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On Thermal States of (1+1)–dimensional Quantum Systems

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Abstract:
(1+1)–dimensional thermal systems will be investigated and their relation to vacuum theories will be explained. Moreover, consequences for the linit T $\to$ 0 and for supersymmetry will be discussed.

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1. Introduction

In a recent paper J. Yngvason and the author [BY99] investigated KMS–states on a dynamical system $(\mathcal{A}, \alpha_t)$. The main aim of this investigation consisted in looking for consequences of Tomita's modular theory [To67,Ta70]. Let ω_β be a β–KMS–state of the dynamical system, let $\mathcal{M}$ be the von Neumann algebra generated by the GNS–representation of $\mathcal{A}$, and $U(t)$ the induced representation of the dynamical group α_t. It was assumed that the cyclic vector Ω is the only vector invariant under $U(t)$. Since ω_β is a KMS–state, the group $U(t)$ is up to a scalefactor the modular group of the pair $(\mathcal{M}, \Omega)$. The precise relation is

$$\Delta_{\mathcal{M}}^{\mathrm{i}t} = U(-\beta t), \tag{1.1}$$

where the sign is a consequence of the different convention in mathematics and physics. Since Ω is the only vector invariant under the modular group it follows that $\mathcal{M}$ is a factor.

Moreover it was assumed that there exists a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ fulfilling

(i) $\mathrm{Ad}\, U(t)\mathcal{N} \subset \mathcal{N}$ for $t \geq 0$.

(ii) $\underset{t}{\cup} \mathrm{Ad}\, U(t)\mathcal{N}$ is dense im $\mathcal{M}$.

This implies by a Reeh–Schlieder type argument [RS61] the cyclicity of Ω for $\mathcal{N}$. Moreover, $\mathcal{N}$ fulfils the condition of half–sided modular inclusion, which implies [Wie93,97] that there exists a one–parametric group $V(s)$ with non–negative spectrum. This group has the following properties:

$$\begin{aligned} V(s)\Omega &= \Omega, \qquad s \in \mathbb{R}, \\ U(t)V(s)U(-t) &= V(\mathrm{e}^{\frac{2\pi}{\beta}t}s), \\ \mathrm{Ad}\, V(1)\mathcal{M} &= \mathcal{N}, \\ J_{\mathcal{M}}V(s)J_{\mathcal{M}} &= V(-s). \end{aligned} \tag{1.2}$$

Since there exists a half–sided translation it follows that $\mathcal{M}$ is of type III [Bch98].

This theory develops interesting features if one looks at two–dimensional models which factorize in light cone coordinates. In this situation the time translation $U(t)$ factorizes also. Therefore, the β–KMS–state is a product state and induces β–KMS–states on every factor. This implies that the group $V(s)$ factorizes into two groups with non–negative spectrum.

$$\begin{aligned} U(t) &= U^+(t) \otimes U^-(t), \\ V(s) &= V^+(s) \otimes V^-(s), \\ \mathcal{M} &= \mathcal{M}^+ \overline{\otimes} \mathcal{M}^-, \\ \Delta^{it}_{\mathcal{N}} &= \Delta^{it}_{\mathcal{N}^+} \overline{\otimes} \Delta^{it}_{\mathcal{N}^-}, \\ \mathcal{H} &= \mathcal{H}^+ \overline{\otimes} \mathcal{H}^-. \end{aligned} \tag{1.3}$$

$\mathcal{N}^\pm$ are defined by $V^\pm(1)\mathcal{M}^\pm V^\pm(-1)$, where $V^+(s)$ is identified with $V^+(s)\otimes \mathbb{1}$ and similar definition for $V^-(s)$.

This tensor product structure inspired B. Schroer and H.-W. Wiesbrock [SW99] to associate with $\mathcal{M}^+$ and its commutant in $\mathcal{H}^+$ a quantum field theory on the line. In this case the theory becomes a vacuum theory and $V^+(s)$ the translation. The algebra $\mathcal{M}^+$ is identified with the algebra of the positive half–line and the algebra $\mathcal{N}^+$ with the algebra of the set $[1,\infty)$. The commutant of $\mathcal{M}^+$ is identified with the algebra of the negative half–line. By standard construction one obtains a theory of local observables on the line provided Ω^+ is also cyclic for the algebras $\mathcal{M}^+([a,b])$. In this setting the algebra $\mathcal{M}$ becomes the algebra of the forward light cone C^+, and the thermal behaviour of the original theory is a kind of Unruh–effect for the forward light cone.

Inspired by the above results one would like to characterize those two–dimensional quantum systems and its thermal states, which can be reduced to the above structure. Moreover, one likes to understand the situation if the theory can not be written as a tensor product.

2. Assumptions and the tensor product case

We start with a C^*–dynamical system $(\mathcal{A}, \alpha_t)$ and assume that ω_β is a β–KMS–state. Let $(\pi_\omega, U(t), \Omega)$ be the GNS–representation of ω_β. We assume that Ω is the only $U(t)$ invariant vector in the representation space $\mathcal{H}_\omega$. This implies the von Neumann algebra $\mathcal{M} := \pi(\mathcal{A})$" is a factor. In addition we assume that there exists a proper von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ which has the properties

$$\begin{aligned} &\Omega \text{ is cyclic for } \mathcal{N}, \\ &\mathrm{Ad}\, U(t)\mathcal{N} \subset \mathcal{N} \text{ for } t > 0, \\ \bigcap_t\, &\mathrm{Ad}\, U(t)\mathcal{N} = \mathbb{C}\mathbb{1}, \\ &\{\bigcup_t \mathcal{N}\}'' = \mathcal{M}. \end{aligned} \tag{2.1}$$

Using modular theory one easily checks that the third and fourth condition are equivalent.

