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Weakening Holonomy

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WEAKENING HOLONOMY

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These notes are based on a I talk I gave at the Erwin Schrödinger International Institute for Mathematical Physics, Vienna, on the 20th October, 1999. This is work in progress, partly based on joint work with F. M. Cabrera and M. D. Monar and partly results of my Ph. D. student Richard Cleyton. It is a pleasure to thank the Erwin Schrödinger Institute and the organisers of the program on Holonomy Groups in Differential Geometry for their kind hospitality.

1. Introduction

Suppose (M, g) is a Riemannian manifold. One fundamental piece of data determined by g is the restricted holonomy group Hol. If we assume that Hol acts irreducibly on TM, which is the case if M is complete and irreducible, then the main classification theorem implies that either (M, g) is locally isometric to a symmetric space K/Hol or Hol is one of $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$, $Sp(n)\,Sp(1)$, G_2 or $Spin(7)$ (see [3]). Studying the geometries determined by these holonomy groups one finds that if $M \neq SO(n)$ or $U(n)$, then g is automatically Einstein. This may be restated as follows.

Theorem 1.1. *Suppose G is a proper connected subgroup of $SO(n)$ that acts irreducibly on $\mathbb{R}^n$ and if n is even suppose that $G \neq U(n/2)$. Let (M, g) be an n-dimensional Riemannian manifold with structure group G. If M admits a torsion-free G-connection then g is Einstein.*

A natural question is:

> Are there weaker conditions than the existence of a torsion-free connection that imply useful restrictions on the curvature?

In 1971, Gray [9] provided one such notion which he called "weak holonomy". He studied this idea for the groups G that act transitively on the sphere. For the groups $SO(n)$, $Sp(n)$, $Sp(n)\,U(1)$, $Sp(n)\,Sp(1)$ and $Spin(7)$, the weak holonomy condition implies that the holonomy group reduces and we obtain no new geometries. However, for $U(n)$, $SU(n)$ and G_2 Gray found that the weak holonomy condition does give

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Group	**Geometry**
$U(n)$, $SU(n)$	Nearly Kähler, i.e., $(\nabla_X J)(X) = 0$ Einstein if $n = 3$
G_2	$d\varphi = \lambda{*}\varphi$, $\lambda \neq 0$ Einstein with $s > 0$
$Spin(9)$	Not Einstein

TABLE 1. The geometries determined by weak holonomy groups acting transitively on a sphere when the holonomy does not reduce to that group.

new structures. Very recently Th. Friedrich has shown that the group $Spin(9)$ also occurs as a weak holonomy group [7]. These results are summarised in Table 1.

As the table indicates, the only new examples of Einstein structures are provided by nearly Kähler six-manifolds and seven-dimensional manifolds with weak holonomy G_2. Many examples of the latter are known. For example each Aloff-Wallach space $SU(3)/\,U(1)_{k,\ell}$, given by embedding $U(1)$ in $SU(3)$ via

$$\exp(i\theta) \mapsto \operatorname{diag}(\exp(ik\theta), \exp(i\ell\theta), \exp(-i(k+\ell)\theta)),$$

carries a homogeneous metric with weak holonomy G_2 [1]. Also non-homogeneous examples can be constructed from non-homogeneous 3-Sasakian metrics in dimension 7 by using the results of [8]. In the case of nearly Kähler six-manifolds that are not Kähler, the only examples known are 3-symmetric spaces, so these are homogeneous. Moroianu & Semmelmann have a proof that there are no other homogeneous examples [10].

As far as I know Gray's condition has not been studied for other G-structures. This may be because his definition is not particularly easy to work with. More in the spirit of Gray's other work would be to look for G-structures which admit a connection whose torsion is 'simple'. This is the approach I wish to take.

2. Torsion and Curvature

Fix a closed connected Lie subgroup G of $SO(n)$. Suppose that M is an n-dimensional manifold with a reduction of its structure group to G. Let g be the corresponding Riemannian metric and write ∇ for the Levi-Civita connection.

