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1 Introduction

The laplacian. Let Δ be the Laplace operator in $\mathbb{R}^n$ ($n = 1, 2, 3, \ldots$). We say that a real-valued function u is harmonic on an open subset of $\mathbb{R}^n$ if $\Delta u = 0$ there, and subharmonic if $\Delta u \geq 0$. A domain in $\mathbb{R}^n$ is an open and connected set. The classical maximum principle for subharmonic functions can be given the following formulation. Let D be a bounded domain in $\mathbb{R}^n$, and u a function continuous on the closure of D. We then have the implication

$$0 \leq \Delta u|_D \quad \text{and} \quad u|_{\partial D} \leq 0 \implies u|_D \leq 0. \tag{MP:Δ}$$

Moreover, unless $u|_D = 0$, the conclusion can be sharpened to $u|_D < 0$.

The bilaplacian. It is natural to try to extend the maximum principle to higher order elliptic operators: let us focus on the simplest example, the bilaplacian Δ^2. In the same way that physically, the laplacian corresponds to a membrane, the bilaplacian corresponds to a plate (there is also a connection with creeping flow). In view of the nature of the boundary data for the Dirichlet problem, the maximum principle we are looking for necessarily will involve two inequalities along the boundary of the subdomain D, one for the functions, and another for the normal derivatives. We first need some notation. A real-valued function u on a domain Ω is *biharmonic* provided that $\Delta^2 u = 0$ there, and *sub-biharmonic* if $\Delta^2 u \leq 0$ (one should think of Δ as a negative operator, which is the reason why the inequality is switched as compared with the definition of subharmonic functions). In the following we shall restrict our attention to the case of the plane $\mathbb{R}^2$, which is identified with $\mathbb{C}$, the complex plane. Around 1900, it was known – more or less – that a variant of a maximum principle can be formulated for circular disks. Let D be a circular disk and u a C^1-smooth function on the closure of D. The maximum principle reads

$$\Delta^2 u|_D \leq 0, \quad u|_{\partial D} \leq 0, \quad \text{and} \quad \frac{\partial u}{\partial n}\Big|_{\partial D} \leq 0 \implies u|_D \leq 0, \tag{MP:Δ^2}$$

where the normal derivative is calculated in the interior direction. Actually, unless $u|_D = 0$, we have $u|_D < 0$. Let Γ_D denote the Green function for the Dirichlet problem associated with Δ^2 on D: for fixed $\zeta \in D$, the function $\Gamma_D(\cdot, \zeta)$ vanishes along with its normal derivative on ∂D, and $\Delta^2 \Gamma_D(\cdot, \zeta)$ equals the unit point mass at ζ. The above maximum principle (MP:Δ^2) then expresses the following three basic facts:

$$0 < \Gamma_D(z, \zeta), \qquad (z, \zeta) \in D \times D, \tag{1.1}$$

$$0 < \Delta_z \Gamma_D(z, \zeta), \qquad (z, \zeta) \in \partial D \times D, \tag{1.2}$$

and

$$\frac{\partial}{\partial n(z)} \Delta_z \Gamma_D(z, \zeta) < 0, \qquad (z, \zeta) \in \partial D \times D. \tag{1.3}$$

These properties are easily verified by computation. In fact, with the normalizations used here, the Green function $\Gamma = \Gamma_{\mathbb{D}}$ for the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is expressed by

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log \left| \frac{z-\zeta}{1-z\bar{\zeta}} \right|^2 + (1-|z|^2)(1-|\zeta|^2), \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D}.$$

It should be mentioned that a local analysis of the behavior near the boundary shows that (1.2) is an immediate consequence of (1.1), at least if we replace the sign "$<$" with "$\leq$". The connection between (MP:Δ^2) and (1.1)–(1.3) is apparent from the symmetry of Γ_D together with Green's formula:

$$u(z) = \int_D \Gamma_D(z,\zeta)\, \Delta^2 u(\zeta)\, d\Sigma(\zeta) + \frac{1}{2} \int_{\partial D} \Big(\Delta_\zeta \Gamma_D(z,\zeta) \frac{\partial u}{\partial n}(\zeta) - \frac{\partial}{\partial n(\zeta)} \Delta_\zeta \Gamma_D(z,\zeta)\, u(\zeta) \Big)\, d\sigma(\zeta), \qquad z \in D, \quad (1.4)$$

where $d\Sigma$ is area measure, normalized by the factor π^{-1}, and $d\sigma$ is one-dimensional Lebesgue measure, normalized by the factor $(2\pi)^{-1}$.

Notation: Throughout the paper, the word *positive* is normally given the weakest possible sense. So, for instance, a function f is positive if $0 \leq f$ holds pointwise, and if we wish to express that $0 < f$ holds everywhere, we say that the function is *strictly* positive. However, when we deal with individual real numbers x, we adhere to the usual standard and say that x is positive if $0 < x$. Unless explicitly stated otherwise, normal derivatives are calculated in the interior direction. Moreover, we have normalized the laplacian Δ acting over the plane: it is the operator

$$\Delta_z = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \qquad z = x + iy \in \mathbb{C}.$$

The plan of Hadamard. Jacques Hadamard, in his treatise on *plaques élastiques encastrées* ([14], pp. 515–641), suggests the possibility of a maximum principle of the type (MP:Δ^2) for more general subdomains D. In fact, he writes: "Du moins, cette proposition, comme l'inégalité $\Gamma_A^B > 0$, paraît incontestable pour tout contour convexe". That is, the Green function for Δ^2 should be positive for a much larger collection of domains than the disks, including all convex regions with smooth boundary. This was, however, later shown not to be the case, by Duffin, Lœwner, and Garabedian. In fact, it follows from Paul Garabedian's work [11] that (1.1)–(1.2) both fail when D is an ellipse, provided that the ratio of the major axis to the minor axis exceeds a certain critical value $\epsilon_0 \approx 1.5933$. Further calculations along Garabedian's lines show that (1.3) fails much sooner, namely when that ratio exceeds another critical value, $\epsilon_1 \approx 1.1713$. The conclusion we can draw from this is that within the family of ellipses, we cannot deviate very far from circles and keep the maximum principle (MP:Δ^2). The Almansi formula, which expresses each biharmonic function locally as $f + |z|^2 g$, where f, g are harmonic, suggests that circles are special for Δ^2. This intuitive feeling is further corroborated by Charles Lœwner's work [31], where it is shown that the only coordinate transformations (suitably modified) preserving the biharmonic functions are of Mœbius type.

Conformal transformations and weights. Let us see how the biharmonic operator transforms under analytic coordinate changes. For a smooth function u in some domain in the plane, and a holomorphic mapping ϕ,

$$\Delta |\phi'|^{-2} \Delta(u \circ \phi) = |\phi'|^2\, (\Delta^2 u) \circ \phi,$$

wherever the expressions make sense. This suggests switching from the bilaplacian to the more general operators $\Delta|\phi'|^{-2}\Delta$. In fact, we shall consider operators of the form $\Delta\,\omega^{-1}\Delta$, where ω is a *logarithmically subharmonic* weight function – this amounts to the squared Laplace-Beltrami operator on a general hyperbolic Riemannian manifold (see below). The term logarithmically subharmonic means that ω takes values in the interval $[0,+\infty[$, and that $\log\omega$ is subharmonic. These operators $\Delta\,\omega^{-1}\Delta$ form a conformally invariant class, because if ω is logarithmically subharmonic, then so is $\omega\circ\phi\,|\phi'|^2$. A real-valued function u is said to be *sub-ω-biharmonic* if $\Delta\,\omega^{-1}\Delta u\le 0$, and *$\omega$-biharmonic* if $\Delta\,\omega^{-1}\Delta u=0$. We fix a bounded simply connected domain Ω in $\mathbb{C}$ – our universe from now on – and suppose ω is defined and logarithmically subharmonic there. Let D be a precompact subdomain of Ω. The problem of determining when we have a maximum principle like (MP:Δ^2) generalizes to the question of determining for which subdomains D we have a maximum principle

$$\Delta\omega^{-1}\Delta u|_D\le 0,\quad u|_{\partial D}\le 0,\quad\text{and}\quad \frac{\partial u}{\partial n}\Big|_{\partial D}\le 0\implies u|_D\le 0, \qquad (\text{MP:}\Delta\omega^{-1}\Delta)$$

whereby u is assumed smooth on the closure of D, and the boundary ∂D is smooth as well (C^1-smoothness is appropriate here). The experience with Δ^2 suggests that we should look for the appropriate analogue of the circular disks. We first turn to some geometric aspects. Endow the domain Ω with the Riemannian metric $ds_\omega(z)=\sqrt{\omega(z)}\,|dz|$, that is,

$$ds_\omega(z)^2=\omega(z)\left(dx^2+dy^2\right),\qquad z=x+iy\in\Omega.$$

The induced area measure is $\omega\,d\Sigma$. The property of ω that

$$0\le\Delta\log\omega(z),\qquad z\in\Omega,$$

means that the Riemannian manifold obtained is *hyperbolic*, in the sense of having negative Gaussian curvature. The distribution $\mu=\Delta\log\omega$ is identified with a positive Borel measure on Ω, which in fact is independent of the particular choice of coordinates for the hyperbolic manifold, and represents the local distribution of negative curvature. The Laplace-Beltrami operator on the manifold is given by

$$\triangle=\frac{1}{\omega(z)}\,\Delta.$$

The weighted biharmonic operator $\Delta\,\omega^{-1}\Delta$ then corresponds to the bi-laplace-beltramian $\triangle^2$. The *energy integral* associated with $\Delta\,\omega^{-1}\Delta$ on the subdomain D is expressed by

$$\mathcal{E}_\omega(u)=\int_{\mathbb{D}}|\triangle u(z)|^2\,\omega(z)\,d\Sigma(z)=\int_{\mathbb{D}}|\Delta u(z)|^2\,\frac{d\Sigma(z)}{\omega(z)}.$$

The ω-biharmonic functions minimize this energy under given Dirichlet boundary data.

A circular disk D centered at z_0 with radius r is uniquely determined by the mean value property

$$\int_D h(z)\,d\Sigma(z)=r^2h(z_0),$$

with h ranging over all bounded harmonic functions on D, as was proved by Bernard Epstein [43]. A precompact subdomain D of the fixed "universal" domain Ω is said to be an *ω-disk* – centered at $z_0\in D$ with "radius" r, $0<r<+\infty$ – provided that

$$\int_D h(z)\,\omega(z)\,d\Sigma(z)=r^2h(z_0),$$

holds for all bounded harmonic functions h on D. In the case of a constant weight, we recover the circular disks. The ω-disks turn out to be uniquely determined by the parameters z_0 and r, just like the circles, and they are simply connected. They are the result of a physical process, a *Hele-Shaw flow* on the hyperbolic manifold (see below). We feel that the maximum principle (MP:$\Delta\omega^{-1}\Delta$) holds for ω-disks D. The proof of this statement, however, remains to be found. Nevertheless, *we have found the weaker principle*

$$\Delta\,\omega^{-1}\Delta u|_D \leq 0, \quad u|_{\partial D} = 0, \quad \text{and} \quad \frac{\partial u}{\partial n}\Big|_{\partial D} \leq 0 \implies u|_D \leq 0, \qquad (\mathrm{MP}'{:}\Delta\omega^{-1}\Delta)$$

with u smooth on D. As we pull the coordinates back to the unit disk, we require of the weight ω to have $\mathbb{D}$ as an ω-disk for $z_0 = 0$ and $r = 1$: we say that ω is *reproducing for the origin provided that*

$$\int_{\mathbb{D}} h(z)\,\omega(z)\,d\Sigma(z) = h(0) \tag{1.5}$$

holds for all bounded harmonic functions h on $\mathbb{D}$. The Green function for the Dirichlet problem associated with the weighted biharmonic operator $\Delta\,\omega^{-1}\Delta$ on the unit disk $\mathbb{D}$ is denoted by Γ_ω. The main result – equivalent to (MP$'$:$\Delta\omega^{-1}\Delta$) – is the following.

THEOREM 1.1 *Suppose ω is a logarithmically subharmonic weight on $\mathbb{D}$ which reproduces for the origin. Then $0 \leq \Gamma_\omega|_{\mathbb{D}\times\mathbb{D}}$.*

The statement of the theorem is false if we keep the reproducing property but scrap the logarithmic subharmonicity: there are simple radial weights that provide counterexamples. Also, if we instead drop the reproducing property and keep the logarithmic subharmonicity, the result is false, as is apparent from Garabedian's work on ellipses [11].

The weighted Hele-Shaw flow. Suppose the bounded simply connected universal domain Ω has as boundary a C^∞-smooth Jordan curve, and that the logarithmically subharmonic weight ω is C^∞-smooth and strictly positive on $\overline{\Omega}$, and real analytic in the interior Ω. We recall that a precompact subdomain D is said to be an ω-disk with center z_0 and radius r provided that $z_0 \in D$ and

$$r^2 h(z_0) = \int_D h(z)\,\omega(z)\,d\Sigma(z), \tag{1.6}$$

holds for all bounded harmonic functions h on D. It is natural to ask for existence and uniqueness of such generalized disks D. It turns out that we have uniqueness and existence in a certain interval $0 < r < \rho(z_0)$, and non-existence for larger r simply because then D necessarily expands beyond the boundary $\partial\Omega$ (the latter statement slightly exaggerates what we actually prove). This suggests writing $D = D(z_0, r; \omega)$ to indicate the determining parameters. For $0 < r < \rho(z_0)$, $D(z_0, r; \omega)$ is simply connected, and the boundary is a real analytic Jordan curve; moreover, $D(z_0, r; \omega)$ grows with the parameter r.

We prove the above statements as follows. We first assume that D has the additional property that

$$r^2 u(z_0) \leq \int_D u(z)\,\omega(z)\,d\Sigma(z) \tag{1.7}$$

holds for all bounded subharmonic functions u on D, which allows us to interpret the problem of finding D in terms of an obstacle problem for the laplacian, which has a unique solution (compare with [12]). We call subdomains D satisfying (1.7) *weighted Hele-Shaw flow domains*, because it is possible to interpret them as arising from a Hele-Shaw flow on the hyperbolic manifold Ω with Riemannian metric ds_ω. The usual Hele-Shaw flow (with $\omega = 1$, and r^2 replaced by a time parameter t) models how the free boundary evolves between an incompressible viscous Newtonian fluid and vacuum, which occupy the space between two parallel, narrowly separated infinitely extended surfaces, as fluid is injected at a constant rate at the source point z_0. We first prove that the weighted Hele-Shaw flow domains are simply connected and have boundaries that are real analytic Jordan curves. The proof is based on Makoto Sakai's fundamental work on the regularity of free boundaries [42], which guarantees that the topological situation is fairly uncomplicated. The positivity of the biharmonic Green function on the unit disk comes in at a crucial point in the argument. It remains to see why any subdomain D with (1.6) necessarily has (1.7). This is achieved by a simple argument due to Sakai and refined by Gustafsson [13].

Sketch of the proof. A few words should be said about the proof of the main result, the positivity of the Green function Γ_ω, given that ω is logarithmically subharmonic and reproduces for the origin. It is important to note that since $\Delta\omega^{-1}\Delta\Gamma_\omega(\cdot,\zeta)$ is a unit point mass at ζ, it follows that

$$\Delta_z\Gamma(z,\zeta) = \omega(z)\big(G(z,\zeta) + H_\omega(z,\zeta)\big), \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

where

$$G(z,\zeta) = \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2$$

is the Green function for the laplacian, and the function $H_\omega(z,\zeta)$ is harmonic in z. We shall call the kernel function H_ω the *harmonic compensator*, because it solves the balayage problem

$$\int_{\mathbb{D}} \big(G(z,\zeta) + H_\omega(z,\zeta)\big)\, h(z)\,\omega(z)\,d\Sigma(z) = 0, \qquad \zeta \in \mathbb{D},$$

for all bounded harmonic functions h on $\mathbb{D}$. If the Green function Γ_ω is positive, then a local analysis near the boundary shows that $\Delta_z\Gamma_\omega(z,\zeta)$ is positive on $\mathbb{T}\times\mathbb{D}$, and using the harmonicity of H_ω in the first variable, it follows that H_ω is positive throughout $\mathbb{D}\times\mathbb{D}$. It is much less obvious that if the harmonic compensator is positive for a certain family of weight functions of the same type as ω, then we can go the other way around and obtain the positivity of Γ_ω. This is done with the help of a variational technique due to Hadamard (see [14, 20]), along domains given by the weighted Hele-Shaw flow $D(r)$, for $0 < r \le 1$, which starts at $z_0 = 0$ and ends with $D(1) = \mathbb{D}$.

The harmonic compensator is related to the reproducing kernel function Q_ω for the space $HP^2(\mathbb{D},\omega)$ obtained as the closure of the harmonic polynomials with respect to the norm of $L^2(\mathbb{D},\omega)$,

$$\|f\|_\omega = \left(\int_{\mathbb{D}} |f(z)|^2\omega(z)\,d\Sigma(z)\right)^{\frac{1}{2}}.$$

More precisely,

$$\Delta_\zeta H_\omega(z,\zeta) = -\omega(\zeta)\,Q_\omega(z,\zeta).$$

If the harmonic compensator is positive, then a local study of the behavior near $\mathbb{T}\times\mathbb{T}$ reveals that $Q_\omega|_{\mathbb{T}^2\setminus\delta(\mathbb{T})} \le 0$, where $\delta(\mathbb{T}) = \{(z,z) : z \in \mathbb{T}\}$ denotes the diagonal. We are led to search for some kind of reverse implication. We first study the reproducing kernel function K_ω for the space $P^2(\mathbb{D},\omega)$ which is obtained as the closure of the (holomorphic) polynomials with respect to the norm of $L^2(\mathbb{D},\omega)$. It is shown that $Q_\omega = 2\,\mathrm{Re}\,K_\omega - 1$, so that the information obtained for K_ω can be readily converted to information about Q_ω.

This identity reflects the fact that under the reproducing condition on ω, the analytic polynomials and the antiananlytic polynomials vanishing at the origin are perpendicular to each other in the Hilbert space $HP^2(\mathbb{D},\omega)$. We obtain a representation formula for K_ω,

$$K_\omega(z,\zeta)=\frac{1-z\bar{\zeta}L_\omega(z,\zeta)}{(1-z\bar{\zeta})^2},\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

whereby L_ω is the reproducing kernel for some Hilbert space of analytic functions on $\mathbb{D}$, which we suggest to call the *deficiency space* for $P^2(\mathbb{D},\omega)$. Using this representation, we find that that Q_ω is negative on $\mathbb{T}^2\setminus\delta(\mathbb{T})$, and in fact that

$$Q_\omega(z,\zeta)\le-\left(\frac{1}{\omega(z)}+\frac{1}{\omega(\zeta)}\right)\frac{1}{|z-\zeta|^2},\qquad (z,\zeta)\in\mathbb{T}\times\mathbb{T}\setminus\delta(\mathbb{T}),\tag{1.8}$$

and just as previously this allows us to go backwards, to obtain the positivity of the harmonic compensator, by means of a variational technique along the weighted Hele-Shaw flow. We use the Hadamard variational method for the laplacian to write the Green function G as a negative integral of a product of two Poisson kernels for the flow domains $D(r)$ over r, $0<r<1$. Noting that $H_\omega(\cdot,\zeta)$ is the orthogonal harmonic projection (with respect to the weight) of the function $-G(\cdot,\zeta)$, we find that it suffices to show that the harmonic projection of a positive harmonic function on a flow region $D(r)$, with $0<r<1$, extended to vanish on $\mathbb{D}\setminus D(r)$, is positive throughout $\mathbb{D}$. This is precisely what the estimate (1.8) permits us to do.

Connection with the Bergman spaces. For $0<p<+\infty$, the Bergman space $A^p(\mathbb{D})$ consists of all holomorphic functions $f:\mathbb{D}\to\mathbb{C}$ with bounded norm

$$\|f\|_{A^p}=\left(\int_{\mathbb{D}}|f(z)|^p\,d\Sigma(z)\right)^{\frac{1}{p}}<+\infty,$$

and there is a corresponding collections of inner functions: a function $\varphi\in A^p(\mathbb{D})$ is inner in $A^p(\mathbb{D})$ if

$$\int_{\mathbb{D}}h(z)\,|\varphi(z)|^p\,d\Sigma(z)=h(0),$$

for all bounded harmonic functions h on $\mathbb{D}$. These inner functions have been studied rather extensively in recent years, primarily because of their use for the factorization of functions with respect to zeros and their relevance for operator theory (see, exempli gratia, Hedenmalm [16], Duren, Khavinson, Shapiro, Sundberg [6, 7], and Aleman, Richter, Sundberg [2]; one should compare with the more classical $A^{-\infty}$ theory of Korenblum [29]). They are analogous to the classical inner functions (Blaschke products, singular inner functions, and products of the two) which play a vital rôle in the function theory of the Hardy spaces $H^p(\mathbb{D})$, for $0<p<+\infty$; we recall that a holomorphic function $f:\mathbb{D}\to\mathbb{C}$ is in $H^p(\mathbb{D})$ if

$$\|f\|_{H^p}=\sup_{0<r<1}\left(\int_{\mathbb{T}}|f(r\zeta)|^p\,d\sigma(\zeta)\right)^{\frac{1}{p}}<+\infty.$$

For an inner function φ in $A^p(\mathbb{D})$, $|\varphi|^p$ is logarithmically subharmonic and reproduces for the origin, so by our main theorem, the Green function $\Gamma_{|\varphi|^p}$ for the weighted biharmonic operator $\Delta|\varphi|^{-p}\Delta$ is positive. As we apply this result to the Bergman spaces $A^p(\mathbb{D})$, we arrive at the following. Given a zero sequence A in $\mathbb{D}$ for the space $A^p(\mathbb{D})$, let M_A be the subspace of all functions in $A^p(\mathbb{D})$ that vanish at all points of A, with multiplicities as prescribed by the sequence. Subspaces of the type M_A are referred to as *zero-set subspaces*. Let φ_A be the function that maximizes $|\varphi(0)|$, given that φ vanishes on A and has norm 1 (this does not define φ_A uniquely, because we can always multiply by a unimodular constant, but this is the only obstruction; if A contains the origin, we need to maximize the first non-vanishing derivative at the origin). The function φ_A is an inner function in $A^p(\mathbb{D})$, and it has no extraneous zeros; in fact, it generates M_A as an invariant subspace [2] (see below for a definition of the term invariant subspace). Duren, Khavinson, Shapiro, and Sundberg coined the term *canonical zero divisors* for these functions φ_A. For two zero sequences A and B, such that A is contained in B, it follows from the main theorem that

$$||\varphi_A f||_{A^p} \le ||\varphi_B f||_{A^p}, \tag{1.9}$$

for all holomorphic functions f on $\mathbb{D}$. This means that the canonical zero divisors are monotonic with respect to Korenblum domination along the lattice of zero-set subspaces: following Boris Korenblum [30], we say that given two function F and G in $A^p(\mathbb{D})$, G *dominates* F, written $F \prec G$, provided

$$||Fq||_{A^p} \le ||Gq||_{A^p}$$

holds for all polynomials q. The relation $\varphi_A \prec \varphi_B$ for $A \subset B$ was conjectured by Hedenmalm in [20, 21] as well as in Problem 12.13 in the Havin-Nikolski problem book [15]. A consequence of this result is the following. We say that a closed subspace M of $A^p(\mathbb{D})$ is *invariant* provided that $Sf \in M$ whenever $f \in M$, where S is the shift operator: $Sf(z) = zf(z)$. A particularly simple collection are the *zero-set subspaces*, as described above. These have the property that they have index 1 (with the exception of the trivial invariant subspace $\{0\}$), meaning that the dimension of M/SM is 1. *If an invariant subspace M with index 1 contains a zero-set subspace, then M itself is a zero-set subspace.*

It is interesting to note that in the Dirichlet space, the domination relation between the corresponding functions φ_A, φ_B is reversed [1, 39].

Higher dimensions. We wish to point out that the above results have been obtained in dimension $n = 2$ only. It is not clear what the appropriate generalization to higher dimensional $\mathbb{R}^n$, for $n = 3, 4, 5, \ldots$, should look like. A serious obstacle is that we do not have the artillery of conformal mappings any more. These problems deserve further investigation.

2 Bergman spaces and kernel functions

Let Ω be a bounded domain in the complex plane $\mathbb{C}$ (a domain is a connected open set). For an area summable function $\omega : \Omega \to [0, +\infty[$ which is positive on a set of positive area-measure (we call ω a *weight*), we let $L^2(\Omega, \omega)$ be the Hilbert space of complex-valued Borel measurable functions on Ω which are square summable with respect to the measure $\omega d\Sigma$: the norm is expressed by

$$||f||_\omega = \left(\int_\Omega |f|^2 \omega \, d\Sigma \right)^{\frac{1}{2}}, \qquad f \in L^2(\Omega, \omega).$$

As a Hilbert space, $L^2(\Omega, \omega)$ is equipped with an inner product

$$\langle f, g \rangle_\omega = \int_\Omega f \, \bar{g} \, \omega \, d\Sigma, \qquad f, g \in L^2(\Omega, \omega).$$

Bergman spaces: the general setting. Let $\mathfrak{S}$ be a complex-linear vector space whose elements are continuous functions on Ω, with the property that the elements of $\mathfrak{S}$ are square summable on Ω with respect to the measure $\omega\, d\Sigma$. Since elements of $L^2(\Omega,\omega)$ are really equivalence classes of functions on Ω, two functions being identified if they coincide except on a null set with respect to $\omega\, d\Sigma$, we cannot be certain that it is possible to identify $\mathfrak{S}$ with a linear subspace of $L^2(\Omega,\omega)$. This is however the case if ω is a *Bergman $\mathfrak{S}$-weight function* on Ω, which requires that for all $f \in \mathfrak{S}$,

$$|f(z)| \le C(K)\, \|f\|_\omega, \qquad z \in K,$$

where K is an arbitrary compact subset of Ω, and $C(K)$ is some positive constant which depends on K. Under this assumption, we may form the completion of $\mathfrak{S}$ with respect to the norm $\|\cdot\|_\omega$, which we denote by $\mathfrak{S}^2(\Omega,\omega)$. The space $\mathfrak{S}^2(\Omega,\omega)$ consists of continuous functions on Ω, and we can regard it as a closed subspace of $L^2(\Omega,\omega)$. Clearly, it gets easier for ω to be a Bergman $\mathfrak{S}$-weight function if the generating space $\mathfrak{S}$ gets smaller. If ω is a Bergman $\mathfrak{S}$-weight function, then the point evaluation functionals at points of Ω are continuous. The representation theorem for bounded linear functionals on a Hilbert space then shows that to each $\lambda \in \Omega$, there is a unique element $K^{\mathfrak{S}}_\omega(\cdot,\lambda;\Omega)$ in $\mathfrak{S}^2(\Omega,\omega)$, such that

$$f(\lambda) = \langle f, K^{\mathfrak{S}}_\omega(\cdot,\lambda;\Omega)\rangle_\omega, \qquad f \in R^2(\Omega,\omega).$$

The function $K^{\mathfrak{S}}_\omega(z,\zeta;\Omega)$, with $(z,\zeta) \in \Omega\times\Omega$, is called the *Bergman $\mathfrak{S}$-kernel function for the weight* ω on Ω. The space $\mathfrak{S}^2(\Omega,\omega)$ is separable because $L^2(\Omega,\omega)$ is, and hence it has a countable orthonormal basis $\varphi_1,\varphi_2,\varphi_3,\ldots$. One shows that the Bergman $\mathfrak{S}$-kernel function has the representation

$$K^{\mathfrak{S}}_\omega(z,\zeta) = \sum_{n=1}^{\infty} \varphi_n(z)\,\bar\varphi_n(\zeta), \qquad (z,\zeta) \in \Omega\times\Omega,$$

whence it follows that the complex conjugate of $K^{\mathfrak{S}}_\omega(z,\zeta)$ equals $K^{\mathfrak{S}}_\omega(\zeta,z)$.

Analytic Bergman spaces. Let P denote the algebra of polynomials. Then the elements of P are square summable on Ω with respect to the measure $\omega\, d\Sigma$, given the assumptions on ω and Ω. We may then use $\mathfrak{S} = P$ in the above setting: we speak of ω as a *Bergman polynomial weight function* when it is a Bergman $\mathfrak{S}$-weight, we write $P^2(\Omega,\omega)$ for the space $\mathfrak{S}^2(\Omega,\omega)$, and call it the *polynomial Bergman space* for the weight ω, and finally, we call the associated reproducing kernel function $K^{\mathfrak{S}}_\omega = K^P_\omega$ the *Bergman polynomial kernel* for the weight ω. The functions in $P^2(\Omega,\omega)$ are holomorphic on Ω, and if Ω is multiply connected, they extend holomorphically across the interior holes (this follows from the maximum principle).

Let $R(\overline{\Omega})$ denote the algebra of rational functions (ratios of polynomials) with poles off the closure of Ω. We may then use $\mathfrak{S} = R(\overline{\Omega})$ in the above setting. So, if ω is an $R(\overline{\Omega})$-weight, we express this in words as being a *Bergman rational weight function* on Ω, and we write $R^2(\Omega,\omega)$ for the space $\mathfrak{S}^2(\Omega,\omega)$, which we call the *rational Bergman space* for the weight ω on Ω. The reproducing kernel function $K^{\mathfrak{S}}_\omega(z,\zeta;\Omega)$ is written in symbols as $K^R_\omega(z,\zeta;\Omega)$, and we call it the *Bergman rational kernel* for the weight ω on Ω.

Another choice of spanning space $\mathfrak{S}$ is the following. Let $\mathfrak{S}$ consist of all holomorphic functions on Ω which are square integrable with respect to the measure $\omega\, d\Sigma$. With this setting, we say that ω is a *Bergman weight function* on Ω if it is a Bergman $\mathfrak{S}$-weight. Given that ω has this property, the space $\mathfrak{S}^2(\Omega,\omega)$ equals $\mathfrak{S}$ (we shall denote it by $A^2(\Omega,\omega)$), and we call it the *Bergman space* with weight ω on Ω. The reproducing kernel function $K_\omega^{\mathfrak{S}}(z,\zeta;\Omega)$ is written in symbols as $K_\omega^A(z,\zeta;\Omega)$, and called the *Bergman kernel* for the weight ω on Ω.

For a Bergman weight ω on Ω, $P^2(\Omega,\omega)$ equals the closure of the polynomials in $A^2(\Omega,\omega)$, and $R^2(\Omega,\omega)$ equals the closure of the rational functions in $R(\overline{\Omega})$.

When we consider the constant weight $\omega(z)\equiv 1$, we drop the indication of the weight, and speak of the polynomial Bergman space $P^2(\Omega)$, the rational Bergman space $R^2(\Omega)$, and the Bergman space $A^2(\Omega)$. On the unit disk $\mathbb{D}$, all three spaces coincide: $P^2(\mathbb{D}) = R^2(\mathbb{D}) = A^2(\mathbb{D})$.