Using the last condition and the fact that, up to a scaling factor, $U(t)$ is the modular group of $\mathcal{M}$ one concludes by a Reeh–Schlieder [RS61] type argument that Ω is also cyclic for $\mathcal{N}$. From these assumptions J. Yngvason and the author [BY99] concluded, that there exists a one–parametric continuous unitary group $V(t)$ with non–negative spectrum which fulfils

$$\begin{aligned} V(s)\Omega &= \Omega, \\ \mathrm{Ad}\,\Delta^{it}_{\mathcal{M}} V(s) &= V(\mathrm{e}^{-2\pi t}s), \\ J_{\mathcal{M}} V(s) J_{\mathcal{M}} &= V(-s), \\ \mathrm{Ad}\, V(1)\mathcal{M} &= \mathcal{N}. \end{aligned} \tag{2,2}$$

Inspired by the two–dimensional example treated in [BY99] we will assume, there exist two groups $V^+(s), V^-(s)$, both with non–negative spectrum. These groups shall fulfil the following relations

2.1 Assumptions:

$$\begin{aligned} V^{\pm}(s)\Omega &= \Omega, \\ V^+(t)V^-(s) &= V^-(s)V^+(t), \quad s,t \in \mathbb{R}^2, \\ V^+(s)V^-(s) &= V(s), \\ \mathrm{Ad}\, V^{\pm}(s)\mathcal{M} &\subset \mathcal{M}, \quad \text{for} \quad s \geq 0, \\ \mathrm{Ad}\, V^{\pm}(s)\mathcal{M} &\neq \mathcal{M}, \quad \text{for} \quad s > 0. \end{aligned}$$

The last relation implies that $V^+(s)$ are half–sided translations. This implies by [Bch92]

$$\begin{aligned} \mathrm{Ad}\,\Delta^{it}_{\mathcal{M}} V^{\pm}(s) &= V^{\pm}(\mathrm{e}^{-2\pi t}s), \\ J_{\mathcal{M}} V^{\pm}(s) J_{\mathcal{M}} &= V^{\pm}(-s). \end{aligned} \tag{2.3}$$

Using the groups $V^{\pm}(s)$ and their properties we show

2.2 Lemma:

Let $V^{\pm}(s)$ be as above, then the following sets coincide

(i) $\bigcap_s \mathrm{Ad}\, V^{\pm}(s)\mathcal{M}$,

(ii) *the set of elements $A \in \mathcal{M}$ which commute with $V^{\pm}(s)$ for all s,*

(iii) $\{\bigcup_s \mathrm{Ad}\, V^{\pm}(s)\mathcal{M}'\}'$, *(iv) the set of weak limit points of* $\lim_{s\to\infty} \mathrm{Ad}\, V^{\pm}(s)A, \quad A \in \mathcal{M}$.

Proof: The set $\mathcal{M}^{\mp}$ of $V^{\pm}(s)$ invariant elements is contained in every $\mathrm{Ad}\, V^{\pm}(s)\mathcal{M}$. Hence the sets (ii) are contained in the sets (i). From $\mathrm{Ad}\, \Delta_{\mathcal{M}}^{it} V^{\pm}(s) = V^{\pm}(e^{2\pi t}s)$ we see that the sets (i) are modular covariant subalgebras of $\mathcal{M}$. On the other hand $V^{\pm}(s)$ map the sets (i) onto itself. Hence the restriction of $V^{\pm}(s)$ and $\Delta_{\mathcal{M}}^{it}$ to the Hilbert spaces, generated by the application of the sets (i) to the cyclic vector, commute. Therefore, the relation $\mathrm{Ad}\, \Delta_{\mathcal{M}}^{it} V^{\pm}(s) = V^{\pm}(e^{2\pi t}s)$ can only hold if $V^{\pm}(s)$, restricted to the corresponding Hilbert spaces, are constant. Since Ω is separating for the sets (i), we conclude that the sets (i) are contained in the sets $\mathcal{M}^{\mp}$.

If $A \in \mathcal{M}^{\mp}$ then it commutes with $\mathcal{M}'$ and also with $V^{\pm}(s)$. Hence the algebras $\mathcal{M}^{\mp}$ are contained in the sets (iii). Conversely let A be contained in the sets (iii). Choose $B_1, B_2 \in \mathcal{M}'$ and look at the function

$$F(s) := (\Omega, B_1 V^{\pm}(s) A V^{\pm}(-s) B_2 \Omega).$$

Using the commutation of A with $V^{\pm}(s) B_2 V^{\pm}(-s)$ we obtain

$$F(s) = (\Omega, B_1 V^{\pm}(s) A V^{\pm}(-s) B_2 V^{\pm}(s)\Omega) = (\Omega, B_1 B_2 V^{\pm}(s) A \Omega).$$

This implies that $F(s)$ has a bounded analytic extension into the upper complex half–plane. By the same method we can bring A to the left and obtain

$$F(s) = (\Omega, A V^{\pm}(-s) B_1 B_2 \Omega).$$

By this relation $F(s)$ has a bounded analytic extension into the lower complex half–plane. Both continuations together imply that $F(s)$ is a bounded entire analytic function and hence constant. Therefore, we get

$$(\Omega, B_1 V^{\pm}(s) A V^{\pm}(-s) B_2 \Omega) = (\Omega, B_1 A B_2 \Omega).$$

Since Ω is cyclic for $\mathcal{M}'$ we see that the sets (iii) are contained in $\mathcal{M}^{\mp}$.

