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Conditional Expectation and Half–sided Translations

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Abstract:
The relations between conditional expextations and half–sided translations and half–sided modular inclusions will be investigated.

1. Introduction and results

In several applications one uses modular covariant subalgebras and the conditional expectations $\mathcal{E}$ which are associated with them (see Takesaki [8]). At the same time there often exist half–sided translations for the original algebra. In this situation one is interested in knowing the result of the application of the conditional expectation on the half–sided translations. In this paper we want to answer this question. One example, where this situation appears, is the investigation of Anosov dynamical systems, see, e.g. [7]. Another case is the investigation of tensor products, see e.g. [3].

We start with a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ and assume that $\mathcal{M}$ has a cyclic and separating vector Ω. The modular operator and the modular conjugation of the pair $(\mathcal{M}, \Omega)$ will be denoted by (Δ, J). In addition we assume, that there exists a $\pm$half–sided translation, i.e. a one–parametric continuous unitary $U(t)$ group with non–negative generator such that one has $U(t)\Omega = \Omega \; \forall t \in \mathrm{I\!R}$ and $\mathrm{Ad}\, U(t)\mathcal{M} \subset \mathcal{M}$ for $t \geq 0$ or for $t \leq 0$. The algebras $\mathrm{Ad}\, U(t)\mathcal{M}$ will be denoted by $\mathcal{M}(t)$. Moreover, we assume that $\mathcal{N} \subset \mathcal{M}$ is a modular covariant subalgebra of $\mathcal{M}$. E denotes the projection onto $[\mathcal{N}\Omega]$. Since the algebra $\mathcal{N}$ is covariant under the modular action it follows that E commutes with Δ. The restriction of $\mathcal{N}, \mathcal{E}$ and Δ to $E\mathcal{H}$ will be denoted by $\widehat{\mathcal{N}}, \widehat{\mathcal{E}}$, and $\widehat{\Delta}$ respectively. The main questions are:
When does $U(t)$ commute with E?
What can we say if $U(t)$ does not commute with E?

We want to show the following results:

1.1 Theorem.

Let $\mathcal{M}, \mathcal{N}, \mathcal{E}, E$ be as above and $U(t)$ be a +half–sided translation. Then the following statements are equivalent:

1. *The group $U(t)$ commutes with E.*
2. *One has $\mathrm{Ad}\, U(t)\widehat{\mathcal{N}} \subset \widehat{\mathcal{N}}$ for $t \geq 0$.*
3. *For every $t \geq 0$ $\widehat{\mathcal{E}}(\mathcal{M}(t))$ is a von Neumann algebra.*
4. *For one $t \geq 0$ $\widehat{\mathcal{E}}(\mathcal{M}(t))$ is a von Neumann algebra.*
5. *There exists a von Neumann algebra $\mathcal{P} \subset \mathcal{N}$ with $[\mathcal{P}\Omega] = E$ and $\mathrm{Ad}\, U(t)\mathcal{P} \subset \mathcal{N}$ for one $t > 0$.*

A similar result holds if $U(t)$ is a −half–sided translation.

The second result deals with the general case, namely the situation where $U(t)$ and E do not commute.

1.2 Theorem.

Let $\mathcal{M}, \mathcal{N}, U(t), \mathcal{E}, E$ be as above then there exists a continuous unitary group $V(t)$ on $E\mathcal{H}$ satisfying for $t \geq 0$ the relation

$$\mathrm{Ad}\, V(t)\widehat{\mathcal{N}} = \{\widehat{\mathcal{E}}(\mathrm{Ad}\, U(t)\mathcal{M})\}''.$$

A similar result holds if $U(t)$ is a −half–sided translation.

For the second theorem we will present two different proofs. The second demonstration is applicable for more general situations. In order to formulate it we need some explanation.

If $\mathcal{M}$ is a von Neumann algebra with cyclic and separating vector Ω then we call the anti–linear operator $S_\mathcal{M} := J_\mathcal{M}\Delta_\mathcal{M}^{1/2}$ the Tomita conjugation of $(\mathcal{M}, \Omega)$. In this section we will deal with operators of the same kind, i.e. operators S fulfilling:

(i) S is a densely defined closed anti–linear operator with domain of definition $\mathcal{D}(S)$.
(ii) $S^2 = \mathbb{1}$ on $\mathcal{D}(S)$.
(iii) $\Omega \in \mathcal{D}(S)$ and $S\Omega = \Omega$.

We will call such operators generalized Tomita conjugations.

Since S is closed it has a polar decomposition $S = J\Delta^{1/2}$. Then Δ is invertible and J is a conjugation, i.e.

$$J\Delta J = \Delta^{-1}, \qquad J = J^* = J^{-1}. \tag{1.1}$$

These properties follow from the condition $S^2 = \mathbb{1}$. (See e.g, Bratteli and Robinson [5] Prop.2.5.11.)

With this notation we obtain:

1.3 Theorem.

Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω and let $S_\mathcal{M}$ be the Tomita conjugation of $\mathcal{M}$. Let S be a generalized Tomita conjugation and assume $S_\mathcal{M}$ is an extension of S. Assume in addition that S is an extension of $\Delta_\mathcal{M}^{\mathrm{i}t} S \Delta_\mathcal{M}^{-\mathrm{i}t}$ for $t \leq 0$. Then:

1. *There exists a continuous unitary group $U(t)$ with*

α $U(t)\Omega = \Omega$ *for all* $t \in \mathbb{R}$.

β $U(t)$ *has a non–negative generator.*

2. *Between the modular group of* $\mathcal{M}$ *and* $U(t)$ *exist the relations*

$$\Delta^{\mathrm{i}t}_{\mathcal{M}} U(s) \Delta^{-\mathrm{i}t}_{\mathcal{M}} = U(\mathrm{e}^{-2\pi t} s), \qquad J_{\mathcal{M}} U(t) J_{\mathcal{M}} = U(-t).$$

3. *Define*

$$S_t = \Delta^{\mathrm{i}t}_{\mathcal{M}} S \Delta^{-\mathrm{i}t}_{\mathcal{M}}$$

which is monotonously increasing with t *and set*

$$S_\infty = \lim_{t\to\infty} S_t.$$

Then there holds for $s > 0$

$$U(s) S_\infty U(-s) = S_{-\frac{1}{2\pi}\log s}.$$

Notice: There exists a variant of this theorem which is obtained by replacing everywhere t by $-t$.

