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ESI - Workshop on Geometrical Aspects of Spectral Theory Matrei in East Tyrolia, Austria July 5 - 12 1999

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This workshop was a consequence of 2 activities which took place at the Erwin Schrödinger Institute (ESI) in Vienna, Austria simultaneously in 1998: Spectral Geometry (organized by L. Friedlander and V. Guillemin) and Schrödinger operaters with magnetic fields (organized by I. Herbst, T. Hoffmann-Ostenhof and J. Yngvason). Already during these activities in spring 1998 it became obvious that the 2 topics have a great overlap and that researchers in the one field can profit from discussions with experts in the other field. In fact the overlap in the workshop was even stronger, colleagues who had participated in the Spectral Geometry activity gave talks on Schrödinger operators and colleagues from Schrödinger operators discussed Spectral Geometry. The following abstracts and open problems should document this.

Leonid Friedlander and Thomas Hofmann-Ostenhof

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Some Eigenvalue Comparison Results for Domains in $\mathbb{S}^n$ and for Annular Domains in $\mathbb{R}^n$ [1]

Mark S. Ashbaugh

Abstract

For bounded domains in Euclidean space, various inequalities between the Dirichlet and Neumann eigenvalues of the Laplacian are known. The main contributions are due to Payne (1955), Aviles (1986), and Levine and Weinberger (1986), culminating in the 1991 proof by Leonid Friedlander that the $(k+1)$th Neumann eigenvalue is always less than or equal to the kth Dirichlet eigenvalue. We report on our recent progress toward extending some of these prior results to bounded domains in homogeneous spaces (such as the sphere $\mathbb{S}^n$).

In addition, we present some monotonicity results for the behavior of the first eigenvalue of the Dirichlet Laplacian on a domain with a moveable hole, such as the region in the plane between two circles (nested, but nonconcentric), which we call an *eccentric annular domain.*

The main new results are based on joint work with Lotfi Hermi (Dirichlet-Neumann eigenvalue comparisons for $\mathbb{S}^n$, etc.) and Thierry Chatelain (eccentric annular domains). Earlier joint results with Rafael Benguria and Howard Levine (separately) also make appearances.

Open Problems

We begin by listing several open problems concerning the eigenvalues of the Dirichlet Laplacian on a bounded domain in Euclidean space $\mathbb{R}^n$. We denote the eigenvalues (counting multiplicities) by $\{\lambda_m\}_{m=1}^{\infty}$.

1. A conjecture of Payne, Pólya, and Weinberger [**28**]: For a bounded domain $\Omega \subset \mathbb{R}^n$, show that

$$\frac{\lambda_{m+1}}{\lambda_m} < \left.\frac{\lambda_2}{\lambda_1}\right|_{\text{ball}} \quad \text{for } m \geq 4. \tag{1}$$

Indeed, it would be of interest to find any general bound better than $1 + 4/n$ (this is the bound which was established by Payne, Pólya, and Weinberger in their original paper [**28**]; see also [**27**] for λ_2/λ_1). Inequality (1) was established for $m = 1$ with a nonstrict inequality in

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[**5**], [**7**], and for $m = 2$ and 3 in [**6**] and [**8**], respectively. In fact, [**8**] shows that the $m = 2$ and 3 cases follow from the stronger inequality $\lambda_4/\lambda_2 < (\lambda_2/\lambda_1)|_{\text{ball}}$. This inequality is an easy consequence of the $m = 1$ result and the fact that any eigenfunction for λ_2 has exactly two nodal domains. The $m = 1$ case of (1) (with $\leq$) has often been referred to in the literature as *the Payne-Pólya-Weinberger (PPW) conjecture*. The conjecture above might then be referred to as *the extended PPW conjecture*.

2. A second conjecture of Payne, Pólya, and Weinberger [**28**]: For a bounded domain $\Omega \subset \mathbb{R}^2$, show that

$$\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq \left.\frac{\lambda_2 + \lambda_3}{\lambda_1}\right|_{\text{disk}}. \tag{2}$$

The analogous conjecture for $\Omega \subset \mathbb{R}^n$ reads

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq \left.\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1}\right|_{n\text{-ball}}. \tag{3}$$

In $\mathbb{R}^2$, for example, the right-hand side of (2) is approximately 5.077 (this is the value of $(\lambda_2 + \lambda_3)/\lambda_1$ for a disk, which is twice the value of λ_2/λ_1 for a disk), while the best upper bound currently established for $(\lambda_2 + \lambda_3)/\lambda_1$ for an arbitrary domain is only approximately 5.507 (see [**11**]).

3. Find the optimal upper bound for

$$\frac{\lambda_3}{\lambda_1} \tag{4}$$

among all bounded domains Ω contained in $\mathbb{R}^2$ (or $\mathbb{R}^n$), and find the shape of domain that maximizes it. In $\mathbb{R}^2$ the best upper bound found so far is approximately 3.831 (see [**11**]), while the highest value of λ_3/λ_1 found so far is approximately 3.2 (the exact value here is 35/11 and occurs for the rectangle having sides in the proportion $\sqrt{8} : \sqrt{3}$, which is easily found to give the maximum of λ_3/λ_1 over all rectangles).

A related problem is that of minimizing λ_3 over all domains in $\mathbb{R}^n$ of fixed n-volume (or just over domains in $\mathbb{R}^2$ of fixed area). For some discussion of this problem, see Problems 7 and 8 in [**2**] (but please note that inequalities (27)-(30) have been withdrawn as conjectures, since (27) and (28) are certainly not generally valid, and therefore (29) and (30) are in doubt for all dimensions $n \geq 3$ as well).

4. For bounded *convex* domains Ω contained in $\mathbb{R}^n$, show that

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}, \tag{5}$$

where d denotes the diameter of Ω. Indeed, one might conjecture the same inequality for the Schrödinger operator $H = -\Delta + V(x)$, where the potential V is a convex function on Ω (convexity of V is needed to avoid "double-well" situations). In fact, the positive results discussed below were all established in this setting (i.e., for Schrödinger operators with convex potentials). Versions of inequality (5) with a smaller constant on the right-hand side are known. For example, Yu and Zhong [**38**] obtained the lower bound π^2/d^2 (i.e., (5) but without the factor of 3 on the right). Earlier, Singer, Wong, Yau, and Yau [**31**] had obtained the weaker lower bound $\pi^2/4d^2$. Theirs was the first general result of this type, and was certainly the inspiration for all later results in the area. The best general bound so far obtained is that of Ling [**18**], who obtained the bound $4K(\sigma)^2/d^2$, where $K(\sigma)$ denotes the complete elliptic integral of the first kind and σ is a parameter which can be estimated in terms of quantities occurring in the problem. Since $K(\sigma)$ is always larger than $\pi/2$, Ling's bound is always better than that of Yu and Zhong (and it implies that we can put π^2/d^2 as a *strict* lower bound). For some further work in this area, see [**32**]. Also, the one-dimensional case of (5) (for a Schrödinger operator with a convex potential) was established by Lavine in [**17**].

Inequality (5) was first suggested by van den Berg [**14**] in 1983 in connection with some questions in statistical mechanics (see ineq. (65) on p. 636, but be aware that there is an extra factor of $\frac{1}{2}$ on the right-hand side due to his use of the operator $H = -\frac{1}{2}\Delta$). Later, not knowing of van den Berg's paper, (5) was conjectured by Rafael Benguria and myself [**4**] in connection with our work on lower bounds on eigenvalue gaps for Schrödinger operators which was motivated in part by the work of Singer, Wong, Yau, and Yau. If (5) could be established, it would tell us that the way to minimize the gap $\lambda_2 - \lambda_1$ among convex domains is to take a rectangular parallelopiped having all but one of its dimensions tiny. Indeed, this is the intuition behind the conjecture. For another conjecture having a similar intuition behind it, see Problem 10 in Section 6 of [**9**] (this problem is also listed as Problem 10 in [**2**], and is related to Question 5 on p. 157 of [**26**]). The conjecture (5) above also occurs as Problem 9 in [**9**].

We next turn to conjectures for the first, or *fundamental*, eigenvalue of the biharmonic operator in two classical settings: the vibration of a clamped plate, and the buckling of a clamped plate.