For ω to be a Bergman weight on Ω, what is essentially required is that ω does not vanish too much near the boundary $\partial\Omega$. If the weight ω is logarithmically subharmonic, this is not a problem.

LEMMA 2.1 *If ω is logarithmically subharmonic on Ω, it is a Bergman weight on Ω.*

Proof. Take an interior point $z_0\in\Omega$, and let r, $0<r<+\infty$, be so small that the disk $\mathbb{D}(z_0,r)=\{z\in\mathbb{C}:\,|z-z_0|<r\}$ is precompact in Ω. For a holomorphic function f on Ω, the function $|f|^2\omega$ is subharmonic, and therefore, by the sub-mean value property,

$$|f(z_0)|^2\omega(z_0)\le\frac{1}{r^2}\int_{\mathbb{D}(z_0,r)}|f(z)|^2\omega(z)\,d\Sigma(z)\le r^{-2}\|f\|_\omega^2,$$

so that

$$|f(z_0)|^2\le\frac{1}{r^2\omega(z_0)}\,\|f\|_\omega^2.$$

Taking logarithms, we obtain

$$\log|f(z_0)|\le\log\frac{\|f\|_\omega}{r}+\frac{1}{2}\,\log\frac{1}{\omega(z_0)}. \tag{2.1}$$

The left hand side is subharmonic, so that we can get the estimate

$$\log|f(z_0)|\le\log\frac{\|f\|_\omega}{r}+\frac{1}{2r}\int_{\partial\mathbb{D}(z_0,r)}\log\frac{1}{\omega(z)}\,d\sigma(z). \tag{2.2}$$

and since $\log\omega$ is subharmonic, it is integrable on compact circles in Ω such as $\partial\mathbb{D}(z_0,r)$. Introducing the Poisson kernel in these calculations allows us to get a uniform estimate on compact subsets, which does it. ∎

Harmonic Bergman spaces. So far we only considered Bergman spaces of holomorphic functions. We also need Bergman spaces of harmonic functions.

If we let $\mathfrak{S}$ equal the space *HP* of harmonic polynomials, which are functions of the type $p+\bar{q}$, where p and q are polynomials, we get the concept of *Bergman harmonic polynomial weight* on Ω, by setting it equal to Bergman $\mathfrak{S}$-weight. The corresponding space $\mathfrak{S}^2(\Omega,\omega)$ is written $HP^2(\Omega,\omega)$, and we call it the *harmonic polynomial Bergman space* with weight ω on Ω. The functions in $HP^2(\Omega,\omega)$ are harmonic on Ω, and if Ω is multiply connected, they extend harmonically across the interior holes. The reproducing kernel function $K_\omega^{\mathfrak{S}}$ is written Q_ω^P, and called the *harmonic polynomial Bergman kernel* with weight ω on Ω.

If we instead let $\mathfrak{S}$ equal the space $HR(\overline{\Omega})$ of harmonic rational functions of the type $h=r+\bar{s}$, where r,s are in $R(\overline{\Omega})$, we get the concept of *Bergman harmonic rational weight* on Ω. The associated space $\mathfrak{S}^2(\Omega,\omega)$ is written $HR^2(\Omega,\omega)$, and we call it the *harmonic* *rational Bergman space* with weight ω on Ω. The reproducing kernel function $K_\omega^{\mathfrak{S}}$ is written Q_ω^R, and called the *harmonic rational Bergman kernel* with weight ω on Ω.

The largest choice of spanning space $\mathfrak{S}$ among the harmonic functions is the following. Let $\mathfrak{S}$ consist of all harmonic functions on Ω which are square integrable with respect to the measure $\omega\, d\Sigma$. With this setting, we say that ω is a *Bergman harmonic weight function* on Ω if it is a Bergman $\mathfrak{S}$-weight. The space $\mathfrak{S}^2(\Omega,\omega)$ then equals $\mathfrak{S}$ (we shall denote it by $HL^2(\Omega,\omega)$), and we call it the *harmonic Bergman space* with weight ω on Ω. The reproducing kernel function $K_\omega^{\mathfrak{S}}(z,\zeta;\Omega)$ is written in symbols as $Q_\omega^H(z,\zeta;\Omega)$, and called the *harmonic Bergman kernel* for the weight ω on Ω.

Logarithmically subharmonic reproducing weights on the unit disk. We now specialize to the domain $\Omega=\mathbb{D}$, the unit disk, and discontinue indicating the domain in the expressions for kernel functions for the remainder of this section. Let ω be a logarithmically subharmonic area summable weight ω on $\mathbb{D}$, which is reproducing for the origin, in the sense of the introduction:

$$h(0)=\int_{\mathbb{D}} h(z)\,\omega(z)\,d\Sigma(z)$$

holds for all bounded harmonic functions h on $\mathbb{D}$.

The following assertion is known, but we do not have a reference. The first result of this type can be found in [16].

LEMMA 2.2 *For the above class of weights ω, we have the following growth control:*

$$\omega(z)\le(1-|z|^2)^{-1},\qquad z\in\mathbb{D}.$$

Proof. To see that this estimate is valid, one can proceed as follows. As in [16], one obtains the Carleson measure type condition

$$\int_{\mathbb{D}}|f|^2\omega\,d\Sigma\le\|f\|_{H^2},\qquad f\in H^2(\mathbb{D}).$$

Consider for $\zeta\in\mathbb{D}$ the Mœbius automorphism of the disk

$$\phi_\zeta(z)=\frac{\zeta-z}{1-z\bar\zeta},\qquad z\in\mathbb{D},$$

and note that the above inequality becomes

$$\int_{\mathbb{D}}|f\circ\phi_\zeta|^2\omega\circ\phi_\zeta\,|\phi_\zeta'|^2\,d\Sigma\le\|f\|_{H^2},\qquad f\in H^2(\mathbb{D}).$$

By choosing

$$f(z)=\frac{(1-|\zeta|^2)^{\frac12}}{1-z\bar\zeta},$$

we obtain from the sub-mean value property that

$$(1-|\zeta|^2)\omega(\zeta)\le(1-|\zeta|^2)\int_{\mathbb{D}}\frac{\omega\circ\phi_\zeta(z)\,|\phi_\zeta'(z)|^2}{|1-\phi_\zeta(z)\bar\zeta|^2}\,d\Sigma(z)\le 1,$$

from which the assertion is immediate. ∎

PROPOSITION 2.3 *Under the above assumptions on ω, $HP^2(\mathbb{D},\omega)$ is a Hilbert space of harmonic functions on $\mathbb{D}$, with locally uniformly bounded point evaluations. Denote by $P_0^2(\mathbb{D},\omega)$ the subspace of $P^2(\mathbb{D},\omega)$ consisting of those functions that vanish at the origin, and by $\bar{P}_0^2(\mathbb{D},\omega)$ its image under complex conjugation. Then the harmonic space splits*

$$HP^2(\mathbb{D},\omega) = P^2(\mathbb{D},\omega) \oplus \bar{P}_0^2(\mathbb{D},\omega),$$

the two subspaces on the right hand side being orthogonal. As a consequence, the kernel function for $HP^2(\mathbb{D},\omega)$ has the form

$$Q_\omega^P(z,\zeta) = 2\,\mathrm{Re}\,K_\omega^P(z,\zeta) - 1, \qquad (z,\zeta) \in \mathbb{D}^2.$$

Proof. Let p, q be polynomials. If $q(0) = 0$, then by the reproducing property of ω,

$$\langle p, \bar{q}\rangle_\omega = \int_{\mathbb{D}} p(z)\, q(z)\, \omega(z)\, d\Sigma(z) = 0,$$

and hence $P^2(\mathbb{D},\omega)$ and $\bar{P}_0^2(\mathbb{D},\omega)$ are perpendicular with respect to the inner product of $HP^2(\mathbb{D},\omega)$. Each harmonic polynomial can be written in the form $p + \bar{q}$. By the Pythagorean theorem,

$$\|p + \bar{q}\|_\omega^2 = \|p\|_\omega^2 + \|q\|_\omega^2.$$

If we take a Cauchy sequence of harmonic polynomials $p_j + \bar{q}_j$ (with $q_j(0) = 0$) with respect to the norm $\|\cdot\|_\omega$, then by the above, p_j is a Cauchy sequence in $P^2(\mathbb{D},\omega)$, and q_j a Cauchy sequence in $P_0^2(\mathbb{D},\omega)$. But then there are elements $f \in P^2(\mathbb{D},\omega)$ and g in $P_0^2(\mathbb{D},\omega)$, holomorphic in the disk $\mathbb{D}$, such that $p_j \to f$ and $q_j \to g$. The limit function $h = f + \bar{g}$ is then harmonic in $\mathbb{D}$, and we have

$$\|h\|_\omega^2 = \|f + \bar{g}\|_\omega^2 = \|f\|_\omega^2 + \|g\|_\omega^2.$$

The local boundedness of point evaluations now follows from Lemma 2.1.

The reproducing kernel for $P^2(\mathbb{D},\omega)$ is K_ω^P, and for $\bar{P}_0^2(\mathbb{D},\omega)$ it is $\bar{K}_\omega^P - 1$. It follows from the above direct sum decomposition that Q_ω is the sum of these two kernels. ■

3 Green functions for weighted biharmonic operators

Smooth weights. Let Ω be a finitely connected bounded domain in $\mathbb{C}$ with C^∞-smooth boundary, by which we mean that locally, the boundary is given as the zero level curve of a C^∞-smooth real-valued function with non-vanishing gradient. Also, let μ be strictly positive and C^∞-smooth on the closure $\overline{\Omega}$. We then define the Green function Γ_μ for the biharmonic operator $\Delta\mu^{-1}\Delta$ in the following way. For fixed $\zeta \in \Omega$, it solves the boundary value problem

$$\begin{aligned} \Delta_z \mu(z)^{-1}\Delta_z \Gamma_\mu(z,\zeta) &= \delta_\zeta(z), \qquad z \in \Omega,\\ \Gamma_\mu(\cdot,\zeta)|_{\partial\Omega} &= 0,\\ \frac{\partial}{\partial n}\Gamma_\mu(\cdot,\zeta)\big|_{\partial\Omega} &= 0, \end{aligned}$$

where the normal derivative is taken in the interior direction. The symbol δ_ζ denotes a unit point mass at the point $\zeta \in \mathbb{D}$. The laplacian Δ has the factorization

$$\Delta_z = \frac{\partial^2}{\partial z \partial \bar{z}},$$

where $(z = x + iy)$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$

are the usual Wirtinger derivatives. We shall also use the space-saving notation ∂_z and $\bar{\partial}_z$ for these operators. Locally summable functions f on Ω are to be interpreted as distributions by the dual action

$$\langle \varphi, f \rangle = \int_\Omega \varphi(z)\, f(z)\, d\Sigma(z),$$

where φ is a test function, that is, a compactly supported C^∞-smooth function on Ω. With these normalization settings, the Green function for $\Omega = \mathbb{D}$ and $\mu = 1$ is

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2 + (1-|z|^2)(1-|\zeta|^2).$$

The weighted biharmonic operators $\Delta\,\mu^{-1}\Delta$ and the associated Green functions seem to have been considered for the first time by Paul Garabedian [11]. When we apply a laplacian to Γ_μ, we should get the weight μ times the Green function G for the laplacian plus a harmonic function, that is,

$$\Delta_z \Gamma_\mu(z,\zeta) = \mu(z)\,\big(G(z,\zeta) + H_\mu(z,\zeta)\big), \qquad (z,\zeta) \in \Omega \times \Omega, \tag{3.1}$$

where $H_\mu(z,\zeta)$ is harmonic in the z variable. Let ϕ be a C^2 function on the closure of Ω, Applying Green's formula, we see that the zero Dirichlet boundary conditions on Γ_μ translate into the requirement that

$$\int_\Omega \Delta_z \Gamma_\mu(z,\zeta)\phi(z)\, d\Sigma(z) = \int_\Omega \Gamma_\mu(z,\zeta)\Delta_z\phi(z)\, d\Sigma(z).$$

By applying this identity to $\phi = h$, where h is harmonic, we obtain

$$\int_\Omega h(z)\,\big(G(z,\zeta) + H_\mu(z,\zeta)\big)\,\mu(z)\, d\Sigma(z) = 0, \tag{3.2}$$

and by an approximation argument we have this for all bounded harmonic functions h on Ω. This is the balayage problem mentioned in the introduction. Since the bounded functions are dense in $HL^2(\Omega,\mu)$, the closed subspace of $L^2(\Omega,\mu)$ consisting of functions harmonic on Ω, it follows that $H_\mu(\cdot,\zeta)$ equals the orthogonal projection of the function $-G(\cdot,\zeta)$ to $HL^2(\Omega,\mu)$ in the space $L^2(\Omega,\mu)$. We shall call the kernel H_μ *the harmonic compensator*. We write this as $H_\omega = -Q_\mu\,\mu\,G$, or written out more explicitly,

$$H_\mu(z,\zeta) = -\int_\Omega Q_\mu(z,\xi)\, G(\xi,\zeta)\, \mu(\xi)\, d\Sigma(\xi), \tag{3.3}$$

where $Q_\mu = Q_\mu^H$ is the reproducing kernel for the space $HL^2(\Omega,\mu)$. Here, we think of a kernel $T(z,\zeta)$ as having an operator T associated to it in the fashion

$$Tf(z) = \int_\Omega T(z,\zeta)\, f(\zeta)\, d\Sigma(\zeta),$$

whenever the integral converges. In principle, the operator also determines the kernel, for we obtain the kernel by applying the operator to a unit point mass at an interior point (or by applying it to functions approximating the unit point mass, and taking the limit).

Because of the boundary conditions, the Green function can be recovered through the formula

$$\Gamma_\mu(z,\zeta) = \int_\Omega G(z,\xi)\,\big(G(\xi,\zeta) + H_\mu(\xi,\zeta)\big)\,\mu(\xi)\,d\Sigma(\xi). \tag{3.4}$$

The operators Q_μ and G are self-adjoint, so that taking adjoints, we have the identity $H_\mu^* = -\Gamma\mu\,Q_\mu$. We note that the kernel for the operator H_μ^* is $H_\mu^*(z,\zeta) = H_\mu(\zeta,z)$. The function $H_\mu(z,\cdot)$ then solves Poisson's equation with data $-\mu(\cdot)Q_\mu(\cdot,z)$. The kernel $H_\mu(z,\zeta)$ is harmonic in the z variable, and for $z \in \partial\Omega$, it solves the boundary value problem (compare with [20]; ∂_n is a condensed notation for the interior normal derivative)

$$\begin{aligned} \Delta_\zeta\,\mu(\zeta)^{-1}\Delta_\zeta H_\mu(z,\zeta) &= 0, &\qquad \zeta &\in \Omega,\\ H_\mu(z,\zeta) &= 0, & \zeta &\in \partial\Omega,\\ \partial_{n(\zeta)}\,H_\mu(z,\zeta) &= 2\,\delta_z(\zeta), & \zeta &\in \partial\Omega. \end{aligned}$$

Smoothness properties of kernels. As above, let Ω be finitely connected with C^∞-smooth boundary, and the weight μ be C^∞-smooth on $\overline{\Omega}$, and strictly positive there. For a subset E of the complex plane $\mathbb{C}$, let $\delta(E) = \{(z,z) \in \mathbb{C}^2 : z \in E\}$ be the corresponding diagonal set. In particular, we shall be concerned with the diagonal $\delta(\overline{\Omega})$, the interior diagonal $\delta(\Omega)$, and the boundary diagonal $\delta(\partial\Omega)$. Then, by an elliptic regularity theorem of Louis Nirenberg [34], which says that we have C^∞-smooth solutions locally if the data are that smooth, the kernels G and Γ_μ are C^∞-smooth on $(\overline{\Omega}\times\overline{\Omega})\setminus\delta(\overline{\Omega})$. It follows that the kernels H_μ and Q_μ are C^∞-smooth on $(\overline{\Omega}\times\overline{\Omega})\setminus\delta(\partial\Omega)$.

Let us for the moment replace C^∞-smoothness with C^ω-smoothness – real analyticity – everywhere above (so that the boundary $\partial\Omega$ is real analytic, and the weight μ is real analytic on $\overline{\Omega}$). Another elliptic regularity theorem, this time due to Morrey and Nirenberg [32], then states that locally, solutions are C^ω-smooth if the data have that degree of regularity. We apply it to our Green functions, to get that the kernels G and Γ_μ are C^ω-smooth on $(\overline{\Omega}\times\overline{\Omega})\setminus\delta(\overline{\Omega})$, and that as before, it follows that the kernels H_μ and Q_μ are C^ω-smooth on $(\overline{\Omega}\times\overline{\Omega})\setminus\delta(\partial\Omega)$.

If we instead consider the complex elliptic second order operator $\bar\partial_z\,\mu^{-1}\partial_z$, we have the same regularity theory. The associated Green function G_μ (with zero Dirichlet boundary data) was considered by Garabedian in [11]. He obtained the identity

$$\partial_z\bar\partial_\zeta\,G_\mu(z,\zeta) = \mu(z)\,\mu(\zeta)\,K_\mu(z,\zeta), \qquad (z,\zeta) \in \Omega^2\setminus\delta(\Omega),$$

which carries over the regularity of G_μ to the reproducing kernel K_μ. Here, K_μ stands for K_μ^R or K_μ^A, which are the same because $R^2(\Omega,\mu) = A^2(\Omega,\mu)$ under the given regularity assumptions. To spell these out: μ is assumed C^∞-smooth and strictly positive on $\overline{\Omega}$, and $\partial\Omega$ is assumed C^∞-smooth, too. The regularity of G_μ shows that K_μ is C^∞-smooth on $(\overline{\Omega}\times\overline{\Omega})\setminus\delta(\partial\Omega)$. This also applies to the real analytic situation: if μ is assumed C^ω-smooth and strictly positive on $\overline{\Omega}$, and $\partial\Omega$ is C^ω-smooth as well, it follows that the kernel K_μ is C^ω-smooth on $(\overline{\Omega}\times\overline{\Omega})\setminus\delta(\partial\Omega)$. This latter fact was mentioned and used in [6].

Some consequences of the positivity of Γ_μ. Let us assume Ω and μ are as above, and that we know that Γ_μ is positive on $\Omega\times\Omega$. For fixed $\zeta\in\Omega$, the function $\Gamma_\mu(\cdot,\zeta)$ vanishes together with its normal derivative along $\partial\Omega$, so for it to be positive in the interior it must have a positive second normal derivative, that is, $0 \le \Delta\Gamma(\cdot,\zeta)$ on $\partial\Omega$. By (3.1), this means that $H_\mu(\cdot,\zeta)$ is positive on $\partial\Omega$, so by harmonicity, we even get that $H_\mu(\cdot,\zeta)$ is positive on Ω. By a similar type of argument applied to the second coordinate, we see that the harmonic kernel function $Q_\mu(z,\zeta)$ is negative for $(z,\zeta)\in(\partial\Omega\times\partial\Omega)\setminus\delta(\partial\Omega)$. This consequence was observed by Garabedian [11], and in fact, it was how he disproved the conjecture that the biharmonic Green function is positive for all ellipses. We use it for a different purpose: we need to know what properties to look for in the harmonic kernel function Q_μ to be able to prove that we have a positive Green function Γ_μ.

More general weights. What if the weight is less smooth than assumed previouslyΓ For instance, let us say that we merely know that μ is summable on Ω. It is fairly clear that some additional requirement is needed to be able to define the biharmonic Green function Γ_μ, in view of the formulas (3.1) and (3.3), which suggests that existence of Γ_μ entails that some type of harmonic Bergman space then becomes well-defined for the weight μ. In the setting considered thus far, the spaces $HL^2(\Omega,\mu)$ and $HR^2(\Omega,\mu)$ coincide, and hence the reproducing kernels Q^H_μ and Q^R_μ, as well. There is a need to make a choice here, and we decide to choose the harmonic rational Bergman space $HR^2(\Omega,\mu)$ and its reproducing kernel Q^R_μ. In other words, for Bergman harmonic rational weights μ on Ω, we take (3.3) as the *definition* of the harmonic compensator H_μ, with $Q_\omega = Q^R_\mu$, and then (3.4) is used to *define* the weighted biharmonic Green function Γ_μ. We formalize this in a definition.

DEFINITION 3.1 *Let μ be a Bergman harmonic rational weight on Ω, and let $G = G_\Omega$ stand for the Green function for the laplacian Δ on Ω. The harmonic compensator is the function*

$$H_\mu(z,\zeta) = -\int_\Omega Q^R_\mu(z,\xi)\, G(\xi,\zeta)\mu(\xi)\, d\Sigma(\xi),$$

and the weighted biharmonic Green function is given by

$$\Gamma_\mu(z,\zeta) = \int_\Omega G(z,\xi)\, \big(G(\xi,\zeta) + H_\mu(\xi,\zeta)\big)\, \mu(\xi)\, d\Sigma(\xi).$$

4 The smoothing of weights

Here we show how to obtain the following approximation result.

THEOREM 4.1 *Let ω be logarithmically subharmonic weight on $\mathbb{D}$ which is reproducing for the origin. Then, for each ε, $0 < \varepsilon < +\infty$, there is another logarithmically subharmonic reproducing weight $\widetilde{\omega}$ which is real analytic on the closed disk $\overline{\mathbb{D}}$ and strictly positive there, such that*

$$\int_{\mathbb{D}} \big|\omega(z) - \widetilde{\omega}(z)\big|\, d\Sigma(z) < \varepsilon.$$

The local smoothing of weights. Let $\operatorname{aut}(\mathbb{D})$ denote the automorphism group of $\mathbb{D}$, which consists of all conformal mappings of $\mathbb{D}$ onto itself. If we let ϕ be an element of $\operatorname{aut}(\mathbb{D})$, then we can find $\alpha, \beta \in \mathbb{T}$ and $r \in [0,1[$ such that

$$\phi = R_\alpha \circ \phi_r \circ R_\beta, \tag{4.1}$$

where $R_\alpha(z) = \alpha z$ and $R_\beta(z) = \beta z$ are rotations, and

$$\phi_r(z) = \frac{r - z}{1 - rz}$$

is a kind of reflexion. This decomposition is unique for $r \neq 0$, and for $r = 0$, the mapping ϕ is a rotation, and only the product $\alpha\beta$ can be determined. For complex $\lambda \in \mathbb{D}$, set

$$\phi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z},$$

which decomposes into

$$\phi_\lambda = R_\alpha \circ \phi_r \circ R_{\bar{\alpha}},$$

provided that $\lambda = r\alpha$, with $0 \leq r < 1$ and $\alpha \in \mathbb{T}$. So we can write a general $\phi \in \operatorname{aut}(\mathbb{D})$ as $\phi = R_\beta \circ \phi_\lambda$, with $\beta \in \mathbb{T}$ and $\lambda \in \mathbb{D}$. In this decomposition, both β and λ are uniquely determined. We can then identify $\operatorname{aut}(\mathbb{D})$ with the set $\mathbb{T} \times \mathbb{D}$, which can be visualized as a subset of $\mathbb{R}^4$. Thinking of $\mathbb{R}^4$ as a subset of the complex four-dimensional space $\mathbb{C}^4$, we can define real analytic functions on $\mathbb{T} \times \mathbb{D}$ as those that extend holomorphically to some open subset of $\mathbb{C}^4$ containing $\mathbb{T} \times \mathbb{D}$. We can also think of another complex structure: $\mathbb{T} \times \mathbb{D} \subset \mathbb{C}^3$, viewing $\mathbb{T}$ as a subset of $\mathbb{C}$, and $\mathbb{D}$ as a subset of $\mathbb{R}^2 \subset \mathbb{C}^2$. This complex structure also gives rise to a class of real analytic functions. Fortunately, the two different complex structures induce the same class of real analytic functions.

There is a left and right invariant Haar measure on $\operatorname{aut}(\mathbb{D})$, which in terms of the representation (4.1) takes the form

$$d\phi = \frac{2r dr}{(1 - r^2)^2}\, d\sigma(\alpha)\, d\sigma(\beta),$$

and in terms of the representation $\phi = R_\beta \circ \phi_\lambda$, with $(\beta, \lambda) \in \mathbb{T} \times \mathbb{D}$, it becomes

$$d\phi = \frac{d\Sigma(\lambda)}{(1 - |\lambda|^2)^2}\, d\sigma(\beta).$$

Let $\Phi : \operatorname{aut}(\mathbb{D}) \to]0, +\infty[$ be a real analytic function of the product form

$$\Phi(\phi) = \Phi_1(\beta)\Phi_2(\lambda), \qquad \phi = R_\beta \circ \phi_\lambda,$$

where

$$\Phi_2(\lambda) = \frac{(1 - |\lambda|^2)^N}{N - 1}, \qquad \lambda \in \mathbb{D}, \tag{4.2}$$

for some integer $N = 2, 3, 4, \ldots$, which has integral

$$\int_{\mathbb{D}} \Phi_2(\lambda)\, \frac{d\Sigma(\lambda)}{(1 - |\lambda|^2)^2} = 1,$$

and $\Phi_1 : \mathbb{T} \to]0, +\infty[$ is some real analytic function with integral

$$\int_{\mathbb{T}} \Phi_1(\beta)\, d\sigma(\beta) = 1.$$

For instance, we can take

$$\Phi_1(\beta) = \frac{1 - \varrho^2}{|1 + \varrho\beta|^2}, \qquad \beta \in \mathbb{T}, \tag{4.3}$$

for some real parameter ϱ with $0 < \varrho < 1$. We shall now see that

$$\int_{\operatorname{aut}(\mathbb{D})} \Phi(\phi)\, h \circ \phi^{-1}(0)\, d\phi = h(0) \tag{4.4}$$

for all bounded harmonic functions h on $\mathbb{D}$. For $\phi = R_\beta \circ \phi_\lambda$, we have $\phi^{-1} = \phi_\lambda \circ R_{\bar\beta}$, so that $\phi^{-1}(0) = \lambda$. It follows that the left hand side of (4.4) assumes the form

$$\int_{\mathbb{T}\times\mathbb{D}} \Phi_1(\beta)\,\Phi_2(\lambda)\,h(\lambda)\,d\sigma(\beta)\,\frac{d\Sigma(\lambda)}{(1-|\lambda|^2)^2}$$
$$= \int_{\mathbb{T}} \Phi_1(\beta)\,d\sigma(\beta) \int_{\mathbb{D}} \frac{(1-|\lambda|^2)^{N-2}}{N-1}\,h(\lambda)\,d\Sigma(\lambda) = h(0),$$

for all bounded harmonic functions h on $\mathbb{D}$, as claimed, if we use the mean value property. We shall use the function Φ to regularize ω: consider the function

$$\omega_\Phi(z) = \int_{\mathrm{aut}\,(\mathbb{D})} \Phi(\phi)\,\omega\circ\phi(z)\,|\phi'(z)|^2 d\phi. \tag{4.5}$$

It is strictly positive on $\mathbb{D}$, because with the given choice of the smoothing function Φ, the only way for ω_Φ to vanish at a point $z \in \mathbb{D}$ would be that $\omega\circ\phi(z) = 0$ for almost all ϕ in $\mathrm{aut}\,(\mathbb{D})$, which never happens, given the assumptions on ω. Just as ω, the function ω_Φ is logarithmically subharmonic, because each individual function $\omega\circ\phi\,|\phi'|^2$ occurring in the integral is, and because the logarithmically subharmonic functions form a cone. It is representing as well, as a computation shows:

$$\int_{\mathbb{D}} h(z)\,\omega_\Phi(z)\,d\Sigma(z) = \int_{\mathrm{aut}\,(\mathbb{D})} \Phi(\phi) \int_{\mathbb{D}} h(z)\,\omega\circ\phi(z)\,|\phi'(z)|^2 d\Sigma(z)\,d\phi$$
$$= \int_{\mathrm{aut}\,(\mathbb{D})} \Phi(\phi) \int_{\mathbb{D}} h\circ\phi^{-1}(z)\,\omega(z)\,d\Sigma(z)\,d\phi = \int_{\mathrm{aut}\,(\mathbb{D})} \Phi(\phi)\,h\circ\phi^{-1}(0)\,d\phi = h(0),$$

for all bounded harmonic functions h on $\mathbb{D}$, where we use (4.4). A shift of variables yields the alternative representation

$$\omega_\Phi(z) = (1-|z|^2)^{-2} \int_{\mathrm{aut}\,(\mathbb{D})} \Phi(\phi\circ\phi_z)\,\omega\circ\phi(0)\,|\phi'(0)|^2 d\phi.$$

This is a mean of the various functions $\Phi(\phi\circ\phi_z)$, taken over the variable ϕ, because

$$\int_{\mathrm{aut}\,(\mathbb{D})} \omega\circ\phi(0)\,|\phi'(0)|^2 d\phi = \int_{\mathbb{T}\times\mathbb{D}} \omega(\beta\lambda)\,(1-|\lambda|^2)^2\,\frac{d\sigma(\beta)\,d\Sigma(\lambda)}{(1-|\lambda|^2)^2}$$
$$= \int_{\mathbb{T}\times\mathbb{D}} \omega(\beta\lambda)\,d\sigma(\beta)\,d\Sigma(\lambda) = \int_{\mathbb{T}} \omega(0)\,d\sigma(\beta) = 1.$$

We want ω_Φ to approximate ω in $L^1(\mathbb{D})$ norm, and to be real analytic on $\mathbb{D}$. The first aim is reached by letting Φ have most of its mass concentrated near the unit element of the group $\mathrm{aut}\,(\mathbb{D})$, which in the coordinates $\phi = R_\beta\circ\phi_\lambda$ corresponds to $\lambda = 0$ and $\beta = -1$. That means that the parameter N should be very large for the function Φ_2 given by (4.2) to be concentrated near 0 in $\mathbb{D}$, and that the function Φ_1 given by (4.3) should have most of its mass near the point -1 on the unit circle, which happens if the parameter ϱ is close to 1. It is helpful to know that the contribution of remote elements ϕ to the integral (4.5) defining ω_Φ is small: here we can use the a priori bound in Lemma 2.2 on ω and the fact that the function $\Phi_2(\lambda)$ drops off quickly as λ approaches $\mathbb{T}$. To deal with the second aim, we proceed as follows. For $\phi = R_\beta\circ\phi_\lambda$,

$$\phi\circ\phi_z = R_{\beta\gamma}\circ\phi_{\phi_z(\lambda)}, \qquad \text{with} \quad \gamma = \frac{\phi_\lambda(z)}{\phi_z(\lambda)} = \frac{\lambda\bar z - 1}{1-\bar\lambda z}, \tag{4.6}$$

so that

$$\Phi(\phi \circ \phi_z) = \Phi_1(\beta\gamma)\, \Phi_2(\phi_z(\lambda)), \qquad \phi = R_\beta \circ \phi_\lambda.$$

We shall need the following: the functions $\Phi(\phi \circ \phi_z)$, considered as functions of $z \in \mathbb{D}$, should extend holomorphically to a neighborhood U of $\mathbb{D} \subset \mathbb{R}^2$ in $\mathbb{C}^2$, and be uniformly bounded there. We shall do this locally around an arbitrary point $z_0 \in \mathbb{D}$. A calculation shows that

$$\Phi_2(\phi_z(\lambda)) = \frac{1}{N-1}\, \frac{(1-|z|^2)^N (1-|\lambda|^2)^N}{|1-\bar{\lambda} z|^{2N}}.$$

Real analytic functions on $\mathbb{D}$ are functions of the type $F(z, \bar{z})$, where F is a holomorphic function of two variables in a neighborhood of the anti-diagonal $\{(z,\bar{z}) : z \in \mathbb{D}\}$. A holomorphic extension of $\Phi_2(\phi_z(\lambda))$ is supplied by the formula

$$F(z, z^*) = \frac{1}{N-1}\, \frac{(1-zz^*)^N (1-|\lambda|^2)^N}{(1-\bar{\lambda} z)^N (1-\lambda z^*)^N},$$

which is bounded uniformly in $\lambda \in \mathbb{D}$ provided that z is close to z_0 and z^* is close to $\bar{z}_0$. Let Φ_1 denote not only the real analytic function on $\mathbb{T}$ but also its bounded holomorphic extension to a neighborhood of $\mathbb{T}$; with Φ_1 given by (4.3), the extension is

$$\Phi_1(\beta) = \frac{1-\varrho^2}{(1+\varrho\beta)(1+\varrho/\beta)}, \qquad \beta \in \mathbb{C} \setminus \{-\varrho, -\varrho^{-1}\}.$$

A holomorphic extension of $G(z,\bar{z}) = \Phi_1(\beta\gamma)$, where γ is given by (4.6), is then given by

$$G(z, z^*) = \Phi_1\left(\beta\, \frac{\lambda z^* - 1}{1-\bar{\lambda} z}\right),$$

which is also uniformly bounded in $\lambda \in \mathbb{D}$ provided that z is close to z_0 and z^* is close to $\bar{z}_0$.