The statement of the theorem needs some explanation. By assumption the family $\Delta^{\mathrm{i}t}_{\mathcal{M}} S \Delta^{-\mathrm{i}t}_{\mathcal{M}}$ is increasing with t. Hence the projections onto the graphs are an increasing family of projections which converges strongly. Since all these projections are majorized by the projection onto the graph of $S_{\mathcal{M}}$ the limit is smaller or equal to the majorant.

2. Proofs and Applications

For the proof of the theorems we need the concept of $\mp$half–sided modular inclusions introduced by Wiesbrock [9,10]. This is in some sense the opposite of the concept of $\pm$half–sided translations. Let $\mathcal{N}$ be a subalgebra of $\mathcal{M}$ such that Ω is cyclic for $\mathcal{N}$ and such that $\mathrm{Ad}\,\Delta^{\mathrm{i}t}_{\mathcal{M}}\mathcal{N} \subset \mathcal{N}$ for $t \le 0$, then we say $\mathcal{N}$ fulfils the condition of $-$half–sided modular inclusion with respect to $\mathcal{M}$. If this is the case then exists a half–sided translation $U(t)$ with $\mathcal{N} = \mathrm{Ad}\,U(1)\mathcal{M}$.

Proof of Thm. 1.1. Since $\widehat{\mathcal{E}}(A), A \in \mathcal{M}$ is given by EAE we see that Ω is cyclic for $\widehat{\mathcal{E}}(\mathcal{M}(t))$. Hence the implications 1. $\to$ 2., 1. $\to$ 3. $\to$ 4. and also the implication 1. $\to$ 5. are trivially fulfilled. So it remains to show the converse implications. Since Δ commutes with E we obtain by the relation $\mathrm{Ad}\,\Delta^{\mathrm{i}t} U(s) = U(\mathrm{e}^{-2\pi t} s)$ the implication 4. $\to$ 3. (see [1]). We know by the separability of Ω that the map $\mathcal{N} \to \widehat{\mathcal{N}}$ is an isomorphism of von Neumann algebras. Let α present this isomorphism, then $\alpha^{-1}\widehat{\mathcal{E}}(\mathcal{M}(t)) \subset \mathcal{N}$ is a von Neumann algebra. Let σ^t denote the modular automorphism, then one obtains

$$\sigma^s(\alpha^{-1}\widehat{\mathcal{E}}(\mathcal{M}(t)) = \alpha^{-1}\widehat{\mathcal{E}}(\sigma^s\mathcal{M}(t)) = \alpha^{-1}\widehat{\mathcal{E}}(\mathcal{M}(\mathrm{e}^{-2\pi s} t)).$$

This implies in particular $\alpha^{-1}\widehat{\mathcal{E}}(\mathcal{M}(t_1)) \subset \alpha^{-1}\widehat{\mathcal{E}}(\mathcal{M}(t_2))$ for $t_1 \ge t_2$. Notice that $\widehat{\mathcal{E}}(\mathcal{M}(t)), t > 0$ fulfils the condition of $-$half–sided modular inclusion. Hence by Wiesbrock's result [9,10] exists on $E\mathcal{H}$ a $+$half–sided translation $V(t)$ with $\mathrm{Ad}\,V(t)\widehat{\mathcal{E}}(\mathcal{M}) =$

$\widehat{\mathcal{E}}(\mathcal{M}(t))$. By means of the isomorphism α we can tranport $V(t)$ to the whole Hilbert space. This means there exists an endomorphism γ^t such that $\alpha \circ \gamma^t(\mathcal{N}) = \mathrm{Ad}\, V(t)\mathcal{N}$ holds. This implies in particular $\mathcal{E}(\mathrm{Ad}\, U(t)\mathcal{M}) = \gamma^t(\mathcal{E}(\mathcal{M}))$. Applying this equation to the vacuum vector we obtain for $A \in \mathcal{M}$ the relation $EU(t)A\Omega = V(t)EA\Omega$. Restricting the elements A to $\mathcal{N}$ we obtain $EU(t)E = V(t)$ for $t > 0$. Since $U(t)$ and $V(t)$ are both unitary the last relation can hold only if $U(t)$ commutes with E. This shows 4. $\to$ 1. and also 2. $\to$ 1. Next we show 5. $\to$ 1. Since $\mathcal{P}\Omega$ is dense in $E\mathcal{H}$ and since $U(t)$ is unitary we obtain $U(t)E\mathcal{H} \subset E\mathcal{H}$. Since $\Delta^{\mathrm{i}t}$ commutes with E we get by the known relation between $\Delta^{\mathrm{i}t}$ and $U(s)$ (see [1]) $U(\mathrm{e}^{-2\pi s}t)E\mathcal{H} \subset E\mathcal{H}$ for all $s \in \mathbb{R}$. Using the spectrum condition for $U(t)$ we obtain by analytic continuation that the last inclusion is valid for all arguments of U. Since $U(t)$ is unitary this implies $U(s)E\mathcal{H} = E\mathcal{H}, \forall s \in \mathbb{R}$, which is equivalent to the commutativity. □