If A_l is a weak limit point of $\lim_{s\to\infty} \mathrm{Ad}\, V^{\pm}(s)A$, then by monotonicity A_l is contained in every $\mathrm{Ad}\, V^{\pm}(s)\mathcal{M}$. Hence the sets (iv) are contained in the sets (i). If A is invariant under the action of $V^{\pm}(s)$ then it coincides with its limit. Hence $\mathcal{M}^{\mp}$ is contained in the sets (iv). □

The algebras introduced by the equivalent sets in Lemma 2.2 suggest the following notation

2.3 Definition:

With the assumptions introduced in this section we set

$$\mathcal{M}^{-} = \bigcap_s \mathrm{Ad}\, V^{+}(s)\mathcal{M},$$
$$\mathcal{M}^{+} = \bigcap_s \mathrm{Ad}\, V^{-}(s)\mathcal{M}.$$

The interchange of signs is inspired by conformal field theory. In that case $\mathcal{M}^+$ is the algebra generated by the fields depending only on the light cone coordinate x^+.

A situation, similar to that of two–dimensional conformal field theory, can be characterized as follows:

2.4 Theorem:

Assume $\mathcal{M}^+$ and $\mathcal{M}^-$ commute. If one has $\mathcal{M}^+ \vee \mathcal{M}^- = \mathcal{M}$ then

(i) $\mathcal{M} \cong \mathcal{M}^+ \overline{\otimes} \mathcal{M}^-$.

(ii) The relative commutant of $\mathcal{M}^+$ in $\mathcal{M}$ is $\mathcal{M}^-$ and viceversa.

(iii) ω_β induces on $\mathcal{M}^+$ and on $\mathcal{M}^-$ β–KMS–states.

Proof: In the proof of Lemma 2.2 we observed that $\mathcal{M}^+$ is a modular covariant subalgebra of $\mathcal{M}$. Let $\mathcal{H}^+$ be the closure of $\mathcal{M}^+\Omega$. This space is invariant under $V^+(s)$. Since there is only one vector invariant under $V^+(s)V^-(s)$ it follows that Ω is the only $V^+(s)$ invariant vector in $\mathcal{H}^+$. Hence by the result explained in [Bch98] one knows that Ω is also the only invariant vector of Δ^{it} in $\mathcal{H}^+$. This implies that $\mathcal{M}^+$ is a factor. Let $(\mathcal{M}^+)^c$ be the relative commutant of $\mathcal{M}^+$ in $\mathcal{M}$ then $\mathcal{M}^- \subset (\mathcal{M}^+)^c$ which implies $\mathcal{M}^+ \vee (\mathcal{M}^+)^c = \mathcal{M}$. Hence by a result of Takesaki [Tak72] we obtain

$$\mathcal{M} \cong \mathcal{M}^+ \overline{\otimes} (\mathcal{M}^+)^c.$$

Since the map $\mathcal{M}^+ \overline{\otimes} (\mathcal{M}^+)^c \to \mathcal{M}$ is normal we obtain

$$\mathcal{M} \cong \mathcal{M}^+ \overline{\otimes} \mathcal{M}^-.$$

This is the first statement. The second follows from this formula. Since ω_β is $(\Omega, .\Omega)$ and since $\mathcal{M}^+$ and $\mathcal{M}^-$ are both covariant under the action of $\Delta^{it}_{\mathcal{M}}$ we get that the restriction of ω_β to $\mathcal{M}^+$ and $\mathcal{M}^-$ respectively are β–KMS–states. □

Having a tensor product decomposition of the theory we could try to associate to every factor a theory on the line as it has been done by Schroer and Wiesbrock [SW99]. Since this procedure has been explained in the introduction we will not repeat the arguments. We are more interested in cases where Th. 2.4 fails.

3. Two–dimensional local quantum fields

In addition to the assumption that there exist the two groups $V^+(s)$ and $V^-(s)$ we will assume that we are dealing with a quantum field theory of local observables. So we make the standard assumptions of locality and isotony. By these assumptions the light cone coordinates x^+, x^- have a definite meaning. Following the idea of Schroer and Wiesbrock [SW99] we try to embed our theory into a vacuum theory, so that $\mathbb{R}^2$ is mapped onto V^+. This is done by the transformation

$$\xi^+ = \mathrm{e}^{x^+}, \qquad \xi^- = \mathrm{e}^{x^-}. \tag{3.1}$$

Since we are dealing with a two–dimensional theory, and since the chosen coordinates are light cone coordinates it follows that the transformation (3.1) preserves locality. Therefore, in the new system we have locality inside the forward light cone. With help of the two groups $V^+(\xi^+)$ and $V^-(\xi^-)$ one can transport locality to all of $\mathbb{R}^2$. Since ω_β is a KMS–state it follows that the vector Ω is not only cyclic but also separating for $\mathcal{M}$, in the new setting this algebra is $\mathcal{M}(C^+)$ where C^+ denotes the forward light cone. The two groups $V^+(\xi^+)$ and $V^-(\xi^-)$ generate the translations $V(\xi)$ which fulfils spectrum condition. By the above transformation the modular group of $\mathcal{M}(C^+)$ is again a geometric transformation, namely the dilatations. In this setting the algebra $\mathcal{N}$ becomes the algebra of the cone shifted by the vector $(1,1)$. More precisely we obtain