It follows that with the above choice of Φ, ω_Φ approximates ω in the $L^1(\mathbb{D})$ norm, is reproducing, logarithmically subharmonic, and real analytic on $\mathbb{D}$.

The effect of dilatation. We wish to approximate a given weight ω on $\mathbb{D}$, which is logarithmically subharmonic and reproducing, by a positive weight which is real analytic on the closed disk $\overline{\mathbb{D}}$, and has the same properties. ¿From the previous section we know that we can achieve real analyticity in the interior $\mathbb{D}$. We can therefore assume from the start that ω is real analytic and positive on $\mathbb{D}$.

For r, $0 < r < 1$, let $\omega_r(z) = \omega(rz)$ be the associated dilatation of ω. We shall see that ω_r is *subrepresenting*, that is, that for all positive bounded harmonic functions h on $\mathbb{D}$, we have

$$\int_{\mathbb{D}} h(z)\, \omega_r(z)\, d\Sigma(z) \le h(0). \tag{4.7}$$

Let $P(z,\zeta)$ denote the Poisson kernel

$$P(z,\zeta) = \frac{1-|z|^2}{|1-z\bar{\zeta}|^2}, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{T},$$

and consider, for $\lambda \in \mathbb{D}$, the function

$$\int_{\mathbb{T}} P(\lambda, \alpha)\, \omega(\alpha z)\, d\sigma(\alpha), \qquad z \in \mathbb{D}.$$

As a function of λ, this function is harmonic and equals $\omega(\lambda z)$ for $\lambda \in \mathbb{T}$. As the function $\omega(\lambda z)$ is subharmonic in the variable λ, it follows that

$$\omega(\lambda z) \leq \int_{\mathbb{T}} P(\lambda, \alpha)\, \omega(\alpha z)\, d\sigma(\alpha), \qquad (z, \lambda) \in \mathbb{D} \times \mathbb{D}.$$

We specialize to $\lambda = r$, $0 < r < 1$:

$$\omega_r(z) \leq \int_{\mathbb{T}} P(r, \alpha)\, \omega(\alpha z)\, d\sigma(\alpha), \qquad z \in \mathbb{D}.$$

By the reproducing property of ω, we get for all positive bounded harmonic functions h on $\mathbb{D}$ that

$$\begin{aligned}\int_{\mathbb{D}} h(z)\, \omega_r(z)\, d\Sigma(z) &\leq \int_{\mathbb{T}\times\mathbb{D}} h(z)\, P(r, \alpha)\, \omega(\alpha z)\, d\Sigma(z)\, d\sigma(\alpha) \\ &= \int_{\mathbb{T}} P(r, \alpha) \int_{\mathbb{D}} h(z)\, \omega(\alpha z)\, d\Sigma(z)\, d\sigma(\alpha) = \int_{\mathbb{T}} P(r, \alpha)\, h(0)\, d\sigma(\alpha) = h(0), \end{aligned} \tag{4.8}$$

as asserted above.

Completing subrepresenting weights. We now complete the subrepresenting weight ω_r by adding a suitable small term which makes the sum representing. We consider first the harmonic function

$$P^*[\omega_r](z) = \int_{\mathbb{D}} P(z, \zeta)\, \omega_r(\zeta)\, d\Sigma(\zeta), \qquad z \in \mathbb{D},$$

where we have extended the Poisson kernel to the interior:

$$P(z, \zeta) = \frac{1 - |z\zeta|^2}{|1 - z\bar{\zeta}|^2}, \qquad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$

The function $P^*[\omega_r]$ extends harmonically to a neighborhood of the closed unit disk. One way to see this is to realize that $P^*[\omega_r]|_{\mathbb{T}}$ is the outward normal derivative of the function

$$G[\omega_r](z) = \int_{\mathbb{D}} G(z, \zeta)\, \omega_r(\zeta)\, d\Sigma(\zeta), \qquad z \in \mathbb{D},$$

which solves the problem $\Delta G[\omega_r] = \omega_r$ with boundary data $G[\omega_r]|_{\mathbb{T}} = 0$. Here, $G(\cdot, \cdot)$ denotes the Green function for the laplacian Δ on $\mathbb{D}$. By a classical theorem of Painlevé, the real analyticity of the data ω_r forces the real analyticity of the solution $G[\omega_r]$, also on the boundary (see also [32]). The assertion that $P^*[\omega_r]$ is real analytic and hence harmonic on $\overline{\mathbb{D}}$ is immediate. By the subrepresenting property (4.7) of ω_r, $0 < P^*[\omega_r] \leq 1$ throughout $\mathbb{D}$, and hence we have $0 \leq P^*[\omega_r] \leq 1$ also on $\mathbb{T}$. Let θ be a real parameter with $0 < \theta < 1$, and consider the function $H(z) = 1 - \theta\, P^*[\omega_r](z)$, which is harmonic, bounded above by 1, and positive, in a neighborhood of $\overline{\mathbb{D}}$. Let ϱ, $1 < \varrho < +\infty$, be so close to 1 that H is harmonic on the dilated disk $\varrho\mathbb{D}$. Then the function

$$F(z) = \int_{\mathbb{T}} \frac{(1 - \varrho^{-2})^2}{|1 - \varrho^{-1} z\bar{\zeta}|^4}\, H(\varrho\zeta)\, d\sigma(\zeta), \qquad z \in \varrho\mathbb{D}, \tag{4.9}$$

is real analytic in $\varrho\mathbb{D}$ and F is positive there. Moreover, F is logarithmically subharmonic, and for $z \in \mathbb{D}$,

$$\begin{aligned} P^*[F](z) &= \int_{\mathbb{D}} P(z, \zeta)\, F(\zeta)\, d\Sigma(\zeta) = \int_{\mathbb{D}} P(z, \zeta) \int_{\mathbb{T}} \frac{(1 - \varrho^{-2})^2}{|1 - \varrho^{-1}\zeta\bar{\xi}|^4}\, H(\varrho\xi)\, d\sigma(\xi)\, d\Sigma(\zeta) \\ &= \int_{\mathbb{T}} \int_{\mathbb{D}} P(z, \zeta) \frac{(1 - \varrho^{-2})^2}{|1 - \varrho^{-1}\zeta\bar{\xi}|^4}\, d\Sigma(\zeta)\, H(\varrho\xi)\, d\sigma(\xi) = \int_{\mathbb{T}} P(\varrho^{-1} z, \xi)\, H(\varrho\xi)\, d\sigma(\xi) = H(z). \end{aligned}$$

It follows that the weight

$$\widetilde{\omega}(z) = \theta\,\omega_r(z) + F(z), \qquad z \in \mathbb{D},$$

is logarithmically subharmonic, strictly positive, and real analytic on some neighborhood of $\overline{\mathbb{D}}$. It also has $P^*[\widetilde{\omega}] = 1$, which is another way of expressing that $\widetilde{\omega}$ is reproducing:

$$\int_{\mathbb{D}} h(z)\,\widetilde{\omega}(z)\,d\Sigma(z) = h(0),$$

first for all bounded and positive harmonic functions h, then in a second step, for all bounded harmonic functions h on $\mathbb{D}$.

We now look at the $L^1(\mathbb{D})$ norm proximity to the original weight ω. If the parameter r, $0 < r < 1$, is close to 1, the dilate ω_r is close to ω. Also, if θ is close to 1, the function $\theta\,\omega_r$ still approximates ω well. But this means that $\theta\,P^*[\omega_r](0)$ is close to 1, and as the $L^1(\mathbb{D})$ norm of F equals the difference $1 - P^*[\omega_r](0)$, the modified weight $\widetilde{\omega}$ approximates ω well in $L^1(\mathbb{D})$. ■

5 The approximation of Green functions

Let ω be logarithmically subharmonic on the unit disk $\mathbb{D}$ and reproducing (for the origin). We wish to show that the biharmonic Green function Γ_ω for the fourth order elliptic operator $\Delta\omega^{-1}\Delta$ with vanishing Dirichlet data is positive. By Theorem 4.1, ω can be approximated in the $L^1(\mathbb{D})$ norm by a weight $\widetilde{\omega}$ which in addition to being logarithmically subharmonic and reproducing is real analytic and strictly positive on $\overline{\mathbb{D}}$. We need to show that the corresponding Green functions Γ_ω and $\Gamma_{\widetilde{\omega}}$ are appropriately close. Throughout this section, we write K_ω for the reproducing kernel in the space $P^2(\mathbb{D},\omega)$, dropping the superscript P. We observe that $K_\omega = K_\omega^R$, because the rational functions in $R(\overline{\mathbb{D}})$ (the space of rational functions with poles off $\overline{\mathbb{D}}$) are easily approximated uniformly by polynomials on $\mathbb{D}$. Similarly, we drop the superscripts P and R for the harmonic Bergman kernel, and write Q_ω.

We shall need the following basic estimate of the kernel K_ω, obtained by Hedenmalm in [23].

THEOREM 5.1 *Let ω be logarithmically subharmonic weight which is reproducing for the origin. Then*

$$|K_\omega(z,\zeta)| \le \frac{2}{|1 - z\bar{\zeta}|^2}, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}.$$

THEOREM 5.2 *Let ω and ω_n, for $n = 1,2,3,\ldots$, be logarithmically subharmonic weights which reproduce for the origin. If $\omega_n \to \omega$ in the norm of $L^1(\mathbb{D})$ as $n \to +\infty$, then $\Gamma_{\omega_n}(z,\zeta) \to \Gamma_\omega(z,\zeta)$ pointwise in $\mathbb{D}\times\mathbb{D}$ as $n \to +\infty$.*

Proof. Let μ denote a weight of the same general type as ω and ω_n, and recall that by the reproducing property of μ, we have the following identity of reproducing kernel functions (see Proposition 2.3):

$$Q_\mu(z,\zeta) = 2\,\mathrm{Re}\,K_\mu(z,\zeta) - 1, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}. \tag{5.1}$$

We have the identity

$$\Gamma_\mu(z,\zeta) = \int_{\mathbb{D}} G(\xi,z)\,G(\xi,\zeta)\,\mu(\xi)\,d\Sigma(\xi)$$
$$- \int_{\mathbb{D}\times\mathbb{D}} Q_\mu(\xi,\eta)\,G(\xi,z)\,G(\eta,\zeta)\,\mu(\xi)\,\mu(\eta)\,d\Sigma(\xi)d\Sigma(\eta), \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}. \tag{5.2}$$

Let $F_\mu(\cdot,\zeta)$ be the minimum norm solution in $L^2(\mathbb{D},\mu)$ to $\Delta F_\mu(\cdot,\zeta)=\delta_\zeta$:

$$F_\mu(z,\zeta) = G(z,\zeta) - \int_{\mathbb{D}} Q_\mu(z,\eta) G(\eta,\zeta)\,\mu(\eta)\,d\Sigma(\eta), \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}.$$

The formula for Γ_μ then simplifies:

$$\Gamma_\mu(z,\zeta) = \int_{\mathbb{D}} G(z,\xi)\,F_\mu(\xi,\zeta)\,\mu(\xi)\,d\Sigma(\xi), \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}. \tag{5.3}$$

By Theorem 5.1 and (5.1), we have the estimate

$$|Q_\mu(z,\zeta)| \le 1 + \frac{4}{|1-z\bar\zeta|^2}, \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

so that in view of the weight growth control in Lemma 2.2, a calculation yields

$$\big|F_\mu(z,\zeta) - G(z,\zeta)\big| \le C(\zeta), \qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}, \tag{5.4}$$

for some constant $C(\zeta)$ depending continuously on $\zeta\in\mathbb{D}$, but independent of the particular weight μ. We shall now show that $F_{\omega_n}(\cdot,\zeta)\to F_\omega(\cdot,\zeta)$ in an appropriate norm. By minimality, we have that

$$\|F_\omega(\cdot,\zeta)\|_\omega \le \|F_{\omega_n}(\cdot,\zeta)\|_\omega, \qquad \|F_{\omega_n}(\cdot,\zeta)\|_{\omega_n} \le \|F_\omega(\cdot,\zeta)\|_{\omega_n}.$$

In fact, a calculation shows that

$$\int_{\mathbb{D}} \big|F_\omega(z,\zeta) - F_{\omega_n}(z,\zeta)\big|^2 \omega(z)\,d\Sigma(z) = \int_{\mathbb{D}} \big(|F_{\omega_n}(z,\zeta)|^2 - |F_\omega(z,\zeta)|^2\big)\omega(z)\,d\Sigma(z), \tag{5.5}$$

because the function $F_\omega(\cdot,\zeta)$ is perpendicular to the harmonic functions in $L^2(\mathbb{D},\omega)$. As we interchange the weights, we also have that

$$\int_{\mathbb{D}} \big|F_\omega(z,\zeta) - F_{\omega_n}(z,\zeta)\big|^2 \omega_n(z)\,d\Sigma(z) = \int_{\mathbb{D}} \big(|F_\omega(z,\zeta)|^2 - |F_{\omega_n}(z,\zeta)|^2\big)\omega_n(z)\,d\Sigma(z). \tag{5.6}$$

We add (5.5) and (5.6) together, to get

$$\int_{\mathbb{D}} \big|F_\omega(z,\zeta) - F_{\omega_n}(z,\zeta)\big|^2 \big(\omega(z)+\omega_n(z)\big)\,d\Sigma(z)$$
$$= \int_{\mathbb{D}} \big(|F_\omega(z,\zeta)|^2 - |F_{\omega_n}(z,\zeta)|^2\big)\big(\omega_n(z)-\omega(z)\big)\,d\Sigma(z).$$

By the uniform estimate (5.4) and the $L^1(\mathbb{D})$ convergence $\omega_n\to\omega$, it follows that for fixed $\zeta\in\mathbb{D}$,

$$\int_{\mathbb{D}} \big|F_\omega(z,\zeta) - F_{\omega_n}(z,\zeta)\big|^2 \omega(z)\,d\Sigma(z) \to 0 \qquad \text{as} \quad n\to+\infty,$$

or in other words, $F_{\omega_n}(\cdot,\zeta)\to F_\omega(\cdot,\zeta)$ in the norm of $L^2(\mathbb{D},\omega)$. By (5.3),

$$\Gamma_\omega(z,\zeta)-\Gamma_{\omega_n}(z,\zeta)=\int_{\mathbb{D}} G(z,\xi)\,\big(F_\omega(\xi,\zeta)-F_{\omega_n}(\xi,\zeta)\big)\,\omega(\xi)\,d\Sigma(\xi)$$
$$+\int_{\mathbb{D}} G(z,\xi)\,F_{\omega_n}(\xi,\zeta)\,\big(\omega(\xi)-\omega_n(\xi)\big)\,d\Sigma(\xi),\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}.$$

so that the desired result follows from the uniform estimate (5.4) on $F_{\omega_n}(\cdot,\zeta)$, the $L^1(\mathbb{D})$ convergence $\omega_n\to\omega$, and the growth estimate on the weights in Lemma 2.2. ∎

6 Bergman kernels: structural properties

The general theory of reproducing kernels. In the general theory of reproducing kernel functions (see Saitoh's book [40]), introduced and studied by Mercer, Moore, Aronszajn, Kreĭn, and Schwartz, a complex-valued function K of two variables, say (x,y), defined on some product set $E\times E$, is said to be a *reproducing kernel* if for any finite subset $\{x_1,x_2,\ldots,x_N\}$ of E, we have that the matrix

$$\{K(x_j,x_k)\}_{j,k=1}^N$$

is positive definite, in other words, that

$$0\le\sum_{j,k=1}^N K(x_j,x_k)\,w_j\,\bar w_k$$

holds for all sequences $\{w_j\}_{j=1}^N\in\mathbb{C}^N$. In particular, such a kernel has $0\le K(x,x)$, $K(x,y)=\overline{K}(y,x)$, and

$$|K(x,y)|\le K(x,x)^{\frac12}K(y,y)^{\frac12},\tag{6.1}$$

for all x and y in E. The above definition does not refer to any Hilbert space of functions with bounded point evaluations, which was the way we defined the reproducing kernels for the Bergman spaces back in Section 2. It turns out that if we have a Hilbert space of functions with bounded point evaluations, then its reproducing kernel function has the above positive definiteness property, and that if on the other hand, we have a reproducing kernel function as above, there exists a unique Hilbert space for which it reproduces the point evaluation functionals, by a theorem ascribed to Moore and Aronszajn [40]. For instance, if we take a subspace $\mathfrak{S}$ with the properties assumed in Section 2, and write down an orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ for $\mathfrak{S}^2(\Omega,\omega)$, then by the formula for the kernel in terms of the basis,

$$0\le\sum_{l=1}^\infty\Big|\sum_{j=1}^N\varphi_l(z_j)\,w_j\Big|^2=\sum_{l=1}^\infty\sum_{j,k=1}^N\varphi_l(z_j)\,\bar\varphi_l(z_k)\,w_j\,\bar w_k=\sum_{j,k=1}^N K^{\mathfrak{S}}(z_j,z_k)\,w_j\,\bar w_k.$$

The following general result is of some interest; it is known, but we do not have a reference. An infinite matrix $\{A(j,k)\}_{j,k=0}^\infty$ is said to be *positive definite* if each finite submatrix $\{A(j,k)\}_{j,k=0}^{N-1}$ is positive definite, that is, for any finite sequence of points $\{\alpha_j\}_{j=0}^{N-1}\in\mathbb{C}^N$, we have that

$$0\le\sum_{j,k=1}^{N-1}A(j,k)\,\alpha_j\,\bar\alpha_k.$$

PROPOSITION 6.1 *Let K be a function with a convergent power series expansion on the bidisk $\mathbb{D}^2$,*

$$K(z,\zeta) = \sum_{j,k=0}^{\infty} \widehat{K}(j,k)\, z^j\, \bar{\zeta}^k, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D}.$$

Then K is a reproducing kernel on $\mathbb{D} \times \mathbb{D}$ if and only if the infinite matrix $\{\widehat{K}(j,k)\}_{j,k=0}^{\infty}$ is positive definite.

Proof. Let $\{w_j\}_{j=1}^N \in \mathbb{C}^N$ be arbitrary, and put

$$\alpha_m = \sum_{j=1}^{N} w_j\, z_j^m, \tag{6.2}$$

where $\{z_j\}_{j=1}^N$ is a given sequence of points in $\mathbb{D}$. Then a change of the order of summation shows that

$$\sum_{j,k=1}^{N} K(z_j, z_k)\, w_j \bar{w}_k = \sum_{m,n=0}^{\infty} \widehat{K}(m,n) \sum_{j,k=1}^{N} w_j\, \bar{w}_k\, z_j^m\, \bar{z}_k^n = \sum_{m,n=0}^{\infty} \widehat{K}(m,n)\, \alpha_j\, \bar{\alpha}_k.$$

One implication is immediate: if $\{\widehat{K}(m,n)\}_{m,n=0}^{\infty}$ is positive definite, then K is a reproducing kernel. We turn to the reverse implication. Given a sequence $\{\alpha_m\}_{m=1}^N \in \mathbb{C}^N$, we would like to find points $z_1, \dots, z_N$ in $\mathbb{D}$ and a sequence $\{w_j\}_{j=1}^N \in \mathbb{C}^N$ such that (6.2) holds for $m = 0, 1, 2, \dots, N$, because then the reverse implication also follows from the above identity. This can easily be accomplished by choosing the points equidistantly on a concentric circle of radius r, $0 < r < 1$,

$$z_j = r\, e^{2\pi i j/N}, \qquad j = 1, 2, \dots, N,$$

because then we can use Fourier analysis on finite commutative groups to find expressions for w_j in terms of the α_m's so as to have the desired relation between these two finite sequences. ∎

Reproducing kernels for weighted Bergman spaces. In the rest of the section, we shall be concerned with weights $\omega : \mathbb{D} \to [0, +\infty[$ which are area-summable on $\mathbb{D}$ and meet the following two conditions:

- ω is logarithmically subharmonic on $\mathbb{D}$, and
- ω is reproducing for the origin.

Note that since we are looking at the unit disk, the spaces $P^2(\mathbb{D},\omega)$ and $R^2(\mathbb{D},\omega)$ coincide, and hence their kernels do as well: $K_\omega^R = K_\omega^P$. This is the analytic kernel that we wish to study in detail. To simplify the notation, we shall write K_ω for it.

The following structure result is a well known consequence of the fact that the shift operator $Sf(z) = z\, f(z)$ is contractive on $P^2(\mathbb{D},\omega)$ (see Saitoh [40], p. 135).

THEOREM 6.2 *The function $J_\omega(z,\zeta) = (1 - z\bar{\zeta})\, K_\omega(z,\zeta)$ is the reproducing kernel for a Hilbert space of holomorphic functions on $\mathbb{D}$.*

The similar-looking structure result below is key to our further investigations.

THEOREM 6.3 *The function L_ω defined by the equality*

$$K_\omega(z,\zeta) = \frac{1 - z\bar{\zeta} L_\omega(z,\zeta)}{(1 - z\bar{\zeta})^2}$$

is the reproducing kernel for a Hilbert space of holomorphic functions on $\mathbb{D}$.

We postpone the proof a little. First, we need the following important property of the *shift operator* S, $Sf(z) = zf(z)$, acting on $P^2(\mathbb{D}, \omega)$.

PROPOSITION 6.4 *For any two functions $f, g \in P^2(\mathbb{D}, \omega)$, we have the inequality*

$$\|Sf + g\|_\omega^2 \le 2\big(\|f\|_\omega^2 + \|Sg\|_\omega^2\big).$$

Proof. It is enough to obtain the inequality when f and g are polynomials. Let us first assume the weight ω is ω is C^∞-smooth up to the boundary. For any $\lambda \in \mathbb{C} \setminus \{0\}$, we have that

$$\begin{aligned} 0 \le \Delta_z\Big(\big|g(z) - \lambda^{-1} z^3 f(z)\big|^2 \omega(z)\Big) &= \Delta_z\left(|g(z)|^2\,\omega(z)\right) \\ &- 2\,\mathrm{Re}\,\Big(\bar\lambda^{-1} \Delta_z\big(g(z)\bar z^3\,\bar f(z)\,\omega(z)\big)\Big) + |\lambda|^{-2}\,\Delta_z\left(|z^3 f(z)|^2\,\omega(z)\right), \end{aligned} \tag{6.3}$$

for $z \in \mathbb{D}$, where the first inequality holds because the product of a logarithmically subharmonic function and the modulus-squared of a holomorphic function is again logarithmically subharmonic, and in particular, subharmonic. Substituting $\lambda = z^2$ in (6.3), we obtain

$$\begin{aligned} 0 \le \Delta_z\left(|g(z)|^2\,\omega(z)\right) - 2\,\mathrm{Re}\,\Big(\bar z^{-2} \Delta_z\big(g(z)\bar z^3\,\bar f(z)\,\omega(z)\big)\Big) \\ + |z|^{-4}\,\Delta_z\left(|z^3 f(z)|^2\,\omega(z)\right). \end{aligned} \tag{6.4}$$

We note that none of the three terms on the right hand side has the slightest singularity at the origin, even though it may seem so to the inexperienced eye. By Green's formula,

$$\int_{\mathbb{D}} (1-|z|^2)^2\,\Delta_z\left(|g(z)|^2\,\omega(z)\right) d\Sigma(z) = \int_{\mathbb{D}} \left(4|z|^2 - 2\right) |g(z)|^2\,\omega(z)\,d\Sigma(z). \tag{6.5}$$

A slightly more sophisticated exercise involving Green's formula shows that if $\mathbb{D}(0,\varepsilon)$ stands for a small circular disk about the origin of radius ε, then

$$\begin{aligned} &\int_{\mathbb{D}\setminus\mathbb{D}(0,\varepsilon)} (1-|z|^2)^2 \Big(\bar z^{-2}\,\Delta_z\big(g(z)\bar z^3\,\bar f(z)\,\omega(z)\big)\Big)\,d\Sigma(z) \\ &\qquad = \int_{\mathbb{D}\setminus\mathbb{D}(0,\varepsilon)} \Delta_z\left(\bar z^{-2}(1-|z|^2)^2\right)\left(g(z)\bar z^3\,\bar f(z)\,\omega(z)\right) d\Sigma(z) \\ &\qquad + 2\int_{\partial\mathbb{D}(0,\varepsilon)} \Big((1-|z|^2)^2\,\bar z^{-2}\,\partial_{n(z)}\big(g(z)\bar z^3\,\bar f(z)\,\omega(z)\big) \\ &\qquad - \partial_{n(z)}\big((1-|z|^2)^2\,\bar z^{-2}\big)\,g(z)\bar z^3\,\bar f(z)\,\omega(z)\Big)\,d\sigma(z) \\ &\qquad = \int_{\mathbb{D}\setminus\mathbb{D}(0,\varepsilon)} \bar z\,g(z)\,\bar f(z)\,\omega(z)\,d\Sigma(z) + O(\varepsilon), \end{aligned} \tag{6.6}$$

as $\varepsilon \to 0$, where the normal derivative is taken inward with respect to the disk $\mathbb{D}(0,\varepsilon)$.

We apply Green's formula a third time, and obtain

$$\int_{\mathbb{D}\setminus\mathbb{D}(0,\varepsilon)} (1-|z|^2)^2 \Big(|z|^{-4}\,\Delta_z\big(|z^3 f(z)|^2\,\omega(z)\big)\Big)\,d\Sigma(z)$$

$$= \int_{\mathbb{D}\setminus\mathbb{D}(0,\varepsilon)} \Delta_z\Big(|z|^{-4}(1-|z|^2)^2\Big)\,|z^3 f(z)|^2\,\omega(z)\,d\Sigma(z)$$

$$+ 2\int_{\partial\mathbb{D}(0,\varepsilon)} \Big((1-|z|^2)^2\,|z|^{-4}\,\partial_{n(z)}\big(|z^3 f(z)|^2\,\omega(z)\big)$$

$$- |z^3 f(z)|^2\,\omega(z)\,\partial_{n(z)}\big((1-|z|^2)^2\,|z|^{-4}\big)\Big)\,d\sigma(z)$$

$$= \int_{\mathbb{D}\setminus\mathbb{D}(0,\varepsilon)} \big(4-2\,|z|^2\big)\,|f(z)|^2\,\omega(z)\,d\Sigma(z) + O(\varepsilon), \tag{6.7}$$

as $\varepsilon\to 0$. Putting the terms (6.5)–(6.7) together, using (6.4), we arrive in the limit $\varepsilon\to 0$ at

$$0 \le \int_{\mathbb{D}} \big(4\,|z|^2-2\big)\,|g(z)|^2\omega(z)\,d\Sigma(z)$$
$$- 2\,\mathrm{Re}\int_{\mathbb{D}} \bar z\, g(z)\,\bar f(z)\,\omega(z)\,d\Sigma(z) + \int_{\mathbb{D}} \big(4-2|z|^2\big)\,|f(z)|^2\omega(z)\,d\Sigma(z), \tag{6.8}$$

which expresses in expanded form the inequality we are looking for. We now turn to the explanation of why we can assume ω to be C^∞-smooth. ¿From the previous section, we know that we can approximate ω in the $L^1(\mathbb{D})$-norm with weights of the same type but with a much higher degree of smoothness (C^ω on $\overline{\mathbb{D}}$, in fact). And since we only need to check the above inequality (6.8) for fixed polynomials f, g at a time, the assertion is immediate. ■

In addition to the forward shift S, we shall need the *backward shift* T, as defined by

$$Tf(z) = \frac{f(z)-f(0)}{z}, \qquad z\in\mathbb{D},$$

which we think of as acting on $P^2(\mathbb{D},\omega)$. The composed operator TS is the identity, and ST is given by $STf(z) = f(z)-f(0)$. The forward shift S is a contraction on $P^2(\mathbb{D},\omega)$, and so is ST, because of the reproducing property of the weight ω, which leads to the norm identity

$$\|f\|_\omega^2 = \|f-f(0)\|_\omega^2 + |f(0)|^2, \qquad f\in P^2(\mathbb{D},\omega).$$

The variant of Proposition 6.4 which we shall actually use is the following.

COROLLARY 6.5 *For any two functions $f, g\in P^2(\mathbb{D},\omega)$, we have the inequality*

$$\|Sf+Tg\|_\omega^2 \le 2\big(\|f\|_\omega^2 + \|g\|_\omega^2\big).$$

We are now ready to prove Theorem 6.3.

Proof. Solving for L_ω, we find that

$$L_\omega(z,\zeta) = \frac{1}{z\bar\zeta}\Big(1-(1-z\bar\zeta)^2 K_\omega(z,\zeta)\Big) = \frac{1-K_\omega(z,\zeta)}{z\bar\zeta} + 2\,K_\omega(z,\zeta) - z\bar\zeta\,K_\omega(z,\zeta). \tag{6.9}$$

By the reproducing property of the weight ω, we have that

$$K_\omega(z,0) = K_\omega(0,\zeta) = 1, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}, \tag{6.10}$$

so that by some well-known division properties of holomorphic function on the bidisk $\mathbb{D}^2$, the function $L_\omega(z,\bar\zeta)$ is holomorphic on $\mathbb{D}^2$. As a consequence of Theorem 5.1, the kernel

$$z\bar\zeta\, L_\omega(z,\zeta) = 1 - (1-z\bar\zeta)^2\, K_\omega(z,\zeta), \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

is bounded in modulus by 3, so that by the maximum principle for holomorphic functions of two complex variables,

$$|L_\omega(z,\zeta)| \le 3, \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D}. \tag{6.11}$$

We shall see later that the bound 3 may be replaced by 1, which is best possible.