Proof of Thm. 1.2. Since $U(t)$ is a half–sided translation we obtain for every $t > 0$ the inclusion $\mathrm{Ad}\, \Delta^{\mathrm{i}s}\mathcal{M}(t) \subset \mathcal{M}(t)$ for $s \leq 0$. Since $\Delta^{\mathrm{i}s}$ commutes with E we obtain $\mathrm{Ad}\, \Delta^{\mathrm{i}s}\widehat{\mathcal{E}}(\mathcal{M}(t)) \subset \widehat{\mathcal{E}}(\mathcal{M}(t))$. This implies $\mathrm{Ad}\, \Delta^{\mathrm{i}s}\{\widehat{\mathcal{E}}(\mathcal{M}(t))\}'' \subset \{\widehat{\mathcal{E}}(\mathcal{M}(t))\}''$ for all $t > 0$ and $s \leq 0$. Since $\widehat{\Delta}$ is the modular group of $\widehat{\mathcal{N}} = \widehat{\mathcal{E}}(\mathcal{M})$ it follows that $\{\widehat{\mathcal{E}}(\mathcal{M}(t))\}''$ fulfils the condition of half–sided modular inclusion and hence by Wiesbrock's result [9,10] exist continuous unitary groups $V_t(s)$ on $E\mathcal{H}$ with $\mathrm{Ad}\, V_t(1)\widehat{\mathcal{N}} = \{\widehat{\mathcal{E}}(\mathcal{M}(t))\}''$. It remains to show that these groups coincide except for a scale factor. Since $\Delta^{\mathrm{i}s}$ commutes with E we obtain $\mathrm{Ad}\, \Delta^{\mathrm{i}s}\widehat{\mathcal{E}}(\mathcal{M}(1)) = \widehat{\mathcal{E}}(\mathcal{M}(\mathrm{e}^{-2\pi s}))$. This relation extends to the von Neumann algebras generated by these sets. Since we also have the relation $\mathrm{Ad}\, \widehat{\Delta}^{\mathrm{i}s}V_1(t) = V_1(\mathrm{e}^{-2\pi s}t)$ we obtain the relation $V_1(\mathrm{e}^{-2\pi s}) = V_{\mathrm{e}^{-2\pi s}}(1)$. This shows the theorem. □

We will show that this theorem is also a consequence of Thm. 1.3. For details see Prop. 2.9.

The proof of Thm. 1.3 is a variation of the proof of Wiesbrock's theorem on half–sided modular inclusions [9,10], but some explanations and preparations are needed.

We deal with the situation that we have a generalized Tomita conjugation S and a Tomita conjugation $S_{\mathcal{M}}$ which is an extension of S. This implies the relation $(1+\Delta_{\mathcal{M}})^{-1} \geq (1+\Delta)^{-1}$. This relation can easily be derived by looking at the graphs of S and $S_{\mathcal{M}}$. A consequence of this is that the operator–valued function $C(t) := \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}$ has a bounded analytic extension into the strip $S(0, \frac{1}{2})$. We are interested in determining the value of this function at the upper boundary. We obtain:

2.1 Lemma.

Let S be a generalized Tomita conjugation and $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$ such that the latter is an extension of S. Define $C(t) := \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}$. Then $C(t)$ has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$ and at the upper boundary one has

$$C(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}}C(t)J. \tag{2.2}$$

Moreover, the following estimate holds:

$$\|C(\tau)\| \leq 1.$$

Proof. Since $\Delta_{\mathcal{M}} \leq \Delta$ it follows by standard arguments that $C(t)$ has a bounded extension into the strip $S(0, \frac{1}{2})$. This extension is bounded in norm by 1. Choose $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S_{\mathcal{M}})$ then we have

$$\begin{aligned}(\varphi, C(t+\frac{\mathrm{i}}{2})\psi) &= (\Delta_{\mathcal{M}}^{\frac{1}{2}}\varphi, \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}\Delta^{-\frac{1}{2}}\psi)\\ &= (J_{\mathcal{M}}S_{\mathcal{M}}\varphi, \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi) = (J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi, S_{\mathcal{M}}\varphi).\end{aligned}$$

Since $S^*\psi \in \mathcal{D}(S^*)$ we find $J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi \in \mathcal{D}(S_{\mathcal{M}}^*)$. Hence we obtain

$$= (\varphi, S_{\mathcal{M}}^*J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi).$$

With $S_{\mathcal{M}}^*J_{\mathcal{M}} = J_{\mathcal{M}}S_{\mathcal{M}}$ and the commutation of $S_{\mathcal{M}}$ with $\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ we find

$$= (\varphi, J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}S_{\mathcal{M}}\Delta^{\mathrm{i}t}JS^*\psi).$$

Because $S_{\mathcal{M}}$ is an extension of S, we can replace $S_{\mathcal{M}}$ by S which commutes with $\Delta^{\mathrm{i}t}$. Hence we obtain

$$= (\varphi, J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}SJS^*\psi).$$

With $SJS^* = J$ we get

$$(\varphi, C(t+\frac{\mathrm{i}}{2})\psi) = (\varphi, J_{\mathcal{M}}C(t)J\psi).$$

Since $\mathcal{D}(S_{\mathcal{M}})$ and $\mathcal{D}(S^*)$ are both dense in $\mathcal{H}$ the lemma follows. □

Next we need a generalization of Thm. A in [2].

2.2 Lemma.
Let $S = J\Delta^{1/2}$ be a generalized Tomita conjugation. In addition let V be a unitary operator with
a. $V\mathcal{D}(S) \subset \mathcal{D}(S)$.
b. $V\Omega = \Omega$.
c. For $\psi \in \mathcal{D}(S)$ one has $SV\psi = VS\psi$.
Then:

The operator–valued function

$$\Delta^{-\mathrm{i}t}V\Delta^{\mathrm{i}t} =: V(t)$$

has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$ which fulfils the estimate

$$\|V(t+\mathrm{i}\tau)\| \leq 1. \quad 0 \leq \tau \leq \frac{1}{2}.$$

At the upper boundary $V(z)$ obeys the equation

$$V(t+\frac{\mathrm{i}}{2}) = JV(t)J.$$

Proof. Since S commutes with $\Delta^{\mathrm{i}t}$ it follows that S commutes with $V(t)$. Moreover, since $V\mathcal{D}(S) \subset \mathcal{D}(S)$ it follows by the usual argument that $\Delta^{-\mathrm{i}t}V\Delta^{\mathrm{i}t}$ has a bounded analytic continuation into $S(0, \frac{1}{2})$. Choose $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S)$. Then one has

$$\begin{aligned}(\varphi, V(t + \frac{\mathrm{i}}{2})\psi) &= (\Delta^{1/2}\varphi, \Delta^{-\mathrm{i}t}V\Delta^{\mathrm{i}t}\Delta^{-1/2}\psi) = (JS\varphi, V(t)JS^*\psi)\\ &= (S^*J\varphi, V(t)JS^*\psi) = (SV(t)JS^*\psi, J\varphi) = (V(t)J\psi, J\varphi) = (\varphi, JV(t)J\psi).\end{aligned}$$

This shows the lemma. □

Next we have a look at the expression $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}$ under the assumption of the theorem.

