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DELTA AND SINGULAR DELTA LOCUS FOR ONE DIMENSIONAL SYSTEMS OF CONSERVATION LAWS

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ABSTRACT. A condition for existence of singular and delta shock waves for systems of conservation laws is given in the paper. The systems considered here have fluxes which are linear in one of the dependent variables. The condition obtained here is analogous to the one for the standard Hugoniot locus. Three different solution concept are used in the paper: associated solution in Colombeau sense, limits of nets of smooth functions together with Rankin-Hugoniot conditions and a kind of a measure valued solutions.

1. INTRODUCTION

The aim of this paper is to give some criterion for which certain classes of function nets, called delta and singular shock waves, satisfy the following system of conservation laws

$$u_t + (f_1(u)v + f_2(u))_x = 0 \tag{1}$$

$$v_t + (g_1(u)v + g_2(u))_x = 0, \tag{2}$$

in an approximated sense. Here, f_i, g_i, $i = 1, 2$ are smooth functions, polynomially bounded together with all their derivatives, and $u, v : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ satisfy

$$\begin{aligned} u(x, 0) &= u_0,\ x < 0,\ u(x, 0) = u_1,\ x > 0 \\ v(x, 0) &= v_0,\ x < 0,\ v(x, 0) = v_1,\ x > 0. \end{aligned} \tag{3}$$

A pair of nets $(u_\varepsilon, v_\varepsilon) \in \mathcal{D}'$, $\varepsilon \in (0, 1)$, is a solution in an approximated sense if

$$\begin{aligned} \langle u_{\varepsilon t} + (f_1(u_\varepsilon)v_\varepsilon + f_2(u_\varepsilon))_x, \psi\rangle \to 0 \\ \langle v_{\varepsilon t} + (g_1(u_\varepsilon)v_\varepsilon + g_2(u_\varepsilon))_x, \psi\rangle \to 0 \end{aligned}$$

as $\varepsilon \to 0$, for every $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$. In order to have a well defined composition of functions we use families of smooth function which can be viewed as representatives of Colombeau functions ([1], [9]) or families of piecewise constant functions which are called box approximations. For the first case we will use the space of generalized functions, $\mathcal{G}_g$, and the concept of a approximate solution defined in [10]. In the second case we use the standard Rankin-Hugoniot conditions for systems of conservation laws like in [6].

Roughly speaking, delta and singular shock wave are pairs of nets $(u_\varepsilon, v_\varepsilon)$ which converge to sums of step and delta functions in $\mathcal{D}'$. They will be precisely defined bellow. Let us remark that singular shock waves have similarities to infinite narrow solitons which are introduced by Maslov and Omel'anov [8]. Further results in this direction for systems of conservation laws can be found in the paper [3] from Danilov, Maslov and Shelkovich.

For a given point $(u_0, v_0) \in \mathbb{R}^2$, a delta (singular delta) locus is a subset of $\mathbb{R}^2$ consists of points (u_1, v_1) for which there exists a delta (singular) shock wave joining (u_0, v_0) and (u_1, v_1). This definition is an analogue of the definition of the classical Hugoniot locus in the case of piecewise constant solutions (shock waves) for Riemann problem.

Admissibility condition for such shocks could be the overcompressiveness like in [6], [11], [4] and [5], i.e.

$$\lambda_2(u_0, v_0) > \lambda_1(u_0, v_0) \geq c \geq \lambda_2(u_1, v_1) > \lambda_1(u_1, v_1),$$

where c is a speed of the delta or singular shock wave, λ_1, λ_2 are the eigenvectors for the system, (u_0, v_0) and (u_1, v_1) denotes left and right hand side initial value, respectively. Let us remark that the hyperbolicity condition is not used in the construction of approximate solutions.

Like in the classical theory of Hugoniot locus, the (singular) delta locus can be used for construction of solutions containing also a rarefaction wave on the left or on the right hand side of a singular or delta shock.

Our investigations are motivated by Keyfitz and Kranzer ([6]) who found singular shock wave solutions to the system

$$\begin{aligned} u_t + (u^2 - v)_x &= 0 \\ v_t + \left(\frac{1}{3}u^3 - u\right)_x &= 0, \end{aligned} \tag{4}$$

in the form $u_\varepsilon(x,t) = G_\varepsilon(x-ct) + a\sqrt{\frac{t}{\varepsilon}}\rho\left(\frac{x-ct}{\varepsilon}\right)$, $v_\varepsilon(x,t) = H_\varepsilon(x-ct) + \frac{a^2 t}{\varepsilon}\rho^2\left(\frac{x-ct}{\varepsilon}\right)$, where G_ε and H_ε converge to appropriate step functions defined by the Riemann initial data (3), $\rho^2_\varepsilon := \varepsilon^{-1}\rho^2(\cdot/\varepsilon)$ converges to the delta distribution and ρ^i_ε, $i = 1, 3$ converge to zero in $\mathcal{D}'$ as $\varepsilon \to 0$. This system is included in our general system (1), (2). Our paper shows that their approach is sufficiently general for solving (1), (2).

An example of a system with a delta shock wave solution is

$$\begin{aligned} u_t + (u^2)_x &= 0 \\ v_t + (uv)_x &= 0 \end{aligned} \tag{5}$$

examined in [11]. The solution is of the form $u_\varepsilon(x,t) = G_\varepsilon(x-ct)$, $v_\varepsilon(x,t) = H_\varepsilon(x-ct) + st\delta_\varepsilon(x-ct)$, where G_ε and H_ε are the same as before and δ_ε is a delta net. By adding a viscosity term Tan, Zhang and Zheng ([11]) proved that the limit of vanishing viscosity solutions is a solution of this form. Let us notice that there is no additional "singular" term in u_ε. Here, the delta shock wave will be used to denote that there are no other terms in a solution except approximations of step functions and the delta distribution.

Systems which are to follow have the similar form of solutions as (5), that is they do not contain term added to G_ε and a delta net is a part of v_ε.

Oberguggenberger ([9]) proved that for

$$u_t + (u^2/2)_x = 0 \\ v_t + (uv)_x = 0 \tag{6}$$

the viscosity limit is a delta shock wave.

By using Le Flock and Vol'pert definition of the product, Hayes and Le Flock ([5]) found a delta shock wave solution to

$$u_t + (u^2)_x = 0 \\ v_t + ((u-1)v)_x = 0. \tag{7}$$

They also proved that the vanishing viscosity limit exists and it is equal to the solution of (7).

Finally, Ercole ([4]) proved that the system

$$u_t + f(u)_x = 0 \\ v_t + (g(u)v)_x = 0, \tag{8}$$

with some minor assumptions on f and g, has a delta shock solution as a limit of vanishing viscosity solutions.

The differences between system (4) and systems (5)-(8) are obvious. Only system (4) has second dependent variable in the first flux function and it can not be solved with delta shock waves.

In this paper we will not use the viscosity approximation. The existence and convergence of vanishing viscosity solutions is not a trivial task even for some specific choice of a function in the flux. An example of this one can find in Dafermos-DiPerna paper [2].