We wish to prove that L_ω is a *reproducing* kernel. In other words, we should show that for any finite subset $\{z_1, z_2, \ldots, z_N\}$ of $\mathbb{D}$, it is the case that

$$0 \le \sum_{j,k=1}^{N} L_\omega(z_j, z_k)\, w_j\, \bar w_k \tag{6.12}$$

holds for all sequences $\{w_j\}_{j=1}^N \in \mathbb{C}^N$. By exploiting the reproducing property of the kernel K_ω, we have that

$$L_\omega(z_j,z_k) = \int_{\mathbb{D}\times\mathbb{D}} L_\omega(z,\zeta)\, K_\omega(z_j,z)\, K_\omega(\zeta,z_k)\, \omega(z)\, \omega(\zeta)\, d\Sigma(z)\, d\Sigma(\zeta),$$

where the integral is absolutely convergent because of estimate (6.11) and the bound on K_ω from Theorem 5.1. If we let f be the $H^\infty(\mathbb{D})$ function – $H^\infty(\mathbb{D})$ is the algebra of bounded holomorphic functions on $\mathbb{D}$ – given by the formula

$$f(z) = \sum_{j=1}^{N} \bar w_j\, K_\omega(z,z_j), \qquad z \in \mathbb{D},$$

we see that (6.12) is equivalent to having

$$0 \le \int_{\mathbb{D}\times\mathbb{D}} L_\omega(z,\zeta)\, \bar f(z)\, f(\zeta)\, \omega(z)\, \omega(\zeta)\, d\Sigma(z)\, d\Sigma(\zeta). \tag{6.13}$$

We shall obtain (6.13) for all $f \in H^\infty(\mathbb{D})$. The forward and backward shift operators S and T, acting on $P^2(\mathbb{D},\omega)$, have adjoints S^*_ω and T^*_ω, where the subscript indicates that the adjoint is taken with respect to the inner product of $P^2(\mathbb{D},\omega)$:

$$\langle Sf, g\rangle_\omega = \langle f, S^*_\omega g\rangle_\omega, \qquad \langle Tf, g\rangle_\omega = \langle f, T^*_\omega g\rangle_\omega, \qquad \text{for} \;\; f,g \in P^2(\mathbb{D},\omega).$$

We have from (6.9) and (6.10) that for $f \in H^\infty(\mathbb{D})$,

$$\begin{aligned}
&\int_{\mathbb{D}} L_\omega(z,\zeta)\, f(\zeta)\, \omega(\zeta)\, d\Sigma(\zeta) \\
&\qquad = -\frac{1}{z}\, \langle f, TK_\omega(\cdot,z)\rangle_\omega + 2\, \langle f, K_\omega(\cdot,z)\rangle_\omega - z\, \langle f, SK_\omega(\cdot,z)\rangle_\omega \\
&\qquad = -\frac{1}{z}\, \langle T^*_\omega f, K_\omega(\cdot,z)\rangle_\omega + 2\, \langle f, K_\omega(\cdot,z)\rangle_\omega - z\, \langle S^*_\omega f, K_\omega(\cdot,z)\rangle_\omega \\
&\qquad = -\frac{T^*_\omega f(z)}{z} + 2\, f(z) - z\, S^*_\omega f(z), \qquad z \in \mathbb{D},
\end{aligned}$$

so that since $T_\omega^* f(0) = 0$ – due to the reproducing property of ω – we can condense the above to

$$\int_{\mathbb{D}} L_\omega(z,\zeta)\, f(\zeta)\, \omega(\zeta)\, d\Sigma(\zeta) = -TT_\omega^* f(z) + 2\, f(z) - S\, S_\omega^* f(z), \qquad z \in \mathbb{D}.$$

Integrating also with respect to the z variable, we arrive at

$$\int_{\mathbb{D}\times\mathbb{D}} L_\omega(z,\zeta)\, \bar{f}(z)\, f(\zeta)\, \omega(z)\, \omega(\zeta)\, d\Sigma(z)\, d\Sigma(\zeta)$$
$$= -\langle TT_\omega^* f, f\rangle_\omega + 2\,\langle f, f\rangle_\omega - \langle S\, S_\omega^* f, f\rangle_\omega = -\|T_\omega^* f\|_\omega^2 + 2\,\|f\|_\omega^2 - \|S_\omega^* f\|_\omega^2,$$

which shows that what we in fact need to know is that

$$0 \le 2 - TT_\omega^* - S\, S_\omega^*, \tag{6.14}$$

where the inequality is interpreted in the sense of operator theory (meaning that the operator on the right hand side is self-adjoint and that its spectrum is contained in the interval $[0, +\infty[$). Let $P^2(\mathbb{D},\omega) \oplus P^2(\mathbb{D},\omega)$ be the orthogonal sum of the two spaces, with elements (f,g), $f,g \in P^2(\mathbb{D},\omega)$, and the inner product

$$\langle (f_1, g_1), (f_2, g_2)\rangle_{\omega,\omega} = \langle f_1, f_2\rangle_\omega + \langle g_1, g_2\rangle_\omega.$$

We consider the operator $R : P^2(\mathbb{D},\omega) \oplus P^2(\mathbb{D},\omega) \to P^2(\mathbb{D},\omega)$ given by

$$R(f,g) = 2^{-\frac{1}{2}}\,(Sf + Tg),$$

and note that if $R_\omega^* : P^2(\mathbb{D},\omega) \to P^2(\mathbb{D},\omega) \oplus P^2(\mathbb{D},\omega)$ is the adjoint defined by

$$\langle R(f,g), h\rangle_\omega = \langle (f,g), R_\omega^* h\rangle_{\omega,\omega}, \qquad f,g,h \in P^2(\mathbb{D},\omega),$$

then we have $R_\omega^*(h) = 2^{-\frac{1}{2}}\,(S_\omega^* h, T_\omega^* h)$. It immediately follows that

$$R\, R_\omega^*\, f = \frac{1}{2}\,(SS_\omega^* f + TT_\omega^* f), \qquad f \in P^2(\mathbb{D},\omega),$$

so that the assertion (6.14) can be written as $R\, R_\omega^* \le 1$. This, however, is fulfilled precisely when R_ω^* is a contraction. By Corollary 6.5, the operator R is a contraction, which implies that R_ω^* is a contraction as well. The proof is complete. ■

COROLLARY 6.6 *Let the kernel L_ω be as in Theorem 6.3. Then*

$$|L_\omega(z,\zeta)| < 1, \qquad (z,\zeta) \in \mathbb{D}^2.$$

Proof. The identity

$$K_\omega(z,z) = \frac{1 - |z|^2 L_\omega(z,z)}{(1-|z|^2)^2}, \qquad z \in \mathbb{D},$$

together with the observations that $0 \le K_\omega(z,z)$ and $0 \le L_\omega(z,z)$ shows that $0 \le L_\omega(z,z) \le 1$, because the function $L_\omega(z,z)$ is subharmonic on $\mathbb{D}$. In fact, unless $L_\omega(z,z)$ equals the constant 1 identically on $\mathbb{D}$, we have a strict inequality: $L_\omega(z,z) < 1$. And if $L_\omega(z,z) \equiv 1$, then $L_\omega(z,\zeta) \equiv 1$ too, because a kernel function is determined by its values along the diagonal. But if $L_\omega(z,\zeta) \equiv 1$, then the kernel K_ω must be the Hardy kernel (associated with the space $H^2(\mathbb{D})$), which cannot be, because $H^2(\mathbb{D})$ is not of the type $P^2(\mathbb{D},\omega)$. We conclude that $|L_\omega(z,\zeta)| < 1$, because after all, $L_\omega(z,z)$ is the norm-squared of the point evaluation functional at $z \in \mathbb{D}$ in the Hilbert space of holomorphic functions that can be associated with L_ω. ■

We have achieved an improvement on the estimate of Theorem 5.1.

COROLLARY 6.7 *The reproducing kernel K_ω can be estimated as follows:*

$$|K_\omega(z,\zeta)| \le \frac{1+|z\zeta|}{|1-z\bar\zeta|^2}, \qquad (z,\zeta) \in \mathbb{D}^2.$$

Proof. This is immediate from Corollary 6.6. ■

The kernel L_ω has the following boundary behavior.

THEOREM 6.8 *Suppose our weight ω is continuous on $\overline{\mathbb{D}}$, in which case $1 \le \omega|_{\mathbb{T}}$. Then the diagonal function $L_\omega(z,z)$ has a continuous extension to $\overline{\mathbb{D}}$, and the boundary values are*

$$L_\omega(z,z) = 1 - \frac{1}{\omega(z)}, \qquad z \in \mathbb{T}.$$

Proof. For $\lambda \in \mathbb{D}$, let F_λ be the function

$$F_\lambda(z) = \frac{1-|\lambda|^2}{(1-\bar\lambda z)^2}, \qquad z \in \mathbb{D},$$

which has norm 1 in $P^2(\mathbb{D})$. We fix a point $\zeta \in \mathbb{T}$. As λ approaches ζ from the interior, F_λ tends to 0 uniformly off every fixed neighborhood of the point ζ, and consequently, the measure $|F_\lambda|^2 d\Sigma$ tends to the unit point mass at ζ. We apply this observation to integration against the weight ω, and obtain

$$\int_{\mathbb{D}} |F_\lambda(z)|^2\,\omega(z)\,d\Sigma(z) \to \omega(\zeta) \qquad \text{as} \quad \lambda \to \zeta.$$

On the other hand, we have the estimate

$$\frac{1}{(1-|\lambda|^2)^2} = |F_\lambda(\lambda)|^2 \le K_\omega(\lambda,\lambda) \int_{\mathbb{D}} |F_\lambda(z)|^2\,\omega(z)\,d\Sigma(z),$$

so that

$$\frac{1}{\omega(\zeta)} \le \liminf_{\lambda\to\zeta}\,(1-|\lambda|^2)^2\,K_\omega(\lambda,\lambda) = \liminf_{\lambda\to\zeta}\big(1-|\lambda|^2\,L_\omega(\lambda,\lambda)\big) = 1 - \limsup_{\lambda\to\zeta} L_\omega(\lambda,\lambda),$$

which leads to half of the desired assertion,

$$\limsup_{\lambda\to\zeta} L_\omega(\lambda,\lambda) \le 1 - \frac{1}{\omega(\zeta)}, \qquad \zeta \in \mathbb{T}.$$

For the other half, we use another collection of functions. For $\lambda \in \mathbb{D}$, let G_λ be the function

$$G_\lambda(z) = K_\omega(\lambda, \lambda)^{-\frac{1}{2}} K_\omega(z, \lambda), \qquad z \in \mathbb{D},$$

which has norm 1 in $P^2(\mathbb{D}, \omega)$. By the estimate of the kernel function of Theorem 5.1, and the well-known fact that $K_\omega(\lambda, \lambda) \to +\infty$ as $|\lambda| \to 1$ (this quantity represents the norm-squared of the point evaluation functional, and the space $P^2(\mathbb{D}, \omega)$ contains the Hardy space $H^2(\mathbb{D})$), the function G_λ tends to 0 uniformly off a fixed neighborhood of the point ζ as λ approaches $\zeta \in \mathbb{T}$. In particular, the measure $|G_\lambda|^2 \, \omega \, d\Sigma$ converges to a point mass at ζ as $\lambda \to \zeta$. Using the properties of the Bergman kernel for $P^2(\mathbb{D})$, we have the estimate

$$K_\omega(\lambda, \lambda) = |G_\lambda(\lambda)|^2 = \left| \int_{\mathbb{D}} (1 - \lambda \bar{z})^{-2} \, G_\lambda(z) \, d\Sigma(z) \right|^2 \le (1 - |\lambda|^2)^{-2} \int_{\mathbb{D}} |G_\lambda(z)|^2 \, d\Sigma(z),$$

whereby in the limit,

$$\int_{\mathbb{D}} |G_\lambda(z)|^2 \, d\Sigma(z) \to \frac{1}{\omega(\zeta)} \qquad \text{as} \;\; \lambda \to \zeta.$$

It follows that

$$\begin{aligned} 1 - \liminf_{\lambda \to \zeta} L_\omega(\lambda, \lambda) = \limsup_{\lambda \to \zeta} \left(1 - |\lambda|^2 \, L_\omega(\lambda, \lambda)\right) \\ = \limsup_{\lambda \to \zeta} (1 - |\lambda|^2)^2 \, K_\omega(\lambda, \lambda) \le 1 - \frac{1}{\omega(\zeta)}, \end{aligned}$$

and consequently,

$$1 - \frac{1}{\omega(\zeta)} \le \liminf_{\lambda \to \zeta} L_\omega(\lambda, \lambda), \qquad \zeta \in \mathbb{T}.$$

This provides an alternative demonstration of the inequality $1 \le \omega|_{\mathbb{T}}$, as is seen by observing that a reproducing kernel is positive along the diagonal. ∎

REMARK 6.9 Let Γ be the biharmonic Green function for $\mathbb{D}$,

$$\Gamma(z, \zeta) = |z - \zeta|^2 \log \left| \frac{z - \zeta}{1 - z\bar{\zeta}} \right|^2 + (1 - |z|^2)(1 - |\zeta|^2), \qquad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$

The – by now classical – factorization-type identity found in [7] for weights ω that reproduce for the origin reads

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^2 \omega(z) \, d\Sigma(z) = \int_{\mathbb{D}} |f(z)|^2 \, d\Sigma(z) \\ + \int_{\mathbb{D} \times \mathbb{D}} \Gamma(z, \zeta) \, |f'(z)|^2 \, \Delta\omega(\zeta) \, d\Sigma(z) \, d\Sigma(\zeta), \qquad f \in H^2(\mathbb{D}), \end{aligned}$$

and in view of the fact that $0 < \Gamma(z, \zeta)$ on $\mathbb{D} \times \mathbb{D}$, and the logarithmic subharmonicity of ω which leads to $0 \le \Delta\omega$, we see that

$$\int_{\mathbb{D}} |f(z)|^2 \, d\Sigma(z) \le \int_{\mathbb{D}} |f(z)|^2 \omega(z) \, d\Sigma(z), \qquad f \in H^2(\mathbb{D}).$$

Suppose that ω extends to a continuous function on $\overline{\mathbb{D}}$. Then, by choosing the analytic function f such that $|f|^2 d\Sigma$ approximates a point mass at a point on the boundary $\mathbb{T}$, we see that $1 \le \omega(z)$ on $\mathbb{T}$.

The function L_ω is bounded and sesqui-holomorphic on $\mathbb{D}^2$ – meaning that the function $L_\omega(z,\bar\zeta)$ is a holomorphic function of two variables there – and hence it possesses radial boundary values almost everywhere on the torus $\mathbb{T}^2$, with respect to the usual area measure there. It follows that the kernels K_ω and Q_ω, too, have radial boundary values almost everywhere on $\mathbb{T}^2$: for K_ω, we can use the formula defining L_ω in Theorem 6.3, and for Q_ω, there is the identity of Proposition 2.3:

$$Q_\omega(z,\zeta) = 2\,\mathrm{Re}\,K_\omega(z,\zeta) - 1, \qquad (z,\zeta)\in\mathbb{D}^2. \tag{6.15}$$

The following result will be used later on in the proof of the positivity of the weighted biharmonic Green function Γ_ω. For this reason, we specify explicitly all the requirements on the weight ω.

COROLLARY 6.10 *Let ω be a logarithmically subharmonic reproducing weight on $\mathbb{D}$, which is continuous on $\overline{\mathbb{D}}$. We then have the inequality (almost everywhere)*

$$Q_\omega(z,\zeta) \le -\left(\frac{1}{\omega(z)}+\frac{1}{\omega(\zeta)}\right)\frac{1}{|z-\zeta|^2}, \qquad (z,\zeta)\in\mathbb{T}\times\mathbb{T}\setminus\delta(\mathbb{T}).$$

Proof. Since the kernel L_ω is reproducing for some space, we have

$$|L_\omega(z,\zeta)| \le L_\omega(z,z)^{\frac12}\,L_\omega(\zeta,\zeta)^{\frac12}, \qquad z,\zeta\in\mathbb{D},$$

and in view of Theorem 6.8 and the geometric-arithmetic mean value inequality, we obtain, almost everywhere,

$$|L_\omega(z,\zeta)| \le \left(1-\frac{1}{\omega(z)}\right)^{\frac12}\left(1-\frac{1}{\omega(\zeta)}\right)^{\frac12} \le 1-\frac12\left(\frac{1}{\omega(z)}+\frac{1}{\omega(\zeta)}\right), \qquad (z,\zeta)\in\mathbb{T}^2. \tag{6.16}$$

We write the equation for K_ω in terms of L_ω as

$$K_\omega(z,\zeta) = \frac{1-z\bar\zeta\,L_\omega(z,\zeta)}{(1-z\bar\zeta)^2} = \frac{1}{1-z\bar\zeta}+\frac{z\bar\zeta}{(1-z\bar\zeta)^2}-\frac{z\bar\zeta}{(1-z\bar\zeta)^2}\,L_\omega(z,\zeta), \qquad (z,\zeta)\in\mathbb{D}^2,$$

where we notice the appearance of the Kœbe function

$$\kappa(z) = \frac{z}{(1-z)^2}, \qquad z\in\mathbb{D},$$

which maps $\mathbb{D}$ onto the slit domain $\mathbb{C}\setminus]-\infty,-\frac14]$. It has the boundary values

$$\kappa(z) = -\frac{1}{|1-z|^2}, \qquad z\in\mathbb{T}\setminus\{1\},$$

so that on $\mathbb{T}^2\setminus\delta(\mathbb{T})$, K_ω equals (almost everywhere)

$$K_\omega(z,\zeta) = \frac{1}{1-z\bar\zeta}-\frac{1}{|z-\zeta|^2}-\frac{1}{|z-\zeta|^2}\,L_\omega(z,\zeta), \qquad (z,\zeta)\in\mathbb{T}^2\setminus\delta(\mathbb{T}).$$

The first term on the right hand side has real part $\frac12$. From the identity (6.15), the above representation formula, and (6.16), we see that (almost everywhere)

$$\begin{aligned}Q_\omega(z,\zeta) &= -\frac{2}{|z-\zeta|^2}-\frac{2}{|z-\zeta|^2}\,\mathrm{Re}\,L_\omega(z,\zeta) \le -\frac{2}{|z-\zeta|^2}+\frac{2}{|z-\zeta|^2}\,|L_\omega(z,\zeta)|\\ &= -\left(\frac{1}{\omega(z)}+\frac{1}{\omega(\zeta)}\right)\frac{1}{|z-\zeta|^2}, \qquad (z,\zeta)\in\mathbb{T}^2\setminus\delta(\mathbb{T}),\end{aligned}$$

as asserted. ■

REMARK 6.11 (a) Proposition 6.4 only uses the logarithmic subharmonicity of ω, not the reproducing property.
(b) In the proof of Theorem 6.3, we appeal to Theorem 5.1 mainly for reasons of convenience of exposition. The use of it can be avoided entirely, and then one has a different proof of Theorem 5.1 from [23].
(c) It is possible to interpret the assertion of Therem 6.8 as a statement about the asymptotical behavior of the matrix

$$\{\widehat{K}_\omega(j,k)\}_{j,k=0}^{\infty}.$$

For large indices, the increments of this matrix in the direction of the diagonal (but not necessarily *on* the diagonal) are asymptotically given in terms of the Fourier coefficients of the reciprocal weight $\omega^{-1}|_{\mathbb{T}}$.
(d) If we assume more regularity of ω, say C^∞-smoothness on $\overline{\mathbb{D}}$, then the kernels K_ω and Q_ω are also much smoother, in fact, C^∞-smooth on $\overline{\mathbb{D}}\times\overline{\mathbb{D}}\setminus\delta(\mathbb{T})$, so that the assertion of Corollary 6.10 is valid everywhere on $\mathbb{T}^2\setminus\delta(\mathbb{T})$.
(e) Suppose ω is is C^∞-smooth on $\overline{\mathbb{D}}$ and real analytic near $\mathbb{T}$. It is a natural problem to ask under what additional assumptions the kernel L_ω becomes sesqui-holomorphic on $\overline{\mathbb{D}}\times\overline{\mathbb{D}}$. We recall that sesqui-holomorphic means that the function is holomorphic in the first variable, and anti-holomorphic in the second. One shows that unless $\Delta\log\omega = 0$ along $\mathbb{T}$, the kernel K_ω necessarily develops a logarithmic singularity which prohibits such smoothness of L_ω. An example of this phenomenon is $\omega(z)=\frac{3}{4}(1+|z|^4)$, with kernel

$$K_\omega(z,\zeta)=\frac{2}{3}\left(\frac{1}{(1-z\bar{\zeta})^2}+\frac{1}{1-z\bar{\zeta}}-\frac{z\bar{\zeta}+\log(1-z\bar{\zeta})}{(z\bar{\zeta})^2}\right).$$

7 The weighted Hele-Shaw flow

Let Ω be a finitely connected bounded domain in $\mathbb{C}$ with C^∞-smooth boundary, and fix a point $z_0\in\Omega$. Without loss of generality, we can take $z_0=0$. Let $\omega:\Omega\to]0,+\infty[$ extend to a $\mathbb{C}^\infty$-smooth function on $\overline{\Omega}$, which is strictly positive there: $0<\omega(z)$ for all $z\in\overline{\Omega}$. The Sobolev space $W^2(\Omega)$ consists of all functions in $L^2(\Omega)$ whose distributional partial derivatives up to order 2 are also in $L^2(\Omega)$. By the Sobolev-Morrey imbedding theorem, the functions in $W^2(\Omega)$ are in $C^{0,\alpha}(\overline{\Omega})$, the space of Hölder continuous functions on $\overline{\Omega}$, for each exponent α, $0<\alpha<1$.

Hele-Shaw flow: the weak solution formulation. For positive r, we wish to find open precompact subset $D(r)$ of Ω, with $0\in D(r)$, such that *the reproducing property*

$$r^2h(0)=\int_{D(r)}h(z)\omega(z)\,d\Sigma(z) \tag{7.1}$$

holds for all $h\in W^2(\Omega)$ which are harmonic on $D(r)$. We also require *the moment inequality to hold*, that is,

$$r^2u(0)\le\int_{D(r)}u(z)\omega(z)\,d\Sigma(z), \tag{7.2}$$

for all $u\in W^2(\Omega)$ that are subharmonic on $D(r)$. In this general setting, there may exist several solutions $D(r)$ to (7.1), but only one of them (up to sets with zero area) also has (7.2). The reason is that under (7.2), the set $D(r)$ can be obtained as the non-coincidence set for an obstacle problem (see below). The condition (7.1) requires $r^{-2}\,\omega\,1_{D(r)}\,d\Sigma$ to be a *reproducing measure for* 0, and the condition (7.2) requires it to be a *Jensen measure for* 0. Note that (7.2) actually contains the condition of (7.1), because for harmonic u we may apply the inequality to both u and $-u$. The property (7.1) is a weighted *quadrature identity*, and (7.2) requires $D(r)$ to be a weighted *subharmonic quadrature domain* [41], [43].

Let $G = G_\Omega$ be the Green function for the laplacian on Ω, as usual. We then form the *potential function*

$$U_r(z) = G\big[\omega\, 1_{D(r)} - r^2\delta_0\big](z) = \int_{D(r)} G(z,\zeta)\,\omega(\zeta)\,d\Sigma(\zeta) - r^2 G(z,0), \qquad z \in \Omega,$$

and observe that by (7.1), $U_r(z) = 0$ off the closure of $D(r)$ in Ω. The requirement of (7.1) that the function $G(z,\cdot)$ should be in $W^2(\Omega)$ is not fulfilled, though, but we easily modify it near the singularity – safely away from $D(r)$ – so that it is. As a matter of fact, no matter how complicated the open set $D(r)$ is, the function $\omega\, 1_{D(r)}$ is bounded and Borel measurable in Ω, which by results from singular integral theory leads to the smoothness information that away from the origin in Ω, U_r is in the Sobolev space $W^{2,p}$ of functions whose distributional partial derivatives up to order 2 are in L^p (with respect to area measure), for each finite p, $1 < p < +\infty$. By the Sobolev-Morrey imbedding theorem, U_r is then of smoothness class $C^{1,\alpha}$, for each α, $0 < \alpha < 1$, away from 0 in $\overline{\Omega}$. We use standard notation here: a function is in $C^{1,\alpha}$ if it is continuously differentiable (C^1-smooth), and the first order partial derivatives are Hölder continuous with exponent α. It is actually the case that U_r is in the Sobolev space $W^{2,\infty}$ away from the origin in Ω, and hence of smoothness class $C^{1,1}$, again away from the origin on $\overline{\Omega}$ (see below). For points $z \in \overline{D}(r)$, we can approximate the Green function $G(z,\cdot)$ by W^2-smooth subharmonic functions, and in the limit we have that $0 \le U_r(z)$, by (7.2). For test functions ϕ on Ω (test functions are C^∞-smooth and have compact support), Green's formula has it that

$$\int_{D(r)} \phi(z)\,\omega(z)\,d\Sigma(z) - r^2\phi(0) = \int_\Omega \Delta U_r(z)\,\phi(z)\,d\Sigma(z) = \int_\Omega U_r(z)\,\Delta\phi(z)\,d\Sigma(z), \quad (7.3)$$

where the middle integral is to be interpreted in the sense of distribution theory. As U_r itself has compact support, we can extend the above equality to the class of ϕ in $C^\infty(\overline{\Omega})$, and then an approximation argument shows that we can take an arbitrary $\phi \in W^2(\Omega)$. The inequality (7.2) states that the left hand side of (7.3) is ≥ 0 whenever $0 \le \Delta\phi$ on $D(r)$. In particular, we can take $\phi \in W^2(\Omega)$ which solves $\Delta\phi = -1_{\Omega\setminus D(r)}$, and get

$$\int_{\Omega\setminus D(r)} U_r(z)\,d\Sigma(z) = 0, \quad (7.4)$$

which sharpens the above conclusion that U_r vanishes off the interior of the closure of $D(r)$: $U_r = 0$ almost surely on $\Omega \setminus D(r)$. We shall now see that $0 < U_r(z)$ almost surely on $D(r)$. Let E be the subset of $D(r)$ where $U_r(z) = 0$, and suppose for the moment that E has positive area measure: $0 < \Sigma(E)$. Then, since U_r is in $W^{2,\infty}$ away from the origin, all the partial derivatives of U_r of degree less than or equal to 2 vanish (almost surely) on E ([28] p. 53). In particular, $\Delta U_r = 0$ almost surely on E, which contradicts that $\Delta U_r = \omega\, 1_{D(r)} - r^2\delta_0$, as the weight ω was assumed strictly positive. This permits us to make the following choice of the set $D(r)$:

$$D(r) = \big\{z \in \Omega : 0 < U_r(z)\big\}. \quad (7.5)$$

So far, we assumed that the open set $D(r)$ was known, and defined the potential U_r in terms of it. However, it is actually more natural to first get the function U_r from an obstacle problem, and then obtain the set $D(r)$ from the above relation.

Hele-Shaw flow: the obstacle problem model. For $0 < r < +\infty$, let V_r be the function

$$V_r(z) = G\big[r^2\delta_0 - \omega\big](z) = r^2 G(z,0) - \int_\Omega G(z,\zeta)\,\omega(\zeta)\,d\Sigma(\zeta), \qquad z \in \Omega,$$

which is superharmonic on $\Omega \setminus \{0\}$, vanishes on $\partial\Omega$, and has a negative logarithmic singularity at 0. We let $\widehat{V}_r$ denote the least superharmonic majorant to V_r on Ω. The connection with the Hele-Shaw flow is the following identity:

$$\widehat{V}_r(z) = V_r(z) + U_r(z), \qquad z \in \Omega. \tag{7.6}$$

This follows from the treatment of the subject in Björn Gustafsson's paper [12], where a similar obstacle problem was shown to be equivalent to the Hele-Shaw flow; that Gustafsson's obstacle problem is equivalent to the above follows from the treatment in Kinderlehrer-Stampacchia [28]; see also [5]. We refer to the paper [27] for a rather extensive list of references on the problem set: Hele-Shaw flow, quadrature domains, and obstacle problems.

For a set E of complex numbers, we write $E \Subset \Omega$ to indicate that E is a precompact subset of Ω.

The obstacle problem makes sense for all values of the parameter r, $0 < r < +\infty$, allowing us to obtain the function U_r from the formula (7.6). The domains $D(r)$ as given by (7.5) are then well-defined for all r. We classify them as *Hele-Shaw domains* when $D(r) \Subset \Omega$, and as *generalized Hele-Shaw domains* when $D(r)$ is too big for this to happen. A physical interpretation of the generalized Hele-Shaw flow is that the liquid is allowed to stack up on the boundary $\partial D(r) \cap \partial\Omega$.

The space $C^{1,1}(\overline{\Omega})$ consists of all continuously differentiable functions on $\overline{\Omega}$, whose first order partial derivatives are Lipschitz continuous. It coincides with the Sobolev space $W^{2,\infty}(\Omega)$ of functions in $L^\infty(\Omega)$ whose partial derivatives (taken in the distributional sense) of order less than or equal to 2 are also in $L^\infty(\Omega)$. The functions in the latter space may need to be redefined on a set of zero area measure to fit into the first-mentioned space.

PROPOSITION 7.1 *Fix an r, $0 < r < +\infty$. Then the superharmonic envelope function $\widehat{V}_r$ is in $C^{1,1}(\overline{\Omega})$. It assumes the value $\widehat{V}_r = 0$ on $\partial\Omega$.*

Proof. It follows from the results of Chapters 2 and 4 in Kinderlehrer-Stampacchia [28] that $\widehat{V}_r$ is in $W^{2,p}(\Omega)$, for each p, $1 < p < +\infty$; we could also use Gustafsson's argument in [12] to this end. To get the stronger result with $p = +\infty$, we instead appeal to the results of Chapter 1 in Avner Friedman's book [10], or to the paper [5] by Caffarelli and Kinderlehrer. Perhaps a word should be said about why $\widehat{V}_r$ vanishes on $\partial\Omega$. The function $G[-\omega]$ is a superharmonic majorant to V_r, and it vanishes on $\partial\Omega$. The function $\widehat{V}_r$ is sandwiched between V_r and $G[-\omega]$, which both vanish on $\partial\Omega$, and hence $\widehat{V}_r|_{\partial\Omega} = 0$. ■

PROPOSITION 7.2 *Fix an r, $0 < r < +\infty$. Then the set*

$$D(r) = \big\{z \in \Omega : V_r(z) < \widehat{V}_r(z)\big\}$$

is an open and connected subset of Ω. Moreover, $\widehat{V}_r$ is harmonic on $D(r)$.

