatation tending to 0 with ε.*

For expository accounts of these results, see [Ro2, Ro3]. More recently, Jarosz [Ja2] was able to prove that the nonseparable algebra $H^\infty(\Delta)$ is also stable.

Meanwhile there is currently no known compact set K in the complex plane with nonempty interior such that $C_A(K)$ can be shown to be linearly nonisomorphic to the disk algebra. Conformal mapping theory and the relation $C_A \sim (C_A \oplus C_A \oplus \dots)_{c_0}$ proved by Wojtaszczyk (see [Woj, III.E, §12]) suggest that a compact set K for which $C_A(K)$ (or $P(K)$, or $R(K)$) is proper but is not isomorphic to the disk algebra should not be too simple (if it exists).

10 The Dimension Conjecture

It is natural to ask how linear topological properties of an algebra of analytic functions of several complex variables reflect the geometry of the underlying domain. The oldest problem along these lines is to determine what effect the number of variables (dimension) has.

Dimension Conjecture. *If G_1 and G_2 are bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, with $n \neq m$, then the spaces $A(G_1)$ and $A(G_2)$ are not linearly homeomorphic, nor are $H^\infty(G_1)$ and $H^\infty(G_2)$.*

We discuss briefly some results related to the dimension conjecture. The main references are [B4] and [Pe, § 11].

It is most natural to examine the dimension conjecture first for the polydisks Δ^n and the balls B_n. In [B4] it was proved that $A(\Delta^n)$ is not linearly homeomorphic to $A(\Delta^m)$ if $m \neq n$. The invariant distinguishing the spaces (in fact, their duals) is the behavior of certain vector-valued multiindexed martingales on the measure space $\Gamma^\infty \times \cdots \times \Gamma^\infty$ with natural filtration; the martingales in question must have some additional complex analytic structure. The method in [B4] yields the following.

Theorem 10.1. *Let $U_1, \ldots, U_n$, $V_1, \ldots, V_m$ be strictly pseudoconvex domains with C^2-smooth boundary (the dimension may vary from one domain to another). If $m > n$, then $A(V_1 \times \cdots \times V_m)^*$ does not embed in $A(U_1 \times \cdots \times U_n)^*$ as a closed subspace.*

Theorem 10.1 includes the previously known result that the spaces $A(B_m)$ and $A(\Delta^n)$ are not linearly isomorphic for $m, n \geq 2$. It had been shown that $A(\Delta^n)^*$ does not embed in a direct sum of an L^1-space and a separable space (see [Pe, § 11]), whereas $A(B_m)^*$ is such a direct sum by Theorem 4.3.

Very little is known beyond Theorem 10.1. Currently it is not even known whether the ball algebras $A(B_m)$ are mutually nonisomorphic for $m \geq 2$. They are all distinct from $A(B_1) = A(\Delta) = C_A$. The latter space is a subspace of $C(\Gamma)$ with separable annihilator, whereas $A(B_m)$ for $m \geq 2$ does not embed in $C(K)$ as a subspace with separable annihilator (see [Pe, § 11]).

The series $\{H^\infty(\Delta^n)\}$ seems to be quite similar to $\{A(\Delta^n)\}$; however, in general the method of [B4] is not applicable to $H^\infty(\Delta^n)$. It is known only that $H^\infty(\Delta)$ differs from $H^\infty(\Delta^n)$ for $n \geq 2$. Again, see [B4, Pe] for proofs.

For comparison, we describe the situation concerning the Hardy spaces $H^1(\Gamma^n)$ and

$H^1(\partial B_n)$. The spaces of the first series are not isomorphic to one another (see [B5, B6]), whereas those of the second series are all isomorphic (see [Wol]). More recently, it was shown that for any strictly pseudoconvex domain with smooth boundary the corresponding space H^1 is isomorphic to the classical H^1 in the disk (see [Ar]).

Finally, note that in contrast to the current state of affairs in one complex variable, it is possible to find in several complex variables many examples of nonisomorphic spaces $A(G)$ where the underlying domains G have the same dimension. For instance, from Theorem 10.1 it follows that $A(\Delta^4)$ is not linearly isomorphic to $A(B_2 \times B_2)$.

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