All the systems considered here have a flux which is a linear function in one variable and this will be a general form of a flux for which we find a delta locus. The interpretation of a limit of ϕ_ε^2 where ϕ_ε is a delta net brings difficulties. Colombeau proved in [1] that ϕ_ε^2 defines an element in the space of the generalized functions which is not associated with any classical distribution.

In all the cited papers a delta locus is a subset of $\mathbb{R}^2$ with the non-zero Lebesgue measure. In general, this set is just a curve, but for systems (5)-(8) our procedure gives the same locus as it was already obtained. Singular delta locus is an set with non-zero Lebesgue measure in general.

In the sequel we shall consider only solutions which equals to initial data for every ε (also after regularization when we use smooth nets) at the initial time. This means that the "value" of the Heaviside function in zero does not have any influence, for example.

The paper is organised as follows. In Section 2 the algebra of generalized functions, delta and singular shock wave solution are precisely defined. In Section 3 we give a form of approximate solutions which will be used in finding delta locus's for smooth functions f_i, g_i, $i = 1, 2$. Roughly speaking, the main property of approximations is a distribution of a "mass" of singular part on the left and right-hand side from zero. Properly placed right and left hand sided masses determines the delta locus. For the use of a delta locus in constructions of admissible solutions one has to remove the points for which the overcompressiveness do not hold.

A box approximation solution is used in Section 4 and it gives the same delta locus as in the case of Colombeau generalized functions. Theorem 1 applied to systems (5)-(8) gives the results previously obtained in the quoted papers. System (4) can not be solved by this procedure.

A concept and construction of a solution in a certain measure space is given in Section 5. Approximated solutions constructed in Section 3 converge to this solution.

In Section 6 systems with polynomial fluxes are examined. There exists a broad class of polynomials for which we can find singular delta locus of the non-zero Lebesgue measure modifying u_ε by adding a net which converge to zero but some of its powers do not. In fact we use an idea of [6] adopted for arbitrary polynomials. In contrast to the singular shock wave solution obtained in [6], we have to assume that singular parts of approximations u_ε and v_ε have disjoint supports. This can be omitted if f_1 and g_1 are constants, or linear functions in u.

There are many open problems concerning the system (1), (2). For example one can try to obtain a limit of viscosity self-similar solutions to the system like it was done in Dafermos and DiPerna's paper [2]. Also, one can try to describe singular shock wave solution formally obtained as a net of approximate solutions by using a weighted measure space as it was done by Keyfitz and Kranzer in [6].

2. Definitions

We shall briefly recall some of the definitions of Colombeau algebra given in [10]. Let $\mathbb{R}^2_+ = \mathbb{R}\times(0,\infty)$, $\overline{\mathbb{R}^2_+} = \mathbb{R}\times[0,\infty)$ and $C_b^\infty(\Omega)$ be the algebra of smooth functions on Ω bounded together to all its derivatives. Let $C_{\overline{b}}^\infty(\mathbb{R}^2_+)$ be a set of all functions $u\in C^\infty(\mathbb{R}^2_+)$ such that $u|_{\mathbb{R}\times(0,T)}\in C_b^\infty(\mathbb{R}\times(0,T))$ for every $T>0$. Let us remark that every element of $C_b^\infty(\mathbb{R}^2_+)$ has a smooth extension up to the line $\{t=0\}$, i.e. $C_b^\infty(\mathbb{R}^2_+) = C_b^\infty(\overline{\mathbb{R}^2_+})$. This is also true for $C_{\overline{b}}^\infty(\mathbb{R}^2_+)$.

$\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ is the set of all maps $G:(0,1)\times\mathbb{R}^2_+\to\mathbb{R}$, $(\varepsilon,x,t)\mapsto G_\varepsilon(x,t)$, smooth for every $\varepsilon\in(0,1)$, which satisfy:
For every $(\alpha,\beta)\in\mathbb{N}_0^2$ and $T>0$, there exists $N\in\mathbb{N}$ such that

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha\partial_t^\beta G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{-N}),\ \text{ as } \varepsilon\to 0.$$

$\mathcal{N}_g(\mathbb{R}^2_+)$ is the set of all $G_\varepsilon\in\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ which satisfy:
For every $(\alpha,\beta)\in\mathbb{N}_0^2$, $a\in\mathbb{R}$ and $T>0$

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha\partial_t^\beta G_\varepsilon(x,t)| = \mathcal{O}(\varepsilon^{a}),\ \text{ as } \varepsilon\to 0.$$

The differential algebra $\mathcal{G}_g(\mathbb{R}^2_+)$ of generalized functions is the factor algebra $\mathcal{G}_g(\mathbb{R}^2_+) = \mathcal{E}_{M,g}(\mathbb{R}^2_+)/\mathcal{N}_g(\mathbb{R}^2_+)$.

By G will be denoted a class in $\mathcal{G}_g$ which has a representative $G_\varepsilon\in\mathcal{E}_{M,g}$.

Since $C_{\overline{b}}^\infty(\mathbb{R}^2_+) = C_{\overline{b}}^\infty(\overline{\mathbb{R}^2_+})$ one can define a restriction of a generalized function to $\{t=0\}$ as follows. For given $G\in\mathcal{G}_g(\mathbb{R}^2_+)$, its restriction $G|_{t=0}\in\mathcal{G}_g(\mathbb{R})$ is the class which contain a function $G_\varepsilon(x,0)$. Similarly, $G(x-ct)\in\mathcal{G}_g(\mathbb{R})$ is defined with $G_\varepsilon(x-ct)\in\mathcal{E}_{M,g}(\mathbb{R})$.

Let us note that if f is a smooth function polynomially bounded together with all its derivatives, then the composition $f(G)$, $G\in\mathcal{G}_g$ is well-defined.

Let $u \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Fix an element $\phi \in \mathcal{D}(\mathbb{R})$, $\phi(x) \geq 0$, $x \in \mathbb{R}$, $\int \phi(x)dx = 1$, $\operatorname{supp}\phi \subset [-1,1]$. Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, $x \in \mathbb{R}$. Then

$$\iota_\phi : u \to \text{ class of } u * \phi_\varepsilon$$

defines an embedding of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ into $\mathcal{G}_g(\mathbb{R})$. Obviously, ι_ϕ commutes with derivative. We shall call a generalized function G a step function if it equals $\iota_\phi(u)$, where u is a step function in the usual sense. The delta generalized function is given by a representative ϕ_ε. In the sequel we shall omit the subscript ϕ.

An element $G \in \mathcal{G}_g(\Omega)$ is associated with $u \in \mathcal{D}'(\Omega)$ if for some (and hence every) representative G_ε of G, $G_\varepsilon \to u$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$. It will be denoted by $G \approx u$. Two generalized functions G and H are associated, $G \approx H$ if $G - H \approx 0$.

A generalized function G is of the bounded type if

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |G_\varepsilon(x,t)| = \mathcal{O}(1) \text{ as } \varepsilon \to 0,$$

for every $T > 0$.