If ∇' is any G-connection on M, then the difference $\nabla - \nabla'$ is tensorial and is a one-form with values in (the bundle associated to) the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$. If η is any one-form with values in the Lie algebra $\mathfrak{g}$ of G, then $\nabla' + \eta$ is also a G-connection. It is now easy to see:

Lemma 2.1. *If G is a subgroup of $SO(n)$ and M is a Riemannian manifold with a G-structure, then there is a unique G-connection $\tilde\nabla$ such that the Levi-Civita connection satisfies*

$$\nabla = \tilde\nabla + \xi$$

with ξ an element of $T^*M \otimes \mathfrak{g}^\perp \subset T^*M \otimes \mathfrak{so}(n)$.

Definition 2.2. We call the connection $\tilde\nabla$ of Lemma 2.1 the *natural metric connection* of the G-structure.

The torsion of $\tilde\nabla$ is given by $T^{\tilde\nabla}(X,Y) = \xi_Y X - \xi_X Y$. Moreover, this torsion determines ξ by

$$\begin{aligned} g(\xi_X Y, Z) = \tfrac{1}{2}\big(g(T^{\tilde\nabla}(X,Z),Y) - g(T^{\tilde\nabla}(X,Y),Z) \\ + g(X, T^{\tilde\nabla}(Y,Z))\big). \end{aligned}$$

We will therefore often abuse terminology and refer to ξ as the torsion of $\tilde\nabla$.

Write V for the representation of G on $\mathbb{R}^n$. Then ξ is an element of the bundle associated to the representation $V \otimes \mathfrak{g}^\perp$. Let us assume that ξ lies in a subrepresentation $W \subset V \otimes \mathfrak{g}^\perp$. It is now possible to deduce some restrictions on the Riemann curvature tensor R of M.

The curvature R of the Levi-Civita connection is an element of $S^2(\mathfrak{so}(n))$. Using $\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ we obtain the decomposition

$$S^2(\mathfrak{so}(n)) = S^2(\mathfrak{g}) \oplus S(\mathfrak{g} \otimes \mathfrak{g}^\perp) \oplus S^2(\mathfrak{g}^\perp),$$

and a corresponding splitting of R:

$$R = R^{\mathfrak{g}} + R^m + R^\perp.$$

We can refine this decomposition further. Let $b\colon S^2(\Lambda^2 V) \to \Lambda^4 V$ be the map defined by

$$\begin{aligned} b(\alpha)(X,Y,Z,W) = \alpha(X,Y,Z,W) + \alpha(X,Z,W,Y) \\ + \alpha(X,W,Y,Z). \end{aligned}$$

The space $\mathcal{K}(\mathfrak{g}) := \ker b \cap S^2(\mathfrak{g})$ consists of elements in $S^2(\mathfrak{g})$ that satisfy the Bianchi identity, and so is the space of algebraic curvature tensors whose holonomy lies in G.

We write $S^2(\mathfrak{g}) = \mathcal{K}(\mathfrak{g}) \oplus \mathcal{K}(\mathfrak{g})^\perp$ and use this to make a splitting $R^{\mathfrak{g}} = R^{\mathfrak{g}}_0 + R^{\mathfrak{g}}_1$. Note that b is injective on $\mathcal{K}(\mathfrak{g})^\perp$, so the fact that

R satisfies the Bianchi identity $b(R) = 0$ implies that $R_1^{\mathfrak{g}}$ is uniquely determined by $R^m + R^\perp$ via

$$b(R_1^{\mathfrak{g}}) = -b(R^m + R^\perp).$$

To obtain information on R^m and $R^\perp$, let us locally choose a tensor φ on M (not necessarily of pure type) such that

(a) $\operatorname{Lie}\operatorname{stab}_{SO(n)}\varphi = \mathfrak{g}$, and
(b) $\tilde{\nabla}\varphi = 0$,

where $\operatorname{Lie}\operatorname{stab}_{SO(n)}\varphi$ denotes the Lie algebra of the stabiliser of φ under the action of $SO(n)$. One can find such a φ_0 satisfying condition (a) at a given point. Suppose $\varphi_0 \in \bigoplus_{i=1}^s V^{\otimes r_i}$, for a minimal set of integers $r_1, \dots, r_s$. Let U_i be the trivial submodule of $V^{\otimes r_i}$, for $i = 1, \dots, s$. Then U_i defines a subbundle of tensor algebra of TM and the restriction of $\tilde{\nabla}$ to U_i is flat, so we may locally extend φ_0 to a tensor φ satisfying (b).