2.3 Lemma.

Assume S is an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ for $t \le 0$. Then for the operator–valued function $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t} =: C(t)$ the following holds:

(i) The inclusion properties:
- *α. $C(t)\mathcal{D}(S) \subset \mathcal{D}(S)$ for $t \ge 0$.*
- *β. $C(t)\mathcal{D}(S^*) \subset \mathcal{D}(S^*)$ for $t \le 0$.*
- *γ. $C(t + \frac{\mathrm{i}}{2})\mathcal{D}(S^*) \subset \mathcal{D}(S^*)$ for $t \in \mathbb{R}$.*

(ii) This implies:
- *α. For $\psi \in \mathcal{D}(S)$ one has $SC(t)\psi = C(t)S\psi$ provided $t \ge 0$.*
- *β. For $\varphi \in \mathcal{D}(S^*)$ one has $S^*C(t)\varphi = C(t)S^*\varphi$ if $t \le 0$.*
- *γ. For $\varphi \in \mathcal{D}(S^*)$ one has $S^*C(t + \frac{\mathrm{i}}{2})\varphi = C(t + \frac{\mathrm{i}}{2})S^*\varphi$ for all $t \in \mathbb{R}$.*

Proof. S is for $t \le 0$ an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}$. This implies $\Delta_{\mathcal{M}}^{\mathrm{i}t}\mathcal{D}(S) \subset \mathcal{D}(S) \subset \mathcal{D}(S_{\mathcal{M}})$. Hence we obtain $C(t)\mathcal{D}(S) \subset \mathcal{D}(S)$ for $t \ge 0$. Next choose $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S)$ then we obtain for $t \le 0$:

$$\begin{aligned}(\psi, S\Delta_{\mathcal{M}}^{\mathrm{i}t}S\varphi) &= (\psi, S_{\mathcal{M}}\Delta_{\mathcal{M}}^{\mathrm{i}t}S\varphi)\\ &= (\psi, \Delta_{\mathcal{M}}^{\mathrm{i}t}S_{\mathcal{M}}S\varphi) = (\psi, \Delta_{\mathcal{M}}^{\mathrm{i}t}\varphi) = (\Delta_{\mathcal{M}}^{-\mathrm{i}t}\psi, \varphi).\end{aligned}$$

On the other hand we get

$$(\psi, S\Delta_{\mathcal{M}}^{\mathrm{i}t}S\varphi) = (S\varphi, \Delta_{\mathcal{M}}^{-\mathrm{i}t}S^*\psi).$$

Since the expression is continuous in φ we conclude $\Delta_{\mathcal{M}}^{-\mathrm{i}t}S^*\psi \in \mathcal{D}(S^*)$ and from $S^*\mathcal{D}(S^*) = \mathcal{D}(S^*)$ we get for $t \le 0$ $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\mathcal{D}(S^*) \subset \mathcal{D}(S^*)$. This implies (i),$\beta$. Using Lemma 2.1 we obtain

$$C(t + \frac{\mathrm{i}}{2})\mathcal{D}(S^*) = J_{\mathcal{M}}C(t)J\mathcal{D}(S^*) = J_{\mathcal{M}}C(t)\mathcal{D}(S).$$

Because of $\mathcal{D}(S) \subset \mathcal{D}(S_{\mathcal{M}})$ we obtain by the definition of $C(t)$ the inclusion $C(t + \frac{\mathrm{i}}{2})\mathcal{D}(S^*) \subset J_{\mathcal{M}}\mathcal{D}(S_{\mathcal{M}}) = \mathcal{D}(S_{\mathcal{M}}^*) \subset \mathcal{D}(S^*)$. This shows (i),$\gamma$.

For $t \geq 0$ we obtain from $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\mathcal{D}(S) \subset \mathcal{D}(S) \subset \mathcal{D}(S_{\mathcal{M}})$

$$\begin{aligned} SC(t)\mathcal{D}(S) &= S\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}\mathcal{D}(S) = S_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}\mathcal{D}(S) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}S_{\mathcal{M}}\Delta^{\mathrm{i}t}\mathcal{D}(S) \\ &= \Delta_{\mathcal{M}}^{-\mathrm{i}t}S\Delta^{\mathrm{i}t}\mathcal{D}(S) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}S\mathcal{D}(S) = C(t)S\mathcal{D}(S). \end{aligned}$$

Next we calculate for $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S)$ and $t \leq 0$

$$(\varphi, S^*C(t)\psi) = (\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}\psi, S\varphi) = (\Delta^{\mathrm{i}t}\psi, \Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{\mathrm{i}t}\varphi).$$

As $\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ is the generalized Tomita conjugation with domain $\Delta_{\mathcal{M}}^{\mathrm{i}t}\mathcal{D}(S) \subset \mathcal{D}(S)$ it follows that $(\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t})^*$ is an extension of S^*. This implies

$$= (\Delta_{\mathcal{M}}^{\mathrm{i}t}\varphi, (\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t})^*\Delta^{\mathrm{i}t}\psi) = (\Delta_{\mathcal{M}}^{\mathrm{i}t}\varphi, S^*\Delta^{\mathrm{i}t}\psi) = (\varphi, \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}S^*\psi).$$

This shows (ii),β. Finally

$$S^*C(t+\frac{\mathrm{i}}{2})\mathcal{D}(S^*) = S^*J_{\mathcal{M}}C(t)J\mathcal{D}(S^*).$$

As in the proof of (i),γ we have $J_{\mathcal{M}}C(t)J\mathcal{D}(S^*) \subset \mathcal{D}(S^*_{\mathcal{M}}) \subset \mathcal{D}(S^*)$. Hence we obtain

$$= S^*_{\mathcal{M}}J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}J\mathcal{D}(S^*) = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}S_{\mathcal{M}}\Delta^{\mathrm{i}t}J\mathcal{D}(S^*).$$