The initial data for the Riemann problem are regularized in the following way. Let

$$\Theta_1(x) = \begin{cases} u_0, & x < 0 \\ u_1, & x > 0, \end{cases} \text{ and } \Theta_2(x) = \begin{cases} v_0, & x < 0 \\ v_1, & x > 0. \end{cases}$$

The initial data are now (G, H) where $G_\varepsilon = \Theta_1 * \phi_\varepsilon$ and $H_\varepsilon = \Theta_2 * \phi_\varepsilon$. This means that a solution to system (1), (2) satisfies

$$u_\varepsilon(x,0) = G_\varepsilon(x),\ G_\varepsilon(x) = u_0,\ x < -\varepsilon,\ G_\varepsilon(x) = u_1,\ x > \varepsilon,$$
$$v_\varepsilon(x,0) = H_\varepsilon(x),\ H_\varepsilon(x) = v_0,\ x < -\varepsilon,\ H_\varepsilon(x) = v_1,\ x > \varepsilon,$$

where G_ε and H_ε are of the bounded type, i.e. they are uniformly bounded with respect of ε. Generalized functions of this type are called the generalized step functions. One will see that the results obtained here will be valid for all representatives of step functions which satisfy these conditions. Let us remark that if u_ε is a function satisfying the above conditions, this will be true for every its power u^m, $m \in \mathbb{N}$, also.

Definition 1. *(Generalized functions)* $(U,V) \in \mathcal{G}(\mathbb{R}^2_+)$ *is a singular shock wave solution to (1), (2) and (3) if*
a)

$$U_t + (f_1(U)V + f_2(U))_x \approx 0 \tag{9}$$
$$V_t + (g_1(U)V + g_2(U))_x \approx 0. \tag{10}$$
$$U|_{t=0} = G,\ V|_{t=0} = H. \tag{11}$$

b) $U(x,t) \approx \Theta_1(x - ct)$, *where* c *is a constant (speed of the shock) and* $V(x,t) \approx \Theta_2(x-ct) + s(t)\delta(x-ct)$, *where* c *is the same constant,* $s \in C^1([0,\infty))$, $s(0) = 0$ *and* δ *is Dirac delta function.*

If in addition U *is of the bounded type, the solution is called delta shock wave.*

Definition 2. *(Box approximations) A pair of nets $(u_\varepsilon, v_\varepsilon)$ consisting of piecewise smooth functions is called an approximated singular shock wave solution to (1), (2) and (3) if*
a)

$$\lim_{\varepsilon\to 0}\langle u_\varepsilon, \phi_t\rangle + \langle f_1(u_\varepsilon)v_\varepsilon + f_2(u_\varepsilon), \phi_x\rangle = 0$$
$$\lim_{\varepsilon\to 0}\langle v_\varepsilon, \phi_t\rangle + \langle g_1(u_\varepsilon)v_\varepsilon + g_2(u_\varepsilon), \phi_x\rangle = 0.$$

b) $(u_\varepsilon, v_\varepsilon)$ satisfy (3) for ε small enough.
c) u_ε converge in the distributional sense to $\Theta_1(x-ct)$, where c is a constant (speed of the shock) and Θ_1 a step function. v_ε converge in the distributional sense to $\Theta_2(x-ct)+s(t)\delta(x-ct)$, where c is the same constant, $s\in C^1([0,\infty))$, $s(0)=0$, $\Theta_2)$ is a step function and δ is Dirac delta function.

If in addition u_ε is bounded independently on ε, the solution is called delta shock wave.

3. Generalized functions and delta locus

Theorem 1. *a) Let $f_1 \not\equiv \text{const}$. Then a delta shock wave solution to (9), (10) and (11) exists if $u_0\neq u_1$, $f_1(u_0)\neq f_1(u_1)$ and*

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0} = \frac{g_1(u_0)f_1(u_1) - g_1(u_1)f_1(u_0)}{f_1(u_1)-f_1(u_0)}, \tag{12}$$

where c is the velocity of the delta shock. The set of all pairs of right hand states (u_1, v_1) such that (12) holds is called the delta locus for (1), (2) and (3).
b) If $f_1(u_0)=f_1(u_1)=0$ (specially, if $f_1\equiv 0$) and $g_1\not\equiv \text{const}$, then the delta locus is the set of all points (u_1,v_1) such that $g_1(u_0)\neq g_1(u_1)$.
c) If $f_1\equiv 0$ and $g_1\equiv b\in\mathbb{R}$, then the delta locus is the set of all points (u_1,v_1) such that $b(u_1-u_0)=f_2(u_1)-f_2(u_0)$.

Proof. Let $\phi\in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\phi\subset[-1,1]$, $\int\phi(x)dx=1$, $\phi\geq 0$ and let

$$D_\varepsilon = \frac{D^-}{\varepsilon^{\frac{A-1}{2}}}\phi\left(\frac{x+\frac{A+1}{2}}{\varepsilon^{\frac{A-1}{2}}}\right) + \frac{D^+}{\varepsilon^{\frac{A-1}{2}}}\phi\left(\frac{x-\frac{A+1}{2}}{\varepsilon^{\frac{A-1}{2}}}\right),$$

where $D^- + D^+ = 1$ Let $s_1(t)$, $s_2(t)$ be smooth functions for $t\geq 0$, such that $s_1(0)=s_2(0)=0$ and $A>0$. Since in the general we do not have any information on the behaviour of f_1, f_2, g_1 and g_2 we shall try to find delta shock wave solution in the form

$$u_\varepsilon(x,t) = G_\varepsilon(x-ct),\ v_\varepsilon(x,t) = H_\varepsilon(x-ct) + s(t)D_\varepsilon(x-ct), \tag{13}$$

where D_ε is defined as above and s is from Definition 1. Let us note that D_ε converge to delta function as $\varepsilon\to 0$. After a substitution of (13) into (1) we have

$$G_{\varepsilon t} + (f_1(G_\varepsilon)H_\varepsilon + f_2(G_\varepsilon))_x + s(t)(f_1(G_\varepsilon)D_\varepsilon)_x \approx 0. \tag{14}$$

A missing argument of a function means that it equals $x-ct$. Let $\operatorname{supp}\phi\subset[-X,X]\times[-T,T]$, for some $X,T>0$. Then

$$
\begin{aligned}
&\iint(\partial_t G_\varepsilon(x-ct)+\partial_x(f_1(G_\varepsilon(x-ct))H_\varepsilon(x-ct)+f_2(G_\varepsilon(x-ct)))\psi(x,t)dxdt\\
=&\iint(cG_\varepsilon(x-ct)-f_1(G_\varepsilon(x-ct))H_\varepsilon(x-ct)-f_2(G_\varepsilon(x-ct)))\partial_x\psi(x,t)dxdt\\
=&\int_{-T}^{T}\int_{-X}^{ct-\varepsilon}(cG_\varepsilon(x-ct)-f_1(G_\varepsilon(x-ct))H_\varepsilon(x-ct)-f_2(G_\varepsilon(x-ct)))\partial_x\psi(x,t)dxdt\\
=&\int_{-T}^{T}\int_{ct-\varepsilon}^{ct+\varepsilon}(cG_\varepsilon(x-ct)-f_1(G_\varepsilon(x-ct))H_\varepsilon(x-ct)-f_2(G_\varepsilon(x-ct)))\partial_x\psi(x,t)dxdt\\
=&\int_{-T}^{T}\int_{ct+\varepsilon}^{X}(cG_\varepsilon(x-ct)-f_1(G_\varepsilon(x-ct))H_\varepsilon(x-ct)-f_2(G_\varepsilon(x-ct)))\partial_x\psi(x,t)dxdt\\
=&I_1+I_2+I_3.
\end{aligned}
$$