5. Rayleigh's conjecture for the "vibrating clamped plate" in dimensions $n \geq 4$. For dimension $n = 2$, the characteristic frequencies of vibration of a clamped plate in the shape of Ω are determined by the eigenvalues Γ_i of the *clamped plate eigenvalue problem*

$$\begin{aligned} &\Delta^2 w = \Gamma w \quad \text{in } \Omega \subset \mathbb{R}^2, \\ &w = 0 = \frac{\partial w}{\partial n} \quad \text{on } \partial\Omega. \end{aligned} \tag{6}$$

This problem has only discrete spectrum consisting of positive eigenvalues of finite multiplicity and which accumulate only at infinity. We list them in ascending order with multiplicities included as $\{\Gamma_i\}_{i=1}^{\infty}$. Thus,

$$0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \to \infty. \tag{7}$$

As in the membrane problem, it is often the first eigenvalue which holds the greatest interest, and that is indeed the case here. For the purpose of stating Rayleigh's conjecture, we focus only on Γ_1, and since our main interest is in how it varies with the domain Ω we denote it $\Gamma_1(\Omega)$. Rayleigh's conjecture asserts that among all domains $\Omega \subset \mathbb{R}^2$ of a given area the one giving the minimal value to Γ_1 is the disk, or, in symbols,

$$\Gamma_1(\Omega) \geq \Gamma_1(\Omega^\star), \tag{8}$$

where $\Omega^\star$ denotes the disk of the same area as Ω.

By extension, we continue to refer to the n-dimensional version of this problem as the vibrating clamped plate problem, and to the extension of Rayleigh's conjecture to this setting as *Rayleigh's conjecture*. All the properties spelled out above for the two-dimensional problem are also true of the n-dimensional one. In the n-dimensional setting, one should, of course, understand $\Omega^\star$ as the n-ball having the same n-volume as Ω.

Rayleigh's conjecture for the vibrating clamped plate problem is known to hold in 2 and 3 dimensions. The two-dimensional case (which one might well regard as *the* Rayleigh conjecture for the clamped plate) was established by Nadirashvili in 1993 (see [**19**], and also [**20**], [**21**]), while the three-dimensional case was established by Ashbaugh and Benguria in 1995 (see [**10**]), using a variant of Nadirashvili's approach. The approach of Ashbaugh and Benguria was then applied to the higher-dimensional cases, $n \geq 4$, by Ashbaugh and Laugesen [**13**], who obtained bounds of the form (8) but with an extra (positive, dimension-dependent) factor less than one occurring on the right-hand side. This extra factor presumably makes the inequality nonoptimal (and therefore nonisoperimetric as well). Ashbaugh and Laugesen's paper shows that the Ashbaugh-Benguria approach cannot succeed for dimension $n \geq 4$, but it does not suggest that (8) cannot hold. Indeed, if anything it still suggests that (8) is quite viable. For more discussion of these issues, and for a general overview of the methods which have proved successful so far, see [**12**], [**3**]. Among the prior literature, the works of Szegő [**33**], [**34**] (see also Note F of Pólya and Szegő's book [**29**]) and Talenti [**35**] are especially noteworthy.

6. The Pólya–Szegő conjecture for the critical buckling load of a clamped plate in all dimensions $n \geq 2$. In 2 dimensions, the critical buckling load of a clamped plate in the shape of Ω is determined by

the first eigenvalue of the *buckling problem*

$$\Delta^2 v = -\Lambda \Delta v \quad \text{in } \Omega \subset \mathbb{R}^2,$$
$$v = 0 = \frac{\partial v}{\partial n} \quad \text{on } \partial\Omega. \tag{9}$$

As for the vibrating clamped plate problem, this problem has solely discrete spectrum consisting of positive eigenvalues of finite multiplicity with infinity as the only accumulation point. We denote the eigenvalues $\{\Lambda_i\}_{i=1}^{\infty}$ (with multiplicities included). Again we concentrate on the first eigenvalue, and, to emphasize its dependence on the domain Ω, we denote it by $\Lambda_1(\Omega)$. As in the problem of the vibration of a clamped plate, one can ask if $\Lambda_1(\Omega)$ takes its least value among all domains of a given area at the disk. This question seems to have first been asked by Pólya and Szegő around 1950, and to have first appeared in print under their names in their book on isoperimetric inequalities [**29**] in 1951. On the other hand, it should be noted that Szegő's paper [**33**] dealing with a special case of the conjecture already appeared in 1950. This paper had some technical flaws, which Szegő corrected in 1958 (see [**34**]). The argument of [**33**] was, in fact, included in an addition (Note F) to [**29**]. The jumbled dates occurring here are due to the delayed publication of [**29**], as explained in the preface: the main part of the book was already finished in 1948, but the delay allowed them to include some later material as well (as added "Notes").

The *Pólya-Szegő conjecture for* Λ_1 asserts that among all domains $\Omega \subset \mathbb{R}^2$ of a given area the one giving the least value of Λ_1 is the disk, or, in symbols,

$$\Lambda_1(\Omega) \geq \Lambda_1(\Omega^\star), \tag{10}$$

where $\Omega^\star$ denotes the disk of the same area as Ω.

As before, both the buckling problem for the clamped plate and the Pólya-Szegő conjecture for its first eigenvalue may easily be generalized to n dimensions (we just need to take $\Omega^\star$ as the n-ball having the same n-volume as Ω). We shall also refer to the generalized problem as the buckling problem for the clamped plate, although again the physical motivation centers on the two-dimensional case.

It turns out that the Pólya-Szegő conjecture for the buckling problem is in a less satisfactory state than is Rayleigh's conjecture for the vibrating clamped plate, in that the Pólya-Szegő conjecture remains open for all dimensions $n \geq 2$. In another sense, though, the situation is more or less analogous. Just as in the case of Rayleigh's conjecture, there is a special case that was handled by Szegő [**33**], [**34**] (see also [**29**]). In addition, the approach used by Ashbaugh and Laugesen to establish weaker versions of (8) can be carried over to the buckling problem and yields a weaker version of (10) which has the same form but contains an unwanted dimension-dependent factor on the right-hand side.

Beyond this (and this has no analog for the vibrating clamped plate problem), there is a second, seemingly unrelated, way to arrive at these same nonoptimal bounds. The alternative approach is to combine the inequality of Payne [**22**],

$$\Lambda_1(\Omega) \geq \lambda_2(\Omega) \text{ for } \Omega \subset \mathbb{R}^n \tag{11}$$

(where $\lambda_2(\Omega)$ denotes the second eigenvalue of the Dirichlet Laplacian), with Krahn's inequality for $\lambda_2(\Omega)$,

$$\lambda_2(\Omega) > 2^{2/n}\lambda_1(\Omega^\star) \text{ for } \Omega \subset \mathbb{R}^n. \tag{12}$$

This inequality of Krahn was proved near the end of his longer paper [**16**] on the Faber-Krahn inequality, the main purpose of which was to establish the Faber-Krahn inequality in all dimensions $n \geq 3$. It is an easy consequence of the Faber-Krahn inequality for λ_1 and the fact that any eigenfunction for λ_2 has exactly two nodal domains. The first people to realize that these two inequalities could be combined to yield an explicit lower bound for Λ_1 were Bramble and Payne [**15**], who gave the two-dimensional bound. However, they formulated the inequality directly, rather than as an inequality of the form (10) with a nonoptimal constant. It is not at all clear why the two approaches give rise to exactly the same (nonoptimal) constants in the inequalities for $\Lambda_1(\Omega)$ that they yield, but they do. The papers [**13**], [**12**], [**2**], and [**3**] all contain more discussion and information on the Pólya-Szegő conjecture. In particular, the variation with n of the nonoptimal constants which occur in the best bounds yet proved is of interest. The factor of nonoptimality turns out to go to 1 as n goes to infinity.

Other well-known problems which come to mind are the Pólya conjectures for the Dirichlet and Neumann eigenvalues of the Laplacian, the nodal line conjecture for simply connected domains in the plane, and the Pompeiu problem. There is a large literature on each of these, toward which a start can be made by consulting a number of the references cited below.