Since $S_{\mathcal{M}}$ is an extension of S we get

$$= J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}S\Delta^{\mathrm{i}t}J\mathcal{D}(S^*) = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\mathcal{D}(S^*) = C(t+\frac{\mathrm{i}}{2})S^*\mathcal{D}(S^*).$$

This shows the lemma. □

$C(t)$ has an analytic extension into $S(0,\frac{1}{2})$. For $t \geq 0$ it maps $\mathcal{D}(S)$ into $\mathcal{D}(S)$ and for the rest of the boundary it maps $\mathcal{D}(S^*)$ into $\mathcal{D}(S^*)$. Therefore, we will map $S(0,\frac{1}{2})$ bi–holomorphic onto $S(0,\frac{1}{2})$ in such a way that $\mathbb{R}^+$ ist mapped onto $\mathbb{R}$ and the rest of the boundary is mapped onto $\frac{\mathrm{i}}{2}+\mathbb{R}$. This is achieved by the transformation

$$\zeta = \frac{1}{2\pi}\log(\mathrm{e}^{2\pi z}-1), \qquad z = \frac{1}{2\pi}\log(\mathrm{e}^{2\pi\zeta}+1).$$

We introduce

$$B(t) := C(\frac{1}{2\pi}\log(\mathrm{e}^{2\pi t}+1)), \tag{2.3}$$

then together with Lemma 2.3 holds

$$\begin{aligned} B(t)\mathcal{D}(S) \subset \mathcal{D}(S), \text{ for } t \in \mathbb{R} \quad &\text{and} \quad SB(t)\mathcal{D}(S) = B(t)S\mathcal{D}(S), \\ B(t+\frac{\mathrm{i}}{2})\mathcal{D}(S^*) \subset \mathcal{D}(S^*), \text{ for } t \in \mathbb{R} \quad &\text{and} \quad S^*B(t+\frac{\mathrm{i}}{2})\mathcal{D}(S^*) = B(t+\frac{\mathrm{i}}{2})S^*\mathcal{D}(S^*). \end{aligned} \tag{2.4}$$

The last inclusion is valid with the possible exception of the point $\frac{\mathrm{i}}{2}$. Next we show:

2.4 Lemma.

Define $B(s,t) = \Delta^{-\mathrm{i}s} B(t) \Delta^{\mathrm{i}s}$ with $B(t)$ from Eq. (2.3). $B(s,t)$ has an analytic extension into the tube based on the quadrangle with the corners

$$(\Im m\, s, \Im m\, t) = (0,0),\ (\frac{1}{2}, -\frac{1}{2}),\ (\frac{1}{2}, 0),\ (0, \frac{1}{2}). \tag{2.5}$$

In the domain of holomorphy one has

$$\|B(\sigma,\tau)\| \leq 1.$$

In the four corners $B(\sigma,\tau)$ takes the values

$$\begin{aligned} B(s,t) &= \Delta^{-\mathrm{i}s} B(t) \Delta^{\mathrm{i}s}, \\ B(s + \frac{\mathrm{i}}{2}, t) &= \Delta^{-\mathrm{i}s} J B(t) J \Delta^{\mathrm{i}s}, \\ B(s, t + \frac{\mathrm{i}}{2}) &= \Delta^{-\mathrm{i}s} B(t + \frac{\mathrm{i}}{2}) \Delta^{\mathrm{i}s}, \\ B(s + \frac{\mathrm{i}}{2}, t - \frac{\mathrm{i}}{2}) &= \Delta^{-\mathrm{i}s} J B(t + \frac{\mathrm{i}}{2}) J \Delta^{\mathrm{i}s}. \end{aligned}$$

Proof. For t real we get by Lemma 2.2 in s an analytic extension into $S(0,\frac{1}{2})$ which is bounded in norm by 1. Moreover, we have $B(s+\frac{\mathrm{i}}{2}, t) = JB(s,t)J = \Delta^{-\mathrm{i}s} J B(t) J \Delta^{\mathrm{i}s}$. For s real Lemma 2.1 yields an analytic extension in t into $S(0,\frac{1}{2})$ which is also bounded in norm by 1. Moreover, we have $B(s, t+\frac{\mathrm{i}}{2}) = \Delta^{-\mathrm{i}s} B(t+\frac{\mathrm{i}}{2}) \Delta^{\mathrm{i}s}$. Since J is anti–linear the expression $JB(t)J$ can be analytically continued into $S(-\frac{1}{2}, 0)$ wich is norm–bounded by 1. At the lower boundary one finds $B(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) = \Delta^{-\mathrm{i}s} J B(t+\frac{\mathrm{i}}{2}) J \Delta^{\mathrm{i}s}$. Using the Malgrange-Zerner theorem (see [6]) we obtain the statement of the lemma. □

Now we are prepared for the first crucial step:

2.5 Proposition.

Between the group $\Delta^{\mathrm{i}s}$ and the operator–valued function $B(t)$ exist the relations

$$\Delta^{\mathrm{i}s} B(t) \Delta^{-\mathrm{i}s} = B(t-s) \quad \text{and} \quad JB(t)J = B(t + \frac{\mathrm{i}}{2}).$$

Proof. Choose $\psi \in \mathcal{D}(S)$ and $\varphi \in \mathcal{D}(S^*)$ and define the two functions

$$\begin{aligned} F^+(s,t) &= (\varphi, B(s,t)\psi) = (\varphi, \Delta^{-\mathrm{i}s} B(t) \Delta^{\mathrm{i}s} \psi), \\ F^-(s,t) &= (S\psi, B(s,t)^* S^* \varphi) = (S\psi, \Delta^{-\mathrm{i}s} B(t)^* \Delta^{\mathrm{i}s} S^* \varphi). \end{aligned}$$

By Lemma 2.4 $F^+(s,t)$ has a bounded analytic extension into the tube given by Eq. (2.5) and $F^-(s,t)$ into the conjugate complex of that domain, which is also the negative of the domain given by Eq. (2.5). By Eq. (2.4) we obtain for real s,t

$$\begin{aligned} F^+(s,t) &= (S^*S^*\varphi, \Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}\psi) = (S\Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}\psi, S^*\varphi) \\ &= (\Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}S\psi, S^*\varphi) = F^-(s,t). \end{aligned}$$