3.1 Lemma

The transformation Eq. (3.1) *sends the lines $x_1 =$ const. onto radial lines $\xi^+ = \alpha\xi^-$, $0 < \alpha < \infty$. The lines $x_0 =$ const. are sent onto hyperboloids $\xi^+\xi^- =$ const. > 0.*

The proof of these statements is trivial.
The fact that the modular group of the forward light cone coincides with the dilatations has drastic consequences.

3.2 Theorem:

Assume we are dealing with a quantum field theory of local observables in the vacuum sector which fulfils
duality if the dimension of the Minkowski space is two,
Bisognano–Wichmann property if the dimension is 4.

Assume the vector Ω is cyclic and separating for the algebra $\mathcal{M}(C^+)$, the algebra of the forward light cone. Let (Δ, J) denote the modular operator and the modular conjugation of the pair $(\mathcal{M}(C^+), \Omega)$. Assume the modular group acts like a dilatation, i.e.

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{M}(D_{a,b}) = \mathcal{M}(D_{\mathrm{e}^{-2\pi t}a,\mathrm{e}^{-2\pi t}b}), \qquad a, b \in C^+, \tag{3,2}$$

where $D_{a,b}$ denotes the double cone $(a+C^+)\cap(b+C^-)$. Then one has timelike commutation, i.e. $\mathcal{M}(D_1)$ and $\mathcal{M}(D_2)$ commute if the two double cones are timelike separated.

Proof: We want to show that the commutant of $\mathcal{M}(C^+)$ is $\mathcal{M}(C^-)$. We obtain the commutant of $\mathcal{M}(C^+)$ by transforming this algebra with J, the modular conjugation of $\mathcal{M}(C^+)$. Since we know

$$J\Delta^{1/2}A\Omega = A^*\Omega, \qquad A \in \mathcal{M}(C^+),$$

we have to show, that to every $A \in \mathcal{M}(C^+)$ exists $\hat{A} \in \mathcal{M}(C^-)$ with

$$\Delta^{1/2}A\Omega = \hat{A}\Omega.$$

This solves our problem because Ω is separating for $J\mathcal{M}(C^+)J$. To this end we have to use the modular group $\Delta^{\mathrm{i}t}$. Let us first assume we are dealing with a scalar Wightman field. In this case we know

$$\Delta^{\mathrm{i}t}A(x)\Omega = A(\mathrm{e}^{-2\pi t}x)\Omega.$$

For $x \in C^+$ this expression can be analytically continued into the strip $S(-\frac{1}{2}, 0)$. We obtain

$$\Delta^{\frac{1}{2}} A(x)\Omega = \Delta^{\mathrm{i}(-\frac{\mathrm{i}}{2})} A(x)\Omega = A(\mathrm{e}^{\mathrm{i}\pi} x)\Omega = A(-x)\Omega.$$

Since $-x$ belongs to C^- we obtain the result in this situation.

The proof of the general situation needs some preparations. We start with

3.3 Lemma:

Let $\sigma_C^t(A) = \Delta^{\mathrm{i}t}_{\mathcal{M}(C+)} A \Delta^{-\mathrm{i}t}_{\mathcal{M}(C+)}$. If σ_C^t acts on $\mathcal{M}(C^+)$ as dilatations, then it acts on every local algebra as dilatations.

Proof: Since the algebra of any domain is generated by the algebra of double cones, it is sufficient to show the lemma for algebras of double cones. Let $D_{a,b}$, $b \in a + C^+$ be a double cone. Then exists a vector x such that $a + x \in C^+$. We write $Da, b = D_{a+x,b+x} - x$ and get $\mathcal{M}(D_{a,b}) = \mathrm{Ad}\, U(-x)\mathcal{M}(D_{a+a,b+a})$. With this we obtain

$$\begin{aligned}
\sigma_C^t(\mathcal{M}(D_{a,b})) &= \sigma_C^t \circ \mathrm{Ad}\, U(-x)(\mathcal{M}(D_{a+x,b+x}) \\
&= \mathrm{Ad}\, U(-\mathrm{e}^{-2\pi t} x)\sigma_C^t(\mathcal{M}(D_{a+x,b+x}) = \\
\mathrm{Ad}\, U(-\mathrm{e}^{-2\pi t} x)\mathcal{M}(D_{\mathrm{e}^{-2\pi t}(a+x),\mathrm{e}^{-2\pi t}(b+x)}) &= \mathcal{M}(D_{\mathrm{e}^{-2\pi t}(a+x),\mathrm{e}^{-2\pi t}(b+x)} - \mathrm{e}^{-2\pi t} x) \\
&= \mathcal{M}(D_{\mathrm{e}^{-2\pi t} a,\mathrm{e}^{-2\pi t} b}).
\end{aligned}$$