Many of the problems given or mentioned above and also several additional problems are discussed in [**9**] (see Section 6), [**2**], and [**3**] (see Section 4). More problems can be found in Payne's papers [**23**], [**24**], [**25**], [**26**]. There are also the extensive problem collections of Yau [**36**], [**37**] and the list of Arnol'd et al. [**1**] that might be consulted. Obviously, the problems suggested by the other participants at the Matrei workshop are of great interest as well.

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Asymptotics for the spectrum of the Dirichlet Laplacian on horn-shaped regions and ζ functions on cross-sections

Michiel van den Berg and M. Lianantonakis

Abstract

Let Ω be an open bounded set in $\mathbb{R}^{m-1}$ with a piecewise smooth boundary, and starshaped with respect to $(0,\dots,0) \in \mathbb{R}^{m-1}$. We extend a result of G. V. Rozenbljum for the spectrum of the Dirichlet Laplace operator for $\{(x,y) \in \mathbb{R}^m : y \in (1+x)^{-\alpha}\Omega, x > 0\}$ in $\mathbb{R}^m$, where $\alpha > 0$. For $2m-1 > \alpha > \left(1-m+[(m-1)(9m-17)]^{1/2}\right)/2 \geq \alpha > 0$ we obtain two-term asymptotics and a remainder estimate for the Dirichlet counting function. For $\alpha \geq 2m-1$ or $m > 2$ and $\left(1-m+[(m-1)(9m-17)]^{1/2}\right)/2 \geq \alpha > 0$ we recover Rozenbljum's result for the leading term of the Dirichlet counting function together with an improved remainder estimate.

Open Problem

Let ϕ, $\|\phi\|_2 = 1$, be the first eigenfunction of the Dirichlet Laplace operator on an open, bounded and convex set D, with inradius ρ and diameter d. Show that

$$\|\phi\|_\infty \leq c_m \rho^{(1-3m)/6} d^{-1/6},$$

with c_m independent of D.

The conjecture is based on a result on the asymptotic behaviour for the L^∞ norm of the first eigenfunction ϕ of the Dirichlet Laplace operator on a conic sector over a geodesic disc B_η in $\mathbb{S}^{m-1}$ as $\eta \to 0$.

An inequality between Dirichlet and Neumann eigenvalues

Leonid Friedlander

Abstract

Let Ω be a centrally symmetric, bounded domain with a C^2 boundary. Denote by λ the smallest eigenvalue of the Dirichlet Laplacian that corresponds to an odd eigenfunction, and let μ be the smallest positive eigenvalue of the Neumann Laplacian that corresponds to an even eigenfunction. D.Jerison and N.Nadirashvili conjectured that, if the domain Ω is convex, then $\lambda > \mu$. We proved this inequality under a less restrictive assumption, namely that the boundary of Ω is of non-negative mean curvature.

Open Problem

For closed manifolds and for manifolds with smooth boundary, the heat trace asymptotics is well known, and its coefficients can be computed (at least, in principle). This is not the case for manifolds with corners. Several authors obtained the formula for the free term of the heat trace asymptotics for a planar polygon (the Laplacian is taken with the Dirichlet boundary conditions):

$$\sum_\gamma \pi^2 - \gamma^2 \frac{2}{4}\pi\gamma;$$

the sum is taken over all angles of the polygon. This formula is rather non-trivial, and I do not know any good interpretation of it. (It is just a result of rather tedious computations.)

I think that it would be very interesting to find a way of computing the coefficients in the heat trace expansion for manifolds with corners in any dimension. Probably, the understanding of the expansion for a three-dimensional polyhedron would be the key to solving the general problem. Anyway, it is a good starting point.

Nodal sets for superconducting states in a non simply connected domain.

Bernard Helffer

Abstract

Motivated by a paper by J. Berger and K. Rubinstein and in continuation of our previous work (in collaboration with Maria and Thomas Hoffmann-Ostenhof and M. Owen), we analyze the nodal sets of some extrema of the following Ginzburg-Landau functional $G_{\lambda,k}$ ($\lambda > 0, k > 0$) defined in a nonsimply connedted domain $\Omega \subset \mathbb{R}^2$ and for pairs (u, A) such that

$$u \in H^1(\Omega, \mathbb{C}) \quad , \quad A \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2) \quad , \quad \mathrm{curl}\, A = 0$$

by

$$G_{\lambda,k}(u, A) = \int_\Omega \lambda(-|u|^2 + |u|^4) + |(\nabla - iA)u|^2 \, dx_1 \, dx_2 + k^2 \lambda^{-1} \int_{\mathbb{R}^2} |\,\mathrm{curl}\, A - H_e|^2 \, dx_1 \, dx_2.$$

Here H_e is a C_0^∞ function on $\mathbb{R}^2$ representing the external magnetic field. As classical , the analysis of the extrema goes through the analysis of solutions of the corresponding Euler-Lagrange equation which is called in this context Ginzburg-Landau equation. This equation admits always (i. e. for all $\lambda > 0$) the so called normal solution: $u = 0, A = A_e$ where A_e is sucht that $\mathrm{curl}\, A_e = H_e$. We show that for λ near the lowest eigenvalues $\lambda^{(1)}$ of the Neumann realization of the magnetic Laplacian in Ω: $\Delta_{A_e} = -(\nabla - iA_e)^2$, other bifurcated solutions exist and we analyze their nodal sets.

Holonomic constraints in classical and quantum mechanics

R. Froese and I. Herbst

Abstract

We constrain a system of non-relativistic particles moving in $\mathbb{R}^n$ to a smooth submanifold M by imposing a large force which draws the system into M. Thus we consider a Hamiltonian

$$H(\lambda) = \frac{1}{2}\langle p, p\rangle + V(x) + \lambda W(x),$$

where $W = 0$ on M and $W > 0$ off M. We consider the limiting behavior of classical and quantum orbits as $\lambda \nearrow \infty$. Much work has

been done on the classical case when the given initial conditions converge (as $\lambda \nearrow \infty$) to those with finite energy. (See [RU], [Ta], [A], [G], [BS].) But in quantum mechanics the uncertainty principle makes infinite energy initial conditions (in the directions orthogonal to the manifold) more natural. We consider this situation in both classical and quantum mechanics [FH].

The quantum problem divides into two parts: energy considerations and dynamics. The problem of energy has been considered in a general setting in [HS1, HS2] and in many different particular situations (see, for example, [S], [DE], [FK]). Computations show the existence of new potential terms in the Hamiltonian resulting from intrinsic and mean curvatures of M (see [To], [dC1], [dC2], [AD]). The last paper gives references to the path integral literature. The most interesting aspect of our treatment of the quantum problem is seeing how the quantum dynamics, $e^{-itH(\lambda)}$, incorporates the averaging over fast variables familiar from classical dynamics [FH]. In another very different context a related averaging procedure arises in the very long time behavior (t/ϵ^2) of classical periodic orbits perturbed by a vanishingly small $(\epsilon\dot{W})$ random perturbation [F].

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Open Problem: Decay Patterns of Solutions to the Schrödinger Equation

I. Herbst

Consider an L^2 solution of the Schrödinger equation

$$(-\Delta + V)\psi = \lambda\psi$$

in $\mathbb{R}^n$. Define the rate of exponential decay, $\alpha_\psi(\omega)$, in direction $\omega \in S^{n-1}$ as

$$\alpha_\psi(\omega) = \sup\{\beta \geq 0 : \ \exp(\beta|x|)\psi \in L^2(C) \text{ for some open cone } C \text{ containing } \omega\}.$$

Let $\alpha_\psi(x) = \alpha_\psi\left(\frac{x}{|x|}\right)|x|$.

The Agmon metric gives a lower bound for $\alpha_\psi(x)$ in many situations [A],

$$\alpha_\psi(x) \geq \rho(x),$$

and if $\psi > 0$ outside a compact set there tends to be equality in (1) [CS]. Although it is probably true that (1) is actually an equality generically [H], there are many situations where (1) is far from optimal [FHHOHO]. At this point it is not known what controls the decay rate of eigenfunctions in the generalized n-body problem, even in two dimensions. But there is a conjecture in [FH1].

Consider a simpler problem. Suppose $-\Delta\psi = \lambda\psi$ in an open cone C in $\mathbb{R}^n$ (no boundary conditions are imposed). If $n = 2$ and $\lambda \geq 0$ there are results [FH2] similar to the trigonometric convexity known for analytic functions [T]. If $n = 2$ and $\lambda < 0$ the results are similar but more complex [FH2]. But as far as I know, for $n \geq 3$ there is virtually nothing known of a general nature, even if $\lambda = 0$.