Moreover, one obtains with Eq. (2.4) and Lemma 2.4

$$\begin{aligned} F^+(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) &= (S^*S^*\varphi, \Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}\psi) = (S\Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}\psi, S^*\varphi) \\ &= (\Delta^{-\mathrm{i}s}JS^*B(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}\psi, S^*\varphi) = (\Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})S^*J\Delta^{\mathrm{i}s}\psi, S^*\varphi) \\ &= (\Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}S\psi, S^*\varphi) = F^-(s-\frac{\mathrm{i}}{2}, t+\frac{\mathrm{i}}{2}). \end{aligned}$$

Using the edge of the wedge theorem we obtain a function which is periodic, i.e.

$$F(s,t) = F(s+n\mathrm{i}, t-n\mathrm{i}), \qquad n \in \mathbb{Z}.$$

The discontinuity which might exist at $\frac{\mathrm{i}}{2}$ is harmless, becase the boundary values coincide in the sense of distributions, for details see e.g. [4]. Since $F(\sigma,\tau)$ is bounded by $\max\{\|\psi\|\|\varphi\|, \|S\psi\|\|S^*\varphi\|\}$ the function must be constant in the direction of periodicity, i,e.

$$F(s,t) = F(s+z, t-z), \qquad z \in \mathbb{C}.$$

Choosing $z=-s$ and inserting the expression for F we obtain:

$$(\varphi, \Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}\psi) = (\varphi, B(t+s)\psi).$$

For $s=\frac{\mathrm{i}}{2}$ and $z=-\frac{\mathrm{i}}{2}$ one finds

$$(\varphi, JB(t)J\psi) = (\varphi, B(t+\frac{\mathrm{i}}{2})\psi).$$

Since $\mathcal{D}(S)$ and $\mathcal{D}(S^*)$ are both dense in $\mathcal{H}$ we obtain the statement of the proposition. □

The last result is the basis of the following

2.6 Proposition.

The operator–valued function $C(t)$ is a commutative family of unitary operators. Moreover, there exists a continuous unitary group $U(s)$ with non–negative generator such that

$$C(t) = U(\mathrm{e}^{2\pi t}-1) \tag{2.6}$$

holds.

The proof of this statement is based on the last proposition and it is an exact copy of the corresponding part of the proof of [2] Thm. 4.1. Therefore it does not need to be repeated here.

Proof of Theorem 1.3. The first statement of the theorem is the content of Proposition 2.6. We know that $C(t)$ fulfils the cocycle relation, which we use in the form $\Delta^{-\mathrm{i}s}C(t)\Delta^{\mathrm{i}s} = C(s+t)C(s)^*$. Inserting Eq.(2.6) we find

$$\Delta_{\mathcal{M}}^{-\mathrm{i}s}U(\mathrm{e}^{2\pi t}-1)\Delta_{\mathcal{M}}^{\mathrm{i}s} = U(\mathrm{e}^{2\pi(s+t)}-1)U(-\mathrm{e}^{2\pi s}+1) = U(\mathrm{e}^{2\pi s}(\mathrm{e}^{2\pi t}-1)).$$

Since $U(t)$ fulfils the spectrum condition the last equation can analytically be continued to arbitrary arguments. This shows the first part of statement 2. From (2.6) we obtain $C(\frac{\mathrm{i}}{2}) = U(-2)$. Hence we obtain $J_{\mathcal{M}} = C(\frac{\mathrm{i}}{2})J = U(-2)J$. If we insert Eq. (2.3) into the second expression of Proposition 2.5 we get

$$\mathrm{Ad}\, JC(\frac{1}{2\pi}\log(\mathrm{e}^{2\pi t}+1)) = C(\frac{1}{2\pi}\log(-\mathrm{e}^{2\pi t}+1)).$$

Using Eq. (2.6) this reads $\mathrm{Ad}\, JU(\mathrm{e}^{2\pi t}) = U(-\mathrm{e}^{2\pi t})$. With the above expression for $J_{\mathcal{M}}$ we obtain

$$\mathrm{Ad}\, J_{\mathcal{M}}U(\mathrm{e}^{2\pi t}) = \mathrm{Ad}\,\{U(-2)J\}U(\mathrm{e}^{2\pi t}) = U(-\mathrm{e}^{2\pi t}).$$

By analytic continuation we obtain the second relation of statement 2. Finally with $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{\mathrm{i}t}S = S_t$ and $\mathrm{Ad}\,\Delta^{\mathrm{i}t}S = S$ we obtain $\mathrm{Ad}\, C(-t)S = S_t$. Inserting Eq. (2.6) we find $\mathrm{Ad}\, U(\mathrm{e}^{-2\pi t}-1)S = S_t$. With $S_\infty = \lim_{t\to\infty} S_t = \lim_{t\to\infty}\mathrm{Ad}\, U(\mathrm{e}^{-2\pi t}-1)S$ we get $S_t = \mathrm{Ad}\, U(\mathrm{e}^{-2\pi t})S_\infty$ or $\mathrm{Ad}\, U(s)S_\infty = S_{-\frac{1}{2\pi}\log s}$, $s>0$. This proves the theorem. □

From Thm. 6.2.2 one can draw several conclusions. We start with the following result:

2.7 Corollary.

Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω and let $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$. Let S be a generalized Tomita conjugation and assume $S_{\mathcal{M}}$ is an extension of S. Assume also that S is an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ for $t \le 0$. If we have in addition

$$S_{\mathcal{M}} = \lim_{t\to\infty} S_t,$$

then S is the Tomita conjugation of a von Neumann algebra $\mathcal{N}$ which has Ω as cyclic and separating vector. Moreover, on has

$$\mathcal{N} = U(1)\mathcal{M}U(-1).$$

2.8 Remark.

Unfortunately I could not show that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, although it is suggested by the fact that $S_{\mathcal{M}}$ is an extension of $S_{\mathcal{N}}$. Up to now one needs additional information in order to conclude that $\mathcal{N}$ is a subalgebra of $\mathcal{M}$.