This shows the lemma. □

Notice that the assumptions of the theorem imply that there exists a PCT–operator Θ. (See [BCH92] for the two–dimensional case and [GL95] for the other case.) Next we show

3.4 Lemma:

Let W denote a wedge the edge of which contains $\{0\}$. Let (Δ_C, J_C) be the modular operator and modular conjugation of the algebra of the forward light cone and (Δ_W, J_W) be those of the algebra of the wedge, and Θ be the PCT–operator. Then $\Delta_C^{\mathrm{i}t}$ commutes with $\Delta_W^{\mathrm{i}s}$, J_W and Θ and $\Delta_W^{\mathrm{i}t}$ commutes with J_C and Θ. Moreover, Ω is cyclic and separating for $\mathcal{M}(C^-)$ and Δ_C^{-1} is the modular operator of $\mathcal{M}(C^-)$.

Proof: From the last lemma we know that $\mathrm{Ad}\, \Delta_C^{\mathrm{i}t}$ maps $\mathcal{M}(W)$ onto itself. Hence it commutes with $\Delta_W^{\mathrm{i}s}$ and with J_W. (See [BR79] Thm. 3.2.18.) Since C^+ is invariant under Lorentz transformations it follows that $\Delta_C^{\mathrm{i}t}$ and J_C commute with the Lorentz transformations. By assumption $\Delta_W^{\mathrm{i}t}$ coincides with the Lorentz boosts of the wedge and hence it commutes with $\Delta_C^{\mathrm{i}t}$ and J_C. Since Θ coincides with $J_W U(R_W(\pi))$, where $R_W(\pi)$ is the rotation which maps W into itself we conclude that Θ commutes with $\Delta_C^{\mathrm{i}t}$ and with $\Delta_W^{\mathrm{i}t}$. From $\Theta\mathcal{M}(C^+)\Theta = \mathcal{M}(C^-)$ we conclude that $\Theta\Delta_C\Theta = \Delta_C^{-1}$ is the modular operator of $\mathcal{M}(C^-)$. □

Proof of the Theorem: Let $x \in C^+$ be a timelike vector and let $W_x = W + x$. Then exists a double cone D such that $D \subset C^+ \cap W_{\lambda x}$ for λ in some interval containing 1. Choose $A, B \in \mathcal{M}(D)$ and set $\lambda = \mathrm{e}^{-2\pi t}$ and denote by $U(x)$ the translations. Define

$$\begin{aligned}
F_1(s,t) &= (A\Omega, \Delta_C^{\mathrm{i}t}\Delta_{W_x}^{\mathrm{i}s} B\Omega) = (\Delta_C^{-\mathrm{i}t} A\Omega, \Delta_{W_x}^{\mathrm{i}s} B\Omega), \\
F_2(s,t) &= (A\Omega, \Delta_{W_{\lambda x}}^{\mathrm{i}s}\Delta_C^{\mathrm{i}t} B\Omega) = (\Delta_{W_{\lambda x}}^{-\mathrm{i}s} A\Omega, \Delta_C^{\mathrm{i}t} B\Omega).
\end{aligned}$$

From $\Delta^{\mathrm{i}s}_{W_{\lambda x}} = \operatorname{Ad} U(\lambda x)\Delta^{\mathrm{i}s}_{W}$ we obtain

$$F_1(s,t) = F_2(s,t).$$

By choice of A, B the function $F_1(s,t)$ can analytically be continued into $S(-\frac{1}{2},0) \times S(-\frac{1}{2},0)$. We obtain

$$\begin{aligned} F_1(s-\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) &= (\Delta_C^{-\mathrm{i}t} J_C A^*\Omega, \Delta^{\mathrm{i}s}_{W_x} J_{W_x} B^*\Omega) \\ &= (\Theta A^*\Omega, \Theta J_C \Delta_C^{\mathrm{i}t}\Delta^{\mathrm{i}s}_{W_x} J_{W_x} B^*\Omega). \end{aligned}$$

Let I_2 be the interval such that $D + (\lambda-1)x \subset W_x$, then for $\lambda \in I_2$ the function $F_2(s,t)$ can in s be analytically continued into the strip $S(-\frac{1}{2},0)$. We obtain

$$F_2(s-\frac{\mathrm{i}}{2}, t) = (U((\lambda-1)x)\Delta^{-\mathrm{i}s}_{W_x} J_{W_x} U((1-\lambda)x)A^*\Omega, \Delta_C^{\mathrm{i}t} B\Omega).$$

With $J_{W_x} = \operatorname{Ad} U(x) J_W$ we obtain $J_{W_x} = J_{W_x} U((\lambda-1)x) = U((\lambda-1)P_W x)J_{W_x}$, where P_W denotes the reflection in the characteristic two–plane of the wedge W. With this we get

$$F_2(s-\frac{\mathrm{i}}{2}, t) = (U((\lambda-1)x)\Delta^{-\mathrm{i}s}_{W_x} U((1-\lambda)P_W x)J_{W_x} A^*\Omega, \Delta_C^{\mathrm{i}t} B\Omega).$$