At first let us notice that

$$|I_2|\leq\int_{-T}^{T}2\varepsilon CC_\psi dt\to 0,\ \text{as } \varepsilon\to 0,$$

since all of G_ε, H_ε and ψ are bounded independently on ε. Next,

$$
\begin{aligned}
I_1+I_3=&\int_{-T}^{T}\int_{-X}^{ct-\varepsilon}(cu_0-f_1(u_0)v_0-f_2(u_0))\partial_x\psi(x,t)dxdt\\
&+\int_{-T}^{T}\int_{-X}^{ct-\varepsilon}(cu_1-f_1(u_1)v_0-f_2(u_1))\partial_x\psi(x,t)dxdt\\
&=(cu_0-f_1(u_0)v_0-f_2(u_0))\int_{-T}^{T}\psi(ct-\varepsilon,t)dt\\
&-(cu_1-f_1(u_1)v_1-f_2(u_1))\int_{-T}^{T}\psi(ct+\varepsilon,t)dt\\
&\to-(c[u]-[f_1(u)v+f_2(u)])\delta|_{x=ct},\ \text{as } \varepsilon\to 0.
\end{aligned}
$$

This means that the sum of first two members of (14) is associated to $\delta(x-ct)$ multiplied by a constant, i.e.

$$G_{\varepsilon t}+(f_1(G_\varepsilon)H_\varepsilon+f_2(G_\varepsilon))_x\approx-(c[u]-[f_1(u)v+f_2(u)])\delta.$$

In the sequel we shall omit this type of calculations. The function $(f_1(G_\varepsilon)D_\varepsilon)_x$ is a derivative of a product of approximations for step and delta function. This product is associated with the delta function multiplied by a constant, and the constant depend of a representative of the delta function. This constant as well as $(c[u]-[f_1(u)v+f_2(u)])$ has to be zero. $(c[u]-[f_1(u)v+f_2(u)])=0$ gives the Rankine-Hugoniot conditions for G_ε and H_ε of the first equation, i.e.

$$c=\frac{f_1(u_1)v_1+f_2(u_1)-f_1(u_0)v_0-f_2(u_0)}{u_1-u_0}.$$

Now we shall find a representative of the delta function such that $f_1(G_\varepsilon)D_\varepsilon \approx 0$. Let us note that $\operatorname{supp} D \subset [-A,-1] \cup [1,A]$, A is a real number greater than 1, $\int D(y)dy = 1$ and $D_\varepsilon(y) = \varepsilon^{-n} D(y/\varepsilon)$. The condition on supports ensures that the product of a step function and δ will not depend on the regularization of the initial data. It will depend only on values of a step function away of zero, $\int_{-\infty}^{0} D(y)dy$ and $\int_0^{-\infty} D(y)dy$. Thus, we have the following system

$$\int_{-A}^{-1} D(y)dy + \int_1^A D(y)dy = 1$$

$$f_1(u_0)\int_{-A}^{-1} D(y)dy + f_1(u_1)\int_1^A D(y)dy = 0.$$

The solution is given by

$$D^- := \int_{-a}^{-1} D(y)dy = \frac{f_1(u_1)}{f_1(u_1)-f_1(u_0)},\ D^+ := \int_1^b D(y)dy = \frac{-f_1(u_0)}{f_1(u_1)-f_1(u_0)},$$

if $f_1(u_1) \neq f_1(u_0)$. In the case when $f_1 \equiv \text{const} \neq 0$, we shall find singular shock solutions of system (1), (2) but for some special forms of f_2, g_1 and g_2. There are no delta shock solutions of this form (only classical shocks, i.e. $s = 0$). If $f_1 \equiv 0$, then D^- and D^+ can be choosen such that

$$cD'_\varepsilon \approx (g_1(G_\varepsilon)D_\varepsilon)_x.$$

The only neccesary condition is that $g_1(u_0) \neq g_1(u_1)$. If g_1 is also a constant then c equals to this constant. The set of all u_1 for which this is true determines the delta locus.

When we substitute u_ε and v_ε into (2),

$$\begin{aligned} &H_{\varepsilon t} + s'(t)D_\varepsilon - cs(t)D_{\varepsilon t} + (g_1(G_\varepsilon)H_\varepsilon + g_2(G_\varepsilon) + g_1(G_\varepsilon)s(t)D_\varepsilon)_x \\ =&H_{\varepsilon t} + (g_1(G_\varepsilon)H_\varepsilon + g_2(G_\varepsilon))_x + s'(t)D_\varepsilon - cs(t)D_{\varepsilon t} + s(t)(g_1(G_\varepsilon)D_\varepsilon)_x. \end{aligned}$$

Since G_ε and H_ε are representatives of classical step functions, it follows that

$$H_{\varepsilon t} + (g_1(G_\varepsilon)H_\varepsilon + g_2(G_\varepsilon))_x \approx -\alpha\delta,$$

where α is Rankine-Hugoniot deficit ([6]) given by

$$\alpha = c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)).$$

By putting $\alpha = 0$ one can see that (u_1, v_1) has to belong to the standard Rankine-Hugoniot locus (then $s = 0$, since only the term $s'(t)D_\varepsilon$ is associated with the delta function multiplied by a constant and all others are associated with the derivative of the delta function multiplied by some constant). In the case $\alpha \neq 0$ the condition $s'(t) = \alpha$ should be satisfied, which implies that $s(t) = \alpha t$ and

$$cD'_\varepsilon \approx (g_1(G_\varepsilon)D_\varepsilon)_x.$$

This implies that

$$c = \frac{g_0(u_1)f_1(u_1) - g_1(u_1)f_1(u_0)}{f_1(u_1) - f_1(u_0)} \tag{15}$$

(since the representative of delta function D_ε (i.e. D^- and D^+) is already determinated). That means that the delta locus is defined as the set of all (u_1, v_1) such that (15) holds.

Remark. In the case when $f_1 \equiv \text{const} \neq 0$ the delta locus is the empty set.