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Multiplicity of eigenvalues of 2-dimensional Laplacians

Thomas Hoffmann-Ostenhof

Abstract

This is joint work with M. Hoffmann-Ostenhof, P. Michor and N. Nadiashvili, as well as joint work with B. Helffer, M. Hoffmann-Ostenhof and N. Nadirashvili. The spectrum of a 2-dimensional Laplacian on a closed compact surface with genus zero is considered. It is shown that the multiplicity m of the k-th eigenvalue λ_k (counting with multiplicity) satisfies for $k \geq 3$, $m(\lambda_k) \leq 2k-3$. This result has appeared in M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili. "On the multiplicity of eigenvalues of the Laplacian on surfaces" Ann. of Global Anal. and Geometry 17, 43-48, (1999). Thereby we could also include a potential. The same result holds also for membranes with Dirichlet boundary conditions. This is to appear in GAFA 1999 and is an ESI Preprint by T.Hoffmann-Ostenhof, P. Michor and N. Nadirashvili. Furthermore the situation is discussed when for the membrane case a $\mathbb{Z}_n$ symmetry is present, this means the Hamiltonian is invariant with repect to a rotation by $2\pi/n$. Then the groundstate eigenvalues in the various symmetry sectors can be ordered and have multiplicity either 1 or 2. While for the case without

symmetry the method of proof relies on the combination of a suitable version of Eulers polyhedral formula, an analysis of nodal sets and Courants nodal theorem in order to get some topological obstructions one has to use in addition variational methods and perturbation arguments for for the groundstate eigenvalues in symmetry sectors. This work is presently being written up in collaboration with B. Helffer, M. Hoffmann-Ostenhof and N. Nadiarashvili. It has also consequences for Aharanov-Bohm Hamiltonians and twodimensional periodic Hamiltonians defined on a strip.

Open Problems

Problem 1: Consider for a 2-dimensional Laplacian the function

$$\mathcal{M}(k) = \sup_M m(\lambda_k)$$

where M either denotes a surface of genus zero or a membranes. It is known from the results above that $\mathcal{M}(k)$ grows at most linearly for k large. The question is what is the asymptotics of $\mathcal{M}(k)$. The standard sphere show a $\sqrt{k}$ growth.

Problem 2: Consider a membrane D with smooth boundary. For the second eigenfunction u_2 define $\mathcal{N}(u_2) = \overline{\{x \in D : u_2 = 0\}}$. Question: Prove or disprove that

$$\frac{\text{Length of } \mathcal{N}(u_2)}{\text{Perimeter of } \partial D} \leq 1/\pi.$$

This would mean that for the disk this ratio is maximised.

Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator

T. Kappeler and B. Mityagin

Abstract

Consider the Schrödinger equation $-y'' + Vz = \lambda z$ for a complex valued potential V of period 1 in the weighted Sobolev space H^w of 2-periodic functions $f : \mathbb{R} \to \mathbb{C}$,

$$H^w \equiv H^w_{\mathbb{C}} := \{f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{i\pi kx} \mid \|f\|_w < \infty\}$$

where

$$\|f\|_w := (2 \sum_k w(k)^2 \; |\hat{f}(k)|^2)^{1/2}$$

and $w = (w(k))_{k\in\mathbb{Z}}$ denotes a symmetric, submultiplicative weight sequence. Denote by $\lambda_n = \lambda_n(V)$ $(n \geq 0)$ the periodic eigenvalues of $-\frac{d^2}{dx^2} + V$ when considered on the interval $[0, 2]$, listed in such a way that $\lambda_{2n}, \lambda_{2n-1} = n^2\pi^2 + 0(1)$ and by $\mu_n = \mu_n(V)$ $(n \geq 1)$ the Dirichlet eigenvalues of $-\frac{d^2}{dx^2} + V$ considered on $[0, 1]$ listed in such a way that $\mu_n = n^2\pi^2 + 0(1)$.

Theorem *There exist (absolute) constants $K_1, K_2 > 0$, so that for any 1-periodic potential V in H^w,*

$$\sum_{n\geq N} w(2n)^2|\lambda_{2n} - \lambda_{2n-1}|^2 \leq K_1(1 + \|V\|_w)^{K_2}$$

and

$$\sum_{n\geq N} w(2n)^2|\mu_n - \lambda_{2n}|^2 \leq K_1(1 + \|V\|_w)^{K_2},$$

where $N := K_1(1 + \|V\|_w)^2$.

Lifshitz tails for random Schrödinger operators with negative singular Poisson potential

Frédéric Klopp

Abstract

This talk is devoted to the description of the low energy behaviour of the density of states of a family of random Schrödinger operators; it is based on the paper [**5**] written in collaboration with L. Pastur. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a function such that

$$V = V_1 + V_2$$

where

H1 for some $C > 0$ and any $x \in \mathbb{R}^d$, $|V_1(x)| \leq Ce^{-|x|/C}$,

H2 the function V_2 is compactly supported and satisfies $V_2 \in L^p(\mathbb{R}^d)$ where $p > p(d)$ and $p(d) = 2$ if $d \leq 2$ and $p(d) = d/2$ if $d \geq 3$,

H3 for some set of positive measure $\mathcal{E}$, $V\big|_{\mathcal{E}} < 0$.

Define the random potential

$$V_\omega(x) = \int_{\mathbb{R}^d} V(x - y)m(\omega, dy) \tag{1}$$

where $m(\omega, dy)$ is a random Poisson measure of concentration μ. V_ω is an ergodic random field on $\mathbb{R}^d$. Consider the random Schrödinger operator

$$H_\omega = -\Delta + V_\omega. \tag{2}$$

One has

THEOREM 1 ([**2**]). *Under the assumptions made above, H_ω is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ ω-almost surely.*

Under our assumptions on V, we know that the almost sure spectrum of H_ω is $\Sigma = \mathbb{R}$ ([**9, 2**]).

The integrated density of states

Let Λ be a cube centered at 0 in $\mathbb{R}^d$. We define $H^D_{\omega,\Lambda}$ to be the Dirichlet restriction of H_ω to Λ. Pick $E \in \mathbb{R}$. Consider the quantity

(3)

$$N_{\omega,\Lambda}(E) = \frac{1}{\mathrm{Vol}(\Lambda)} \sharp\{\text{eigenvalues of } H^D_{\omega,\Lambda} \text{ smaller than or equal to } E\}.$$

Then one has

THEOREM 2 ([**2**]). *Under the assumptions made above, there exists a non random, non decreasing, non negative, right continuous function $N(E)$ such that, ω-almost surely, for all $E \in \mathbb{R}$, E a continuity point of N, $N_{\omega,\Lambda}(E)$ converges to $N(E)$ as Λ exhausts $\mathbb{R}^d$.*

$N(E)$ is the *integrated density of states* (IDS) of H_ω. As N is non decreasing, one can define its distributional derivative dN. It is a positive measure and is supported on the almost sure spectrum of H_ω (see [**2, 9**]). One proves the following result

THEOREM 3. *For $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, we have*

$$(\varphi, dN) = \mathbb{E}(tr(\mathbf{1}_{C(0,1)}\varphi(H_\omega)\mathbf{1}_{C(0,1)})). \tag{4}$$

where $C(0,1)$ is the cube of center 0 and side length 1, $\mathbf{1}_{C(0,1)}$ its characteristic function and $\mathbb{E}(\cdot)$ denotes the expectation with respect to the Poisson process.

Formula (4) is well known under more restrictive assumptions on the potential V_ω i.e. for less singular single site potentials V (see [**9**]).