Now consider the action of the curvature R on φ. We have

$$R.\varphi = R^m.\varphi + R^\perp.\varphi.$$

Moreover, $R.\varphi$ determines R^m and $R^\perp$. On the other hand

$$\begin{aligned} R.\varphi &= \mathbf{a}(\nabla\nabla\varphi) = \mathbf{a}(\nabla(\tilde{\nabla}\varphi + \xi.\varphi)) = \mathbf{a}(\nabla(\xi.\varphi)) \\ &= \mathbf{a}((\tilde{\nabla}\xi).\varphi) + \mathbf{a}(\xi^2.\varphi), \end{aligned}$$

where $\mathbf{a}$ denotes the alternation map. From this we see that if ξ lies in a subrepresentation W of $V \otimes \mathfrak{g}^\perp$, then $R^m + R^\perp$ lies in the representation $V \otimes W + W \otimes W$. Note that $R_1^{\mathfrak{g}} \in V \otimes W + W \otimes W$ too, since $R_1^{\mathfrak{g}}$ is determined by $R^m + R^\perp$.

Write $S_0^2 V$ for the space of trace-free symmetric tensors on V. Then the trace-free Ricci tensor lies in $S_0^2 V$ and the vanishing of this component of R is exactly the Einstein condition. The above discussion now implies that R is Einstein provided $R_0^{\mathfrak{g}}$ and $R_1^{\mathfrak{g}} + R^m + R^\perp$ are both Einstein. We thus have:

Theorem 2.3. *Suppose M is a manifold with structure group $G \leqslant SO(n)$. Let V denote the representation of G on TM and suppose that the torsion ξ lies in the subrepresentation $W \subset V \otimes \mathfrak{g}^\perp$. Then a sufficient condition for M to be Einstein is*

(a) $(V \otimes W + W \otimes W) \cap S_0^2 V = \{0\}$, *and*
(b) *any element of* $\mathcal{K}(\mathfrak{g})$ *is Einstein.*

The simplest case is when the representation W is trivial. This occurs precisely when ξ is invariant under the action of G.

Definition 2.4. We say that the M is a Riemannian manifold with *invariant torsion* if the structure group of M reduces to a proper subgroup G of $SO(n)$ and the torsion ξ of the natural metric connection for this G-structure is invariant under the action of G.

3. Examples

Let us consider some examples of manifolds with invariant torsion and see how they relate to Theorem 2.3. We begin with Gray's weak holonomy structures that are Einstein.

3.1. **Weak Holonomy G_2.** We have $G = G_2$ and V is the irreducible representation on $\mathbb{R}^7$. Then

$$\mathfrak{so}(7) = \Lambda^2 V = \mathfrak{g}_2 \oplus V$$

so $\mathfrak{g}^\perp = V$ and $V \otimes \mathfrak{g}^\perp$ certainly contains a trivial representation. In fact

$$V \otimes \mathfrak{g}^\perp = V \otimes V = \mathbb{R} \oplus V \oplus \mathfrak{g}_2 \oplus S^2_0 V$$

as a sum of irreducible modules.

Taking $\xi \in W = \mathbb{R}$, we have $V \otimes W + W \otimes W = V + \mathbb{R}$ which has no subrepresentation in common with $S^2_0 V$, which is irreducible. Therefore condition (a) of Theorem 2.3 is satisfied.

For condition (b), we have that $\mathcal{K}(\mathfrak{g}_2)$ is the algebraic space of curvature tensors of metrics with holonomy G_2. But all such metrics are Ricci flat and condition (b) holds. (In fact, $\mathcal{K}(\mathfrak{g}_2)$ is an irreducible representation of dimension 77.) Thus for these G_2-structures, $\xi \in \mathbb{R}$ implies that M^7 is Einstein.

The tensor φ in the proof of Theorem 2.3 may be taken to be the fundamental 3-form of the G_2-structure [4]. The condition that ξ lies in $\mathbb{R}$ implies that $\nabla\varphi = \xi.\varphi$ is an invariant tensor in $V \otimes \Lambda^3 V$. But

$$\Lambda^3 V = \mathbb{R} \oplus V \oplus S^2_0 V$$

and so $V \otimes \Lambda^3 V$ contains a unique invariant summand. This is spanned by the four-form $*\varphi$, so we have $d\varphi = \mathbf{a}(\nabla\varphi) = \lambda{*\varphi}$ and the structure has weak holonomy G_2. Conversely, for $\lambda \neq 0$, a metric with weak holonomy G_2 always has invariant torsion $\xi = c\varphi$.