Proof. The function $\widehat{V}_r - V_r$ is continuous, and hence the set $D(r)$ where it is strictly positive is open. By the Perron process, $\widehat{V}_r$ is harmonic on $D(r)$. For, if it were not harmonic on some small circular disk in $D(r)$, we can replace it on the disk by the harmonic function with the same boundary values on the small circle, and obtain a function that is smaller (by the maximum principle), and still superharmonic on Ω. This new function remains a majorant to V_r if the disk is small enough, in violation of the definition of $\widehat{V}_r$ as the smallest superharmonic majorant to V_r. We turn to the assertion that $D(r)$ is connected. It is clear that the origin is an interior point of $D(r)$, because $V_r(z)$ tends to $-\infty$ as z tends to 0. If $D(r)$ is indeed disconnected, then we can find a connectivity component – call it $D_1(t)$ – which does not contain the origin. As the origin is an interior point of $D(r)$, the connected open set $D_1(r)$ is comfortably at a distance from it. Moreover, we have that $\partial D_1(r) \subset \overline{\Omega} \setminus D(r)$, because if a sequence of points of $D_1(r)$ have a limit point in $D(r)$, well, then all point sufficiently near the limit point are in $D_1(r)$ as well, making the point interior for $D_1(r)$. On $D_1(r)$, $\widehat{V}_r$ is harmonic, and on $\partial D_1(r)$, it equals the function V_r. As V_r is superharmonic on $D_1(r)$ (after all, it is superharmonic on $\Omega \setminus \{0\}$), we obtain from the maximum principle that $\widehat{V}_r \le V_r$ on $D_1(r)$, in clear violation of the definition of the set $D(r)$. ■

PROPOSITION 7.3 *Fix an r, $0 < r < +\infty$. We then have that $\Delta U_r = \omega\, 1_{D(r)} - r^2\delta_0$ on Ω, in the sense of distributions.*

Proof. On $D(r)$, $\widehat{V}_r$ is harmonic, and hence $\Delta U_r = \Delta(\widehat{V}_r - V_r) = -\Delta V_r = \omega - r^2\delta_0$ there. On $\Omega \setminus D(r)$, $\widehat{V}_r$ and V_r coincide, and hence their first and second order derivatives coincide almost everywhere there, in view of [28], p. 53, and the regularity result that $\widehat{V}_r$ is in $W^{2,\infty}(\Omega)$ (Proposition 7.1). In particular, $\Delta U_r = 0$ almost everywhere on $\Omega \setminus D(r)$. ■

The following proposition makes the relationship between the obstacle problem and the Hele-Shaw flow explicit. For a set E of complex numbers, we write $E \Subset \Omega$ to indicate that E is a precompact subset of Ω.

PROPOSITION 7.4 *Fix an r, $0 < r < +\infty$. Then the following assertions are valid.*
(a) Suppose $D(r)$ is the non-coincidence set from the obstacle problem with obstacle V_r, as in Proposition 7.2, and that $D(r) \Subset \Omega$. Then the mean value and sub-mean value properties (7.1) and (7.2) hold for $D(r)$.
(b) Suppose, on the other hand, that $D(r)$ is an open precompact subset of Ω with $0 \in D(r)$ for which (7.1) and (7.2) hold. Then, up to a set of zero area measure, $D(r)$ equals the non-coincidence set from the obstacle problem with obstacle V_r.

Proof. The second assertion, (b), was discussed thoroughly in the introduction of this subsection.

We turn to part (a). The function $U_r = \widehat{V}_r - V_r$ has support set $\overline{D}(r)$, which is compact in Ω, and Proposition 7.3 tells us what its laplacian is. We apply Green's formula to test functions as in (7.3), and obtain

$$\int_{D(r)} \phi(z)\,\omega(z)\,d\Sigma(z) - r^2\phi(0) = \int_\Omega \Delta U_r(z)\,\phi(z)\,d\Sigma(z) = \int_\Omega U_r(z)\,\Delta\phi(z)\,d\Sigma(z),$$

first for functions ϕ that are C^∞-smooth on $\overline{\Omega}$, and then, by approximation, we get it for all $\phi \in W^2(\Omega)$. As we apply this identity to harmonic and subharmonic ϕ on $D(r)$, the properties (7.1) and (7.2) follow. ■

PROPOSITION 7.5 *For r, $0 < r < +\infty$, the function $U_r = \widehat{V}_r - V_r$ increases with the parameter r. Moreover, if the weight ω is increased, U_r decreases, for fixed r. When U_r*

increases, the flow domain $D(r,\omega) = \{z \in \Omega : 0 < U_r(z)\}$ also increases. In particular, $D(r,\omega)$ increases with increasing r, and decreases with increasing weight ω.

Proof. Let r, r' be related as follows: $0 < r < r' < +\infty$. We check that $V_r - V_{r'}$ is superharmonic, so that the function $\widehat{V}_{r'} - V_{r'} + V_r$ is superharmonic, too. The latter function also majorizes V_r, and hence $\widehat{V}_r \le \widehat{V}_{r'} - V_{r'} + V_r$. It follows that U_r increases with r.

A similar argument shows that U_r decreases as the weight ω increases. The details are as follows. Let ω' be a bigger weight than ω: $\omega \le \omega'$ on Ω, and let V_r' be the potential associated with ω': $V_r' = G[r^2\delta_0 - \omega']$. The function $V_r' - V_r$ is then superharmonic, because $\Delta(V_r' - V_r) = \omega - \omega' \le 0$. It follows that the function $\widehat{V}_r - V_r + V_r'$ is superharmonic, too, and it clearly majorizes V_r'. It is immediate that $\widehat{V}_r' \le \widehat{V}_r - V_r + V_r'$, which leads to $U_r' \le U_r$ (obvious notation), as asserted. ∎

In the introduction, we claimed that the Hele-Shaw flow was well-defined for all parameter values of r with $0 < r < \rho(0)$, where $0 < \rho(0) < +\infty$. The critical radius parameter $\rho(0)$ is finite because our confining domain Ω is bounded (apply the reproducing property (7.1) to the constant function $h = 1$). We still need to see that $0 < \rho(0)$, as we define $\rho(0)$ to be the infimum of all r with $\overline{D}(r) \cap \partial\Omega \ne \emptyset$. With this definition, $D(r) \Subset \Omega$ for all r with $0 < r < \rho(0)$, so that $D(r)$ arises from the Hele-Shaw flow for these parameter values. The domains $D(r)$ increase with r, so we just need to check that that $D(r)$ is precompact in Ω at least for one value of r, $0 < r < +\infty$. This is accomplished by the following proposition. We need some notation: for a point $w \in \mathbb{C}$ and a positive real parameter ϱ, we let

$$\mathbb{D}(w,\varrho) = \{z \in \mathbb{C} : |z - w| < \varrho\}$$

denote the open circular disk of radius ϱ about w.

PROPOSITION 7.6 *Let m be the minimum value of ω on $\overline{\Omega}$, and M the maximum value. If r, $0 < r < +\infty$, is so small that the circular disk $\mathbb{D}(0, r/\sqrt{m})$ is contained in Ω, then $D(r)$ is sandwiched as follows: $\mathbb{D}(0, r/\sqrt{M}) \subset D(r) \subset \mathbb{D}(0, r/\sqrt{m})$.*

Proof. This follows from Proposition 7.5, by comparing the weight ω with the constant weights m and M, for which the Hele-Shaw flow consists of circular disks about 0. ∎

It is of interest to see how the choice of underlying domain Ω affects the Hele-Shaw flow domains $D(r)$.

PROPOSITION 7.7 *Fix an r, $0 < r < +\infty$. Let Ω' be an open subset of Ω, containing the origin. Let $\widehat{V}_r'$ denote the least superharmonic majorant to $V_r|_{\Omega'}$ on Ω', and put $D'(r) = \{z \in \Omega' : V_r(z) < \widehat{V}_r'(z)\}$. Then we have in general $\widehat{V}_r' \le \widehat{V}_r|_{\Omega'}$, and $D'(r) \subset D(r) \cap \Omega'$. Conversely, we have the following:*
(a) if $D(r) \subset \Omega'$, then $\widehat{V}_r' = \widehat{V}_r|_{\Omega'}$ and $D'(r) = D(r)$, and
(b) if $D'(r) \Subset \Omega'$, then $\widehat{V}_r' = \widehat{V}_r|_{\Omega'}$ and $D'(r) = D(r)$.

Proof. The assertions that $\widehat{V}_r' \le \widehat{V}_r|_{\Omega'}$ and $D'(r) \subset D(r) \cap \Omega'$ are self-evident in view of the definitions of these objects in terms least superharmonic majorants. We turn to the assertion (a), that we have the equalities $\widehat{V}_r' = \widehat{V}_r|_{\Omega'}$ and $D'(r) = D(r)$ provided that $D(r) \subset \Omega'$. Given that $D(r) \subset \Omega'$, we construct a function $\widetilde{V}_r$ on Ω by setting it equal to $\widehat{V}_r'$ on $D(r)$, and V_r on $\Omega \setminus D(r)$. It is clear that $\widetilde{V}_r \le \widehat{V}_r$ on Ω. The function $\widetilde{V}_r$ equals $\widehat{V}_r'$ on Ω', and is therefore superharmonic there; on $\Omega \setminus \overline{D}(r)$ it is also superharmonic, because

V_r is, at least away from 0. We wish to show that $\widetilde{V}_r$ is superharmonic throughout Ω. It is well known that a function is superharmonic on Ω if we have the appropriate mean value inequality on sufficiently small circles about each point of Ω. We just need to check this for points $z_1 \in (\Omega \setminus \Omega') \cap \overline{D}(r) \subset \partial D(r)$. Let ε, $0 < \varepsilon$, be so small that $\mathbb{D}(z_1, \varepsilon) \Subset \Omega$, and calculate, using the superharmonicity of $\widehat{V}_r$,

$$\frac{1}{\varepsilon} \int_{\partial D(z_1,\varepsilon)} \widetilde{V}_r(z)\, d\sigma(z) \leq \frac{1}{\varepsilon} \int_{\partial D(z_1,\varepsilon)} \widehat{V}_r(z)\, d\sigma(z) \leq \widehat{V}_r(z_1).$$

Since $z_1 \in \partial D(r)$, we have $\widehat{V}_r(z_1) = V_r(z_1)$, whence $\widetilde{V}_r(z_1) = \widehat{V}_r(z_1)$, and the mean value property has been established. The minimality of $\widehat{V}_r$ now forces the equality $\widetilde{V}_r = \widehat{V}_r$. The assertion $D'(r) = D(r)$ is immediate.

The assertion (b) is proved in an analogous fashion. ■

Less smooth obstacles. Suppose for the moment that the weight ω is not as smooth as before, say that we only know it is in $L^p(\Omega)$ for some p, $1 < p < +\infty$, and that $0 \leq \omega$ holds throughout Ω. Let us see what conclusions remain from the previous subsection. Clearly, we can still form the potential function V_r, which is of Sobolev class $W^{2,p}$ away from the origin in Ω, and the superharmonic envelope function $\widehat{V}_r$ can also be formed, and it is, by the same arguments from Kinderlehrer-Stampacchia [28], in $W^{2,p}(\Omega)$. The Sobolev-Morrey imbedding theorem shows that $W^{2,p}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$, for some α, $0 < \alpha < 1$ (in fact, for $1 < p < 2$, we can take $\alpha = 2(p-1)/p$). This means that the defining function $U_r = \widehat{V}_r - V_r$ for the sets $D(r)$ is continuous on $\overline{\Omega} \setminus \{0\}$, and hence that the sets $D(r)$ are open, for all r, $0 < r < +\infty$. Propositions 7.2, 7.3, 7.5, and 7.7 hold without changes. If ω is bounded away from 0 locally around the origin, the comparison argument of Proposition 7.6 shows that $D(r) \Subset \Omega$ for sufficiently small positive r. Proposition 7.4 remains valid, modulo the following modification: in part (b), we need to replace "zero area measure" with "zero mass with respect to the measure $\omega\, d\Sigma$". If we assume that $0 < \omega$ holds area-almost everywhere on Ω, then Proposition 7.4 remains valid as it stands.

Continuity properties of the weighted Hele-Shaw flow. We keep the original context, where Ω is a finitely connected bounded domain in $\mathbb{C}$ with C^∞-smooth boundary, and ω is a C^∞-smooth strictly positive weight on $\overline{\Omega}$. The sets $D(r)$ are obtained from the obstacle problem, which is equivalent to the Hele-Shaw flow, as demonstrated in Proposition 7.4. The following continuity property of the weighted Hele-Shaw flow is basic to our investigations.

PROPOSITION 7.8 *Fix an r, $0 < r < \rho(0)$. To each given ε, $0 < \varepsilon$, there exists a $\delta = \delta(\varepsilon)$, $0 < \delta < \rho(0) - r$, such that if r' is confined to the interval $r < r' < r + \delta$, we have the inclusion*

$$D(r') \subset D(r) + \mathbb{D}(0, \varepsilon) = \big\{z + \zeta : \, z \in D(r),\ \zeta \in \mathbb{D}(0,\varepsilon)\big\}.$$

Proof. For the proof, we shall use a positive ε which is somewhat smaller than the one appearing in the formulation of the proposition; precisely how much smaller will be made evident later. Let $D_\varepsilon(r) = D(r) + \mathbb{D}(0,\varepsilon)$ and $D_{2\varepsilon}(r) = D(r) + \mathbb{D}(0, 2\varepsilon)$ be the correspondingly fattened domains (both are open and connected), and suppose ε is so small that $D_{2\varepsilon}(r) \Subset \Omega$. Moreover, let ϖ_ε stand for harmonic measure (supported on the boundary) for the domain $D_\varepsilon(r)$ with respect to the interior point 0. Then, if u

is a subharmonic function on $D_{2\varepsilon}(r)$, which is continuous on $\overline{D}_{2\varepsilon}(r)$, we have from the sub-mean value property of harmonic measure that

$$u(0) \le \int_{\partial D_\varepsilon(r)} u(z)\, d\varpi_\varepsilon(z). \tag{7.7}$$

Let ψ_ε be a real-valued C^∞-smooth function on $\mathbb{C}$, subject to the following restrictions:

- ψ_ε is radial: $\psi_\varepsilon(z) = \psi_\varepsilon(|z|)$,
- $0 \le \psi_\varepsilon$ throughout $\mathbb{C}$,
- $0 < \psi_\varepsilon(z)$ holds if and only if $z \in \mathbb{D}(0, \varepsilon)$, and
- $\int_{\mathbb{C}} \psi_\varepsilon(z)\, d\Sigma(z) = 1$.

We use the function ψ_ε to mollify the harmonic measure ϖ_ε,

$$\nu_\varepsilon(z) = \psi_\varepsilon * \varpi_\varepsilon(z) = \int_{\partial D_\varepsilon(r)} \psi_\varepsilon(z-\zeta)\, d\varpi_\varepsilon(\zeta), \qquad z \in \mathbb{C},$$

producing a positive C^∞-smooth function with support contained in $\overline{D}_{2\varepsilon}(r) \setminus D(r)$. Let $u \in W^2(\Omega)$ be subharmonic on $D_{2\varepsilon}(r)$; by Sobolev's imbedding theorem, u is continuous on $\overline{\Omega}$. From the sub-mean value property for circles and the radial symmetry of the mollifier ψ_ε, we have that

$$u(\zeta) \le \int_{\mathbb{D}(0,\varepsilon)} u(\zeta+z)\, \psi(z)\, d\Sigma(z) = \int_\Omega u(z)\, \psi(z-\zeta)\, d\Sigma(z), \qquad \zeta \in \overline{D}_\varepsilon(r),$$

whence it follows that

$$\int_{\partial D_\varepsilon(r)} u(\zeta)\, d\varpi_\varepsilon(\zeta) \le \int_{\partial D_\varepsilon(r)} \int_\Omega u(z)\, \psi(z-\zeta)\, d\Sigma(z)\, d\varpi_\varepsilon(\zeta) = \int_\Omega u(z)\, \nu_\varepsilon(z)\, d\Sigma(z).$$

As we combine this with (7.7), we arrive at

$$u(0) \le \int_\Omega u(z)\, \nu_\varepsilon(z)\, d\Sigma(z). \tag{7.8}$$

Note that the mollifier ψ_ε can be assumed to be bounded by $\sup_{\mathbb{C}} \psi_\varepsilon \le 2\,\varepsilon^{-2}$, which leads to the same behavior for ν_ε: $\sup_{\mathbb{C}} \nu_\varepsilon \le 2\,\varepsilon^{-2}$. The weight ω is bounded away from 0 on Ω, and so the function

$$\theta(\varepsilon) = \sup_{z\in\mathbb{C}} \frac{\nu_\varepsilon(z)}{\omega(z)}$$

has the asymptotics $\theta(\varepsilon) = O(\varepsilon^{-2})$ as $\varepsilon \to 0$. The weight

$$\omega_\varepsilon(z) = \omega(z)\, 1_{D(r)}(z) + \theta(\varepsilon)^{-1}\, \nu_\varepsilon(z) + \omega(z)\, 1_{\Omega\setminus D_{2\varepsilon}(r)}(z), \qquad z \in \Omega,$$

is smaller than ω, and in view of (7.8) and the moment inequality property (7.2) of $D(r)$,

$$\big(r^2 + \theta(\varepsilon)^{-1}\big)\, u(0) \le \int_{D_{2\varepsilon}(r)} u(z)\, \omega_\varepsilon(z)\, d\Sigma(z),$$

for all $u \in W^2(\Omega)$ that are subharmonic on $D_{2\varepsilon}(r)$. This is a moment inequality for the weight ω_ε, which shows that for the flow associated with that weight, $D_{2\varepsilon}(r)$ is the Hele-Shaw domain – up to sets of zero mass for $\omega_\varepsilon\, d\Sigma$ – for the radial parameter value $\sqrt{r^2 + \theta(\varepsilon)^{-1}}$. In any case, up to sets of zero area, we have that

$$D\big(\sqrt{r^2 + \theta(\varepsilon)^{-1}}, \omega_\varepsilon\big) \subset D_{2\varepsilon}(r).$$

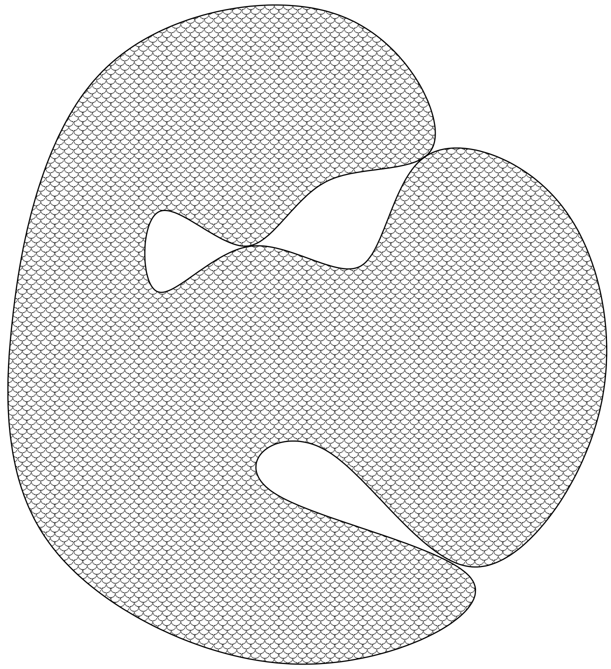

Figure 7.1: A flow domain $D(r)$ with three contact points.

By the comparison principle (Proposition 7.5) – applicable to the less regular weight ω_ε by the remarks of the previous subsection – we have, up to sets of zero area, the inclusion

$$D\big(\sqrt{r^2+\theta(\varepsilon)^{-1}}\big) = D\big(\sqrt{r^2+\theta(\varepsilon)^{-1}}, \omega\big) \subset D\big(\sqrt{r^2+\theta(\varepsilon)^{-1}}, \omega_\varepsilon\big) \subset D_{2\varepsilon}(r).$$

This may not be an actual inclusion, and the reason is that a part of the boundary of $D_{2\varepsilon}(r)$ of zero area might be contained in the set on the left hand side. However, $D_{2\varepsilon}(r) \Subset D_{\varepsilon'}(r)$ whenever $2\varepsilon < \varepsilon'$. The assertion, with ε' in place of ε, is now immediate. ∎

It is a consequence of Proposition 7.8 that the reason why the flow stops at the parameter value $r = \rho(0)$ is that then, the boundary $\partial D(r)$ hits the outer boundary $\partial\Omega$. We also need to know that the flow moves at a positive speed, at least in a situation with fairly regular boundary.

PROPOSITION 7.9 *Fix an r, $0 < r < \rho(0)$, and suppose that the flow domain $D(r)$ is simply connected with C^2-smooth boundary, with the exception of finitely many so-called contact points. Near each contact point, we assume the boundary consists of two C^2-smooth curves tangent to each other at the point, and that $D(r)$ is what remains when we cut out the thin two-sided wedge located between the two curves. To each given δ, $0 < \delta < \rho(0) - r$, there exists an $\varepsilon = \varepsilon(\delta)$, $0 < \varepsilon$, such that – up to sets of zero area – we have the inclusion*

$$D_\varepsilon(r) \subset D(r+\delta),$$

where $D_\varepsilon(r) = D(r) + \mathbb{D}(0,\varepsilon)$.

The geometric situation is illustrated in Figure 7.1.

Proof. For ε with $0<\varepsilon$, note that we can actually write

$$D_\varepsilon(r)=\overline{D}(r)+\mathbb{D}(0,\varepsilon),$$

and suppose ε is so small that $D_\varepsilon(r)\Subset\Omega$. Let ϖ denote the harmonic measure on $\partial D(r)$ corresponding to the domain $D(r)$ and the interior point 0. The C^2-smoothness of $\partial D(r)$ (although somewhat degenerate at the contact points) entails that ϖ is comparable to normalized arc length measure σ, in symbols $\varpi\asymp\sigma|_{\partial D(r)}$, in the sense that there exist real constants A and B, $0<A<B<+\infty$, such that

$$A\,\sigma|_{\partial D(r)}\le\varpi\le B\,\sigma|_{\partial D(r)}.$$

To see this, we can use the conformal invariance of harmonic measure, and the Kellogg-Warschawski theorem on conformal maps ([36], p. 49). Let ψ_ε be the function $\psi_\varepsilon(z)=\varepsilon^{-2}\psi(\varepsilon^{-1}z)$, where

$$\psi(z)=\tfrac{1}{2}\,(1-|z|^2)^{-\frac{1}{2}}\,1_{\mathbb{D}}(z),\qquad z\in\mathbb{C}.$$

The function ψ_ε is positive and supported on $\overline{\mathbb{D}}(0,\varepsilon)$, and it has $L^1(\mathbb{C})$-norm 1; it will serve as a mollifier for our purposes. It should be observed that it is also in $L^p(\mathbb{C})$ for $1<p<2$. We form the convolution

$$\nu_\varepsilon(z)=\psi_\varepsilon*\varpi(z)=\int_{\partial D(r)}\psi_\varepsilon(z-\zeta)\,d\varpi(\zeta),\qquad z\in\mathbb{C},$$

and, as we shall see later, the mollifier ψ_ε is tailored in such a way that the function ν_ε is bounded away from 0 on the open subset $\partial D(r)+\mathbb{D}(0,\varepsilon)$ of Ω, which constitutes a sort of "snake" around $\partial D(r)$; the function ν_ε vanishes off the snake. To be more precise, we have that there exist two real constants A and B (not the same as above), independent of ε, with $0<A<B<+\infty$, such that

$$\frac{A}{\varepsilon}\le\nu_\varepsilon(z)\le\frac{B}{\varepsilon},\qquad z\in\partial D(r)+\mathbb{D}(0,\varepsilon).\tag{7.9}$$

For the moment we shall assume that the above estimate is valid, and indicate how to proceed to obtain the desired assertion. We consider the function

$$\theta(\varepsilon)=\inf\left\{\frac{\nu_\varepsilon(z)}{\omega(z)}:\,z\in D_\varepsilon(r)\setminus D(r)\right\},$$

and observe that $D_\varepsilon(r)\setminus D(r)$ is contained in the snake $\partial D(r)+\mathbb{D}(0,\varepsilon)$, so that by estimate (7.9) and the fact that $0<\inf_\Omega\omega$, this function has the asymptotics $\theta(\varepsilon)=O(\varepsilon^{-1})$ as $\varepsilon\to0$. The weight

$$\omega_\varepsilon(z)=\omega(z)\,1_{D(r)}(z)+\theta(\varepsilon)^{-1}\,\nu_\varepsilon(z)+\omega(z)\,1_{\Omega\setminus D_\varepsilon(r)}(z),\qquad z\in\Omega,$$

is then larger than ω. As in the proof of Proposition 7.8, we have a moment inequality for ν_ε,

$$u(0)\le\int_\Omega u(z)\,\nu_\varepsilon(z)\,d\Sigma(z),$$

valid for all functions $u\in W^2(\Omega)$ that are subharmonic on $D_\varepsilon(r)$. In view of the fact that ν_ε vanishes off the snake $\partial D(r)+\mathbb{D}(0,\varepsilon)$, and hence off $D_\varepsilon(r)$, it follows from the moment inequality (7.2) for $D(r)$ that

$$\left(r^2+\theta(\varepsilon)^{-1}\right)u(0)\le\int_{D_\varepsilon(r)}u(z)\,\omega_\varepsilon(z)\,d\Sigma(z)$$

holds, for all $u \in W^2(\Omega)$ that are subharmonic on $D_\varepsilon(r)$. This is a moment inequality for the weight ω_ε, which shows that for the flow associated with that weight, $D_\varepsilon(r)$ is the Hele-Shaw domain – up to sets of zero area – for the radial parameter value $\sqrt{r^2+\theta(\varepsilon)^{-1}}$. By the comparison principle (Proposition 7.5) – applicable to the less regular weight ω_ε by the remarks of the previous subsection – we have, up to sets of zero area, the inclusion

$$D_\varepsilon(r) = D\big(\sqrt{r^2+\theta(\varepsilon)^{-1}}, \omega_\varepsilon\big) \subset D\big(\sqrt{r^2+\theta(\varepsilon)^{-1}}, \omega\big) = D\big(\sqrt{r^2+\theta(\varepsilon)^{-1}}\big).$$

The assertion of the proposition is immediate from this.