The asymptotics of the IDS

To describe the asymptotic behavior of $N(E)$ near $-\infty$, we will need to define an auxiliary operator. For $g \in \mathbb{R}$, define

$$H(g) = -\Delta + gV. \tag{5}$$

Under our assumptions on V, V is relatively form bounded with respect to $-\Delta$ with relative bound 0. Hence, $H(g)$ admits a unique self-adjoint extension. Let $\sigma(H(g))$ denote its spectrum. It is lower semi-bounded. The infimum of $\sigma(H(g))$ i.e. the ground state energy of $H(g)$ will be denoted by $E(g)$. Let φ_g be the respective ground state i.e. the unique positive normalized eigenfunction of $H(g)$ associated to energy $E(g)$

([**10, 11**]). In the sequel it will often be more convenient to work with $E_-(g) = -E(g)$ instead of $E(g)$ itself. From assumption H3, one easily infers that $E_-(g) \to +\infty$ when $g \to +\infty$. Moreover E_- is strictly increasing in a neighborhood of $+\infty$. Let g be an inverse of E_- in a neighborhood of $+\infty$. g is strictly increasing.

In the regular (classical) case, it was found that g is governing the first term asymptotic of $\log N$ (cf [**9, 8**]). In the singular (quantum) case, the singular set of V will play a special part in the asymptotics. To measure this role, we introduce the notion of *asymptotic ground state* i.e.

DEFINITION 1. Let $g \in (1, +\infty) \mapsto \psi_g \in H^1(\mathbb{R}^d)$. We will say that ψ_g is an *asymptotic ground state* if and only if

- the vector ψ_g is normalized.
- $\exists g_0 > 1$, $l_0 > 0$ such that $\forall g \geq g_0$, $\operatorname{supp} \psi_g \subset C(0, l_0)$ (where $C(x, l)$ denotes the cube of center x and side length l).
-

$$\frac{|\langle (H(g) - E(g))\psi_g, \psi_g\rangle|}{|E(g)|} \to 0 \text{ as } g \to +\infty. \tag{6}$$

We prove the existence of an asymptotic ground state. For $a \in \mathbb{R}^d$, we define the translation τ_a by $\tau_a V(x) = V(x - a)$ and we define

$$\mathcal{A}_{\psi_g} = \left\{ \alpha > 0;\ \lim_{g \to +\infty} \sup_{|a| \leq g^{-\alpha}} \left[\frac{g|\langle (\tau_a V - V)\psi_g, \psi_g\rangle|}{E_-(g)} \right] = 0 \right\}$$

If $\mathcal{A}_{\psi_g} \neq \emptyset$, then we define $\alpha^*(\psi_g) := \inf \mathcal{A}_{\psi_g}$. Moreover, we define $\mathcal{A}$ to be the union of all $\mathcal{A}_{\psi_g}$. We prove that $\mathcal{A} \neq \emptyset$. We define

$$\alpha^* := \inf \mathcal{A}. \tag{7}$$

Then we prove

THEOREM 4. *Under the assumptions H1, H2 and H3, for sufficiently large E, one has*

$$\begin{aligned} &-(1 + \alpha^* d) g(E) \log g(E)(1 + o(1)) \\ &\qquad \leq \log N(-E) \leq -g(E) \log g(E)(1 + o(1)). \end{aligned} \tag{8}$$

One may complain that Theorem 4 is somewhat imprecise in that it only gives a two sided estimate. But, as we will see below, this is in some way unavoidable as the true asymptotic does not only depend on g but also on the singular set of the negative part of V. More precisely, as can be seen from Theorem 6 (and from the proof of Theorem 4), the asymptotics of the IDS depends on the way the eigenfunction associated to the lowest eigenvalue for the operator $-\Delta + gV$ concentrates near the singular set of the negative part of V as g becomes large. In general the correction also depends on the geometry of the singular set. For example, if the singular set is a segment (e.g. a dislocation), one can see that neither the lower nor the upper bound given by Theorem 4 are sharp. The two sided estimate (8) can be made more precise if we know more on V.

The first and simplest example we give is the case when V is bounded from below, reaches its minimum at a single point, say 0, and is continuous near 0. Then one easily proves that $\alpha^* = 0$ and the upper and lower bounds in (8) coalesce to give

$$\log N(-E) \underset{E\to+\infty}{\sim} -g(E)\log g(E). \tag{9}$$

We will now give other results that, we think, enclose most of the physically relevant examples.
Let v_- be the essential infimum of V and assume that V is bounded from below, say

H1' $-\infty < v_- < 0$.

It is easy to show that $g(E) \sim E/|v_-|$ when $E \to +\infty$. We obtain

THEOREM 5. *Under the assumptions H1, H2 and H1', one has*

$$\log N(-E) \underset{E\to+\infty}{\sim} -g(E)\log g(E) \underset{E\to+\infty}{\sim} \frac{E}{v_-}\log E. \tag{10}$$

Here and in the sequel $a \sim b$ will always mean $a = b(1+o(1))$.

This result extends (9) removing the continuity assumption near the minimum.

Consider now an example a bit more singular. In this case, $d = 2$ and

$$V_2(x) = \log_- |x|, \quad x \in \mathbb{R}^2$$

where, for $a \geq 0$, $\log_- a = \min\{\log a, 0\}$. Using the inequality $\log_- |x| + \log R \leq \log_- R|x| \leq \log |x|$ for $0 < R < 1$ and the variational principle for the ground state energy, one shows that, in this case,
$E_-(g) \underset{g\to+\infty}{\sim} g/2\log g$ hence $g(E) \underset{E\to+\infty}{\sim} 2E/\log E$. One also shows that $\alpha^* = 0$ for this single site potential. Therefore, Theorem 4 tells us that

$$\log N(-E) \underset{E\to+\infty}{\sim} -g(E)\log g(E) \underset{E\to+\infty}{\sim} -2E.$$

Hence, the asymptotic formula (9) is also valid for certain mildly singular potentials.

Another case where one can find an asymptotic for $\log N$ is when V has only power law singularities. Let q be a positive integer and pick q positive exponents $(\nu_i)_{i=1,\dots,q}$ and q functions $(h_i(\theta))_{i=1,\dots,q}$ continuous on the sphere $\mathbb{S}^{d-1}$. For $1 \leq i \leq q$, consider the potentials

$$V_i(x) = \frac{h_i(\theta(x))}{|x|^{\nu_i}} \text{ where } \theta(x) = \frac{x}{|x|} \tag{11}$$

Assume that

$$0 < \nu_i < \begin{cases} 1 \text{ if } d = 1, 2, \\ 2 \text{ if } d \geq 3. \end{cases} \tag{12}$$

Then V_i is relatively form bounded with respect to $-\Delta$ with relative bound 0 and we can consider the operators $H^i = -\Delta + V_i$ with form domain $H^1(\mathbb{R}^d)$. For $1 \leq i \leq q$, E_i denotes the ground state energy of H^i. Now we assume that

H1"

- there exists q distinct points $(x_i)_{i=1,\dots,q}$ in $\mathbb{R}^d$ and q continuous compactly supported function $(W_i)_{i=1,\dots,q}$ such that $W_i(0) = 1$ such that

$$V_2(x) = \sum_{i=1}^{q} W_i(x - x_i)V_i(x - x_i) = \sum_{i=1}^{q} \tau_{x_i}(W_i V_i)(x). \tag{13}$$

- for some $1 \leq i_0 \leq q$, we have $E_{i_0} < 0$.

Notice that assumption H1" implies assumptions H2 and H3. Define

$$\nu^\dagger = \sup \{\nu_i;\ 1 \leq i \leq q \text{ such that } E_i < 0\}, \tag{14}$$

$$E_- = \sup \left\{|E_i|;\ 1 \leq i \leq q \text{ such that } E_i < 0 \text{ and } \nu_i = \nu^\dagger\right\}, \tag{15}$$

$$\alpha^\dagger = \frac{1}{2 - \nu^\dagger}. \tag{16}$$

In this case, we compute $\alpha^* = \alpha^\dagger$ and prove

Theorem 6. *Under the assumptions H1 and H1", one has*

$$\begin{aligned} \log N(-E) &\underset{E\to+\infty}{\sim} -(1 + \alpha^\dagger d) g(E) \log g(E) \\ &\underset{E\to+\infty}{\sim} -\left(1 + \frac{d - \nu^\dagger}{2}\right)\left(\frac{E}{E_-}\right)^{1-\nu^\dagger/2} \log\left(\frac{E}{E_-}\right) \end{aligned} \tag{17}$$

Assumptions H1 and H1" include most physically interesting cases as, for example, the 3-dimensional attractive screened Coulomb potential $V(x) = -\dfrac{e^{-|x|}}{|x|}$. In this case we have $\alpha^* = \alpha^\dagger = 1$ and $E_- = 1$ (see [**6**]); thus

$$\log N(-E) \underset{E\to+\infty}{\sim} -2\sqrt{E} \log E.$$

There is another physically interesting case that has not been discussed here: it is the case of point potentials [**4**].