3.2. **Nearly Kähler Six-Manifolds.** Let $U(n)$ act irreducibly on $V = \mathbb{R}^{2n}$. Then V is the real representation underlying $\Lambda^{1,0}$ and we write $V = [\Lambda^{1,0}]$. In this case $\mathfrak{u}(n)^\perp = [\Lambda^{2,0}]$, so in order for $V \otimes \mathfrak{u}(n)^\perp$ to have a trivial summand we need an isomorphism of $[\Lambda^{1,0}]$ with $[\Lambda^{2,0}]$. For the dimensions of these two representations to be equal we have to have $n = 3$. However, even in that case the centre of $U(3)$ acts on these two representations with different weights. We therefore conclude that there is a trivial summand only with respect to the action of $SU(3)$.

So we must take $G = SU(3)$ and $V = [\Lambda^{1,0}]$. We then have $V \otimes \mathfrak{g}^\perp$ contains $W = 2\mathbb{R}$. For this choice of W, condition (a) of Theorem 2.3 is satisfied. Condition (b) is also satisfied, as metrics of holonomy $SU(3)$ are Ricci-flat. Therefore, an $SU(3)$-structure with invariant torsion $\xi \in 2\mathbb{R}$ is Einstein.

For $\xi \neq 0$, these are exactly the nearly Kähler six-manifolds that are not Kähler. Notice that the structure group of such a manifold always reduces from $U(3)$ to $SU(3)$ as $d\omega + i{*}d\omega$ trivialises $\Lambda^{3,0}$.

3.3. **Holonomy Representations.** Suppose G acts irreducibly on V via the holonomy representation of a Riemannian metric. Assume that $G \neq SO(\dim V)$. Looking at each individual case, one can see that the only times where $\mathfrak{g}^\perp$ contains a copy of V are (i) $G = SU(3)$, $V = [\Lambda^{1,0}]$ and (ii) $G = G_2$, $V = \mathbb{R}^7$. We thus have:

Proposition 3.1. *Suppose M is a Riemannian manifold with non-zero invariant torsion. If the structure group G acts on TM via a holonomy representation, then M is either a six-dimensional nearly Kähler manifold or M is a seven-dimensional manifold with weak holonomy G_2.*

3.4. **Representations of $SU(2)$.** Let us consider the case when $G = SU(2)$ and V is an irreducible representation of G. This implies that $V \otimes \mathbb{C} = S^k\mathbb{C}^2$, the kth symmetric power of $\mathbb{C}^2$, for some integer k. This representation only admits an invariant metric if k is even. The condition that $V \otimes \mathfrak{su}(2)^\perp$ contains a trivial representation then implies that $k \equiv 2 \pmod 4$ and that $k \neq 2$. One can now check that conditions (a) and (b) of Theorem 2.3 are satisfied. Therefore, if they exist, such structures will give an Einstein metric in dimensions $4r + 3$ for $r > 0$.

Two examples can be easily found. For $r = 1$, the space $M^7 = Sp(2)/Sp(1)$, with $Sp(1)$ embedded maximally in $Sp(2)$ has complexified isotropy representation $S^6\mathbb{C}^2$ and the only invariant metric is a structure with invariant torsion $SU(2)$. Similarly, for $r = 2$, $M^{11} = G_2/SU(2)$, again with $SU(2)$ maximally embedded, is isotropy irreducible and carries an Einstein metric with invariant torsion. We will see later that these are the only examples that arise from this family of representations of $SU(2)$.

3.5. **Homogeneous Spaces.** Let $M = K/G$ be a reductive homogeneous space with K and G semi-simple and compact. Write $\mathfrak{k} = \mathfrak{g} + \mathfrak{p}$, then $T_eM = \mathfrak{p}$ and the negative $\langle\cdot,\cdot\rangle$ of the Killing form induces a positive definite $\mathfrak{g}$-invariant inner product on $\mathfrak{p}$ and hence a Riemannian metric on M. The canonical connection on M is a G-connection with torsion $\xi(X,Y,Z) = \langle [X,Y], Z\rangle$, for left-invariant vector fields X, Y and Z.