4. Box approximations and delta locus

Theorem 1 is also true for the box approximations. In this case $D_\varepsilon(y)$ consists of two boxes,

$$D_\varepsilon(y) = \begin{cases} \dfrac{D^-}{\varepsilon}, \; y \in [-\varepsilon, 0] \\ \dfrac{D^+}{\varepsilon}, \; y \in [0, \varepsilon] \\ \quad 0, \text{ otherwise.} \end{cases}$$

After the substitution of u and v into (1), by using the standard Rankine-Hugoniot conditions for step functions, we have

$$\begin{aligned} &(-c[G] + [f_1(G)H + f_2(G)])\delta_0 + s(t)([f_1(G)D_\varepsilon]_{-\varepsilon}\delta_{-\varepsilon} \\ &+[f_1(G)D_\varepsilon]_0\delta_0 + [f_1(G)D_\varepsilon]_\varepsilon\delta_\varepsilon) = 0, \end{aligned}$$

where $\langle \delta_a, \psi \rangle = \psi(a)$ and $[\cdot]_\xi$ denotes a jump in a point ξ. Again the same condition for the speed holds,

$$c = \frac{[f_1(G)H + f_2(G)]}{[G]},$$

and

$$\begin{aligned} \langle [f_1(G)D_\varepsilon]_{-\varepsilon}\delta_{-\varepsilon}, \psi \rangle &= f_1(u_0)D^-\varepsilon^{-1}\psi(-\varepsilon) = f_1(u_0)D^-\varepsilon^{-1}(\psi(0) - \varepsilon\psi'(y_\varepsilon)) \\ &= f_1(u_0)D^-\varepsilon^{-1}\psi(0) - f_1(u_0)D^-\psi'(y_\varepsilon), \end{aligned}$$

where y_ε is some number in $[-\varepsilon, 0]$. Similarly,

$$\langle [f_1(G)D_\varepsilon]_\varepsilon\delta_\varepsilon, \psi \rangle = -f_1(u_1)D^+\varepsilon^{-1}\psi(0) - f_1(u_1)D^+\psi'(\tilde{y}_\varepsilon),$$

$\tilde{y}_\varepsilon \in [0, \varepsilon]$. Finally,

$$\langle [f_1(G)D_\varepsilon]_0\delta_0, \psi \rangle = (-f_1(u_0)D^- + f_1(u_1)D^+)\varepsilon^{-1}\psi(0).$$

Then

$$\begin{aligned} &\langle [f_1(G)D_\varepsilon]_{-\varepsilon}\delta_{-\varepsilon} + [f_1(G)D_\varepsilon]_0\delta_0 + [f_1(G)D_\varepsilon]_\varepsilon\delta_\varepsilon, \psi \rangle \\ =& - f_1(u_0)D^-\psi'(y_\varepsilon) + f_1(u_1)D^+\psi'(\tilde{y}_\varepsilon) \to (-f_1(u_0)D^- - f_1(u_1)D^+)\psi'(0) \\ =&(-f_1(u_0)D^- - f_1(u_1)D^+)\langle \delta', \psi \rangle, \end{aligned}$$

as $\varepsilon \to 0$. This gives the following two conditions

$$\begin{aligned} &D^- + D^+ = 1 \\ &f_1(u_0)D^- + f_1(u_1)D^+ = 0. \end{aligned}$$

These conditions are the same as in the previous case.

In the similar way as above one can obtain the third condition which defines the delta locus

$$g_1(u_0)D^- + g_1(u_1)D^+ = c.$$

The analysis of possible cases is the same as for smooth aproximations.

In both cases it seems that one splits delta function in two parts, left delta, δ^-, and right one, δ^+, such that its product with piecewise continuous and bounded functions is defined by

$$\delta^*(y)f(y) = \lim_{y\to 0^*} f(y), \text{ where } * \text{ is } - \text{ or } +.$$

In the case when system (1), (2) is hyperbolic, admissible part of the delta locus is the set of (u_1, v_1) such that $\lambda_2(u_1,v_1) \le c \le \lambda_1(u_0,v_0)$ where $\lambda_1 < \lambda_2$ are the eigenvalues of (1), (2).

Remark. Eigenvalues of (1), (2) equals

$$\lambda_{1,2}(u,v) = \frac{f_1'(u)v + f_2'(u) + g_1(u)}{2} \pm \frac{\sqrt{(f_1'(u)v + f_2'(u) - g_1(u))^2 + 4f_1(u)(g_1'(u)v + g_2'(u))}}{2}$$

If one uses the box approximation then one can see that the hyperbolicity of the system means $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, for $u = u_i$, $v = v_i$, $i = 1,2$ and for v equals some number depending on ε which tends to infinity as $\varepsilon \to 0$. The last will be true if $f_1'(u_i) \neq 0$, $i = 1,2$. If one uses smooth approximation, then the question of hyperbolicity is much more difficult and highly depends on functions in the fluxes.

Overcompressivity condition can be written in the following explicit form

$$\begin{aligned}
&\frac{f_1'(u_0)v_0 + f_2'(u_0) + g_1(u_0)}{2} - \frac{\sqrt{(f_1'(u_0)v_0 + f_2'(u_0) - g_1(u_0))^2 + 4f_1(u_0)(g_1'(u_0)v_0 + g_2'(u_0))}}{2}\\
\ge &\frac{g_1(u_0)f_1(u_1) - g_1(u_1)f_1(u_0)}{f_1(u_1) - f_1(u_0)}\\
\ge &\frac{f_1'(u_1)v_1 + f_2'(u_1) + g_1(u_1)}{2} + \frac{\sqrt{(f_1'(u_1)v_1 + f_2'(u_1) - g_1(u_1))^2 + 4f_1(u_1)(g_1'(u_1)v_1 + g_2'(u_1))}}{2}
\end{aligned}$$

5. Measure valued solutions

In order to explain more simply the concept of solution we use the self similar solutions to (1) and (2), i.e. $u = u(x/t)$, $v = v(x/t)$. In that case the system is

$$-yu'(y) + (f_1(u(y))v(y) + f_2(u(y)))' = 0 \tag{16}$$

$$-yv'(y) + (g_1(u(y))v(y) + g_2(u(y)))' = 0, \tag{17}$$

$$u(-\infty) = u_0,\ u(\infty) = u_1,\ v(-\infty) = v_0,\ v(\infty) = v_1. \tag{18}$$

Our aim is to find a solution (u,v), where u is an element of the space of piecewise continuous functions having the right and left hand limits in all points in $\mathbb{R}$, $P_L(\mathbb{R})$, while $v = (v_0.\dots,v_{m+1})$ belonges to the space ${}_{y_0,\dots,y_m}\mathcal{M}(\mathbb{R}) = \mathcal{M}((-\infty,y_0]) \times$

$\mathcal{M}([y_0, y_1]) \times \cdots \times \mathcal{M}([y_m, \infty))$, where y_i, $i = 0, \ldots, m$ are some real numbers. In this case the products $f_1(u)v$ and $g_1(u)v$ exist and belong to the space ${}_{y_0,\ldots,y_m}\mathcal{M}(\mathbb{R})$. Denote by $\mathcal{M}_F(\mathbb{R})$ the union of all ${}_{y_0,\ldots,y_m}\mathcal{M}(\mathbb{R})$, where $\{y_1, \ldots, y_m\}$ is a finite set of real numbers defined above.