Proof of the Corollary. With $S_\infty = \lim_{t\to\infty} S_t$ we know from Thm. 6.2.2 the relation $S = U(1)S_\infty U(-1)$. With $S_\infty = S_{\mathcal{M}}$ it follows $S = U(1)S_{\mathcal{M}}U(-1)$. Since $\mathcal{M}\Omega$ is a core for $S_{\mathcal{M}}$ it follows with $\mathcal{N} = U(1)\mathcal{M}U(-1)$ that $\mathcal{N}\Omega$ is a core for S. Hence the corollary is proved. □

In connection with conditional expectations one can conclude that the algebra $\mathcal{N}$, described in Corollary 2.7, is a subalgebra of $\mathcal{M}$.

2.9 Proposition.

Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω. Assume $\mathcal{N}$ is a modular covariant subalgebra of $\mathcal{M}$ and $\mathcal{E}$ the associated conditional expectation. Denote by $\widehat{\mathcal{N}}$ resp. $\widehat{\mathcal{E}}$ the restriction of $\mathcal{N}$ resp. $\mathcal{E}$ to the cyclic subspace of $\mathcal{N}$. Assume $V(t)$ is a +half–sided translation for $\mathcal{M}$. Then:
(i) $\mathcal{E}(V(t)\mathcal{M}V(-t))$ is dense in the von Neumann algebra $\{\mathcal{E}(V(t)\mathcal{M}V(-t))\}''$.
(ii) There exists a +half–sided translation for $\widehat{\mathcal{N}} = \widehat{\mathcal{E}}(\mathcal{M})$ with

$$U(t)\widehat{\mathcal{N}}U(-t) = \{\widehat{\mathcal{E}}(V(t))\mathcal{M}V(-t))\}''.$$

Proof. From the relation $\widehat{\mathcal{E}}(V(t)\mathcal{M}V(-t))\Omega = EV(t)\mathcal{M}\Omega$ we see that $\widehat{\mathcal{E}}(V(t)\mathcal{M}V(-t))\Omega$ is dense in $E\mathcal{H}$. Let $S_{-\frac{1}{2\pi}\log t}$ be the map $EV(t)AV(-t)\Omega \longrightarrow EV(t)A^*V(-t)\Omega$. Since $J_{\mathcal{M}}\widehat{\mathcal{N}}J_{\mathcal{M}}$ is the commutant of $\widehat{\mathcal{N}}$ in $E\mathcal{H}$ it follows that $S_{-\frac{1}{2\pi}\log t}$ is pre–closed. Denote the closure again by $S_{-\frac{1}{2\pi}\log t}$. Since $V(t)\mathcal{M}V(-t) \subset V(t_0)\mathcal{M}V(-t_0)$ for $t \geq t_0$ we obtain with $\Delta^{\mathrm{i}t}_{\mathcal{M}}V(s)\Delta^{-\mathrm{i}t}_{\mathcal{M}} = V(\mathrm{e}^{-2\pi t}s)$ and with $\Delta^{\mathrm{i}t}_{\widehat{\mathcal{N}}} = \Delta^{\mathrm{i}t}_{\mathcal{M}}|E\mathcal{H}$ that $S_{\widehat{\mathcal{N}}}$ is an extension of S_0 which is an extension of $\Delta^{\mathrm{i}t}_{\widehat{\mathcal{N}}}S_0\Delta^{-\mathrm{i}t}_{\widehat{\mathcal{N}}}$ for $t \leq 0$. Hence the family $\{S_t\}$ fulfils the conditions of Thm. 1.3. Consequently exists a +half–sided translation $U(t)$ of $\widehat{\mathcal{N}}$ with

$$S_t = U(\mathrm{e}^{2\pi t})S_{\widehat{\mathcal{N}}}U(-\mathrm{e}^{2\pi t}).$$

Since $\{EV(\mathrm{e}^{2\pi t})A\Omega; A \in \mathcal{M}\}$ is a core for S_t there exists an operator B affiliated with $\mathcal{N}$ such that $U(\mathrm{e}^{2\pi t})BU(-\mathrm{e}^{2\pi t})\Omega = EV(\mathrm{e}^{2\pi t})AV(-\mathrm{e}^{2\pi t})\Omega$ holds. (See [BR79] Prop. 2.9.5.) Since Ω is separating for $\widehat{\mathcal{N}}$ we obtain $U(\mathrm{e}^{2\pi t})\hat{B}U(-\mathrm{e}^{2\pi t}) = EV(\mathrm{e}^{2\pi t})AV(-\mathrm{e}^{2\pi t})E$ which implies $\|B\| \leq \|A\|$. Hence we get $EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E \subset U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}U(-\mathrm{e}^{2\pi t})$. The sets $EV(\mathrm{e}^{2\pi t})\mathcal{M}\Omega$ and $U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}\Omega$ are both a core for S_t which implies that $EV(\mathrm{e}^{2\pi t})\mathcal{M}\Omega$ is dense in $U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}\Omega$ in the graph topology of S_t. Since the graph topology of S_t is stronger than the Hilbert space topology we get the density in the Hilbert space topology. As Ω is separating and since $EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E$ is convex we conclude that $EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E$ is strongly dense in $U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}U(-\mathrm{e}^{2\pi t})$. Hence the theorem is proved. □

An application of Thm. 1.1 can be found in [7] Thm. (3.10). Another problem is the following: Let $U(t)$ be a half–sided translation for $\mathcal{M}$ and $\mathcal{N}$ a modular covariant subalgebra of $\mathcal{M}$. Assume $U(t)$ does not commute with the conditional expectation $\mathcal{E}$:

$\mathcal{M} \to \mathcal{N}$. Can one find modular covariant subalgebras $\tilde{\mathcal{N}}^{\pm}$ with $\mathcal{N} \subset \tilde{\mathcal{N}}^{+}$ and $\tilde{\mathcal{N}}^{-} \subset \mathcal{N}$ in such a way that $U(t)$ commute with the corresponding conditional expectations? The answer is the following:

2.10 Lemma:

Let $U(t)$ be a half–sided translation for $\mathcal{M}$ and $\mathcal{N}$ a modular covariant subalgebra of $\mathcal{M}$. Assume $U(t)$ does not commute with the conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$. Then there exists a minimal modular covariant subalgebra $\tilde{\mathcal{N}}^{+}$ of $\mathcal{M}$ such that $\mathcal{N} \subset \tilde{\mathcal{N}}^{+}$ and $U(t)$ commute with the conditional expectation $\mathcal{E}^{+} : \mathcal{M} \to \tilde{\mathcal{N}}^{+}$. There exists also a maximal modular covariant subalgebra $\tilde{\mathcal{N}}^{-}$ of $\mathcal{M}$ such that $\mathcal{N} \supset \tilde{\mathcal{N}}^{-}$ and $U(t)$ commutes with the conditional expectation $\mathcal{E}^{-} : \mathcal{M} \to \tilde{\mathcal{N}}^{-}$.