If $\mathfrak{p}$ is an irreducible $\mathfrak{g}$-module then $M = K/G$ is isotropy irreducible. These spaces have been classified by Wolf [13]. One can check directly that condition (b) is satisfied for all these spaces. However, with $W = \mathbb{R}$, it is not always the case that $V = \mathfrak{p}$ satisfies condition (a), even though K/G is well-known to be Einstein.

3.6. **Three-Sasakian Manifolds.** 3-Sasakian manifolds give another class of Einstein manifolds with invariant torsion. However, in this neither condition (a) nor condition (b) is satisfied.

4. General Results

Some general results may be obtained by studying conditions (a) and (b) of Theorem 2.3 in more detail.

First we note that for any representation W in $V \otimes \mathfrak{g}^{\perp}$ will have $\mathbb{R}$ as a subrepresentation of $W \otimes W$. Thus condition (a) implies that $S^2_0 V$ does not contain a trivial representation. It is straightforward to check that V is then forced to be irreducible.

For irreducible representations V, condition (b) is easy to satisfy given the current state of knowledge of the holonomy classification. Let us consider Berger's approach to the holonomy problem [2] as explained by Bryant [5] and Schwachhöfer [11].

Definition 4.1. Let G be a subgroup of $SO(n)$. Define the *Berger algebra* $\underline{\mathfrak{g}}$ of G by

$$\underline{\mathfrak{g}} = \{\, R(X,Y) : R \in \mathcal{K}(\mathfrak{g}),\ X, Y \in V \,\}$$

It is easy to show

Lemma 4.2. *The Berger algebra* $\underline{\mathfrak{g}}$ *is an ideal of* $\mathfrak{g}$*, i.e.,* $\underline{\mathfrak{g}} \triangleleft \mathfrak{g}$*, and* $\mathcal{K}(\underline{\mathfrak{g}}) = \mathcal{K}(\mathfrak{g})$.

Berger two necessary conditions for $\mathfrak{g}$ to be a holonomy algebra, if G acts irreducibly on $\mathbb{R}^n$. The first is that $\mathcal{K}(\mathfrak{h})$ should be strictly smaller than $\mathcal{K}(\mathfrak{g})$ for any proper subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This may be rephrased as $\mathfrak{g} = \underline{\mathfrak{g}}$. The second criteria comes from consideration of the possible covariant derivatives of curvature tensors. It turns out that this second condition merely distinguishes holonomy groups which can only occur for symmetric spaces from the others. The work on existence of metrics with non-symmetric holonomies now implies that each algebra satisfying Berger's first criterion is the holonomy algebra of some torsion-free connection.

Theorem 4.3. *$\mathfrak{g}$ is a holonomy algebra of an irreducible Riemannian manifold if and only if $\underline{\mathfrak{g}} = \mathfrak{g}$.*

We may thus calculate the space $\mathcal{K}(\mathfrak{g})$ by considering all ideals of $\mathfrak{g}$ and comparing them with holonomy representations.

Corollary 4.4. *Suppose G is a proper subgroup of $SO(n)$ acting irreducibly on $V = \mathbb{R}^n$. Then $\mathcal{K}(\mathfrak{g})$ consists only of Einstein tensors unless n is even and $\mathfrak{g} = \mathfrak{u}(n/2)$.*

Proof. If G is simple then either $\mathfrak{g}$ is a holonomy algebra or $\underline{\mathfrak{g}} = \{0\}$ and there is nothing to prove.

Suppose G is not simple and that $\{0\} \neq \underline{\mathfrak{g}} \neq \mathfrak{g}$. If $\underline{\mathfrak{g}}$ acts irreducibly on V then the only possibility we have to rule out is $\underline{\mathfrak{g}} = \mathfrak{u}(n/2)$, if n is even. However $\mathfrak{u}(n/2)$ is maximal in $\mathfrak{so}(n)$, and so the fact that the containments $\underline{\mathfrak{g}} < \mathfrak{g} < \mathfrak{so}(n)$ are strict, rule out this case.