Let $w = (w_0, \ldots, w_{m+1}) \in_{y_0,\ldots,y_m} \mathcal{M}(\mathbb{R})$,

$$w_i = \overline{w}_i + \overline{\gamma}_{i-1}\delta^+(\cdot - y_{i-1}) + \gamma_i\delta^-(\cdot - y_i),\ i = 1, \ldots, m,$$
$$w_0 = \overline{w}_0 + \gamma_0\delta^-(\cdot - y_0),\ w_{m+1} = \overline{w}_{m+1} + \overline{\gamma}_m\delta^+(\cdot - y_m),$$

where $\overline{w}_i$ are the measures with the zero mass in the points $\{y_0, \ldots, y_m\}$.

The mapping $\iota : w \mapsto \iota(w) = W \in \mathcal{D}'(\mathbb{R})$ is defined in the following way.

$$\iota(w) := \overline{W} + (\gamma_0 + \overline{\gamma}_0)\delta(\cdot - y_0) + \cdots + (\gamma_m + \overline{\gamma}_m)\delta(\cdot - y_m),$$

where $\overline{W}$ is the image of $\overline{w} = \overline{w}_0 + \cdots + \overline{w}_{m+1} \in \mathcal{M}(\mathbb{R})$ by the usual injection of the space of measures into the space of distributions. Thus, $\iota(w)$ is a sum of images of two measures. The first one is a measure which has zero mass at points $y_0, \ldots, y_m$ and equals $w_0 + \cdots + w_{m+1}$ when restricted to $\mathbb{R} \setminus \{y_0, \ldots, y_m\}$. The second one is a linear combination of delta measures in the points $\{y_0, \ldots, y_m\}$.

We say that $(u, v) \in P_L(\mathbb{R}) \times \mathcal{M}_F(\mathbb{R})$ is a solution to (16), (17) if

$$-y\iota(u)' + \iota(f_1(u)v + f_2(u))' = 0$$
$$-y\iota(v)' + \iota((g_1(u)v + g_2(u)))' = 0$$

in $\mathcal{D}'$.

With such interpretation, for given (u_0, v_0), the delta locus will be the set of all pairs $(u_1, v_1) \in \mathbb{R}^2$ such that there exists a solution in the above sense which is constant except in one point.

Now we will find the delta locus (with the above definition) for system (16), (17) and (18). Suppose that $y_0 = c$ is the point of discontinuity and let

$$u(y) = \begin{cases} u_0,\ y < c \\ u_1,\ y > c \end{cases},\ v = w + \gamma\delta^- + \overline{\gamma}\delta^+,\ \text{where } w(y) = \begin{cases} v_0,\ y < c \\ v_1,\ y > c \end{cases}.$$

Then

$$\begin{aligned} f_1(u)v + f_2(u) &= f_1(u)w + f_2(u) + f_1(u)(\gamma\delta^- + \overline{\gamma}\delta^+) \\ &= f_1(u)w + f_2(u) + \gamma f_1(u_0)\delta^- + \overline{\gamma} f_1(u_1)\delta^+. \end{aligned}$$

After the injection of the above term in $\mathcal{D}'$, equation (16) becomes

$$-yu' + (f_1(u)w + f_2(u))' + (\gamma f_1(u_0) + \overline{\gamma} f_1(u_1))\delta' = 0.$$

The term $-yu' + (f_1(u)w + f_2(u))' = \text{const} \cdot \delta$ determines the shock wave speed c as usual, i.e. const $= 0$. So, the term $(\gamma f_1(u_0) + \overline{\gamma} f_1(u_1))\delta'$ has to be zero, i.e. we have

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0} \tag{19}$$

and

$$\gamma f_1(u_0) + \overline{\gamma} f_1(u_1) = 0. \tag{20}$$

Equation (19) defines the speed of the shock and (20) gives the first condition on γ and $\overline{\gamma}$.

Equation (17) is transformed in the following way. The first term, $yv = yw + y\gamma\delta^- + y\overline{\gamma}\delta^+$, is mapped into $yw + y(\gamma + \overline{\gamma})\delta \in \mathcal{D}'$. The second term, $g_1(u)v + g_2(u) = g_1(u)w + g_2(u) + g_1(u_0)\gamma\delta^- + g_1(u_1)\overline{\gamma}\delta^+$, is mapped into $g_1(u)w + g_2(u) + (g_1(u_0)\gamma + g_1(u_1)\overline{\gamma})\delta \in \mathcal{D}'$. After the substitution of these elements of $\mathcal{D}'$ into (17) we obtain

$$\begin{aligned}
&-yw' + (g_1(u)w + g_2)' - c(\gamma + \overline{\gamma})\delta'(y - c) + (\gamma + \overline{\gamma})\delta(y - c)\\
&+ (g_1(u_0)\gamma + g_1(u_1)\overline{\gamma})\delta'(y - c) = 0.
\end{aligned}$$

Since $-yw' + (g_1(u)w + g_2)' = -\alpha\delta(y - c)$, where α is Rankine-Hugoniot deficit, $\alpha = c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0))$, γ and $\overline{\gamma}$ satisfy

$$\gamma + \overline{\gamma} = \alpha \tag{21}$$

$$g_1(u_0)\gamma + g_1(u_1)\overline{\gamma} = c. \tag{22}$$

The solution to system (20), (21) and (22) in respect of γ and $\overline{\gamma}$ exists if

$$\begin{vmatrix} f_1(u_0) & f_1(u_1) & 0 \\ 1 & 1 & -\alpha \\ g_1(u_0) & g_1(u_1) & -c \end{vmatrix} \neq 0.$$

The result is the same as in the case when the approximations are used. One can see that $s = \gamma + \overline{\gamma}$, $D^- = \gamma/s$, and $D^+ = \overline{\gamma}/s$.

6. Singular shocks

In the case when the fluxes f_1, f_2, g_1 and g_2 are polynomials it is often possible to find a singular shock solution to system (1)-(3) for a larger set of the initial data. This will be done by adding some terms in u_ε beside G_ε. When all functions in the flux are polynomials we can control behaviour of these additional terms in u_ε. Suppose that the maximal degree of all polynomials in the fluxes equals m. Let

$$f_1(y) = \sum_{i=0}^{m} \overline{a}_i y^i,\ f_2(y) = \sum_{i=0}^{m} a_i y^i,\ g_1(y) = \sum_{i=0}^{m} \overline{b}_i y^i,\ g_2(y) = \sum_{i=0}^{m} b_i y^i.$$

We will try to find solutions in the form

$$\begin{aligned}
u_\varepsilon(x,t) &= G_\varepsilon(x - ct) + s_1(t)d_\varepsilon(x - ct)\\
v_\varepsilon(x,t) &= H_\varepsilon(x - ct) + s_2(t)D_\varepsilon(x - ct),
\end{aligned} \tag{23}$$

where s_i, $u = 1, 2$ and D_ε are of the same form as before. A net d_ε is a representative of the generalized function associated with zero, and it will be described in the proof of the following theorem.