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Open Problems

Horst Knörrer

Problem:

Let $a_1, \cdots, a_n$; $b_1, \cdots, b_n$ be real numbers. For a permutation $\pi \in S_n$ set

$$\epsilon(\pi) = \begin{cases} \operatorname{sign}\pi & \text{if } a_{\pi(1)} \le b_1,\ a_{\pi(1)} + a_{\pi(2)} \le b_2,\ \cdots, a_{\pi(1)} + \cdots + a_{\pi(n)} \le b_n \\ 0 & \text{otherwise} \end{cases}$$

Here $\operatorname{sign}\pi$ is the signum of the permutation π. Is there a constant C independent of n, $a_1, \cdots, a_n$; $b_1, \cdots, b_n$ such that

$$\Big| \sum_{\pi \in S_n} \epsilon(\pi) \Big| \le C^n \sqrt{n!}$$

Problem:

Let Γ be a lattice in $\mathbb{R}^2$. For a real valued analytic function on the torus $\mathbb{R}^2/\Gamma$ (= periodic potential) and $k \in \mathbb{R}^2$ let $E_n(k; V)$ be the n^{th} Floquet

eigenvalue of $-\Delta+V$, that is the n^{th} eigenvalue of the operator $-\Delta+V$ on the space $\{\psi \in H^2_{loc}(\mathbb{R}^2) \,|\, \psi(x+\gamma) = e^{\imath\, k\cdot\gamma}\,\psi(x) \text{ for all } \gamma \in \Gamma\}$.
We say that two periodic potentials V_1, V_2 are Floquet isospectral if $E_n(k, V_1) = E_n(k, V_2)$ for all n and all $k \in \mathbb{R}^2$. V_1 and V_2 are trivially Floquet isospectral, if $V_2(x) = V_1(\pm x + t)$ for some vector $t \in \mathbb{R}^2$, or if there are perpendicular vectors $v, w \in \mathbb{R}^2$ and pairs of onedimensional isospectral periodic potentials p_1, p_2 and q_1, q_2 such that $V_1(x) = p_1(x \cdot v) + q_1(x \cdot w)$ and $V_2(x) = p_2(x \cdot v) + q_2(x \cdot w)$.

Question: Are there nontrivial pairs of Floquet isospectral two dimensional periodic potentialsΓ

Some known Results:
[G.Eskine, J.Ralston, E.Trubowitz: On isospectral periodic potentials in $\mathbb{R}^n$, CPAM 37, 715-753 (1984)] show the existence of "large" sets of periodic potentials V to which there are no nontrivial Floquet isospectral potentials. The case that $-\nabla\cdot\nabla+V$ acts on a topologically nontrivial Hermitian line bundle over $\mathbb{R}^2/\Gamma$ is discussed in [V.Guillemin: Inverse spectral results on two dimensional tori. J.American Math. Society 3, 375-387 (1990)]. Any potential that is isospectral to a constant potential is itself constant (see e.g. [H.Knörrer, E.Trubowitz: A directional compactification of the complex Bloch variety. Comm. Math. Helvetici 65, 114-149 (1990)]). Observe that this is not the case for complex valued potentials; any potential of the form

$$V(x_1, x_2) = \sum_{m,n} \hat{V}(m,n)\, e^{2\pi\imath(mx_1+nx_2)}$$

with $\hat{V}(m,n) = 0$ unless $m > 0$, $|n/m| \le 1/2$ is Floquet isospectral to the potential 0.
There is also a preprint by O.Veliev on inverse problems for periodic potentials in dimension ≥ 2.

Problem:

For a lattice Γ in $\mathbb{R}^2$ set

$$\sigma(\Gamma) = \{|\gamma|^2 \,|\, \gamma \in \Gamma\}$$

We say that Γ has arbitrary long gaps in its spectrum, if for every $\ell > 0$ there is an interval I of length ℓ in $\mathbb{R}_+$ such that $I \cap \sigma(\Gamma) = \emptyset$. Let L be the set of all lattices, and X the set of all $\Gamma \in L$ that have arbitrary long gaps in their spectrum.

Question: Does $L \setminus X$ have measure zeroΓ

Some known Results:
[Th.Kappeler: On double eigenvalues of Schrödinger operators on two–dimensional tori. J. Functional Analysis 115, 166-183 (1993)] shows

that rational lattices lie in X, and that X is a set of second Baire category in L.

Mathematics of photonic crystals

Peter Kuchment

Abstract

The burgeoning young area of research on photonic crystals is a dream of an applied mathematician: it is of paramount practical importance, most of its problems are precisely formulated in exact mathematical terms and remain unresolved, and resolving them requires a wide range of mathematical tools ranging from analytic operator functions to several complex variables, to algebraic geometry, to PDEs, to probability theory, to numerics. A photonic crystal is an artificial periodic dielectric medium that is an optical analog of semiconductors. Its main property is existence of a "complete band gap", i.e. of an interval of frequencies for which electromagnetic waves cannot propagate in the medium. Creation of such a material promises to bring about a technological revolution. The talk provides a brief overview of the mathematical component of the photonic crystal problem. The major part of it deals with various aspects of spectral theory of differential and pseudodifferential operators and operator pencils (Maxwell operator, its scalar counterparts, Dirichlet to Neumann operators on graphs, ODEs on graphs, etc.).

New bounds on the constants $L_{\gamma,d}$ appearing in the Lieb-Thirring inequalities

Ari Laptev

Abstract

We show how a matrix version of the Buslaev-Faddeev-Zakharov trace formulae for a one-dimensional Schrödinger operator leads to Lieb-Thirring inequalities with sharp constants $L^{cl}_{\gamma,d}$ with $\gamma \geq 3/2$ and arbitrary $d \geq 1$.

Improved estimates on the constants $L_{\gamma,d}$, for $1/2 < \gamma < 3/2$, $d \in N$ in the inequalities for the eigenvalue moments of Schrödinger operators are established.

Open Problems

Let H be a Schrödinger operator

$$H = -\Delta - V$$

in the space of vector functions

$$L^2(\mathbb{R}^d, \mathbb{C}^m) = \{u : \int_{\mathbb{R}^d} \|u(x)\|^2_{\mathbb{C}^m}\, dx < \infty\},$$

where V is a non-negative Hermitian $m \times m$ matrix-valued potential satisfying $\|V\|_{\mathbb{C}^m} \in L^{d/2}(\mathbb{R}^d)$. Let $N(H)$ be the number of negative eigenvalues of H.

1. Prove or disprove the following matrix version of the Cwikel-Lieb-Rosenblum (CLR) inequality: that is if $d \geq 3$ and $m \geq 2$ then there is a constant $M_{0,d} < \infty$ independent of m such that

$$N(H) \leq M_{0,d} \int \operatorname{tr} V^{d/2}\, dx.$$

2. Let $L_{0,d}$ be the corresponding constant appearing in the scalar case.

Is it true that $M_{0,d} = L_{0,d}$ if the inequality $M_{0,d} < \infty$ is provedΓ

A simple analytic proof of the glueing formula for the analytic torsion in the presence of a general (nonunimodular) flat bundle

Matthias Lesch

Abstract

In recent years there has been considerable progress in understanding the mechanism behind the celebrated Cheeger–Müller Theorem on the equality of analytical and combinatorial torsion. Both torsions are numerical invariants of a compact manifold and a representation of its fundamental group. The combinatorial torsion is well–defined only if the representation is unimodular. In the present work I will restrict myself to finite–dimensional representations, though it should be mentioned that there is considerable interest in infinite–dimensional representations [**3**].

The Cheeger–Müller theorem states that for unitary representations the analytical torsion (an invariant of the spectrum of the Laplacians on forms) equals the combinatorial torsion (a combinatorial invariant) [**4, 6**]. For more general representations there is a defect between these two invariants which, in principle, can be calculated [**1**]. If the

boundary of the manifold is nonempty, the defect is nontrivial already for unitary representations [**5**].