We turn to the technical work of verifying the estimate (7.9). Since $\varpi \asymp \sigma|_{\partial D(r)}$, it suffices to obtain the estimate (7.9) with μ_ε in place of ν_ε, where

$$\mu_\varepsilon(z) = \psi_\varepsilon * \sigma|_{\partial D(r)}(z) = \int_{\partial D(r)} \psi_\varepsilon(z-\zeta)\, d\sigma(\zeta), \qquad z \in \mathbb{C}.$$

Let $\gamma(z,r,\varepsilon)$ be the curve $\{\zeta \in \mathbb{C} : z + \varepsilon\zeta \in \partial D(r)\}$, which is a magnified and translated version of the boundary $\partial D(r)$. A change of variables yields the identity

$$\mu_\varepsilon(z) = \frac{1}{2\varepsilon}\int_{\mathbb{D}\cap\gamma(z,r,\varepsilon)} \frac{d\sigma(\zeta)}{(1-|\zeta|^2)^{\frac12}}, \qquad z \in \mathbb{C},$$

with the usual agreement that the integral over the empty set is 0. The requirement that $z \in D(r) + \mathbb{D}(0,\varepsilon)$ is equivalent to having $\mathbb{D}\cap\gamma(z,r,\varepsilon) \neq \emptyset$, so that we need to show that

$$A \le \int_{\mathbb{D}\cap\gamma(z,r,\varepsilon)} \frac{d\sigma(\zeta)}{(1-|\zeta|^2)^{\frac12}} \le B \tag{7.10}$$

holds for some positive constants A, B, whenever the integration is over a non-empty set. The set $\mathbb{D}\cap\gamma(z,r,\varepsilon)$ then consists of finitely many curve segments, each of which enters at some point of $\mathbb{T}$ and exits at another. We need only be concerned with very small ε, in which case the curve segments of $\mathbb{D}\cap\gamma(z,r,\varepsilon)$ are pretty much straight lines, being blow-ups of C^2-smooth curves. As a matter of fact, unless we are blowing up near a contact point, there is only one curve, and near a contact point, we have two, so "finitely many" can be replaced by "one or two". If there are two curves, it is enough to obtain an estimate (7.10) for each of them, so it is enough to treat the case of a single curve segment. The curvature of the curve segment $\gamma^\sharp = \mathbb{D}\cap\gamma(z,r,\varepsilon)$ is uniformly of the size $O(\varepsilon)$ as $\varepsilon \to 0$. We recall that if we parametrize $\mathbb{D}\cap\gamma(z,r,\varepsilon)$ by $\zeta = \gamma(t)$, where t runs over a bounded open interval I of $\mathbb{R}$, in such a way that the parametrization is at constant speed 1, $|\dot\gamma(t)| \equiv 1$, then the curvature is expressed by $|\ddot\gamma(t)|$ (we use dots to indicate differentiation with respect to t). We fix the parameter interval $I =]t_0, t_1[\subset \mathbb{R}$ by requiring that $t_0 < 0 < t_1$ and that $\gamma(0) = \min_{t\in I}|\gamma(t)|$ (there is only one point on $\gamma^\sharp$ of minimal distance to 0, at least for small ε). We calculate

$$\frac{d^2}{dt^2}\left(|\gamma(t)|^2\right) = 2\,|\dot\gamma(t)|^2 + 2\,\mathrm{Re}\,\ddot\gamma(t)\bar\gamma(t) = 2 + 2\,\mathrm{Re}\,\ddot\gamma(t)\bar\gamma(t), \qquad t \in]t_0, t_1[,$$

where the right hand side is $2 + O(\varepsilon)$ as $\varepsilon \to 0$. Hence, for small ε,

$$\frac32 \le \frac{d^2}{dt^2}\left(|\gamma(t)|^2\right) \le \frac52, \qquad t \in]t_0, t_1[.$$

The first derivative of $|\gamma(t)|^2$ vanishes for $t = 0$, so that an integration of the previous estimate yields

$$\frac32\, t \le \frac{d}{dt}\left(|\gamma(t)|^2\right) \le \frac52\, t, \qquad t \in [0, t_1[,$$

with an analogous estimate in the remaining interval $]t_0, 0]$. At the right endpoint t_1, the curve intersects the unit circle, and we have $|\gamma(t_1)| = 1$. Another integration starting from t_1 gives as result that

$$\frac{3}{4}(t_1^2 - t^2) \le 1 - |\gamma(t)|^2 \le \frac{5}{4}(t_1^2 - t^2), \qquad t \in [0, t_1].$$

We have an analogous estimate on the remaining interval $[t_0, 0]$. Since

$$\int_0^{t_1} \frac{dt}{(t_1^2 - t^2)^{\frac{1}{2}}} = \int_{t_0}^{0} \frac{dt}{(t_0^2 - t^2)^{\frac{1}{2}}} = \int_0^1 \frac{dt}{(1 - t^2)^{\frac{1}{2}}} = \tfrac{1}{2}\pi,$$

it follows that the integral expression

$$\int_{\gamma^\sharp} \frac{d\sigma(\zeta)}{(1 - |\zeta|^2)^{\frac{1}{2}}} = \int_{t_0}^{t_1} \frac{dt}{(1 - |\gamma(t)|^2)^{\frac{1}{2}}}$$

is kept between $2\pi/\sqrt{5}$ and $2\pi/\sqrt{3}$, which does it. ■

The local existence of a Schwarz function. We now suppose in addition to the previous context that the weight ω is real analytic on Ω. This will allow us to show that the boundary $\partial D(r)$ is much better behaved than if we kept the C^∞-smoothness. To be more precise, we shall establish the local existence of a *Schwarz function*: for each point $z_1 \in \Omega \cap \partial D(r)$, there is an open neighborhood $N(z_1)$ and a continuous function $S : N(z_1) \cap \overline{D}(r) \to \mathbb{C}$ which is holomorphic on $N(z_1) \cap D(r)$, such that $S(z) = \bar{z}$ on $N(z_1) \cap \partial D(r)$. The function S is the local Schwarz function. The weight ω, being real analytic, has a convergent power series expansion near z_1:

$$\omega(z) = \sum_{m,n=0}^{\infty} \widehat{\omega}(m,n)\,(z - z_1)^m (\bar{z} - \bar{z}_1)^n.$$

Let

$$W(z) = \sum_{m,n=0}^{\infty} \frac{\widehat{\omega}(m,n)}{(m+1)(n+1)}\,(z - z_1)^{m+1} (\bar{z} - \bar{z}_1)^{n+1}$$

for z near z_1, and observe that $\Delta W(z) = \omega(z)$ there. By Proposition 7.1, the function $\partial_z U_r(z)$ is of regularity class $C^{0,1}$ away from 0 on $\overline{\Omega}$, in particular on $N(z_1)$, provided the given neighborhood is small. Let R be the function

$$R(z) = \bar{z}_1 + \frac{1}{\omega(z_1)}\,\partial_z\big(W(z) - U_r(z)\big),$$

which is well defined near z_1 and of regularity class $C^{0,1}$ there. The $\bar{\partial}$ derivative of R is, in view of Proposition 7.3,

$$\bar{\partial}_z R(z) = \frac{1}{\omega(z_1)}\,\bar{\partial}_z \partial_z\big(W(z) - U_r(z)\big) = \frac{1}{\omega(z_1)}\,\Delta_z\big(W(z) - U_r(z)\big) = \frac{\omega(z)}{\omega(z_1)}\,1_{\Omega\setminus D(r)}(z),$$

for z near z_1. In particular, if $N(z_1)$ is a small neighborhood of z_1, the function R is holomorphic on $D(r) \cap N(z_1)$. The function U_r vanishes on $\overline{\Omega} \setminus D(r)$, and as 0 is the lowest value of this function in $C^{1,1}$ (away from the origin), the gradient ∇U_r must vanish at all interior points, that is, on the set $\Omega \setminus D(r)$. It follows that $\partial_z U_r(z) = 0$ on $\Omega \setminus D(r)$, so that

$$R(z) = \bar z_1 + \frac{1}{\omega(z_1)}\,\partial_z W(z) = \bar z_1 + \frac{1}{\omega(z_1)} \sum_{m,n=0}^{\infty} \frac{\widehat{\omega}(m,n)}{n+1}\,(z-z_1)^m (\bar z - \bar z_1)^{n+1}$$
$$= \bar z_1 + \bar z - \bar z_1 + O(|z-z_1|^2) = \bar z + O(|z-z_1|^2), \quad (7.11)$$

where $O(|z-z_1|^2)$ stands for a real analytic function of the given magnitude. Let us write $T(z,\bar z)$ for the real analytic function near z_1 expressed by the right hand side of (7.11), with notation that emphasizes the separate dependence of z and $\bar z$; we think of T as a holomorphic function of two complex variables near $(z_1, \bar z_1) \in \mathbb{C}^2$. We recapture what we know about the function R: for some small neighborhood $N(z_1)$ of $z_1 \in \Omega \cap \partial D(r)$, we have that R is Lipschitz continuous there; moreover, on $N(z_1) \cap D(r)$, R is holomorphic, and on $N(z_1) \setminus D(r)$, $R(z) = T(z,\bar z) = \bar z + O(|z-z_1|^2)$. By the implicit function theorem, if $N(z_1)$ is small, there exists a Lipschitz continuous function S on $N(z_1)$ such that $R(z) = T(z, S(z))$, which is then holomorphic on $N(z_1) \cap D(r)$. This is the sought-after Schwarz function. The criterion that allows us to invoke the implicit function theorem is that $\bar\partial_z T(z_1, \bar z_1) = 1 \neq 0$.

In his Acta paper [42], Makoto Sakai mentions that the above construction of a Schwarz function is possible. We have merely filled in the details.

Sakai's work on the regularity of boundaries with a Schwarz function. Sakai shows in [42] that in view of the previous subsection, we have a classification of the points of $\partial D(r) \cap \Omega$. There are:
• *isolated points*,
• *regular boundary points*, where nearby $\partial D(r)$ is a real analytic curve, and $D(r)$ is situated on one side of the curve,
• *interior regular boundary points*, where nearby $\partial D(r)$ is an infinite (closed) subset of a real analytic curve, and $D(r)$ is on both sides of the curve,
• *regular contact points*, where nearby $\partial D(r)$ consists of two real analytic curves, tangent to each other at the point, and $D(r)$ is the complement of the thin wedgelike set between the curves,
• *cusp points*, where nearby $\partial D(r)$ is the image of a real analytic curve under a second degree polynomial mapping, which is such that it produces a cusp at the point in question; the set $D(r)$ is located to the one side of the cusp, with the cusp pointing inward toward $D(r)$.

We shall apply this to the case $D(r) \Subset \Omega$. Then $\partial D(r) \subset \Omega$, and Sakai's classification applies to all boundary points. It is certainly possible for $D(r)$ to be multiply connected, but the holes have to be pretty well-behaved. In fact, there may be at most finitely many holes with nonempty interior, and the rest of the holes are of the types finitely many (subsets of) interior real analytic arcs, and finitely many isolated points (the ones not accounted for already). See Figure 7.2 for an illustration of what $D(r)$ may look like.

Logarithmically subharmonic weights. Assume in addition to the previous setting – with $\partial\Omega$ C^∞-smooth and ω strictly positive and C^∞-smooth on $\overline{\Omega}$ and real analytic on Ω – that Ω is simply connected and ω logarithmically subharmonic. We shall show that then whenever $D(r) \Subset \Omega$, then $D(r)$ *is simply connected and has a boundary that is a real analytic Jordan curve.*

The following lemma will prove instrumental.

LEMMA 7.10 *Let Y be a C^∞-smooth real-valued function on $\overline{\mathbb{D}} \setminus \{0\}$, with a logarithmic singularity at the origin, such that $\Delta^2 Y = \Delta\delta_0 - \mu$ on $\mathbb{D}$, where $\mu \in C^\infty(\overline{\mathbb{D}})$ has $0 \le \mu$*

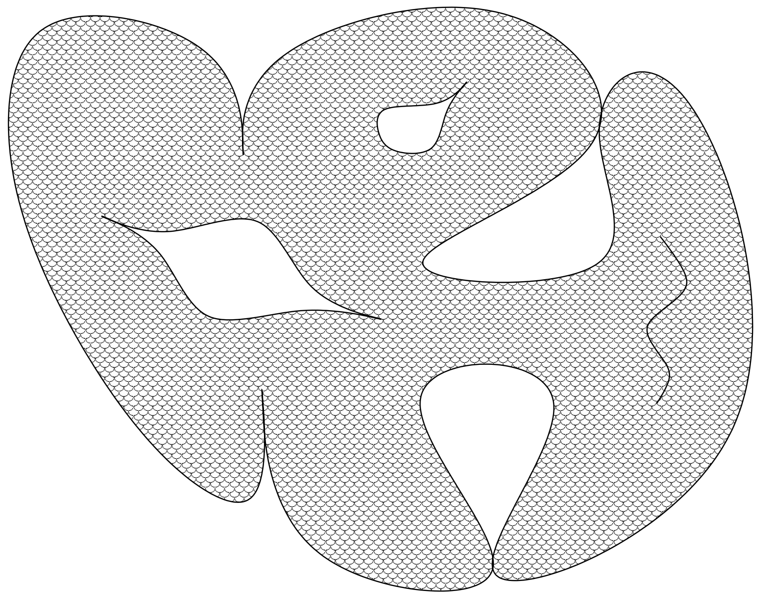

Figure 7.2: A generic flow domain $D(r)$ according to Sakai

on $\mathbb{D}$. *Suppose that* $Y|_{\mathbb{T}}=0$, *and that*

$$\frac{\partial}{\partial n}Y \le 0 \quad \text{on } \mathbb{T},$$

where the normal derivative is in the interior direction. Then

$$Y(z) \le \log|z|^2 + 1 - |z|^2 < 0, \qquad z \in \mathbb{D}.$$

Proof. We use the biharmonic Green function

$$\Gamma(z,\zeta) = |z-\zeta|^2 \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2 + (1-|z|^2)(1-|\zeta|^2), \qquad (z,\zeta)\in\mathbb{D}^2,$$

and the associated harmonic compensator function

$$H(z,\zeta) = (1-|\zeta|^2)\,\frac{1-|z\zeta|^2}{|1-z\bar{\zeta}|^2}, \qquad (z,\zeta)\in\overline{\mathbb{D}}\times\mathbb{D},$$

to represent the function Y:

$$Y(z) = \int_{\mathbb{D}} \Gamma(z,\zeta)\,\Delta^2 Y(\zeta)\,d\Sigma(\zeta) + \frac{1}{2}\int_{\mathbb{T}} H(\zeta,z)\,\partial_n Y(\zeta)\,d\sigma(\zeta),$$

where we condensed the notation for the normal derivative, and think of the integration in the sense of distributions. As the harmonic compensator is positive, the expression on the right hand side only gets larger if we drop the second term:

$$\begin{aligned} Y(z) &\le \int_{\mathbb{D}} \Gamma(z,\zeta)\,\big(\Delta\delta_0(\zeta)-\mu(\zeta)\big)\,d\Sigma(\zeta) \\ &= \Delta_\zeta\Gamma(z,0) - \int_{\mathbb{D}}\Gamma(z,\zeta)\,\mu(\zeta)\,d\Sigma(\zeta) \le \Delta_\zeta\Gamma(z,0) = \log|z|^2+1-|z|^2, \end{aligned}$$

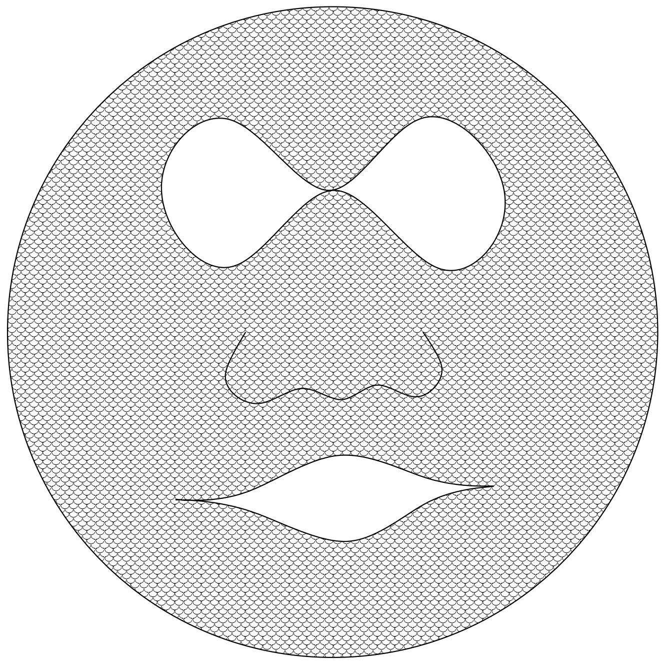

Figure 7.3: The mapped flow domain B

where we have also used the fact that Γ is positive. The proof is complete. ■

Logarithmically subharmonic weights: ruling out holes and cusps. We now assume Ω to be *simply connected*, and fix an r, $0 < r < \rho(0)$. Then $D(r) \Subset \Omega$, and Sakai's classification applies to all boundary points of $D(r)$. Let $\widehat{D}(r)$ stand for the simply connected domain obtained from $D(r)$ by adding all the interior holes, both the ones with nontrivial interior and the ones that are parts of arcs as well as the isolated points. Then the boundary $\partial\widehat{D}(r)$ is a closed real analytic curve, with the exception of finitely many contact and cusp points. Let $\phi : \mathbb{D} \to \widehat{D}(r)$ be the Riemann mapping which sends 0 onto 0, the "center" of the generalized disk $D(r)$ (it is unique up to rotations of $\mathbb{D}$). By the regularity of the boundary, ϕ extends analytically to a neighborhood of $\overline{\mathbb{D}}$, with the cusp points corresponding to simple zeros of ϕ'. We then consider the domain $B = \phi^{-1}(D(r))$, and note that $\mathbb{D} \setminus B$ is a compact subset of $\mathbb{D}$. In general, B may look like what is illustrated by Figure 7.3. *We shall prove that* $B = \mathbb{D}$. Introducing the weight $\nu = r^{-2}\,\omega \circ \phi\,|\phi'|^2$, which is logarithmically subharmonic on $\mathbb{D}$, we obtain from (7.1) that

$$h(0) = \int_B h(z)\,\nu(z)\,d\Sigma(z), \tag{7.12}$$

for all harmonic functions on B of the form $h = g \circ \phi$, with $g \in W^2(\Omega)$ harmonic on $D(r)$. We also have the corresponding inequality for subharmonic functions. The new weight ν is real analytic on $\overline{\mathbb{D}}$, with zeros at the finitely many points of $\mathbb{T}$ corresponding to the cusps; elsewhere, it is strictly positive.

We can interpret the domain B as appearing from an obstacle problem. After all, the function $V_r \circ \phi$ has a least superharmonic majorant on $\mathbb{D}$, and one checks that $\widehat{V}_r \circ \phi$ is that majorant (by Proposition 7.7, with $\Omega' = \widehat{D}(r)$, we have that on $\widehat{D}(r)$, $\widehat{V}_r$ is the least superharmonic majorant to V_r; the conformal invariance of the operation of taking the least superharmonic majorant then proves the claim). We calculate that

$$\Delta(V_r \circ \phi)(z) = \Delta V_r(\phi(z))\,|\phi'(z)|^2 = r^2\delta_0(z) - \omega\circ\phi(z)\,|\phi'(z)|^2, \qquad z \in \mathbb{D},$$

so that if we define

$$W(z) = G[\delta_0 - \nu](z) = \log|z|^2 - \int_{\mathbb{D}} G(z,\zeta)\,\nu(\zeta)\,d\Sigma(\zeta), \qquad z \in \mathbb{D},$$

where G refers to the Green potential operator and kernel function for the disk $\mathbb{D}$,

$$G(z,\zeta) = G_{\mathbb{D}}(z,\zeta) = \log\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right|^2,$$

we obtain $W = r^{-2}\,V_r\circ\phi - r^{-2}\,P[V_r\circ\phi|_{\mathbb{T}}]$, where in general, $P[f]$ expresses the harmonic extension to $\mathbb{D}$ via the Poisson integral formula of the function f given on the boundary $\mathbb{T}$. Let $\widehat{W}$ stand for the least superharmonic majorant to W on $\mathbb{D}$; by the above, $\widehat{W} = r^{-2}\,\widehat{V}_r\circ\phi - r^{-2}\,P[V_r\circ\phi|_{\mathbb{T}}]$.

For $1 \le t < +\infty$, let

$$W_t(z) = G[t\,\delta_0 - \nu](z) = t\log|z|^2 - \int_{\mathbb{D}} G(z,\zeta)\,\nu(\zeta)\,d\Sigma(\zeta), \qquad z \in \mathbb{D},$$

and let $\widehat{W_t}$ be its least superharmonic majorant on $\mathbb{D}$. We are interested in the open sets

$$B(t) = \left\{z \in \mathbb{D} : W_t(z) < \widehat{W_t}(z)\right\}.$$

By Proposition 7.2, the set $B(t)$ is connected for each t, and by Proposition 7.5, it increases with t. The left end point $t = 1$ corresponds to dropping the parameter: $W_1 = W$ and $B(1) = B$. The function W_t vanishes on the unit circle $\mathbb{T}$. If $W_t(z) \le 0$ throughout $\mathbb{D}$, then the least superharmonic majorant is trivially $\widehat{W_t}(z) \equiv 0$. In this case, we also have $\partial_n W_t(z) \le 0$ on $\mathbb{T}$ (interior normal derivative). Lemma 7.10 provides a converse to this statement: since $\Delta^2 W_t = t\,\Delta\delta_0 - \Delta\nu$, and ν is subharmonic, we obtain from $\partial_n W_t|_{\mathbb{T}} \le 0$ that

$$W_t(z) \le t\left(\log|z|^2 + 1 - |z|^2\right) < 0, \qquad z \in \mathbb{D}; \tag{7.13}$$

it follows that $B(t) = \mathbb{D}$ in this case. In other words,

- $\sup_{\mathbb{D}} W_t \le 0$ if and only if $\sup_{\mathbb{T}} \partial_n W_t \le 0$, and
- if $\sup_{\mathbb{T}} \partial_n W_t \le 0$, then $W_t < 0$ on $\mathbb{D}$, and $B(t) = \mathbb{D}$.

If $\sup_{\mathbb{T}} \partial_n W_t \le 0$ holds for $t = 1$, then we are done, for then $B = B(1) = \mathbb{D}$. So, let us suppose instead that $0 < \sup_{\mathbb{T}} \partial_n W_1$. The formula defining W_t yields

$$\partial_n W_t(z) = -2(t-1) + \partial_n W_1(z), \qquad z \in \mathbb{T},$$

and since $\partial_n W_1$ is in $C^\infty(\mathbb{T})$ and real-valued, there exists a critical value $t = t_1$, $1 < t_1 < +\infty$, such that $0 < \sup_{\mathbb{T}} \partial_n W_t$ for $1 \le t < t_1$ and $\sup_{\mathbb{T}} \partial_n W_t \le 0$ for $t_1 \le t < +\infty$; in fact, $t_1 = 1 + \frac{1}{2}\sup_{\mathbb{T}} \partial_n W_t$.

The set $\mathbb{D}\setminus B(t)$ is called the *coincidence set* for the obstacle problem, and each point of $\mathbb{D}$ where the smallest concave majorant of W_t touches the graph of the function definitely belongs to this set. In particular, any point of $\mathbb{D}$ where W_t attains its maximum is in $\mathbb{D}\setminus B(t)$. Since we know that $\mathbb{D}\setminus B \Subset \mathbb{D}$ from Sakai's classification, and $\mathbb{D}\setminus B(t)$ gets smaller as t increases, it follows that any such maximum point is in the compact $\mathbb{D}\setminus B$. For each t with $1 \le t < t_1$, the function W_t attains a positive maximum on $\mathbb{D}$. Let us say that the maximum is attained at the interior point $z(t)$, which then must belong to $\mathbb{D}\setminus B$. We choose a sequence τ_j, $j = 1, 2, 3, \ldots$, with $1 < \tau_j < t_1$, and limit $\tau_j \to t_1$ as $j \to +\infty$. The points $z_j = z(\tau_j)$ are in $\mathbb{D}\setminus B$, and a subsequence of them tends to a point $z_\infty \in \mathbb{D}\setminus B$ (by compactness). We have that $0 < W_{\tau_j}(z_j)$, so that in the limit $0 \le W_{t_1}(z_\infty)$. But for $t = t_1$, (7.13) holds, which does not permit such a point z_∞ to exist. So, something must be wrong, and that something is the assumption that B was not all of $\mathbb{D}$.

So we now know that $B = \mathbb{D}$, and hence that $D(r)$ is simply connected. We shall now demonstrate that the boundary of $D(r)$ fails to have cusp points. We also know that (7.12) holds for all h of the form $h = g \circ \phi$, with $g \in W^2(\Omega)$ harmonic on $D(r)$. If h is a C^∞-smooth function on $\overline{\mathbb{D}}$, *holomorphic* on $\mathbb{D}$, with the property that it is totally flat (meaning that all derivatives of finite order vanish) at the finitely many points of $\mathbb{T}$ which are mapped onto cusp or contact points under ϕ, then h is of the above form $g \circ \phi$, with a *holomorphic* g. The identity (7.12) is preserved under closure with respect to the norm of $L^1(\mathbb{D})$, and since the above-mentioned class of h is dense in the Bergman space $A^1(\mathbb{D})$, we have (7.12) for all $h \in A^1(\mathbb{D})$. Taking complex conjugates in (7.12), and forming sums, another approximation process yields that (7.12) holds for all h in the harmonic Bergman space $HL^1(\mathbb{D})$. In other words, ν is a *reproducing weight for the origin*. It follows from Remark 6.9 that $1 \le \nu|_{\mathbb{T}}$, and, in particular, that ν has no zeros on $\mathbb{T}$. Consequently, *the simply connected region $D(r) = \widehat{D}(r)$ cannot have any cusps.*

Logarithmically subharmonic weights: ruling out contact points. We turn to the problem of showing that for logarithmically subharmonic weights ω, we do not even have contact points in Sakai's classification of the boundaries $\partial D(r)$ arising from the weighted Hele-Shaw flow.

We continue to work within the set-up of the previous subsection, where we were able to rule out the possibility of holes in the Hele-Shaw domain $D(r)$ for $0 < r < \rho(0)$. Fix an r with $0 < r < \rho(0)$, and note that by the results of the previous subsection, $D(r)$ is simply connected and the boundary $\partial D(r)$ consists of a real analytic curve, with the possible exception of finitely many regular contact points, where locally, the boundary is the union of two real analytic curves tangential to each other at the point, and $D(r)$ is situated on both sides of the thin wedge defined by the two curves. We refer to Figure 7.1 for an illustration of the situation. We claim that *even for slightly larger r', $r < r' < \rho(0)$, the set $D(r')$ will have merged at all the contact points, and that therefore, $D(r')$ cannot be simply connected anymore.* But $D(r')$ has to be simply connected just as $D(r)$, again by the results of the previous subsection, which *makes it impossible for the set $D(r)$ to have contact points to begin with.* We turn to the claim. According to Proposition 7.8, the flow domain $D(r')$ is just a little bigger than $D(r)$ for r' close to r, and the only place where topological changes are possible is in small neighborhoods of the set of contact points. On the other hand, by Proposition 7.9, $D(r')$ does indeed contain a whole little neighborhood of each contact point, at least up to sets of zero area measure. But in view of the known regular nature of the boundary of $D(r')$, we can remove the *proviso*. It remains to show that if the set of contact points for $D(r)$ is non-empty, the domain $D(r')$ now has to possess holes. Let us study an *outer contact point*, where to the one side, we have the unbounded component of $\mathbb{C} \setminus D(r)$, and, to the other, a bounded one. As the contact point fuses, at least part of the bounded component remains, and it now is a hole, because there is no longer a path to the unbounded component. The existence of holes in $D(r')$ is immediate.

In conclusion, we have obtained the following theorem.

THEOREM 7.11 *Let Ω be a bounded domain in $\mathbb{C}$, with C^∞-smooth boundary, and let ω be a C^∞-smooth strictly positive weight function on $\overline{\Omega}$. Let the domains $D(r)$ be defined in terms of least superharmonic majorants, for $0 < r < +\infty$. Then there is a* *number $\rho(0)$, $0<\rho(0)<+\infty$, such that $D(r)\Subset\Omega$ for all r, $0<r<\rho(0)$, and such that this fails for all r, $\rho(0)\le r<+\infty$. The domains $D(r)$ increase continuously with r for $0<r<\rho(0)$, and as $r\to\rho(0)$, a portion of $\partial D(r)$ approches $\partial\Omega$. If in addition Ω is simply connected, and ω is real analytic and logarithmically subharmonic on Ω, then for each r, $0<r<\rho(0)$, the domain $D(r)$ is simply connected, and its boundary $\partial D(r)$ is a real analytic Jordan curve.*

Hele-Shaw flow domains and ω-disks. Fix an r, $0<r<\rho(0)$, and for convenience of notation, drop for the remainder of this subsection the indication of r: we write U and D in place of U_r and $D(r)$. Suppose D^* is a precompact subdomain of Ω containing the origin, such that the reproducing property

$$r^2\,h(z_0)=\int_{D^*}h(z)\,\omega(z)\,d\Sigma(z)$$

holds for all bounded harmonic functions h on D^*. We need to show that $D^*=D$, where we allow the two sets to differ by an area-null set. We proceed as in [13], recalling that D is simply connected and has smooth boundary. We shall prove that after a slight regularization of D^*, $\partial D^*\subset\overline{D}$, which leads to the conclusion that $D^*\subset D$. From the observation that D^* and D both have $\omega\,d\Sigma$-area r^2, the desired equality $D^*=D$ readily follows. We form the potential function

$$U^*(z)=\int_{D^*}G(z,\zeta)\,\omega(\zeta)\,d\Sigma(\zeta)-r^2G(z,0),\qquad z\in\Omega,$$

and observe that it is of class $C^{1,\alpha}$ away from 0, for each α, $0<\alpha<1$. In contrast with the function U, U^* may attain negative values. The function U^* vanishes on $\overline{\Omega}\setminus D^*$. To refine this conclusion, we can run the analogue of the argument surrounding (7.3) with a ϕ such that $\Delta\phi$ equals the sign of U^* on $\Omega\setminus D^*$ (the sign function takes on the values $1,-1,0$) and vanishes on D^*. The conclusion is that $U^*=0$ area-almost everywhere on $\Omega\setminus D^*$. As a consequence, $\nabla U^*=0$ holds area-almost everywhere on $\Omega\setminus D^*$. The set where at least one of U^* and ∇U^* differs from 0 is an open subset of the closure of D^*, and if we include it in D^*, we have at most increased D^* by an area-null set. So let us do that, and notice that D^* still is connected and precompact. The difference function U^*-U is subharmonic on D^*; in fact,

$$\Delta\left(U^*(z)-U(z)\right)=\omega(z)\left(1_{D^*}(z)-1_D(z)\right),\qquad z\in\Omega,$$

holds in the sense of distributions. It is negative on the boundary, for U^* vanishes there:

$$U^*(z)-U(z)=-U(z)\le 0,\qquad z\in\partial D^*.$$

By the maximum principle, therefore, $U^*-U\le 0$ on D^*. In particular, $U^*\le 0$ on $D^*\setminus D$. We also know that $U^*=0$ on $\Omega\setminus D^*$. Suppose we can find a point $z_1\in\partial D^*\setminus\overline{D}$. Then $U^*(z_1)=0$ and $U^*\le 0$ in some small neighborhood $N(z_1)$ of z_1. The function U^* is subharmonic on $\Omega\setminus\{0\}$, in particular on $N(z_1)$, and hence, by the maximum principle, $U^*=0$ on $N(z_1)$. But then $\Delta U^*=0$ on $N(z_1)$ as well, which does not square with the definition of U^*, according to which $\Delta U^*=\omega\,1_{D^*}-r^2\delta_0$. The contradiction obtained shows that $\partial D^*\subset\overline{D}$. The argument is complete.

Classical Hele-Shaw flow. We still need to understand analytically how fast the simply connected domains $D(r)$ grow with the parameter r. Let r,r' be related as follows: $0<r'<r<\rho(0)$. Then $D(r')\subset D(r)\Subset\Omega$, and according to (7.1), we have

$$h(0)=\frac{1}{r^2-(r')^2}\int_{D(r)\setminus D(r')}h(z)\,\omega(z)\,d\Sigma(z),$$

first for all harmonic functions h on $D(r)$ with extensions to $W^2(\Omega)$, and after an approximation argument, using the structure of $D(r)$ obtained in Theorem 7.11, for all bounded harmonic functions h on $D(r)$. As $r' \to r$, the area of the difference set $D(r) \setminus D(r')$ tends to zero, and we expect the measure

$$\frac{1}{r^2 - (r')^2} 1_{D(r)\setminus D(r')}(z)\,\omega(z)\,d\Sigma(z) \tag{7.14}$$

to converge in the weak-star topology of measures to a probability measure ϖ_r supported on $\partial D(r)$ with the reproducing property

$$h(0) = \int_{\partial D(r)} h(z)\,d\varpi_r(z),$$

for all bounded harmonic functions h on $D(r)$. Luckily, there is only one measure with this property, namely *harmonic measure*. More precisely, ϖ_r must be the harmonic measure for the point 0 with respect to the domain $D(r)$. Let us make the simplifying assumption that $D(r)$ is simply connected, so that there exists a Riemann map $\phi_r : \mathbb{D} \to D(r)$ taking 0 to 0. By the invariance of harmonic measure, $d\varpi_r(\phi_r(z))$ is the harmonic measure for the origin with respect to the unit disk, which is known to coincide with $d\sigma(z)$, normalized arc length measure on $\mathbb{T}$. It follows from the above considerations that for r' close to r, $D(r)$ should be a simply connected domain, too, such that $\mathbb{D} \setminus \phi_r^{-1}(D(r'))$ is an annular band of variable width – with outer boundary $\mathbb{T}$ – the width at a given point $\zeta \in \mathbb{T}$ being given by $(r - r')\,\varrho_r(\zeta) + o(r - r')$, where the second term expresses a small error, and the local width function $\varrho_r(\zeta)$ is expressed by

$$\varrho_r(\zeta) = \frac{r}{\omega(\phi_r(\zeta))\,|\phi_r'(\zeta)|^2}, \qquad \zeta \in \mathbb{T}.$$

The classical Hele-Shaw problem – the terminology is from Richardson [38] – involves having the constant weight $\omega(z) \equiv 1$, as it corresponds to having constant distance in the narrow channel between the two confining plates in the physical model. As such it was treated by Vinogradov and Kufarev [49]. The method is to use the formulas for conformal mappings of nearly circular domains, as described in Nehari's book ([33], pp. 263–265), which, by the way, relies on Hadamard's variational formula for the Green function for the laplacian [14]. One obtains the integro-differential equation (this involves deciding more precisely which conformal mapping ϕ_r we want, because the condition that $\phi_r(0) = 0$ only specifies ϕ_r uniquely up to rotation of $\mathbb{D}$)

$$\frac{d\phi_r}{dr}(z) = rz\,\phi_r'(z) \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\,\frac{d\sigma(\zeta)}{\omega(\phi_r(\zeta))\,|\phi_r'(\zeta)|^2}, \qquad z \in \mathbb{D}. \tag{7.15}$$

Some terminology: a map $\phi : E \to F$, where E and F are closed subsets of $\mathbb{C}$, is said to be *conformal* if $\phi(E) = F$ and ϕ extends to a conformal map from a neighborhood of E onto a neighborhood of F. If we prefer not to mention the set F, we simply say that ϕ is conformal on E.