Theorem 2. *A constant*

$$c = \frac{f_1(u_1)v_1 + f_2(u_1) - f_1(u_0)v_0 - f_2(u_0)}{u_1 - u_0} = \frac{[f_1(u)v + f_2(u)]}{[u]}$$

will be called a speed of the singular shock, and

$$\alpha = c(v_1 - v_0) - (g_1(u_1)v_1 + g_2(u_1) - g_1(u_0)v_0 - g_2(u_0)) = c[v] - [g_1(u)v + g_2]$$

will be called Rankin-Hugoniot deficit.

A singular shock wave solution to (9)-(11) in the form (23) exists in the following cases.
a) Let m *be an even number. Then there must be possible to find* $\sigma > 0$ *such that there exists a solution* $(D^-, d^-) \in \mathbb{R} \times \mathbb{R}_+$ *to the system*

$$\begin{aligned} &[f_1(u)]\alpha D^- + [v]\overline{a}_m \sigma d^- = a_m\sigma + f_1(u_1)\alpha + v_1\overline{a}_m\sigma \\ &[g_1(u)]\alpha D^- + [v]\overline{b}_m \sigma d^- = b_m\sigma + g_1(u_1)\alpha + v_1\overline{b}_m\sigma - c\alpha. \end{aligned} \tag{23}$$

b) Let m *be an odd number. One of the two following conditions has to hold.*
(i) There exists $\sigma \in \mathbb{R} \setminus \{0\}$ *such that there exists a solution* $(D^-, d^-) \in \mathbb{R}^2$ *to system (23).*
(ii) There exists $\sigma > 0$ *such that there exists a solution* $(D^-, d^-) \in \mathbb{R} \times \mathbb{R}_+$ *to the system*

$$\begin{aligned} &\alpha[f_1(u)]D^- + (\overline{a}_{m-1}[u] + m\overline{a}_m[uv] + ma_m[u])\sigma d^- \\ =&\sigma(\overline{a}_m v_1 + m\overline{a}_m u_1 v_1 + m a_m u_1) + \alpha f_1(u_1) + a_{m-1}\sigma \\ &\alpha[g_1(u)]D^- + (\overline{b}_{m-1}[u] + m\overline{b}_m[uv] + mb_m[u])\sigma d^- \\ =&\sigma(\overline{b}_m v_1 + m\overline{b}_m u_1 v_1 + m b_m u_1) + \alpha g_1(u_1) + b_{m-1}\sigma - c\alpha. \end{aligned} \tag{24}$$

The set of all points $(u_1, v_1) \in \mathbb{R}^2$ *for which there exists a singular shock wave solution is called the singular delta locus.*

Proof.
a) Let $\phi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\phi \subset [-1,1]$, $\int \phi(x)dx = 1$, $\phi \geq 0$ and let

$$d_\varepsilon = \left(\frac{d^-}{\varepsilon\frac{b-a}{2}}\phi\left(\frac{x + \frac{a+b}{2}}{\varepsilon\frac{b-a}{2}}\right)\right)^{1/m} + \left(\frac{d^+}{\varepsilon\frac{b-a}{2}}\phi\left(\frac{x - \frac{a+b}{2}}{\varepsilon\frac{b-a}{2}}\right)\right)^{1/m},$$

where $(d^-)^m + (d^+)^m = 1$. Let $s_1(t)$, $s_2(t)$ be smooth functions for $t \geq 0$, $s_1(0) = s_2(0) = 0$, and let D_ε be of the same form as in Theorem 1.

If G_ε is a representative of a generalized function of the bounded type (some power of a step function, for example), then

$$\left|\int G_\varepsilon(x - ct)d_\varepsilon^j(x - ct)\psi(x,t)dxdt\right| \leq 2C_G C_\psi \int \varepsilon^{1-j/m}\phi^{1/m}(y)dydt$$

The last term converge to zero as $\varepsilon \to 0$, for $j < m$. Since the multiplication with a smooth function preserves association, there holds

$$(G_\varepsilon + s(t)d_\varepsilon)^j \approx G_\varepsilon^j,\ j < m \text{ and } (G_\varepsilon + s(t)d_\varepsilon)^m \approx G_\varepsilon^m + s^m(t)d_\varepsilon^m.$$

Let $G_\varepsilon(y)$ be a generalized step function which equals u_0 for $y<-\varepsilon$ and equals u_1 for $y>\varepsilon$ One can see that $G_\varepsilon d_\varepsilon^m \approx (u_0 d^- + u_1 d^+)\delta(x-ct)$ from the construction of d_ε. Also, the construction of D_ε i d_ε implies that $D_\varepsilon d_\varepsilon \approx 0$.

If $\Gamma(y)=\sum_{i=0}^m \gamma_i y^i$, then

$$\Gamma(G_\varepsilon + s(t)d_\varepsilon) \approx \Gamma(G_\varepsilon) + s^m(t)\gamma_m d_\varepsilon^m.$$

After substitution of $U_\varepsilon = G_\varepsilon + s_1(t)d_\varepsilon$ and $V_\varepsilon = H_\varepsilon + s_2(t)D_\varepsilon$ in system (9)-(11), one can see that from the first equation that $c=\frac{[f_1(u)v+f_2(u)]}{[u]}$ (Rankin-Hugoniot condition) and from the second that $s_2(t)=\alpha t$, where $\alpha = c[v]-[g_1(u)v+g_2]$ is a Rankin-Hugoniot deficit. Further on, after association procedure like in the proof of Theorem 1 and grouping coefficients in respect of δ and δ' one finds that $s_1(t)=\tilde{s}t^{1/m}$ and (recall that $D^-+D^+=(d^-)^m+(d^+)^m=1$)

$$\begin{aligned}
&[f_1(u)]\alpha D^- + [v]\overline{a}_m \tilde{s}^m d^- = a_m\tilde{s}^m + f_1(u_1)\alpha + v_1\overline{a}_m\tilde{s}^m\\
&[g_1(u)]\alpha D^- + [v]\overline{b}_m \tilde{s}^m d^- = b_m\tilde{s}^m + g_1(u_1)\alpha + v_1\overline{b}_m\tilde{s}^m - c\alpha.
\end{aligned}$$

This proves the first part of the theorem.

b) (i) The proof of the assertion is the same as the one of a). The only difference is that $\tilde{s}^m$ has not to be positive.