A completely different approach to the Cheeger–Müller Theorem, however only in the case of a trivial representation, was invented by S. Vishik [**7**]. J. Brüning and the author jointly extended Vishik's method to nonlocal well–posed boundary value problems to obtain a new proof of the glueing formula for the η–invariant [**2**].

This is the starting point of the present work. I showed that the method of loc. cit. can be used to give a rather straightforward proof of the glueing formula for the analytic torsion. I do not impose any restrictions on the representation of the fundamental group except finite–dimensionality. This generalizes the above mentioned result of Vishik and also later work of Burghelea, Kappeler and Friedlander. As an application I can present a Cheeger–Müller type theorem on manifolds with boundary which is in the spirit of [**1**]. The manifold with boundary case is nontrivial since for non–unitary representations of the fundamental group the 'doubling trick' of [**5**] is not applicable.

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Open Problems

Let $C \subset \mathbb{CP}^N$ be an algebraic variety in complex projective space. Locally, C is the common zero locus of a set of homogeneous polynomials. We denote by $\Sigma := \operatorname{sing} C$ the singular locus of C and we equip $C \setminus \Sigma$ with a hermitian metric induced from such a metric on $\mathbb{CP}^N$ (e.g. the Fubini–Study metric).

1. $\dim_{\mathbb{C}} C = 1$: C is a complex algebraic curve and it is well–known (cf. [**2**]) that $C \setminus \Sigma$ is compact surface with isolated asymptotically cone–like singularities. Hence the conical analysis invented by J. Cheeger [**3, 4**] applies and one can show that the basic results of Spectral Geometry hold for $C \setminus \Sigma$. As these basic results we consider

- The Gauß–Bonnet Theorem [**7**] and the index theorem for the $\overline{\partial}$–operator [**2**].
- The discreteness of the Laplacian, more precisely, the Friedrichs extension of the Laplacian [**4**].
- The complete short–time asymptotics of the heat trace of the Laplacian [**1**].

Furthermore, in [**1**] it is shown that the spectrum of the Laplacian detects whether Σ is nonempty or not.

2. $\dim_{\mathbb{C}} C > 1$: The problem I want to address here is: What can one say about the spectral theory of the Laplacian in higher dimensionsΓ Only some partial results are known:

- The L^2-Stokes Theorem in the complex case and the fact that it is wildly wrong for real varieties (cf. e.g. [**8**],[**5**]).
- Discreteness of the Laplacian on functions [**6**] and, for varieties with isolated singularities, on forms except in degree $\dim_{\mathbb{C}} C$ [**5**].
- Hodge Theory [**8**].

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Smooth perturbation theory of unbounded operators

Peter W. Michor

Abstract

All presented results can be found in:

[1] Kriegl, Andreas; Michor, Peter W.: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, Volume: 53, American Mathematical Society, Providence, 1997.

or:

[2] Alekseevky, Dmitri; Kriegl, Andreas; Losik, Mark; Michor, Peter W.: Choosing roots of polynomials smoothly, Israel J. Math 105 (1998), p. 203-233.

Theorem 0.1. *Let $t \mapsto A(t)$ be a smooth curve of unbounded self-adjoint operators in a Hilbert space with common domain of definition and with compact resolvent. Then the eigenvalues of $A(t)$ may be arranged in such a way that each eigenvalue is C^1.*

Suppose moreover that no two of the continuously chosen eigenvalues meet of infinite order at any $t \in \mathbb{R}$ if they are not equal. Then the eigenvalues and the eigenvectors can be chosen smoothly in t, on the whole parameter domain.

That $A(t)$ is a smooth curve of unbouded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$ with the same domains V, where the adjoint operator $A(t)^*$ is defined by $\langle A(t)u, v\rangle = \langle u, A(t)^* v\rangle$ for all v for which the left hand side is bounded as function in $u \in H$. Moreover we require that $t \mapsto \langle A(t)u, v\rangle$ is smooth for each $u \in V$ and $v \in H$.

With the help of the catesian closed calculus for locally convex spaces as explained in [1] one can show that in turn the following mappings are smooth:

$t \mapsto A(t)u$ is smooth $\mathbb{R} \to H$ for each $u \in V$.

$t \mapsto A(t)$ is smooth $\mathbb{R} \to L(V_t, H)$, with the topology of uniform convergence on compact parts of smooth curves in V_t, where the Hilbert space V_t is V with inner product $\langle u, v\rangle_t = \langle u, A(t)v\rangle$.

$t, z, u \mapsto (A(t)-z)^{-1}u$ is smooth into H for $t \in \mathbb{R}$, z in the resolvent set, and $u \in H$.

The proof of the theorem (which was not presented) at later stages also involves results about choosing roots of polynomials smoothly. These were presented in some details. See [2].

Schrödinger Operators on Graphs and Symplectic Geometry

Sergey P. Novikov

Abstract

Since 1997 (Uspekhi Math Nauk-Russia Math Surveys, 1997,n 6) the present author published a series of works dedicated to the Spectral Theory of the Schrodinger Operators on Graphs (discrete and continuous) (see the volume of Conference dedicated to the 60th birthday of V.Arnold, "Schrodinger Operators on Graphs and Symplectic Geometry", Fields Institute, Toronto, 1999, and the Asian Math Journal, volume dedicated to the 70th birthday of Mikio Sato, "Schrodinger Operators on Graphs and Topology", December 1998).

The 1-homology-valued symplectic form ("Symplectic Wronskian") on the spaces of solutions has been constructed. This quantity plays fundamental role in the construction of the Scattering Theory for the Graphs with tails. In particular, the unitarity properties of scattering are elementary topological and symplectic phenomena. Nonlinear systems also were considered recently in the joint work with A.Schwarz (Uspekhi, 1999, n 1).

CRITICAL METRICS FOR SPECTRAL ZETA FUNCTIONS

Kate Okikiolu

Abstract

Let M be a closed compact n-dimensional manifold with n odd. For a metric g on M, let Δ be the Laplace-Beltrami operator with eigenvalues $0 < \lambda_1 \leq \lambda_2 \dots$ and corresponding L^2-orthonormalized eigenfunctions $\phi_1, \phi_2, \dots$. The *spectral zeta function* for Δ is given by

$$Z(s) \ = \ \sum_{j=1}^{\infty} \lambda_j^{-s}$$

when $\Re s > n/2$. It extends to a meromorphic function for $s \in \mathbb{C}$. The determinant of Δ is defined by $\det' \Delta = e^{-Z'(0)}$. For k fixed, $Z(k)$ defines a functional on the space of metrics. When k is an integer we consider metrics of a given volume and give a description of the critical metrics for $Z(k)$ in terms of the Schwartz kernel of Δ^{-k}. We show that when $k < n/2$, every critical metric for $(-1)^{k+(n+1)/2} Z(k)$ has finite index. Similarly, every critical metric for $(-1)^{(n-1)/2} \det' \Delta$ has finite

index, so $\det' \Delta$ behaves like $Z(1)$ in this respect. When $k > n/2$ the series $Z(k)$ is summable and the behavior of $Z(k)$ close to a critical metric g_0 depends on the local zeta function

$$Z(k,x) \;=\; \sum_{j=1}^{\infty} \lambda_j^{-k} |\phi_j(x)|^2$$

at g_0. In particular, if $Z(k,x) < (2/n)Z(k)$ for some x, then $Z(k)$ has an essential saddle point at g_0, that is neither $Z(k)$ nor $-Z(k)$ has finite index. It is not known whether there is a critical metric satisfying this condition. On the other hand, if $Z(k,x) > (2/n)Z(k)$ then $-Z(k)$ has finite index. This condition holds for all homogeneous spaces.

Absolute Continuity of Periodic Schrodinger Operators

Zhongwei Shen

Abstract

This talk concerns the Schrödinger operator $-\Delta + V(\mathbf{x})$ in $\mathbb{R}^d$, $d \geq 3$, with periodic potential V. Under the assumption $V \in L^{d/2}_{loc}(\mathbb{R}^d)$, it is shown that the spectrum of $-\Delta + V(\mathbf{x})$ is purely absolutely continuous. The condition on the potential V is optimal in the context of L^p spaces. The proof relies on certain uniform Sobolev inequalities on the d-torus. We also establish the absolute continuity of $-\Delta + V(\mathbf{x})$ with certain periodic potential V in the weak-$L^{d/2}$ space.