Vinogradov and Kufarev [49] obtained the result below in a special case (when the weight ω is the square of the modulus of the derivative of a conformal mapping on $\overline{\mathbb{D}}$). The more modern approach, due to Reissig and von Wolfersdorfer [37], appeals to the Nishida-Nirenberg nonlinear Cauchy-Kovalevskaya theorem, and it is more easily modified to lead the statement below.

THEOREM 7.12 *Suppose that the weight ω is defined and strictly positive in a neigborhood of the closed unit disk $\overline{\mathbb{D}}$, and that it is C^∞-smooth on $\mathbb{D}$, and real analytic* *on a neighborhood of $\mathbb{T}$. Let us take as initial function for $r = 1$ the identity mapping $\phi_1(z) = z$. Then there exists a small interval $I(\varepsilon) =]1-\varepsilon, 1+\varepsilon[$, with $0 < \varepsilon$, such that the following is true. There is a solution ϕ_r of (7.15) for $r \in I(\varepsilon)$, with the property that the mapping $(r, z) \mapsto \phi_r(z)$ extends to a holomorphic function of two complex variables on a neighborhood of $I(\varepsilon) \times \overline{\mathbb{D}}$. Moreover, for $r \in I(\varepsilon)$, the mappings ϕ_r are all conformal from $\overline{\mathbb{D}}$ onto $\overline{D}(r)$, where $D(r)$ is a simply connected domain $D(r)$ whose boundary is a real analytic Jordan curve. For each pair of parameter values $r, r' \in I(\varepsilon)$, with $r < r'$, we have that $D(r) \Subset D(r')$. Finally, if ω has the reproducing property on the unit disk*

$$h(0) = \int_{\mathbb{D}} h(z)\,\omega(z)\,d\Sigma(z),$$

for all bounded harmonic functions h on $\mathbb{D}$, then it also has the analogous reproducing property on $D(r)$, for $r \in I(\varepsilon)$:

$$r^2 h(0) = \int_{D(r)} h(z)\,\omega(z)\,d\Sigma(z),$$

for all bounded harmonic functions h on $D(r)$.

It should be pointed out that in the context of the above theorem, the measure (7.14) does converge to the harmonic measure ϖ_r in the weak-star topology of the Borel measures on $\overline{\Omega}$.

Proof. We indicate the modifications needed for the approach of [37]. Let $\mathfrak{H}_+$ denote the *Herglotz transform*,

$$\mathfrak{H}_+[f](z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, f(\zeta)\,d\sigma(\zeta), \qquad z \in \mathbb{D},$$

where f is assumed to be in $L^1(\mathbb{T})$.

Let θ, $0 < \theta < 1$, be so close to 1 that near the unit circle $\mathbb{T}$, ω is of the form $\omega(z) = \omega^\sharp(z, \bar{z})$, where $\omega^\sharp$ is a holomorphic function of two variables on the product domain obtained from two annuli,

$$\big(\mathbb{D}(0,\theta^{-1}) \setminus \overline{\mathbb{D}}(0,\theta)\big) \times \big(\mathbb{D}(0,\theta^{-1}) \setminus \overline{\mathbb{D}}(0,\theta)\big).$$

Now let f be defined and holomorphic on a neighborhood of $\overline{\mathbb{D}}$, and assume it is zero-free. We can then let F be the primitive of its reciprocal,

$$F(z) = \int_0^z \frac{d\xi}{f(\xi)}, \qquad z \in \overline{\mathbb{D}},$$

and note that the composition $\omega \circ F$ makes sense on $\overline{\mathbb{D}}$ as long as F maps $\overline{\mathbb{D}}$ into $\mathbb{D}(0, \theta^{-1})$, which is expected to be the case provided f does not deviate far from the constant function 1. Let $\mathfrak{T}$ be the non-linear operator

$$\mathfrak{T}[f](z) = \mathfrak{H}_+\left[\frac{|f|^2}{\omega \circ F}\right](z), \qquad z \in \mathbb{D},$$

and put

$$\mathfrak{L}[f](z) = z\,f'(z)\,\mathfrak{T}[f](z) - f(z)\,\frac{d}{dz}\big(z\,\mathfrak{T}[f](z)\big), \qquad z \in \mathbb{D}.$$

The evolution equation (7.15) for the Hele-Shaw flow can be expressed in terms of the function

$$\psi_r(z) = \frac{1}{\phi_r'(z)}, \qquad z \in \mathbb{D},$$

using that

$$\int_0^z \frac{d\xi}{\psi_r(\xi)} = \int_0^z \phi_r'(\xi)\, d\xi = \phi_r(z),$$

and it leads to the initial value problem

$$\frac{d}{dr}\,\psi_r = r\,\mathfrak{L}[\psi_r], \qquad \psi_1 = 1. \tag{7.16}$$

We shall see that the function ψ_r is characterized uniquely by this equation. The non-linear Cauchy-Kovalevskaya theorem due to Nishida-Nirenberg involves setting up a Banach scale of spaces, and we make the same choice as Reissig and von Wolfersdorfer: the spaces X_t, for $0 \le t \le 1$; here, X_t consists of all functions holomorphic and continuous up to the boundary on the disk $\mathbb{D}(0, \lambda_t)$, with $\lambda_t = (1-t)\lambda_0 + t\lambda_1$, for some fixed values λ_0, λ_1, $1 < \lambda_0 < \lambda_1 < +\infty$, close to 1. We use the uniform norm on $\mathbb{D}(0,\lambda_t)$ as norm on X_t. Suppose f is close to the constant function 1 in the norm of X_t: $\|f - 1\|_{X_t} < \delta$ for some small δ, $0 < \delta < 1$. It follows that the function F is close to the coordinate function z, because they coincide at the origin, and their derivatives are close:

$$\left\| \frac{1}{f} - 1 \right\|_{X_t} < \delta\,(1-\delta)^{-1}.$$

In particular, if δ is small, and λ_1 is close to 1, the image of the annulus $\overline{\mathbb{D}}(0,\lambda_t) \setminus \mathbb{D}$ under F is contained inside the other annulus $\mathbb{D}(0, \theta^{-1}) \setminus \overline{\mathbb{D}}(0, \theta)$, so that $\omega^\sharp(F(z), \bar{F}(z^*))$ is holomorphic and continuous up to the boundary as a function of z in $\overline{\mathbb{D}}(0,\lambda_t) \setminus \mathbb{D}$. Here, for $z \in \mathbb{C}$, we use the notation $z^* = \bar{z}^{-1}$ for the point reflected in the unit circle $\mathbb{T}$. Changing the path of integration, as permitted by Cauchy's formula, we write

$$\mathfrak{T}[f](z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}(0,\lambda_t)} \frac{\zeta + z}{\zeta - z}\, \frac{f(\zeta)\, \bar{f}(\zeta^*)}{\omega^\sharp(F(z), \bar{F}(z^*))}\, \frac{d\zeta}{\zeta}, \qquad z \in \mathbb{D}(0, \lambda_t).$$

This provides an analytic extension of $\mathfrak{T}[f]$ to $\mathbb{D}(0,\lambda_t)$ given that $f \in X_t$ is sufficiently close to the constant function 1. As in [37], the non-linear operator $\mathfrak{T}$ satisfies

$$\|\mathfrak{T} f\|_{X_t} \le K, \quad \text{and}$$
$$\|\mathfrak{T} f - \mathfrak{T} g\|_{X_t} \le L\, \|f - g\|_{X_t},$$

for some positive constants K, L, whenever f and g belong to the ball $\{f \in X_t : \|f - 1\|_{X_t} < \delta\}$, provided δ is small. And as in the Reissig-von Wolfersdorfer paper [37], these properties of the operator $\mathfrak{T}$ imply that the operator $r\,\mathfrak{L}$ satisfies all the requirements of the abstract non-linear Cauchy-Kovalevskaya theorem, due mainly to Nishida-Nirenberg. It follows that the initial value problem (7.16) has a unique C^1 solution with values along the Banach scale, on an interval around 1. The proof of Nishida's theorem [35] is based on the well-known Picard iterative process from the theory of ordinary differential equations. Taking into account the simple real analytic dependence on the parameter r in the initial value problem (7.16), and the real analyticity of the $\mathfrak{L}[f]$ with respect to f, as an operator acting on the given scale of Banach algebras X_t, an analysis of the Picard scheme shows that the solution is real analytic in r near 1, with values in the Banach scale. This is an existence statement, so we can take the values in a fixed space, namely the largest space in the scale: X_0.

We turn to the further properties of ϕ_r. For r close to 1, the function ϕ_r' is close to the constant 1 uniformly on a fixed neighborhood of $\overline{\mathbb{D}}$, and hence ϕ_r is univalent on $\overline{\mathbb{D}}$. We could also have used the lemma on p. 187 of [49], which in addition guarantees that $D(r) \Subset D(r')$ for $r < r'$. For r close to 1, the mapping ϕ_r maps $\mathbb{D}$ onto $D(r)$, which differs from $\mathbb{D}$ by an annular band of variable width, where the width at a point $\zeta \in \mathbb{T}$ is given by the expression

$$(r-1)\,\varrho(\zeta) + o(r-1),$$

the little ordo being taken as $r \to 1$. The width function $\varrho(\zeta)$ is obtained as follows:

$$\varrho(\zeta) = \operatorname{Re}\left(\bar{\zeta}\,\frac{d}{dr}\,\phi_r(\zeta)\Big|_{r=1}\right) = \frac{1}{\omega(\zeta)}, \qquad \zeta \in \mathbb{T}.$$

Consequently, for a function h harmonic on $\overline{\mathbb{D}}$,

$$\frac{d}{dr}\int_{D(r)} h(z)\,\omega(z)\,d\Sigma(z)\Big|_{r=1} = 2\,h(0),$$

and after having applied an appropriate conformal transformation, we get the corresponding statement for general r, near 1:

$$\frac{d}{dr}\int_{D(r)} h(z)\,\omega(z)\,d\Sigma(z) = 2r\,h(0).$$

Integrating this equality with respect to r, we arrive at

$$r^2\,h(0) = \int_{D(r)} h(z)\,\omega(z)\,d\Sigma(z),$$

for h harmonic on a neighborhood of $\overline{\mathbb{D}}$ and r close to 1. An approximation argument extends this identity to all bounded harmonic h on $D(r)$. The proof is complete. ■

Since the reproducing property (7.1) alone determines the Hele-Shaw flow domains for logarithmically subharmonic ω, we have the following theorem.

THEOREM 7.13 *In the context of Theorem 7.11, with Ω simply connected and the logarithmically subharmonic weight ω strictly positive and real analytic on $\overline{\Omega}$, there is a continuous sequence of conformal maps $\phi_r : \mathbb{D} \to D(r)$, for $0 < r < \rho(0)$, which depends in a real analytic fashion on the parameter r. Each ϕ_r extends to a conformal mapping from a neighborhood of $\overline{\mathbb{D}}$ onto a neighborhood of $\overline{D}(r)$. To a given r_0, $0 < r_0 < \rho(0)$, there is a small open interval $I(r_0)$ around it such that all the functions ϕ_r extend as conformal maps to one and the same neighborhood of $\overline{\mathbb{D}}$ for $r \in I(r_0)$. The conformal maps ϕ_r satisfy the evolution equation (7.15).*

8 Hadamard's variational formula

As before, we let Ω be a simply connected bounded domain in $\mathbb{C}$ with C^∞-smooth boundary. Moreover, ω is a real analytic weight function on Ω with a C^∞ extension to $\overline{\Omega}$, which is strictly positive there, and we suppose that $\log\omega$ is subharmonic on Ω. By the previous section on the weighted Hele-Shaw flow, we have, given a point $z_0 \in \Omega$, a real analytic continuous sequence of simply connected domains $D(r)$, indexed by r, $0 < r < \rho(z_0)$, generalized disks of "radius" r about z_0, whose boundaries are real analytic Jordan curves, determined by the reproducing property (7.1), with the origin 0 replaced by z_0.

The variational formula for the weighted biharmonic Green function. Let $\Gamma_{\omega,r}$ be the Green function for the weighted biharmonic operator $\Delta\omega^{-1}\Delta$ on $D(r)$:

$$\begin{aligned}\Delta_z\,\omega(z)^{-1}\Delta_z\Gamma_{\omega,r}(z,\zeta)=\delta_\zeta(z),&\qquad z\in D(r),\\ \Gamma_{\omega,r}(z,\zeta)=0,&\qquad z\in\partial D(r),\\ \partial_{n(z)}\Gamma_{\omega,r}(z,\zeta)=0,&\qquad z\in\partial D(r).\end{aligned}$$

It is intuitively clear that the Green function $\Gamma_{\omega,r}$ varies continuously with the parameter r; in fact, this can be made rigorous, for instance with the methods of Section 5. We shall derive a variational formula, originally found by Hadamard in 1908, which describes the development quantitatively. We follow the pattern from Hedenmalm's 1994 paper [20]. By the elliptic regularity theorem of Morrey-Nirenberg [32] (for details, see Section 3), the Green function $\Gamma_{\omega,r}$ extends real analytically to a neighborhood of the set $\overline{D}(r)\times\overline{D}(r)\setminus\delta(\overline{D}(r))$. In particular, for fixed $\zeta\in\overline{D}(r)$, $\Gamma_{\omega,r}$ solves the differential equation $\Delta\,\omega^{-1}\Delta\Gamma_{\omega,r}(\cdot,\zeta)=\delta_\zeta$ on a neighborhood of $\overline{D}(r)\setminus\{\zeta\}$. We consider two parameter values r,r' with $0<r<r'<\rho(z_0)$, and note that by (3.4) and (3.2), with obvious notations (for instance, G_r is the Green function for Δ on $D(r)$),

$$\Gamma_{\omega,r}(z,\zeta)=\int_{D(r)}\big(G_r(\xi,z)+H_{\omega,r}(\xi,z)\big)\big(G_{r'}(\xi,\zeta)+H_{\omega,r'}(\xi,\zeta)\big)\,\omega(\xi)\,d\Sigma(\xi),$$

for $(z,\zeta)\in D(r)\times D(r)$, and for $(z,\zeta)\in D(r')\times D(r')$, we have

$$\Gamma_{\omega,r'}(z,\zeta)=\int_{D(r')}\big(G_r(\xi,z)+H_{\omega,r}(\xi,z)\big)\big(G_{r'}(\xi,\zeta)+H_{\omega,r'}(\xi,\zeta)\big)\,\omega(\xi)\,d\Sigma(\xi).$$

As $r<r'$, we have the inclusion $D(r)\subset D(r')$, so that forming the difference of the above relations, we obtain

$$\begin{aligned}&\Gamma_{\omega,r'}(z,\zeta)-\Gamma_{\omega,r}(z,\zeta)\\&\qquad=\int_{D(r')\setminus D(r)}\big(G_r(\xi,z)+H_{\omega,r}(\xi,z)\big)\big(G_{r'}(\xi,\zeta)+H_{\omega,r'}(\xi,\zeta)\big)\,\omega(\xi)\,d\Sigma(\xi),\end{aligned}\tag{8.1}$$

whereby $(z,\zeta)\in D(r)\times D(r)$, but, if r' is sufficiently close to r, so that we can use the elliptic regularity, we can take $(z,\zeta)\in D(r')\times D(r')$. From the subsection on the classical Hele-Shaw flow in Section 7, we know that as $r'\to r$, the measure

$$\frac{1}{(r')^2-r^2}\,1_{D(r')\setminus D(r)}(z)\,\omega(z)\,d\Sigma(z)$$

converges (in the weak-star topology) to the harmonic measure ϖ_r for the point z_0 in the domain $D(r)$, which is supported on $\partial D(r)$. Dividing both sides of (8.1) by $r'-r$, and taking the limit as $r'\to r$, we find that since the Green function G_r vanishes when one of the variables is on the boundary $\partial D(r)$,

$$\frac{d}{dr}\,\Gamma_{\omega,r}(z,\zeta)=2r\int_{\partial D(r)}H_{\omega,r}(\xi,z)\,H_{\omega,r}(\xi,\zeta)\,d\varpi_r(\xi).\tag{8.2}$$

Some further explanation is needed here. We need to know *a priori* that the kernels $\Gamma_{\omega,r}$, $H_{\omega,r}$, and G_r depend fairly smoothly on the parameter r. It is convenient to use the conformal map $\phi_r:\mathbb{D}\to D(r)$ to pull back the situation to the unit disk. We then have the identity

$$\Gamma_{\omega_r}(z,\zeta)=\Gamma_{\omega,r}(\phi_r(z),\phi_r(\zeta)),\qquad(z,\zeta)\in\mathbb{D}\times\mathbb{D},$$

whereby $\omega_r = \omega \circ \phi_r \, |\phi_r'|^2$, and the left hand side expresses the Green function on the unit disk for the weighted biharmonic operator $\Delta \, \omega_r^{-1} \Delta$. By Theorems 7.12 and 7.13, the weight $\omega_r(z)$ is real analytic (and strictly positive) in the coordinates (z, r) on a neighborhood of $\overline{\mathbb{D}} \times]0, \rho(z_0)[$. The Green function G_r is easily expressed in terms of the Green function $G = G_{\mathbb{D}}$ for the laplacian on the unit disk,

$$G(z, \zeta) = G_r(\phi_r(z), \phi_r(\zeta)), \qquad (z, \zeta) \in \mathbb{D} \times \mathbb{D},$$

and this identity gives us fairly complete regularity information for G_r. As far as Γ_{ω_r} is concerned, we can turn to the proof of the elliptic regularity theorem of Morrey-Nirenberg [32], which gives us more quantitative information concerning the domain of convergence for the power series used to represent the real anaytic functions. It can be shown that the kernel $\Gamma_{\omega_r}(z, \zeta)$ is real analytic in the coordinates (z, ζ, r) on a neighborhood of the product set

$$\left(\overline{\mathbb{D}}^2 \setminus \delta(\overline{\mathbb{D}})\right) \times]0, \rho(z_0)[,$$

which then leads to the analogous information that $H_{\omega_r}(z, \zeta)$ is real analytic in the coordinates (z, ζ, r) on a neighborhood of the slightly bigger set

$$\left(\overline{\mathbb{D}}^2 \setminus \delta(\mathbb{T})\right) \times]0, \rho(z_0)[,$$

This justifies the limit process leading up to (8.2).

We want to turn the differential equation (8.2) into an integral equation. Note that when one of the variables z, ζ is on the boundary $\partial D(r)$, and the other is in the interior $D(r)$, the Green function for $\Delta \omega^{-1} \Delta$ vanishes: $\Gamma_{\omega,r}(z, \zeta) = 0$. As we integrate (8.2) with respect to r, the following formula emerges:

$$\Gamma_{\omega,r}(z, \zeta) = \int_{\max\{R(z), R(\zeta)\}}^{r} \int_{\partial D(\varrho)} H_{\omega,\varrho}(\xi, z) \, H_{\omega,\varrho}(\xi, \zeta) \, d\varpi_\varrho(\xi) \, 2\varrho \, d\varrho, \tag{8.3}$$

for $(z, \zeta) \in D(r) \times D(r)$. Here, $R(z)$ stands for the parameter value of ϱ for which the boundary of $D(\varrho)$ reaches the point z:

$$R(z) = \inf \left\{ \varrho : \, z \in D(\varrho) \right\}.$$

We know from Section 3 that if the Green function $\Gamma_{\omega,r}$ is positive on $D(r) \times D(r)$, then the corresponding harmonic compensator $H_{\omega,r}$ is positive on $\partial D(r) \times D(r)$, and since the latter is harmonic in the first variable, it is then positive throughout $D(r) \times D(r)$. Hadamard's variational formula (8.3) provides a kind of converse to the first implication: *if all the harmonic compensators for the subdomains $D(\varrho)$, $0 < \varrho < r$, are positive, then the Green function $\Gamma_{\omega,r}$ is positive.*

The variational formula for the Green function for the laplacian. We turn to Hadamard's better-known variational formula for G_r ([33], p. 46), which has important applications to the theory of conformal mappings. Let P_r be given by

$$P_r(z, \zeta) = -\tfrac{1}{2} \, \partial_{n(\zeta)} \, G_r(z, \zeta), \qquad (z, \zeta) \in D(r) \times \partial D(r),$$

the normal derivative being taken with respect to the boundary $\partial D(r)$ in the interior direction. This function then serves as a Poisson kernel on $D(r)$. For instance, we have the identity

$$d\varpi_r(z) = P_r(z_0, z) \, d\sigma(z), \qquad z \in \partial D(r).$$

The variational formula states that

$$\frac{d}{dr}\,G_r(z,\zeta)=-2r\int_{\partial D(r)}P_r(z,\xi)\,P_r(\zeta,\xi)\,\frac{d\varpi_r(\xi)}{\omega(\xi)},\qquad(z,\zeta)\in D(r)\times D(r),$$

and in integral form, it becomes

$$G_r(z,\zeta)=-\int_{\max\{R(z),R(\zeta)\}}^{r}\int_{\partial D(\varrho)}P_\varrho(z,\xi)\,P_\varrho(\zeta,\xi)\,\frac{d\varpi_\varrho(\xi)}{\omega(\xi)}\,2\varrho\,d\varrho,\tag{8.4}$$

for $(z,\zeta)\in D(r)\times D(r)$. We may combine this with equation (3.3), to get

$$\begin{aligned}H_{\omega,r}(\zeta,z)=\int_{D(r)}\int_{\max\{R(z),R(\eta)\}}^{r}\int_{\partial D(\varrho)}&Q_{\omega,r}(\zeta,\eta)\\&\times P_\varrho(z,\xi)\,P_\varrho(\eta,\xi)\,\frac{d\varpi_\varrho(\xi)}{\omega(\xi)}\,2\varrho\,d\varrho\,\omega(\eta)\,d\Sigma(\eta),\end{aligned}$$

which transforms to

$$\begin{aligned}H_{\omega,r}(\zeta,z)=\int_{R(z)}^{r}\int_{\partial D(\varrho)}\int_{D(\varrho)}&Q_{\omega,r}(\zeta,\eta)\,P_\varrho(\eta,\xi)\,\omega(\eta)\,d\Sigma(\eta)\\&\times P_\varrho(z,\xi)\,\frac{d\varpi_\varrho(\xi)}{\omega(\xi)}\,2\varrho\,d\varrho,\qquad(z,\zeta)\in D(r)\times D(r).\end{aligned}\tag{8.5}$$

9 Positivity of the weighted biharmonic Green function

We continue the presentation from the previous section on Hadamard's variational formula. We recapture: Ω is a simply connected bounded domain in $\mathbb{C}$ with C^∞-smooth boundary, and the weight ω is real analytic on Ω and has a C^∞ extension to $\overline{\Omega}$, which is strictly positive there; we also suppose that $\log\omega$ is subharmonic on Ω. The domains $D(r)$, indexed by r, $0<r<\rho(z_0)$, are the generalized "disks" about a fixed point $z_0\in\Omega$ of radius r arising from the weighted Hele-Shaw flow, and these constitute a real analytic continuous sequence of simply connected domains, whose boundaries are real analytic Jordan curves. It is a consequence of formula (8.5) that if we can prove that

$$0\le\int_{D(\varrho)}Q_{\omega,r}(\zeta,\eta)\,P_\varrho(\eta,\xi)\,\omega(\eta)\,d\Sigma(\eta),\qquad(\xi,\zeta)\in\partial D(\varrho)\times D(r),\tag{9.1}$$

whenever $0<\varrho<r<\rho(z_0)$, then the harmonic compensator $H_{\omega,r}$ is positive on $D(r)\times D(r)$. In the above integral, the function $Q_\omega(\zeta,\cdot)$ is harmonic on $\overline{D}(r)$, and in particular, bounded there, and the Poisson kernel $P(\cdot,\xi)$ is area summable on $D(\varrho)$. We conclude that the integral in (9.1) makes sense.

We shall obtain the following result, which is equivalent to (9.1).

THEOREM 9.1 *Fix ϱ,r such that $0<\varrho<r<\rho(z_0)$. Let h be a positive harmonic function on $D(\varrho)$, and define*

$$h_r(z)=\int_{D(\varrho)}Q_{\omega,r}(z,\xi)\,h(\xi)\,\omega(\xi)\,d\Sigma(\xi),\qquad z\in D(r).$$

Then h_r is positive on $D(r)$.

Proof. It suffices to obtain the result under the proviso that h is harmonic and strictly positive on $\overline{D}(\varrho)$. Since $Q_{\omega,r}(z_0,\cdot)=r^{-2}$ (this is a consequence of the reproducing property of the flow domain $D(r)$), the value of the function h_r at the center point z_0 is

$$h_r(z_0)=\frac{1}{r^2}\int_{D(\varrho)} h(\xi)\,\omega(\xi)\,d\Sigma(\xi)=\frac{\varrho^2}{r^2}\,h(z_0),$$

which is positive. We split the proof into three parts.

Part 1: continuity of h_r in r. The function h_r is the orthogonal projection of $h\,1_{D(\varrho)}$ – interpreted to vanish on $D(r)\setminus D(\varrho)$ – onto the harmonic subspace $HL^2(D(r),\omega)$ in $L^2(D(r),\omega)$. From the smoothness of the harmonic compensator $H_{\omega,r}$ in the r variable alluded to above, and the corresponding fact for the weighted harmonic Bergman kernel $Q_{\omega,r}$ as deduced from the identity

$$Q_{\omega,r}(z,\zeta)=-\omega(z)^{-1}\,\Delta_\zeta H_{\omega,r}(z,\zeta),$$

it is immediate that $h_r(z)$ is real analytic in the coordinates (z,r) on (a neighborhood of) the set

$$\big\{(z,r):\,z\in\overline{D}(r),\ r\in]\varrho,\rho(z_0)[\big\}\cup\big\{(z,r):\,z\in D(r),\ r\in[\varrho,\rho(z_0)[\big\}.$$

We need to investigate the continuity of $h_r(z)$ near the left end-point $r=\varrho$. By the reproducing property of the flow domains with respect to the weight, we have that

$$\int_{D(\varrho)} Q_{\omega,r}(z,\xi)\,\omega(\xi)\,d\Sigma(\xi)=\varrho^2\,Q_{\omega,r}(z,z_0)=\frac{\varrho^2}{r^2},\qquad z\in D(r),$$

and hence

$$h_r(z)-\frac{r^2}{\varrho^2}\,h(z)=\int_{D(\varrho)} Q_{\omega,r}(z,\xi)\,\big(h(\xi)-h(z)\big)\,\omega(\xi)\,d\Sigma(\xi),\qquad z\in D(r),\tag{9.2}$$

provided r is so close to ϱ that h is defined as a harmonic function on $D(r)$. We have that $h(\xi)-h(z)=O(|z-\xi|)$ for z,ξ in some fixed neighborhood of $\overline{D}(\varrho)$, so that part of the singularity of the kernel $Q_{\omega,r}$ is neutralized by the appearance of this factor on the right hand side of (9.2). As before, let $\phi_r:\mathbb{D}\to D(r)$ be the Riemann map taking 0 onto z_0, and let ω_r stand for the pulled-back weight on the unit disk,

$$\omega_r(z)=r^{-2}\,\omega\circ\phi_r(z)\,|\phi_r'(z)|^2,$$

which is reproducing for the origin as well as logarithmically subharmonic. ¿From the conformal invariance of the reproducing property of the weighted harmonic Bergman kernel, the following identity can be deduced:

$$r^2\,Q_{\omega,r}\big(\phi_r(z),\phi_r(\zeta)\big)=Q_{\omega_r}(z,\zeta)=2\,\mathrm{Re}\,K_{\omega_r}(z,\zeta)-1,\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}.\tag{9.3}$$

We apply Theorem 5.1 to K_{ω_r}, and obtain as a result that

$$r^2\,\big|Q_{\omega,r}\big(\phi_r(z),\phi_r(\zeta)\big)\big|=\big|Q_{\omega_r}(z,\zeta)\big|\le 1+\frac{4}{|1-z\bar\zeta|^2},\qquad (z,\zeta)\in\mathbb{D}\times\mathbb{D}.\tag{9.4}$$

We rewrite (9.2) in terms of the variable ζ, $\phi_r(\zeta)=\xi$, and get

$$\begin{aligned}&h_r\circ\phi_r(z)-\frac{r^2}{\varrho^2}\,h\circ\phi_r(z)\\&\quad=r^2\int_{\phi_r^{-1}(D(\varrho))} Q_{\omega,r}\big(\phi_r(z),\phi_r(\zeta)\big)\,\big(h(\phi_r(\zeta))-h(\phi_r(z))\big)\,\omega_r(\zeta)\,d\Sigma(\zeta),\qquad z\in\mathbb{D},\end{aligned}$$

where $\phi_r^{-1}(D(\varrho)) \subset \mathbb{D}$. Given the estimates mentioned previously, *it is easily deduced from this identity that* $h_r \circ \phi_r \to h \circ \phi_\varrho$ *uniformly on* $\mathbb{D}$ *as* $r \to \varrho$. In particular, since we assume h to be strictly positive on $\overline{D}(\varrho)$, it follows that $h_r \circ \phi_r$ *is uniformly (in* r*) strictly positive on* $\overline{\mathbb{D}}$ *for* r *in some short interval* $]\varrho, \varrho+\delta]$, *with* $0 < \delta$.