Since $U(x)$ fulfils the spectrum condition and since $-P_W x$ belongs to C^+ the last expression for $F_2(s-\frac{\mathrm{i}}{2},t)$ has in t an analytic continuation into the strip $S(-\frac{1}{2},0)$. We obtain with $\lambda(t-\frac{\mathrm{i}}{2}) = -\lambda(t)$

$$F_2(s-\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) = (U(-(\lambda+1)x)\Delta^{-\mathrm{i}s}_{W_x} U((1+\lambda)P_W x)J_{W_x} A^*\Omega, \Delta_C^{\mathrm{i}t} J_C B^*\Omega).$$

Looking at the set $(t \in I_2) \times (\Im m\, s = -\frac{\mathrm{i}}{2})$ we see by the Malgrange–Zerner theorem that $F_2(s,t)$ has an analytic continuation in both variables into some set, which has $(s \in \mathbb{R}) \times (t \in I_2)$ as boundary points. Therefore, F_1 coincides with F_2 in the domain of analyticity. Using $F_1(-\frac{\mathrm{i}}{2},-\frac{\mathrm{i}}{2}) = F_2(-\frac{\mathrm{i}}{2},-\frac{\mathrm{i}}{2})$ we obtain by the cyclicity of Ω for $\mathcal{M}(D)$ the equation

$$\begin{aligned} \Theta J_C J_W U((P_W-1)x) &= \Theta U(x) J_W U(-x) U(2(1-P_W)x) J_C \\ &= \Theta J_W U((1-P_W)x) J_C = \Theta J_W J_C U((P_W-1)x). \end{aligned}$$

This relation implies that J_W and J_C commute. Hence J_C commutes also with Θ, since it commutes with the rotation.

We know that $\Theta J_C \Theta = J_C$ is the modular conjugation of $\mathcal{M}(C^-)$. Let $A \in \mathcal{M}(C^+)$ and $B = J_C A^* J_C$ then $B\Omega$ and $B^*\Omega$ belong to the domain of $\Delta_C^{1/2}$. Hence by [BR79] Prop. 2.5.9 exists an operator $\widehat{A}\eta\mathcal{M}(C^-)$ with $B\Omega = \widehat{A}\Omega$. This implies $\Theta J_C A^*\Omega = \Theta\widehat{A}\Theta\Omega$. Hence ΘJ_C maps $\mathcal{M}(C^+)\Omega$ onto $\mathcal{M}(C^+)\Omega$. Since ΘJ_C is unitary and maps Ω onto itself we conclude by [BR79] Thm. 3.2.18 that ΘJ_C defines an automorphism of $\mathcal{M}(C^+)$. Hence $\mathcal{M}(C^-)$ is the commutant of $\mathcal{M}(C^+)$. □

Under special additional assumptions the result of Thm.3.2 has been shown by Buchholz and Fredenhagen [BF77].

4. Exit from the conformal trap

In the last section we treated two–dimensional theories of local observables and considered thermal states such that the Assumptions 2.1 are fulfilled. We saw that this can only happen for a massless non–interacting theory. If one wants to treat interacting field theories one must change Assumptions 2.1. We want to keep the two groups $V^{\pm}(s)$ but we have to replace the requirements.

4.1 Assumptions:

$$\begin{aligned} V^{\pm}(s)\Omega &= \Omega, \\ V^+(t)V^-(s) &= V^-(s)V^+(t), \quad s,t \in \mathbb{R}^2, \\ V^+(s)V^-(s) &= V(s), \\ \mathrm{Ad}\, V^+(s)\mathcal{M} &\subset \mathcal{M}, \quad \text{for} \quad s \geq 0, \\ \mathrm{Ad}\, V^-(s)\mathcal{M} &\subset \mathcal{M}, \quad \text{for} \quad s \leq 0, \\ \mathrm{Ad}\, V^+(s)\mathcal{M} &\neq \mathcal{M}, \quad \text{for} \quad s > 0, \\ \mathrm{Ad}\, V^-(s)\mathcal{M} &\neq \mathcal{M}, \quad \text{for} \quad s < 0. \end{aligned}$$

Also with help of Assumptions 4.1 one can connect the thermal representation with a vacuum theory. This is done by the transformation

$$\xi^+ = \mathrm{e}^{x^+}, \qquad \xi^- = -\mathrm{e}^{-x^-}. \tag{4.1}$$

By this change $\mathbb{R}^2$, descibed by the x–variables, is mapped onto the right wedge, described by the ξ–variables. Also in this situation the transformation (4.1) preserves locality. Therefore, one has locality in the right wedge. Again with help of the groups $V^+(\xi^+)$ and $V^+(\xi^-)$ the locality can be transported to all of $\mathbb{R}^2$ in the ξ–variables. By this manipulation we obtain a quantum field theory in the vacuum sector.

4.2 Lemma:

The transformation (4.1) *sends the lines* $x_1 =$ *const. onto hyperboloids* $\xi^+\xi^- = -m^2$ *and lines* $x_0 =$ *const. onto radial lines* $\xi^+ = -\alpha\xi^-$, $\alpha > 0$.