On the Bethe-Sommerfeld conjecture for the polyharmonic operator

Alexander V. Sobolev

Abstract

Let $H = H^{(l)} = (-\Delta)^l + V$ be the polyharmonic operator in $L^2(\mathbb{R}^d)$, $d \geq 2$, perturbed by a real-valued potential V periodic with respect to a lattice $\Gamma \subset \mathbb{R}^d$. The spectrum of H is known to consists of a union of closed intervals called *spectral bands*, possibly separated by spectrum free intervals called *spectral gaps*. We prove that the spectrum of the operator H has finitely many spectral gaps under suitable conditions on the order l and dimension d. To state the result in the precise form introduce some notation. For each $\lambda \in \mathbb{R}$ define the quantity $\mathfrak{m}(\lambda)$ to

be the number of spectral bands covering the point λ. Also, define

$$\mu = \begin{cases} 1, & d \neq 1 \mod 4, \\ 3, & d = 1 \mod 4. \end{cases}$$

Then the following two theorems hold.

THEOREM 0.2. *Suppose that the real-valued function V is bounded. If $4l > d + \mu$, then*

$$\mathfrak{m}(\lambda) \geq C\lambda^{\frac{d-\mu}{4l}}, \tag{1}$$

for all sufficiently large values of λ, with a positive constant C independent of V and λ.

THEOREM 0.3. *Suppose that V is an infinitely smooth periodic function. Then the estimate* (1) *holds under the condition $4l > d + \mu - 2$.*

Open Problem

Let $\Gamma \subset \mathbb{R}^d$, $d \geq 2$ be a lattice with a fundamental domain $\mathcal{Q}$. Denote by $\#(\mathbf{k}, \rho)$ the number of the lattice points inside the ball $B(\mathbf{k}, \rho)$ of radius $\rho > 0$ centered at the point $-\mathbf{k} \in \mathbb{R}^d$. We are interested in the variation of the number $\#(\mathbf{k}, \rho)$ when both $\mathbf{k}$ and the radius ρ change. It is clear that as $\rho \to \infty$, the leading term of $\#(\mathbf{k}, \rho)$ is given by the volume $\mathrm{vol}(\rho)$ of the ball $B(0, \rho)$. It is also known that for large ρ

$$\int_{\mathcal{Q}} |\#(\mathbf{k}, \rho) - \mathrm{vol}(\rho)| d\mathbf{k} \geq c\rho^{\frac{d-\mu}{2}}, \tag{2}$$

with a positive constant c and the number

$$\mu = \begin{cases} 1, & d \neq 1 \mod 4, \\ 3, & d = 1 \mod 4. \end{cases}$$

This estimate immediately leads to the point-wise bounds

$$\begin{cases} \max_{\mathbf{k}} \#(\mathbf{k}, \rho) \geq & \mathrm{vol}(\rho) + c\rho^{\frac{d-\mu}{2}}, \\ \min_{\mathbf{k}} \#(\mathbf{k}, \rho) \leq & \mathrm{vol}(\rho) - c\rho^{\frac{d-\mu}{2}}, \end{cases} \tag{3}$$

and hence

$$\max_{\mathbf{k}} \#(\mathbf{k}, \rho) - \min_{\mathbf{k}} \#(\mathbf{k}, \rho) \geq c\rho^{\frac{d-\mu}{2}}. \tag{4}$$

If one assumes that the lattice is *rational*, then the latter estimate is known to hold with the exponent $d - 2$ instead of $(d - \mu)/2$.

Questions: Is it possible to prove either of the bounds (2) or (3) with $\mu = 1$ for all dimensionsΓ

Can one find an estimate of the form (4) with an exponent greater than $(d - \mu)/2$ *without* the assumption that the lattice is rationalΓ

Hearing analytic plane domains with the symmetry of an ellipse

Steven Zelditch

Abstract

Let D be the class of real analytic plane domains with the symmetry of an ellipse, i.e. an up-down symmetry and a right-left symmetry. Assume that at least one symmetry axis is non-degenerate as a bouncing ball orbit of the billiard flow. We will sketch the proof of the following theorem: Two domains in the class D with the same Dirichlet spectrum are isometric.

Open Problems

Problem 1: Metrics with maximal multiplicities.

Suppose that g is a metric on S^2 with the property that the multiplicities of its distinct eigenvalues are precisely the same as for the standard metric, i.e. $m_k = 2k + 1$. Is g the standard metricΓ

Discussion: This problem was posed by S.T. Yau after some work of M. Engman and S.Y. Cheng. In the paper, 'Maximally degenerate Laplacians'. Ann. Inst. Fourier (Grenoble) 46 (1996), I proved that g must be a Zoll metric (all geodesics closed) with the property that its Laplacian Δ_g is isopectral to $\Delta_0 + S$, where Δ_0 is the standard Laplacian, and S is a smoothing operator. I also gave an 'integral geometry' condition on g. In 'Fine structure of Zoll spectra'. J. Funct. Anal. 143 (1997), 415–460. I further showed that the projections Π_k onto eigenspaces were asymptotic to all orders to the standard projections. Thus, the spectral theory of Δ_g seems to have 'infinite order contact' with the spectral theory of Δ_0. The problem whether g is the standard metric remains open. There are analogous problems on higher dimensional spheres, etc.

Problem 2: Sup-norms of eigenfunctions.

Hörmander proved in the 60's that the sup norm $||\phi_\lambda||_\infty$ of an eigenfunction $\Delta\phi_\lambda = \lambda^2\phi_\lambda$ of the Laplacian on a compact Riemannian manifold (M, g) satisfies: $||\phi_\lambda||_\infty << \lambda^{\frac{n-1}{2}}$. This estimate is sharp in the sense that it is achieved for $(M, g) = (S^n, g_0)$, the standard metric on the n-sphere.

The proof is that $N(\lambda, x) := \sum_{j:\lambda_j \leq \lambda} |\phi_{\lambda_j}(x)|^2 = C_n\lambda^n + R(\lambda, x)$ where $R(\lambda, x) << \lambda^{n-1}$ uniformly in x. Observe that $|\phi_{\lambda_j}(x)|^2 << R(\lambda, x)$ to conclude the sup-norm estimate. Here, $<<$ means 'bounded by a constant independent of x.'

Question 1: For which manifolds M does there exist a metric g achieving this boundΓ

There should exist topological restrictions. Heuristically, the point at which ϕ_λ achieves its sup-norm should be a point of recurrence for the geodesic flow. Roughly speaking, there should exist a 'point' such that almost all geodesics leaving that point return to that point at a fixed time. This poses well-known topological conditions on M (cf. the book of Besse, Manifolds all of whose geodesics are closed.)

At the opposite extreme, there exist metrics (e.g. flat metrics on irrational tori) for which $||\phi_\lambda||_\infty << 1$.

Question 2: For which M does there exist a g such that $||\phi_\lambda||_\infty << 1$ Must (M, g) be a flat torusΓ

In a forthcoming paper with J. Toth, we will prove that the answer is yes for (at least broad classes) of metrics with completely integrable geodesic flow.

Breit-Wigner approximations

Vesselin Petkov and Maciej Zworski

Abstract

For operators with a discrete spectrum, $\{\lambda_j^2\}$, the counting function of λ_j's, $N(\lambda)$, trivially satisfies $N(\lambda+\delta) - N(\lambda-\delta) = \sum_j \delta_{\lambda_j}((\lambda-\delta, \lambda+\delta])$. In scattering situations the natural analogue of the discrete spectrum is given by resonances, $\lambda_j \in \mathbb{C}_+$, and of $N(\lambda)$, by the scattering phase, $s(\lambda)$. The relation between the two is now non-trivial and we prove that

$$s(\lambda+\delta) - s(\lambda-\delta) = \sum_{|\lambda_j-\lambda|<\epsilon} \omega_{\mathbb{C}_+}(\lambda_j, [\lambda-\delta, \lambda+\delta]) + \mathcal{O}(\delta)\lambda^{n-1} ,$$

where $\omega_{\mathbb{C}_+}$ is the harmonic measure of the upper of half plane and δ can be taken dependent on λ. This provides a precise high energy version of the Breit-Wigner approximation, and relates the properties of $s(\lambda)$ to the distribution of resonances close to the real axis.

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