Part 2: the derivative of $h_r \circ \phi_r$. The derivative of the composition $h_r \circ \phi_r$ with respect to the parameter r is, by the chain rule,

$$\frac{d}{dr} h_r \circ \phi_r(z) = \frac{\partial h_r}{\partial r} \circ \phi_r(z) + 2\,\mathrm{Re}\left(\frac{\partial h_r}{\partial z} \circ \phi_r(z)\, \frac{d\phi_r}{dr}(z)\right), \tag{9.5}$$

where the partial derivatives with respect to r and z correspond to thinking of the function h_r as a function of two variables: $h_r(z) = h(z, r)$. The derivative of ϕ_r with respect to r is supplied by formula (7.15), which simplifies to

$$\frac{d\phi_r}{dr}(z) = \frac{z}{r}\,\phi_r'(z) \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, \frac{d\sigma(\zeta)}{\omega_r(\zeta)} = \frac{z}{r}\,\phi_r'(z)\; \mathfrak{H}_+\left[\frac{1}{\omega_r}\right](z), \qquad z \in \mathbb{D}, \tag{9.6}$$

where the symbol $\mathfrak{H}_+$ stands for the Herglotz transform. We would like to find a way to express the partial derivative $\partial_r h_r$. Let r', $\varrho < r' < r$ be so close to r that $h_{r'}$ extends harmonically and boundedly to $D(r)$. Then, from the reproducing property of the weighted harmonic Bergman kernel, we have

$$h_{r'}(z) = \int_{D(r)} Q_{\omega,r}(z,\xi)\, h_{r'}(\xi)\, \omega(\xi)\, d\Sigma(\xi), \qquad z \in D(r). \tag{9.7}$$

On the other hand, again by the reproducing property,

$$\int_{D(r')} Q_{\omega,r}(z,\xi)\, Q_{\omega,r'}(\xi,\zeta)\, \omega(\xi)\, d\Sigma(\xi) = Q_{\omega,r}(z,\zeta), \qquad (z,\zeta) \in D(r) \times D(r'),$$

so that

$$\begin{aligned}
\int_{D(r')} & Q_{\omega,r}(z,\xi)\, h_{r'}(\xi)\, \omega(\xi)\, d\Sigma(\xi) \\
&= \int_{D(r')} Q_{\omega,r}(z,\xi) \int_{D(\varrho)} Q_{\omega,r'}(\xi,\zeta)\, h(\zeta)\, \omega(\zeta)\, d\Sigma(\zeta)\, \omega(\xi)\, d\Sigma(\xi) \\
&\qquad = \int_{D(\varrho)} Q_{\omega,r}(z,\zeta)\, h(\zeta)\, \omega(\zeta)\, d\Sigma(\zeta) = h_r(z), \qquad z \in D(r).
\end{aligned} \tag{9.8}$$

Forming the difference between (9.7) and (9.8), we obtain

$$h_r(z) - h_{r'}(z) = -\int_{D(r)\setminus D(r')} Q_{\omega,r}(z,\xi)\, h_{r'}(\xi)\, \omega(\xi)\, d\Sigma(\xi), \qquad z \in D(r). \tag{9.9}$$

In view of the observation made following Theorem 7.12, to the effect that the measure

$$\frac{1}{r^2-(r')^2}\, 1_{D(r)\setminus D(r')}(z)\, \omega(z)\, d\Sigma(z)$$

converges weak-star to the harmonic measure ϖ_r on the boundary for domain $D(r)$ with respect to the interior point z_0 as $r' \to r$, it follows from (9.9) that

$$\frac{\partial h_r}{\partial r}(z) = -2r \int_{\partial D(r)} Q_{\omega,r}(z,\xi)\, h_r(\xi)\, d\varpi_r(\xi), \qquad z \in D(r).$$

Shifting the coordinates back to the unit disk, we obtain, keeping in mind (9.3),

$$\frac{\partial h_r}{\partial r}\circ\phi_r(z) = -\frac{2}{r}\int_{\mathbb{T}} Q_{\omega_r}(z,\zeta)\, h_r\circ\phi_r(\zeta)\, d\sigma(\zeta), \qquad z\in\mathbb{D}. \tag{9.10}$$

By the Poisson integral formula for harmonic functions in $\mathbb{D}$, we have the representation

$$h_r\circ\phi_r(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-z\bar\zeta|^2}\, h_r\circ\phi_r(\zeta)\, d\sigma(\zeta), \qquad z\in\mathbb{D},$$

which after an application of one of the two Wirtinger differential operators leads to

$$\phi_r'(z)\,\frac{\partial h_r}{\partial z}\circ\phi_r(z) = \int_{\mathbb{T}} \frac{\bar\zeta}{(1-z\bar\zeta)^2}\, h_r\circ\phi_r(\zeta)\, d\sigma(\zeta), \qquad z\in\mathbb{D}. \tag{9.11}$$

In view of (9.6), (9.10), and (9.11),the identity (9.5) becomes

$$\frac{d}{dr}\,h_r\circ\phi_r(z) = \\ \frac{2}{r}\int_{\mathbb{T}}\left\{\mathrm{Re}\left(\mathfrak{H}_+\left[\frac{1}{\omega_r}\right](z)\,\frac{z\bar\zeta}{(1-z\bar\zeta)^2}\right) - Q_{\omega_r}(z,\zeta)\right\} h_r\circ\phi_r(\zeta)\, d\sigma(\zeta), \qquad z\in\mathbb{D}. \tag{9.12}$$

As in the proof of Corollary 6.10, we notice the appearance of the Kœbe function. *Suppose for the moment that for some value of the parameter r, $\varrho<r<\rho(z_0)$, the real analytic function $h_r\circ\phi_r|_{\mathbb{T}}$ vanishes along with its (tangential) derivative at some point $z_1\in\mathbb{T}$.* Then $h_r\circ\phi_r(z)=O(|z-z_1|^2)$ as z approaches z_1 along $\mathbb{T}$, which counterbalances the singularities of the Kœbe function and the weighted harmonic Bergman kernel, as estimated by (9.4), at least when $z\in\mathbb{D}$ approaches the boundary point z_1 radially. Taking into account the well-known boundary behavior of the Kœbe function, we obtain in the limit that (the real part of the Herglotz transform is the Poisson integral, with well-known boundary values)

$$\frac{d}{dr}\,h_r\circ\phi_r(z_1) = -\frac{2}{r}\int_{\mathbb{T}}\left\{\frac{1}{\omega_r(z_1)}\,\frac{1}{|\zeta-z_1|^2} + Q_{\omega_r}(z_1,\zeta)\right\} h_r\circ\phi_r(\zeta)\, d\sigma(\zeta). \tag{9.13}$$

If, in addition, $0\le h_r\circ\phi_r$ on $\overline{\mathbb{D}}$, well, then, by invoking Corollary 6.10, which states that

$$Q_{\omega_r}(z_1,\zeta)\le -\left(\frac{1}{\omega_r(z_1)}+\frac{1}{\omega_r(\zeta)}\right)\,\frac{1}{|\zeta-z_1|^2}, \qquad \zeta\in\mathbb{T}\setminus\{z_1\},$$

we can assert that

$$0<\frac{2}{r}\int_{\mathbb{T}}\frac{1}{\omega_r(\zeta)}\,\frac{1}{|\zeta-z_1|^2}\,h_r\circ\phi_r(\zeta)\,d\sigma(\zeta)\le\frac{d}{dr}\,h_r\circ\phi_r(z_1). \tag{9.14}$$

The leftmost inequality holds because $h_r\circ\phi_r$ cannot vanish identically – after all, we know that $0<h_r(z_0)=h_r\circ\phi_r(0)$.

Part 3: the finishing argument. Consider the function

$$\mathfrak{h}(r)=\min\left\{h_r(z):\, z\in\overline{D}(r)\right\}=\min\left\{h_r\circ\phi_r(z):\, z\in\overline{\mathbb{D}}\right\}, \qquad \varrho<r<\rho(z_0),$$

which, by the results of Part 1, extends continuously to the interval $[\varrho,\rho(z_0)[$, and is positive at the left end-point: $0<\mathfrak{h}(\varrho)$. We shall demonstrate that $0<\mathfrak{h}(r)$ holds for all $r\in[\varrho,\rho(z_0)[$, which is actually slightly stronger than what is needed. We argue by contradiction, and assume $\mathfrak{h}(r) \leq 0$ for some $r \in]\varrho, \rho(z_0)[$. Forming the infimum over all such r, we find a parameter value $r_1 \in]\varrho, \rho(z_0)[$ with $\mathfrak{h}(r_1) = 0$, such that $0 < \mathfrak{h}(r)$ holds for all $r \in [\varrho, r_1[$. By the maximum principle, this means that there exists a point $z_1 \in \mathbb{T}$, such that $h_{r_1} \circ \phi_{r_1}(z_1) = 0$, and that $0 \leq h_{r_1} \circ \phi_{r_1}$ elsewhere on $\overline{\mathbb{D}}$. The point z_1 is precisely of the type considered in Part 2, so that by (9.14),

$$0 < \frac{d}{dr} h_r \circ \phi_r(z_1)\Big|_{r=r_1}.$$

We immediately see that $h_r \circ \phi_r(z_1) < 0$ for r, $\varrho < r < r_1$, sufficiently close to r_1, and hence $\mathfrak{h}(r) < 0$ for such r. This contradicts the minimality of r_1, and completes the proof. ■

¿From the previous section on Hadamard's variational formula, we then have the following corollary.

COROLLARY 9.2 *Fix r with $0 < r < \rho(z_0)$. Then both $H_{\omega,r}$ and $\Gamma_{\omega,r}$ are positive on $D(r) \times D(r)$.*

In the corollary, we need in fact not the entire assumption that ω is logarithmically subharmonic throughout Ω: it can be weakened to requiring $\log \omega$ to be subharmonic on $D(r)$. This leads immediately to the following result, where ω is a weight on the unit disk.

COROLLARY 9.3 *Suppose ω is a logarithmically subharmonic and reproducing (for the origin) weight on $\mathbb{D}$, real analytic on $\overline{\mathbb{D}}$, and strictly positive on $\overline{\mathbb{D}}$ as well. Then the weighted biharmonic Green function Γ_ω is positive on $\mathbb{D} \times \mathbb{D}$.*

In view of the sections on approximation of weights and Green functions, (Sections 4 and 5, and more to the point, Theorems 4.1) and 5.2), we can remove the regularity assumptions in the above corollary.

COROLLARY 9.4 *Suppose ω is a logarithmically subharmonic and reproducing (for the origin) weight on $\mathbb{D}$. Then the weighted biharmonic Green function Γ_ω is positive on $\mathbb{D} \times \mathbb{D}$.*

The above corollary was obtained earlier by Shimorin [48] in the special case of a radial weight; see also Hedenmalm [20, 22].

10 Applications to the Bergman spaces

Applications to the Bergman spaces $A^p(\mathbb{D})$. The study of the kernel function in the context of the Bergman spaces was initiated by Stefan Bergman [4]. However, in the first couple of attempts toward a factorization theory for the Bergman spaces – by Charles Horowitz [25], [26], and Boris Korenblum [29] – it played a subordinate rôle, if used at all. The kernel function later reappeared in the work of Hedenmalm [16]. Given a zero sequence A for the Hilbert Bergman space $A^2(\mathbb{D})$ on the unit disk (which for simplicity avoids the origin), he considered the invariant subspace M_A of all functions in $A^2(\mathbb{D})$ that vanish on A (counting multiplicities), and formed the function

$$\varphi_A(z) = K_A(0,0)^{-\frac{1}{2}} K_A(z,0), \qquad z \in \mathbb{D},$$

where K_A denotes the reproducing kernel for M_A. The function φ_A has norm 1 in $A^2(\mathbb{D})$, and has largest value in modulus at the origin among all functions in the closed unit ball of

M_A; for this reason, such functions are sometimes called *extremal functions*. Hedenmalm showed that φ_A is an expansive multiplier, that is,

$$\|f\|_{A^2} \le \|\varphi_A f\|_{A^2}, \qquad f \in A^2(\mathbb{D}).$$

For infinite zero sequences A, it may happen that the right hand side attains the value $+\infty$. Furthermore, the function φ_A has no extraneous zeros in $\mathbb{D}$, and it is a contractive divisor,

$$\|f/\varphi_A\|_{A^2} \le \|f\|_{A^2}, \qquad f \in M_A.$$

In [6], [7], the quartet Duren-Khavinson-Shapiro-Sundberg generalized Hedenmalm's results to the Bergman spaces $A^p(\mathbb{D})$, $0 < p < +\infty$. In the context of $A^2(\mathbb{D})$, the main idea is to write the kernel function K_A in the form

$$K_A(z,\zeta) = b_A(z)\,\overline{b}_A(\zeta)\,K_{|b_A|^2}(z,\zeta), \qquad (z,\zeta) \in \mathbb{D}\times\mathbb{D},$$

where b_A is the Blaschke product for A, assuming the sequence A meets the Blaschke condition. For general p, $0 < p < +\infty$, we then set

$$\varphi_A(z) = K_{|b_A|^p}(0,0)^{-\frac{1}{p}}\,b_A(z)\,K_{|b_A|^p}(z,0)^{\frac{2}{p}}, \qquad z \in \mathbb{D},$$

where as it happens, the weighted Bergman kernel function fails to have zeros, so that it is all right to take fractional powers of it. These functions φ_A can be defined for all zero sequences, not just the for ones that satisfy the Blaschke condition, and have factorization properties analogous to what was the case for $p = 2$. An important observation is that the weight $|\varphi_A|^p$ is a logarithmically subharmonic and reproducing for the origin. Multiplication by φ_A is an isometry $P^p(\mathbb{D}, |\varphi_A|^p) \to A^p(\mathbb{D})$ (evident notation), so that the question whether for two zero sequences A and B,

$$\|\varphi_A f\|_{A^p} \le \|\varphi_B f\|_{A^p}$$

holds for all polynomials f, becomes a matter of whether the injection mapping

$$P^p(\mathbb{D}, |\varphi_B|^p) \to P^p(\mathbb{D}, |\varphi_A|^p)$$

is a contraction.

We have the following theorem.

THEOREM 10.1 *Let A and B be two zero sequences for $A^p(\mathbb{D})$, such that A is contained in B. Then*

$$\|\varphi_A f\|_{A^p} \le \|\varphi_B f\|_{A^p}, \qquad f \in A^p(\mathbb{D}).$$

Proof. For finite sequences A and B, the functions φ_A and φ_B are holomorphic in a neighborhood of $\overline{\mathbb{D}}$, and we consider the function $\Phi_{B,A}$ which solves the boundary value problem

$$\begin{aligned} \Delta\Phi_{B,A}(z) &= |\varphi_B(z)|^p - |\varphi_A(z)|^p, \qquad && z \in \mathbb{D}, \\ \Phi_{B,A}(z) &= 0, && z \in \mathbb{T}. \end{aligned}$$

¿From an application of Green's formula, as in [16], [6], [7], we see that the fact that the right hand side – the function $|\varphi_B|^p - |\varphi_A|^p$ – annihilates the harmonic functions in $L^2(\mathbb{D})$ translates to the additional boundary data

$$\partial_{n(z)}\,\Phi_{B,A}(z) = 0, \qquad z \in \mathbb{T}.$$

Dividing the differential equation by $|\varphi_A(z)|^2$, and then afterward applying another laplacian, we find that it solves

$$\Delta \frac{1}{|\varphi_A(z)|^p} \Delta \Phi_{B,A}(z) = \Delta \left|\frac{\varphi_B(z)}{\varphi_A(z)}\right|^p, \qquad z \in \mathbb{D},$$

which is positive on $\mathbb{D}$. In view of the given boundary data, we may write the function $\Phi_{B,A}$ as an integral in terms of the weighted biharmonic Green function $\Gamma_{|\varphi_A|^p}$:

$$\Phi_{B,A}(z) = \int_{\mathbb{D}} \Gamma_{|\varphi_A|^p}(z,\zeta)\, \Delta_\zeta \left|\frac{\varphi_B(\zeta)}{\varphi_A(\zeta)}\right|^p d\Sigma(\zeta), \qquad z \in \mathbb{D},$$

which is then positive. The importance of the potential function $\Phi_{B,A}$ comes from the fact that Green's formula yields the identity

$$\|\varphi_B f\|_{A^p}^p - \|\varphi_A f\|_{A^p}^p = \int_{\mathbb{D}} \Phi_{B,A}(z)\, \Delta_z |f(z)|^p\, d\Sigma(z), \qquad z \in \mathbb{D},$$

for polynomials f, which yields the desired inequality in this case, because we can approximate functions in $A^p(\mathbb{D})$ by polynomials, and because the functions φ_A and φ_B are bounded on $\mathbb{D}$. Setting $g = \varphi_B f$, we conclude that

$$\left\|\frac{\varphi_A}{\varphi_B}\, g\right\|_{A^p} \le \|g\|_{A^p}, \tag{10.1}$$

for all $g \in A^p(\mathbb{D})$ that vanish on B. Now let A and B be arbitrary zero sequences, and form finite subsequences $A' \subset A$ and $B' \subset B$, with $A' \subset B'$. Then the above inequality holds with A and B replaced by A' and B', respectively, and we apply it to g vanishing on B. Letting A' grow up to A, and B' up to B, $\varphi_{A'} \to \varphi_A$ and $\varphi_{B'} \to \varphi_B$ in $A^p(\mathbb{D})$, and Fatou's lemma delivers the above inequality for arbitrary A and B, which implies the assertion of the theorem. ∎

COROLLARY 10.2 ($p = 2$) *Let A be and B be two zero sequences for $A^2(\mathbb{D})$, and suppose that $B \setminus A$ consists of a single point $\alpha \in \mathbb{D}$. Then the quotient $\varphi_\alpha = \varphi_B/\varphi_A$ is a bounded holomorphic function on $\mathbb{D}$, and it only vanishes at the point α in $\mathbb{D}$. Moreover, if b_α is the Blaschke factor corresponding to the point $B \setminus A$, then $1 \le |\varphi_\alpha/b_\alpha|$ holds throughout $\mathbb{D}$. In particular, $\varphi_\alpha(\mathbb{D})$ covers the whole disk $\mathbb{D}$.*

Proof. The function φ_α is given by the formula

$$\varphi_\alpha(z) = \left(1 - K_{|\varphi_A|^2}(\alpha,\alpha)^{-1}\right)^{-\frac{1}{2}} \left(1 - \frac{K_{|\varphi_A|^2}(z,\alpha)}{K_{|\varphi_A|^2}(\alpha,\alpha)}\right), \qquad z \in \mathbb{D},$$

which, in view of Theorem 5.1, shows that φ_α is bounded on $\mathbb{D}$, at least for $\alpha \in \mathbb{D}\setminus\{0\}$. A closer analysis of what happens as $\alpha \to 0$ reveals that it is bounded also for $\alpha = 0$. Using a peaking function argument as in [16], we obtain $1 \le |\varphi_\alpha/b_\alpha|$ on $\mathbb{D}$, as a consequence of Theorem 10.1, at least for finite sequences A. Approximating general zero sequences by finite ones, the assertion follows for general A. ∎

The following answers a question raised by Korenblum.

COROLLARY 10.3 *Let B be a zero sequence for $A^p(\mathbb{D})$ and M an invariant subspace in $A^p(\mathbb{D})$. Suppose M has index 1, that is, the dimension of the quotient space M/SM is 1, where S is the operator of multiplication by z. Then if $M_B \subset M$, the subspace M is of the form $M = M_A$, for some smaller zero sequence A.*

Proof. We follow the scheme from Hedenmalm [18]. Let A be the sequence of common zeros (counting multiplicities) of the functions in M. We are to show that $M = M_A$. According to Theorem 5.2 and Proposition 5.4 of Aleman-Richter-Sundberg [2], it suffices to obtain that $\varphi_A \in M$, because φ_A generates M_A as an invariant subspace. From the assumption $M_B \subset M$, we have that $\varphi_B \in M$. Then, because M has index 1, we may divide out superfluous zeros in φ_B, one by one, and remain in M. So, if q is a polynomial whose zeros constitute a finite subset of $B \setminus A$, the function φ_B/q is also in M. Let A' be a finite subsequence of A, and B' one of B, with $A' \subset B'$. Then the function $\varphi_{B'}/\varphi_{A'}$ can be factored as an invertible element of $H^\infty(\mathbb{D})$ times a polynomial q with zeros at $B' \setminus A'$, and hence

$$\varphi_{A',B',B} = \frac{\varphi_{A'}}{\varphi_{B'}}\,\varphi_B \in M.$$

Meanwhile, by (10.1),

$$\left\| \frac{\varphi_{A'}}{\varphi_{B'}}\,\varphi_B \right\|_{A^p} \leq \|\varphi_B\|_{A^p} = 1.$$

The function $\varphi_{A',B',B}$ tends to φ_A as A' grows to A and B' grows to B, and the limit element φ_A has norm 1, so that nothing is lost in Fatou's lemma. This means that $\varphi_{A',B',B}$ tends to φ_A in norm, and the conclusion $\varphi_A \in M$ follows. The proof is complete. ■

Applications to weighted Bergman spaces. The main theorem enables us to develop a factorization theory for the spaces $P^2(\mathbb{D},\omega)$, where the weight ω is assumed to be logarithmically subharmonic and reproducing for the origin. We say that a function $\varphi \in P^2(\mathbb{D},\omega)$ is a $P^2(\mathbb{D},\omega)$-*inner function* provided that the weight $|\varphi|^2\omega$ is reproducing for the origin.

THEOREM 10.4 *Assume ω is logarithmically subharmonic and reproducing for the origin. Let $\varphi \in P^2(\mathbb{D},\omega)$ be a $P^2(\mathbb{D},\omega)$-inner function. Then $\|f\|_\omega \leq \|\varphi f\|_\omega$ for all polynomials f. In fact, we have the norm identity*

$$\|\varphi f\|_\omega^2 = \|f\|_\omega^2 + \int_{\mathbb{D}\times\mathbb{D}} \Gamma_\omega(z,\zeta)\,|\varphi'(z)|^2 |f'(\zeta)|^2\, d\Sigma(z)\, d\Sigma(\zeta), \qquad f \in P^2(\mathbb{D}, |\varphi|^2\omega).$$

Proof. The proof is really in the same vein as that of, for instance, Theorem 10.1. A slightly different approach is needed, though, because of the lack of smoothness assumption on the weight, in which case Definition 3.1 is used to define the Green function Γ_ω. The calculations are analogous to the ones used in [19] (see also [2]), and therefore omitted. ■

A variant of the above runs as follows.

THEOREM 10.5 *Assume that ω and ω' are two logarithmically subharmonic weights which are reproducing for the origin. Suppose that in addition, both are C^∞-smooth on $\overline{\mathbb{D}}$, and that the quotient ω'/ω is subharmonic. Then $\|f\|_\omega \leq \|f\|_{\omega'}$ holds for all $f \in A^2(\mathbb{D})$.*

We remark that because of the regularity assumptions on ω and ω', the spaces $P^2(\mathbb{D},\omega)$ and $P^2(\mathbb{D},\omega')$ coincide with $A^2(\mathbb{D})$ in the above theorem.

By the general theory of reproducing kernel functions, as found in Saitoh's book [40], Theorem 10.5 leads to the following conclusion.

COROLLARY 10.6 *Under the assumptions of Theorem 10.5, the difference $K_\omega - K_{\omega'}$ is a reproducing kernel on $\mathbb{D}\times\mathbb{D}$. In other words,*

$$\frac{L_{\omega'}(z,\zeta) - L_\omega(z,\zeta)}{(1-z\bar{\zeta})^2}$$

is a reproducing kernel on $\mathbb{D}\times\mathbb{D}$.

11 Directions for further research

The main result of this paper, the positivity of the weighted biharmonic Green function Γ_ω for logarithmically subharmonic reproducing weights ω, was conjectured by Hedenmalm in 1992. Partial results in this direction were found by Hedenmalm [17], [20], [22], [21], by Sergei Shimorin [44], [45], [46], [48], and by Miroslav Engliš [8], [9]. As a matter of fact, the variational technique used in [22] was an inspiration for this work, although the context is different.

It appears likely that the strong maximum principle suggested in the introduction should be true. We formulate this as a conjecture. The normal derivative is as always in the interior direction.

CONJECTURE 11.1 *Let ω be a logarithmically subharmonic reproducing weight on $\mathbb{D}$, which is C^∞-smooth on $\overline{\mathbb{D}}$. Let u and v be two C^∞-smooth real-valued functions on $\overline{\mathbb{D}}$, and suppose that u is sub-ω-biharmonic and v is ω-biharmonic on $\mathbb{D}$. We then have the maximum principle*

$$u|_{\mathbb{T}} \le v|_{\mathbb{T}} \quad \text{and} \quad \frac{\partial u}{\partial n}\Big|_{\mathbb{T}} \le \frac{\partial v}{\partial n}\Big|_{\mathbb{T}} \implies u|_{\mathbb{D}} \le v|_{\mathbb{D}}.$$

In terms of the Green function Γ_ω, what is required is that

$$\frac{\partial}{\partial n(z)} \frac{1}{\omega(z)} \Delta_z \Gamma_\omega(z,\zeta) = -2\, P(z,\zeta) + \frac{\partial}{\partial n(z)} H_\omega(z,\zeta) \le 0, \qquad (z,\zeta) \in \mathbb{T} \times \mathbb{D}.$$

In the special case $\omega(z) \equiv 1$, an explicit calculation yields

$$\frac{\partial}{\partial n(z)} \Delta_z \Gamma(z,\zeta) = -2\,(1-|\zeta|^2) \left(\frac{1}{|1-z\bar\zeta|^2} + \operatorname{Re} \frac{z\bar\zeta}{(1-z\bar\zeta)^2} \right) < 0, \qquad (z,\zeta) \in \mathbb{T} \times \mathbb{D}.$$

In [47], Shimorin showed that one-point zero divisors are univalent functions – just like the individual Blaschke factors – in the weighted spaces $P^2(\mathbb{D},\omega)$, where ω is radial and logarithmically subharmonic. He also showed that, modulo some regularity, a univalent one-point zero divisor is automatically an expansive multiplier; see also [3]. We believe that the following is true.

CONJECTURE 11.2 *Let ω be a logarithmically subharmonic reproducing weight on $\mathbb{D}$. Then, for each $\alpha \in \mathbb{D} \setminus \{0\}$, the one-point zero divisor in $P^2(\mathbb{D},\omega)$,*

$$\varphi_\alpha(z) = \left(1 - K_\omega(\alpha,\alpha)^{-1}\right)^{-\frac{1}{2}} \left(1 - \frac{K_\omega(z,\alpha)}{K_\omega(\alpha,\alpha)}\right), \qquad z \in \mathbb{D},$$

is univalent and maps $\mathbb{D}$ onto a star-shaped domain. Moreover, $|\varphi_\alpha(z)| < 3$ holds for all $z \in \mathbb{D}$.

In fact, in a conversation with Peter Duren, it became clear that if the extended maximum principle holds, as formulated in Conjecture 11.1, then each one-point zero divisor φ_α is univalent and maps $\mathbb{D}$ onto a star-shaped domain, which by Corollary 10.2 contains $\mathbb{D}$. Assuming this for the moment, the geometry of the "corona" $\phi_\alpha(\mathbb{D}) \setminus \mathbb{D}$ should contain information about how close the space $P^2(\mathbb{D},\omega)$ is to the limit case $H^2(\mathbb{D})$. We wish to point out that the focus on one-point zero divisors is not as special at it may seem: each zero-divisor can be written as a product of one-point zero divisors, where each factor is a divisor with respect to a weight that is the original weight times the modulus-squared of product of the previous factors.

We turn to the connection between Conjectures 11.1 and 11.2. We can write $|\varphi_\alpha|^2$ as an integral,

$$|\varphi_\alpha(z)|^2 = 1 + \int_{\mathbb{D}} (G(z,\zeta) + H_\omega(z,\zeta))\, |\varphi'_\alpha(\zeta)|^2\, d\Sigma(\zeta), \qquad z \in \mathbb{D},$$

and assuming that Conjecture 11.1 holds, we obtain that on the unit circle $\mathbb{T}$, $|\varphi_\alpha|^2$ increases in the outward normal direction. Then, assuming that ω is real analytic near $\mathbb{T}$, an application of the Cauchy-Riemann equations to the locally defined function $\log \varphi_\alpha$ shows that the argument of $\varphi_\alpha(e^{i\theta})$ increases with θ. The case of more general weights is obtained by approximation.

A way to reach more detailed information about Γ_ω is to try to dissect the kernel function K_ω further than we did back in Section 6. We should like to obtain further structural information about the kernel L_ω.

In view of Corollary 10.6, one may ask the question whether the difference $L_{\omega'} - L_\omega$ is also a reproducing kernel on $\mathbb{D} \times \mathbb{D}$, under the assumptions of Theorem 10.5. It turns out that it is *not so*, even if we replace the subharmonicity of ω'/ω with logarithmic subharmonicity.

Example. If $L_{\omega'} - L_\omega$ were a reproducing kernel function for all logarithmically subharmonic reproducing weights ω, ω' with ω'/ω logarithmically subharmonic, subject to the condition that both weights are C^∞-smooth on $\bar{\mathbb{D}}$, then it would also be so, by a dilation argument, for the radial weights

$$\omega(z) = \omega_\alpha(z) = (1-\alpha)\,(1-|z|^2)^{-\alpha}, \qquad z \in \mathbb{D},$$

and

$$\omega'(z) = \omega_\beta(z) = (1-\beta)\,(1-|z|^2)^{-\beta}, \qquad z \in \mathbb{D},$$

which are less regular near the boundary, where the parameters α, β range over $0 < \alpha < \beta < 1$. The corresponding Bergman kernels are well-known, and we only write down the formula for α:

$$K_{\omega_\alpha}(z,\zeta) = (1 - z\bar{\zeta})^{-2+\alpha}, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D}.$$

We calculate that along the diagonal,

$$L_{\omega_\beta}(z,z) - L_{\omega_\alpha}(z,z) = \frac{(1-|z|^2)^\alpha - (1-|z|^2)^\beta}{|z|^2}, \qquad z \in \mathbb{D},$$

which is positive, but fails to be subharmonic, which is necessary for it to be the restriction to the diagonal of a reproducing kernel function.

The above example, however, leaves open the possibility that at least one of the expressions

$$\frac{L_{\omega'}(z,\zeta) - L_\omega(z,\zeta)}{1 - z\bar{\zeta}}, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D},$$

and

$$\frac{L_{\omega'}(z,\zeta) - L_\omega(z,\zeta)}{1 - z\bar{\zeta} L_\omega(z,\zeta)}, \qquad (z,\zeta) \in \mathbb{D} \times \mathbb{D},$$

might be a reproducing kernel. We note that if the second one is a reproducing kernel, then so is the first one, because it can be written as a product of the second one and the reproducing kernel J_ω of Theorem 6.2.

Creeping flow. The slow motion of a viscous incompressible fluid (with small Reynolds number), squeezed in between two parallel walls of constant infinitesimal width, is known as *creeping flow*. The motion of the fluid is governed by a real-valued potential function Ψ, which satisfies the biharmonic equation $\Delta^2\Psi = 0$; the velocity of the fluid is given by the expression

$$\left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right),$$

which constitutes a vector perpendicular to the gradient $\nabla\Psi$. Consequently, the level curves of Ψ are the flow lines of the creeping flow. Let the potential function Ψ equal the biharmonic Green function $\Gamma_\Omega(\cdot,\zeta)$, for a fixed $\zeta \in \Omega$, where Ω is a bounded planar domain Ω with smooth boundary. The corresponding flow then involves a torque applied at ζ, and friction at the boundary $\partial\Omega$, so that the velocity vanishes there. There appears a main swirl (vortex) centered at a point near ζ, and it is of interest to know whether there are any other smaller swirls (eddies) located further away from ζ. Apparently, this is the same as asking whether the Green function $\Gamma_\Omega(\cdot,\zeta)$ has more than one local extreme point in Ω. Let us consider the weighted biharmonic Green function Γ_ω for a logarithmically subharmonic reproducing weight ω on the unit disk $\mathbb{D}$. We may think of it as corresponding to a weighted creeping flow. By our main theorem, the function Γ_ω is positive. This suggests that for fixed $\zeta \in \mathbb{D}$, the flow has no eddy immediately near $\mathbb{T}$. The question arises: *is there only one swirl in this situation?*

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