In this setting the thermal state ω_β can be identified with the Hawking–Unruh effect for the algebra of the wedge. Coming back to the thermal representation, we find that the algebra of the forward light cone is a proper subalgebra of $\mathcal{M}$. Therefore, the results of [BY99] apply. This result can be seen as follows: The relation in the vacuum representation for positive ξ^+ is

$$\mathrm{Ad}\, V^+(\xi^+)\mathcal{M}(W_r) = \mathcal{M}(W_r + \xi^+) \subset \mathcal{M}(W_r),$$

where this is a proper inclusion. In the x–coordinates the set $W_r + \xi^+,\ \xi^+ > 0$ becomes

$$x^+ > \log \xi^+.$$

Hence for $\xi^+ \leq 1$ this set contains the forward light cone in the x–variables. Therefore, $\mathcal{M}(C^+)$ is a proper subalgebra for $\mathcal{M}$. The difference to the situation described in the last section is the fact that for positive s the transformation $\mathrm{Ad}\, V^-(s)$ sends $\mathcal{M}$ partly into the commutant of $\mathcal{M}$.

The conditions of this section or of the last section are compatible with the assumption that the theory can be written as a tensor product. One must have

$$\mathcal{M}^+ \vee \mathcal{M}^- = \mathcal{M}, \tag{4.2}$$

where $\mathcal{M}^+$ and $\mathcal{M}^-$ have to commute and are defined as in Def. 2.3. But in a theory with lower mass gap we see that Eq. (4.2) is not fulfilled.

5. Conclusions

If we are dealing with a thermodynamical representation of a two–dimensional quantum system, we obtain an extra structure in the following cases: Let x^+ and x^- be the light cone coordinates. If the algebras associated with the sets

$$x^+ > c^+, \qquad x^- > c^-, \tag{5.1}$$

or

$$x^+ > c^+, \qquad x^- < c^- \tag{5.2}$$

are proper subalgebras of the global algebra $\mathcal{M}$, then the thermal representation can be embedded into a vacuum theory. The original algebra $\mathcal{M}$ is in these situations isomorphic to the algebra of the forward light cone or to the algebra of the right wedge respectively. If the algebras of the sets (5.1) and (5.2) are not proper subalgebras, then the space dimension plays only the role of a parameter. The only exception is the relativistic KMS–condition of Bros and Buchholz [BB94]. But to my knowledge only a few consequences have been drawn from this condition, e.g. [BB98].

If one has neither our conditions nor the relativistic KMS–condition then one is dealing with a usual dynamical system which depends on some further parameter. A thermal representation of a dynamical system $(\mathcal{A}, \alpha_t)$ is called a K–system [Em76] if there exists a proper subalgebra $\mathcal{N} \subset \mathcal{M} = \pi_\omega(\mathcal{A})''$ obeying:
(i) $\mathrm{Ad}\, U(t)\mathcal{N} \subset \mathcal{N}$ for $t \geq 0$, $(t \leq 0)$.
(ii) $\bigcap_t \mathrm{Ad}\, U(t)\mathcal{N} = \mathbb{C}\mathbb{1}$.
(iii) $\{\bigcup_t \mathrm{Ad}\, U(t)\mathcal{N}\}'' = \mathcal{M}$.
In this situation exists a one–parametric unitary group $V(s)$ with non–negative spectrum and the properties:
(i) $\mathrm{Ad}\, V(1)\mathcal{M} = \mathcal{N}$.

(ii) $\bigcap\limits_s \operatorname{Ad} V(s)\mathcal{M} = \mathbb{C}\mathbb{1}$.

(iii) $\{\bigcup\limits_s \operatorname{Ad} V(s)\mathcal{M}\}'' = \mathcal{B}(\mathcal{H})$.

The first line follows from the fact that $\mathcal{N}$ fulfils the condition of half–sided modular inclusion, see Wiesbrock [Wie93,97]. The second line follows from Yngvason and the author [BY99] and the third line from this and the modular theory.

If we are dealing with a dynamical system, which is a K–system in a thermal representation, then one can embed this theory into a ground state representation of a theory on a line. The algebra $\mathcal{M} = \pi_\omega(\mathcal{A})''$ is mapped onto the algebra of the half–line $\mathbb{R}^+$, and the commutant $\mathcal{M}'$ is mapped onto the algebra of $\mathbb{R}^-$. The algebra $\mathcal{N}$ is mapped onto the algebra of the set $[1, \infty)$ and the group $V(s)$ becomes the translation of the line. The original time translation $U(t)$ is the modular group of $\mathcal{M}$ and acts in this setting like a dilatation on the half–line $\mathbb{R}^+$. In case that the cyclic vector Ω_ω is also cyclic for the relative commutant $\mathcal{N}^c = \mathcal{N}' \cap \mathcal{M}$ of $\mathcal{N}$ one has a two–sided K–system. In this situation one also has a representation of the Möbius group. The associated ground state theory is a theory of local observables. Whether or not this ground state theory is also a chiral theory, that means it can be mapped onto a theory on the unit circle, depends on the structure of the algebras $\mathcal{M}([1, \infty))$ and its relative commutant $\mathcal{M}([0, 1])$. For details see [Bch95].

The ground state theories obtained from a K–system are constructed by associating to the commutant some new "reality". A consequence of this construction is the fact that these ground state theories do not describe systems with an energy gap as it appears in superconductivity, superfluidity, Bose–Einstein condensation, etc. The reason is a generalization of a result by Sadowski and Woronowicz [SW71].