(ii) The construction of d_ε in the second case is a slightly different. Let ϕ be as above and

$$\begin{aligned}
d_\varepsilon =& \left(\frac{2d^-}{\varepsilon(b-a)}\phi\left(\frac{x+(a+3b)/4}{\varepsilon(b-a)/4}\right)\right)^{1/(m-1)} - \left(\frac{2d^-}{\varepsilon(b-a)}\phi\left(\frac{x+(3a+b)/4}{\varepsilon(b-a)/4}\right)\right)^{1/(m-1)}\\
&+\left(\frac{2d^+}{\varepsilon(b-a)}\phi\left(\frac{x-(3a+b)/4}{\varepsilon(b-a)/4}\right)\right)^{1/(m-1)} + \left(\frac{2d^+}{\varepsilon(b-a)}\phi\left(\frac{x-(a+3b)/4}{\varepsilon(b-a)/4}\right)\right)^{1/(m-1)},
\end{aligned}$$

where $(d^-)^{m-1}+(d^+)^{m-1}=1$. Like in the previous case, one can see that

$$(G_\varepsilon + s(t)d_\varepsilon)^j \approx G_\varepsilon^j,\ j<m-1 \text{ and } (G_\varepsilon + s(t)d_\varepsilon)^{m-1} \approx G_\varepsilon^{m-1} + s^{m-1}(t)d_\varepsilon^{m-1}$$

for every generalized function G_ε of the bounded type. Because of the specific definition of d_ε one can see that $G_\varepsilon d_\varepsilon^m \approx 0$ if G_ε is a generalized step function. The above means that

$$\Gamma(G_\varepsilon + s_1(t)d_\varepsilon) \approx \Gamma(G_\varepsilon) + \gamma_{m-1}s_1^{m-1}(t)d_\varepsilon^{m-1} + m\gamma_m s_1^{m-1}(t)G_\varepsilon d_\varepsilon^{m-1}.$$

Substituting $U_\varepsilon = G_\varepsilon + s_1(t)d_\varepsilon$ and $V_\varepsilon = H_\varepsilon + s_2(t)D_\varepsilon$ in system (9)-(11) one can see that $c=\frac{[f_1(u)v+f_2(u)]}{[u]}$, $s_2(t)=\alpha t$, where $\alpha=c[v]-[g_1(u)v+g_2]$ like in a). Also, one finds that $s_1(t)=\tilde{s}t^{1/(m-1)}$ and

$$\begin{aligned}
&\alpha[f_1(u)]D^- + (\overline{a}_{m-1}[u] + m\overline{a}_m[uv] + ma_m[u])\tilde{s}^{m-1}d^-\\
=&\tilde{s}^{m-1}(\overline{a}_m v_1 + m\overline{a}_m u_1 v_1 + ma_m u_1) + \alpha f_1(u_1) + a_{m-1}\tilde{s}^{m-1}\\
&\alpha[g_1(u)]D^- + (\overline{b}_{m-1}[u] + m\overline{b}_m[uv] + mb_m[u])\tilde{s}^{m-1}d^-\\
=&\tilde{s}^{m-1}(\overline{b}_m v_1 + m\overline{b}_m u_1 v_1 + mb_m u_1) + \alpha g_1(u_1) + b_{m-1}\tilde{s}^{m-1} - c\alpha.
\end{aligned}$$

This proves the theorem.

In the following corollaries we shall describe singular delta locus in the case $f_1 \equiv a$ where a is a nonzero constant. This is the case when the delta locus is the empty set (see Theorem 1).

Corollary 1. *Let $f_1 \equiv a$, $a \neq 0$ is a constant, and let g_1 be non constant function.*
a) If m is an even number, then (u_1, v_1) is in a delta singular locus if $a_m \neq 0$ (i.e. $\deg f_2 = m$), there exists $\sigma > 0$ such that $\operatorname{sgn}([v]a_m) = \operatorname{sgn}(a\alpha + v_1 a_m \sigma)$ and $[g_1(u)] \neq 0$.
b) If m is an odd number and $a_m \neq 0$, then the delta singular locus is the set $\mathbb{R}^2 \setminus \{u_1 \in \mathbb{R} : [g_1(u)] = 0\}$.
c) Let m be an odd number, $a_m = 0$ and $a_{m-1} \neq 0$ (i.e. $\deg f_2 = m-1$). Then (u_1, v_1) is in a delta singular locus if $a\alpha / a_{m-1} < 0$ and $[g_1(u)] \neq 0$.

Let us note that in the case $\deg f_2 < m-1$ the delta singular locus is the empty set ($\alpha = 0$ is the condition for Rankin-Hugoniot locus).

Proof. In all of these cases, the condition $[g_1(u)] \neq 0$ ensures that there always exists D^- if appropriate d^- exists.
a) The proof easily follows from (24) since D^- is an arbitrary real number.
b) The proof is an immediate consequence of b) (i) in the previous theorem.
c) One can see from the first equation in (24) that $\sigma = -a\alpha / a_{m-1} > 0$.

Corollary 2. *Let $f_1 \equiv a$ and $g_1 \equiv b$, where $a \neq 0$ and b are some constants.*
a) Let a_m and b_m be different from zero. Then

$$\sigma = \frac{((b-c)a_m - ab_m)\alpha}{a_{m-1}b_m - a_m b_{m-1}}$$

has to be greater than zero, and (u_1, v_1) is in delta singular locus if

$$d^- = \frac{\sigma m a_m u_1 + a\alpha + a_{m-1}\sigma}{m a_m [u]} \in [0,1]$$

for σ defined above.
b) Let $a_m \neq 0$ and $b_m = 0$. Then

$$\sigma = \frac{c\alpha - b\alpha}{b_{m-1}}$$

has to be positive, and (u_1, v_1) is in a delta locus if

$$d^- = \frac{\sigma m a_m u_1 + a\alpha + a_{m-1}\sigma}{m a_m [u]\sigma} \in [0,1]$$

for σ defined above.
c) Let $a_m = 0$ and $b_m \neq 0$. Then

$$\sigma = \frac{-a\alpha}{a_{m-1}}$$

has to be positive, and (u_1, v_1) is in a delta locus if

$$d^- = \frac{\sigma m b_m u_1 + b\alpha + b_{m-1}\sigma - c\alpha}{m a_m [u]\sigma} \in [0,1]$$

for σ defined above.

d) Let $a_m = b_m = 0$. Then a delta singular locus exist, and it is $\mathbb{R}^2$, only if

$$\frac{c\alpha - b\alpha}{b_{m-1}} = \frac{-a\alpha}{a_{m-1}} > 0.$$

Proof. In all of these cases we are using the second definition of d_ε and the assertions follow by solving the system of equations (24) with the unknowns σ and d^-.

Remark 1. As in the first section all the calculation can be made for box approximations.

Remark 2. In [6] the singular parts of functions u and v have nonempty intersection (even $d_\varepsilon^2 = D_\varepsilon \approx \delta$). This is possible because in the system (4) there are no multiplication of d_ε and D_ε, i.e. $f_1 = -1$, $g_1 = 0$ are the constants. It is also possible to choose d_ε and D_ε with non-disjoint supports if flux functions are linear in u. In general case, problems with terms $D_\varepsilon d_\varepsilon^j$ can appear. The solution of Keyfitz and Kranzer is the same as in the case c) of Corrolary 2 for $m = 3$, $a = -1$, $b = 0$, $a_{m-1} = 1$, $b_m = 1/3$ and $b_{m-1} = 0$. When all points for which the overcompressibility condition does not hold are extracted, one obtain the same area as them (it is denoted by Q_7 in their paper).

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