If the representation of $\mathfrak{h}_1 := \underline{\mathfrak{g}}$ on V is reducible then $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Depending on the type of the representation V, we may decompose $V \otimes \mathbb{C}$ as a sum of 1, 2 or 4 tensor products of irreducible $\mathfrak{h}_i$-modules over $\mathbb{C}$. If $\mathfrak{h}_i$ is not Abelian, we find that V is the isotropy representation of a Grassmann symmetric space. The space of curvature tensors for such representations are known and the condition $\underline{\mathfrak{g}} = \mathfrak{h}_1$ can not be satisfied. If one $\mathfrak{h}_i$ is Abelian, then a direct calculation shows that there are no non-trivial curvature tensors with values in $\mathfrak{h}_1$. □

We now return to condition (a) of Theorem 2.3. When W is a trivial representation and V is irreducible, (a) is equivalent to $S^2_0 V$ not containing a copy of V.

Lemma 4.5. *If V is irreducible, ξ lies in a trivial submodule of $V \otimes \mathfrak{g}^\perp$ and $S^2_0 V$ has no submodule isomorphic to V, then ξ is a three-form $\xi \subset \Lambda^3 V$.*

Proof. If $V \otimes \mathfrak{g}^\perp$ contains a trivial summand, then we have $V \subset \mathfrak{g}^\perp \subset \Lambda^2 V$. Now, under the action of $SO(n)$, we have

$$V \otimes \Lambda^3 V = V + \Lambda^3 V + U, \tag{4.1}$$

where U is irreducible, and we also have

$$V \otimes S^2_0 V = S^3 V + U. \tag{4.2}$$

Our hypotheses imply that for the action of G, (4.2) contains no trivial submodules. In particular, $U^G = \{0\}$. Therefore, any trivial submodule of (4.1) lies in either V or $\Lambda^3 V$. But V is irreducible, so any trivial module is in $\Lambda^3 V$. □

Thus for V irreducible, condition (a) of Theorem 2.3 forces the torsion to be totally skew. This is interesting, as such a totally skew condition on torsion seems to be natural in physical consideration of for example hyperKähler geometries with torsion [6].

Interestingly, Lemma 4.5 has a converse.

Proposition 4.6. *If M is a Riemannian manifold with a G-structure whose natural metric connection has torsion ξ and ξ is a three-form, then $\tilde{\nabla}\xi$ does not contribute to the curvature of the Levi-Civita connection and condition* (a) *of Theorem 2.3 can be replaced by*

(a′) $W \otimes W \cap S^2_0 V = \{0\}$

Proof. The tensor $\tilde{\nabla}\xi$ is a sum of four tensors $\psi(X, Y, Z, W)$ which are totally skew in their last three entries. The corresponding element of $S^2(\Lambda^2 V)$ is

$$\begin{aligned}\tilde{\psi}(X, Y, Z, W) = \psi(X, Y, Z, W) &- \psi(Y, X, Z, W) \\ &+ \psi(Z, W, X, Y) - \psi(W, Z, X, Y).\end{aligned}$$

Now one can check directly that $\tilde{\psi}$ is skew in its first two indices and $\tilde{\psi}(Y, Z, W, X) = -\tilde{\psi}(X, Y, Z, W)$. Therefore $\tilde{\psi}$ is a four-form and so orthogonal to the kernel of the Bianchi map $b\colon S^2(\Lambda^2 V) \to \Lambda^4 V$. □

As we have already seen, for W trivial and V irreducible, condition (a′) of Proposition 4.6 is satisfied. We therefore have:

Theorem 4.7. *Let (M, g) be a Riemannian manifold with structure group G acting irreducibly and for which the natural torsion is invariant and totally-skew. Suppose that the structure group is not $SO(n)$ or $U(n/2)$. Then g is Einstein.*

In certain cases we can show uniqueness of the Einstein metric.

Theorem 4.8. *Let (M, g) be a complete Riemannian manifold satisfying the hypotheses of Theorem 4.7 with structure group G and tangent representation V. Suppose that the space of invariant three-forms $(\Lambda^3 V)^G$ on M is one-dimensional and that the scalar curvature of M is non-zero. If $\mathfrak{g} \neq \mathfrak{g}_2$, then M is homogeneous and isometric to an isotropy irreducible space.*

Sketch Proof. If the Berger algebra $\underline{\mathfrak{g}}$ is non-trivial then it acts on V preserving ξ. Proposition 3.1 then implies that $\underline{\mathfrak{g}} = \{0\}$, as we have specifically excluded the other cases apart from $\mathfrak{su}(3)$. But for $\mathfrak{su}(3)$ the space of invariant three-forms has dimension 2 rather than 1, so this case does not occur.