5.1 Proposition:

Let π_0 be a ground state representation of a dynamical system. Then the two statements exclude eachother:

(1) *We are dealing with a K–system.*

(2) *The spectrum of the time translation has an energy gap, or the spectrum contains a Lebesgue singular part, except for $\{0\}$.*

Proof: Assume we are dealing with a K–system. Let $\mathcal{N}$ be the subalgebra which defines the K–property. The Reeh–Schlieder argument implies that Ω is cyclic for the algebra $\mathcal{N}$. This implies that the time translations fulfil the condition of half–sided translations. Hence by [Bch92] exists between these translations and the modular group of $\mathcal{N}$ the relation

$$\operatorname{Ad} \Delta_{\mathcal{N}}^{\mathrm{i}t} U(s) = U(\mathrm{e}^{-2\pi t} s).$$

This implies that the spectrum of $U(s)$ does not have any gap and that, except for $\{0\}$, it is absolutely Lebesgue continuous. □

It might be interesting to look at the embedded theories if one tries to go with the temperature T to zero. For this investigation one has to assume that the system we are dealing with is a K–system for every temperature in an interval $T \in (0, T_0)$ and for the same subalgebra of $\mathcal{A}$. In order to be able to look at the limit we have to choose the correct coordinates. Up to now we have worked with the modular groups of $\mathcal{M}$ and $\mathcal{N}$. In this setting the generator of $V(s) = e^{iGs}$ is

$$G = \log \Delta_{\mathcal{N}} - \Delta_{\mathcal{M}}.$$

We have to work with the time translation

$$U(-\beta t) = \Delta_{\mathcal{M}}^{it}.$$

With this notation and $U(t) = e^{iHt}$ the operator G becomes

$$G = \beta H + \log \Delta_{\mathcal{N}}.$$

If $A \in \mathcal{M}$ then the vector valued function $U(t)A\Omega$ has an analytic continuation into the strip $S(0, \beta)$. Hence for $T \to 0$ $U(t)$ fulfils the spectrum condition. In order to avoid two groups fulfilling spectrum condition at $T = 0$ one has to rescale also the group $V(s)$. We define

$$W(u) = V(\frac{u}{\beta}).$$

Writing $W(u) = e^{iFu}$ we get

$$F = H + \frac{1}{\beta} \log \Delta_{\mathcal{N}}.$$

Hence $U(t)$ and $W(t)$ coincide in the limit $T \to 0$. Since we had $\mathcal{N} = \mathrm{Ad}\, V(1)\mathcal{M}$ we get by this rescaling

$$\mathcal{N} = \mathrm{Ad}\, W(\beta)\mathcal{M}.$$

This formula does not make sense at $T = 0$. Hence there is no conflict with Prop. 5.1. By all the manipulations $\mathcal{M}$ remains the algebra of $\mathbb{R}^+$ and $\mathcal{M}'$ remains the algebra of $\mathbb{R}^-$. But it is better to deal with two copies of the real line. The structure of the "real world" is transported with help of the conjugation J to the "second world". Here the order of the time–axis is reversed.

Since at $T = 0$ the group $U(t)$ has a non–negative spectrum it defines an inner automorphism. By assumption there is only one vacuum which implies that $\mathcal{M}$ is a factor of type I. Therefore, in the limit $T = 0$ one obtains a tensor product of $\mathcal{M}$ with its isomorphic copy, which is the limit of the $\mathcal{M}'$. Consequently we obtain two copies of the real line. If one takes the other dimension into account one gets two copies of the Minkowski space. Whether or not this doubling of the space has any relation to the doubling of the space introduced by Connes and Lott [Co91], which they introduced for the description of the standard model, can only be answered by future investigations.

The last question we want to discuss is the problem of supersymmetry. After a decade of discussions on the existence of this supersymmetry in thermal states Buchholz and Ojima [BO97] showed that supersymmetry can only exist in a groundstate representation. We refere to [BO97] for references to earlier papers on this subject. Buchholz and Ojima called this result the collaps of supersymmetry. This is due to the fact that the generator of the time translation is no longer positive in thermal representations. I think there is a way out of this disastrous situation. In order to understand this let us discuss the situation. In all thermal representations the Lorentz boosts are broken. This implies the decoupling of the space translations from the time translations. Mostly, at low temperatures the space translations are also broken. Usually only some discrete spacial symmetries remain. Therefore, one can expect that at most the zero component of the supercharge remains, when passing to thermal representations. Representing the supercharge as a graded derivation one would like to get into contact with some physical quantity. The only restriction of this quantity is the requirement that it has to tend to the energy operator H for $T \to 0$. In our discussion we have met such object, namely the generator of the group $W(u)$. Therefore, my suggestion for the connection of the supercharge with a generator of some group is

$$\frac{1}{2}(\delta\overline{\delta} + \overline{\delta}\delta) = H + \frac{1}{\beta}\log \Delta_{\mathcal{N}}.$$

Here δ denotes the graded derivation associated with the supercharge.

There is also no problem with $W(u)$ acting only as a semi–group on the algebra $\mathcal{M}$. This is due to the fact that the supercharge acts only as graded derivation and not in the integrated form. In standard theories δ maps Bose fields at the point x onto Fermi fields at the same point x and viceversa. That there might appear derivatives does not cause any problem. The algebra of a domain G is generated by field operators smeared with testfunctions vanishing identically outside of G and hence also on the boundary of G. This property is stable under derivations.

The only unsolved problem for this approach to susy is to understand why physical systems should be K–systems for all temperatures except for those of phasetransitions.

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