Proof. Define $\tilde{\mathcal{N}}^{+} = \bigvee_{t\geq 0} \operatorname{Ad} U(t)\mathcal{N}$. This implies by the invariance of $\mathcal{N}$ $\operatorname{Ad} \Delta^{\mathrm{i}t}\tilde{\mathcal{N}}^{+} = \bigvee_{t\geq 0} \operatorname{Ad} U(\mathrm{e}^{-2\pi s}t)\mathcal{N} = \tilde{\mathcal{N}}^{+}$. Hence $\tilde{\mathcal{N}}^{+}$ is a modular covariant subalgebra of $\mathcal{M}$ containing $\mathcal{N}$. Moreover we obtain for $s > 0$ $\operatorname{Ad} U(s)\tilde{\mathcal{N}}^{+} = \bigvee_{t\geq s} \operatorname{Ad} U(t)\mathcal{N} \subset \tilde{\mathcal{N}}^{+}$. Hence $U(t)$ commutes with $\mathcal{E}^{+}$ by Thm. 1.1. From the construction we see that $\tilde{\mathcal{N}}^{+}$ is the minimal modular subalgebra of $\mathcal{M}$ with the stated properties.

For defining $\tilde{\mathcal{N}}^{-}$ we use the commutant of $\mathcal{N}$ and set $\tilde{\mathcal{N}}^{-} = \{\bigvee_{t\leq s} \operatorname{Ad} U(t)\mathcal{N}'\}'$. That this algebra fulfils the stated requirements is shown as before. □

In the last result we have seen, that one can vary the modular covariant subalgebra if $U(t)$ and $\mathcal{E}$ do not commute in such a way that $U(t)$ commutes with the new conditional expectation. But in some situation it might be better to keep the modular subalgebra fixed and try to change the half–sided translation in such a way that one obtains commutation. That this is indeed possible is the content of the next result.

2.11 Lemma:

Let $U(t)$ be a half–sided translation for $\mathcal{M}$ and $\mathcal{N}$ a modular covariant subalgebra of $\mathcal{M}$. Assume $U(t)$ does not commute with the conditional expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$. Then there exists a modified half–sided translation $W(t)$ which commutes with $\mathcal{E}$.

Proof. Let α be the isomorphism $\alpha : \mathcal{N} \to \widehat{\mathcal{N}}$. Define the algebra $\mathcal{P}(t) = \alpha^{-1}\{\widehat{\mathcal{E}}(\mathcal{M}(t))\}''$. This algebra is contained in $\mathcal{N}$ by construction. Since the modular action commutes with α we obtain $\operatorname{Ad} \Delta^{\mathrm{i}s}\mathcal{P}(t) = \mathcal{P}(\mathrm{e}^{\mathrm{i}s}t)$. Defining $\tilde{\mathcal{M}}(t) = \mathcal{P}(t) \vee \mathcal{M}(t)$ we get $\operatorname{Ad} \Delta^{\mathrm{i}s}\tilde{\mathcal{M}}(t) = \tilde{\mathcal{M}}(\mathrm{e}^{\mathrm{i}s}t)$. This shows that $\tilde{\mathcal{M}}(1)$ fulfils the condition of $-$half–sided modular inclusion. Therefore exists a half–sided translation $W(t)$ with $\operatorname{Ad} W(t)\mathcal{M} = \tilde{\mathcal{M}}(t)$. It remains to show that $W(t)$ commuts with $\mathcal{E}$. We know from construction $\mathcal{E}(\mathcal{M}(t)) \subset \mathcal{P}(t)$. Let S_t be the map $A\Omega \to A^*\Omega$, $A \in \mathcal{P}(t) \cup \mathcal{M}(t)$. Since $\mathcal{M}'$ commutes with $\mathcal{P}(t) \cup \mathcal{M}(t)$ it follows that S_t is a closable operator. Denoting its closure again by S_t then it fulfils the conditions of Thm. 1.3. Hence we obtain $\tilde{\mathcal{M}}(t)\Omega =$ closure of $\{\mathcal{P}(t) \cup \mathcal{M}(t)\}\Omega$ in the graph topology of S_t. From this we get $\mathcal{E}(\tilde{\mathcal{M}}(t)\Omega = \mathcal{E}(\text{closure } \{\mathcal{P}(t) \cup \mathcal{M}(t)\}\Omega = E \text{ closure } \{\mathcal{P}(t) \cup \mathcal{M}(t)\}\Omega$. Since with $A \in \{\mathcal{P}(t) \cup \mathcal{M}(t)\}$ also A^* belongs to $\{\mathcal{P}(t) \cup \mathcal{M}(t)\}$ we obtain that the closure commutes with E. But from $E\{\mathcal{P}(t) \cup \mathcal{M}(t)\}E = \mathcal{P}(t)E$ it follows that $\mathcal{E}(\tilde{\mathcal{M}}(t)) = \mathcal{P}(t)$ holds and hence $W(t)$ commutes with $\mathcal{E}$ by Thm. 1.1. □

2.12 Remarks. In special situations it might happen that the algebra $\tilde{\mathcal{N}}^+$ defined in Lemma 2.10 coincides with $\mathcal{M}$ or that the algebra $\tilde{\mathcal{N}}^-$ coincides only with the center of $\mathcal{M}$. It also can happen that the algebra $\tilde{\mathcal{M}}(t)$ defined in the proof of Lemma 2.11 coincides with $\mathcal{M}$. In this case the half–sided translations $W(t)$ are trivial.

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