Now we see that $R_0^{\mathfrak{g}} = 0$ and by Proposition 4.6 $R^m = 0$. Thus R is algebraically determined by the torsion ξ. Write $R = R(\xi^2)$.

As the space of invariant three-forms is one-dimensional, locally the ξ is proportional to a $\tilde{\nabla}$-parallel three-form φ. Write $\xi = f\varphi$. Then $R(\xi^2) = f^2 R(\varphi^2)$, and in particular the scalar curvature $s(\xi^2) = f^2 s(\varphi^2)$. But $s(\xi^2)$ is constant, as g is Einstein, and $s(\varphi^2)$ is constant, since it is parallel for $\tilde{\nabla}$. Therefore, f is constant under the hypothesis that $s(\xi^2) \neq 0$.

We thus have that $\tilde{\nabla}\xi = 0$ and $\tilde{\nabla}R = 0$. By definition this means that $\tilde{\nabla}$ is an Ambrose-Singer connection. Results of Tricerri & Vanhecke [12] imply that M is a homogeneous space with isotropy group $\operatorname{stab} R \cap \operatorname{stab} \xi$. However, this group contains G, and so M is isotropy irreducible. $\square$

Example 4.9. One instructive example might be helpful at this point. As mentioned above, the Aloff-Wallach spaces $M_{k,\ell} = SU(3)/\,U(1)_{k,\ell}$ carry invariant metrics of weak holonomy G_2. However, in dimension 7 we also have the isotropy irreducible space $M^7 = Sp(2)/\,Sp(1)$ with isotropy representation $S^6\mathbb{C}^2$. Theorem 4.8 applies to the $SU(2)$-structure of M^7 and shows that this is the only complete metric with invariant torsion.

Now G_2 has a subgroup $SU(2)$ that acts on the seven-dimensional representation of G_2 as $S^6\mathbb{C}^2$. Remarkably, the space of invariant tensors in $T \otimes \Lambda^2 T^*$ is the same for both groups.

If we look parameters k and ℓ such that $M_{k,\ell}$ carries such an $SU(2)$-structure we find that topologically the only solution is $k = 1$ and $\ell = 4$. Thus $M_{1,4}$ has an invariant metric with weak holonomy G_2 and a reduction of the structure group to the seven-dimensional irreducible representation of $SU(2)$. Theorem 4.8 implies that with respect to the structure group $SU(2)$, $M_{1,4}$ can not be a manifold with invariant torsion, even though it has invariant torsion with respect to G_2. We can see that this is not a contradiction by considering the relations

$$\begin{aligned} \nabla &= \tilde{\nabla}^{\mathfrak{g}_2} + \xi^{\mathfrak{g}_2} \\ &= \tilde{\nabla}^{\mathfrak{su}(2)} + \xi^{\mathfrak{su}(2)} \end{aligned}$$

This implies $\xi^{\mathfrak{su}(2)} = \xi^{\mathfrak{g}_2} + (\tilde{\nabla}^{\mathfrak{g}_2} - \tilde{\nabla}^{\mathfrak{su}(2)})$. The last bracket takes values in $\mathfrak{g}_2 \ominus \mathfrak{su}(2)$ and there is no particular reason for it to vanish. Thus, if $\xi^{\mathfrak{g}_2}$ is invariant, this will not imply that $\xi^{\mathfrak{su}(2)}$ is. However, the converse *is* true, and the $SU(2)$-structure on M^7 is also a metric of weak holonomy G_2.

Giving this result it is therefore interesting to find representations V of G for which the dimension of $(\Lambda^3 V)^G$ is at least 2, as these would give some hope of giving non-homogeneous Einstein structures. It is interesting to remark that there are isotropy irreducible spaces that satisfy this condition. For example, if G is a simple group with Lie algebra not equal to $\mathfrak{su}(2)$ or $\mathfrak{sp}(2)$ then the isotropy irreducible space

$$\frac{SO(\dim G)}{G}$$

has at each point a two-dimensional family of invariant three-forms if G is not of type A_n, $n \geqslant 3$, and a four-dimensional family in these remaining cases. There therefore appears to be a second natural three-form for these representations and it would be interesting to determine that and to see whether non-homogeneous Einstein structures can be constructed.

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