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CORRESPONDENCE SPACES AND TWISTOR SPACES FOR PARABOLIC GEOMETRIES

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ABSTRACT. For a semisimple Lie group G with parabolic subgroups $Q \subset P \subset G$, we associate to a parabolic geometry of type (G, P) on a smooth manifold N the correspondence space $\mathcal{C}N$, which is the total space of a fiber bundle over N with fiber a generalized flag manifold, and construct a canonical parabolic geometry of type (G, Q) on $\mathcal{C}N$.

Conversely, for a parabolic geometry of type (G, Q) on a smooth manifold M , we construct a distribution corresponding to P , and find the exact conditions for its integrability. If these conditions are satisfied, then we define the twistor space N as a local leaf space of the corresponding foliation. We find equivalent conditions for the existence of a parabolic geometry of type (G, P) on the twistor space N such that M is locally isomorphic to the correspondence space $\mathcal{C}N$, thus obtaining a complete local characterization of correspondence spaces.

We show that all these constructions preserve the subclass of normal parabolic geometries (which are determined by some underlying geometric structure) and that in the regular normal case, all characterizations can be expressed in terms of the harmonic curvature of the Cartan connection, which is easier to handle. Several examples and applications are discussed.

1. Introduction

This paper is devoted to the study of relations between different geometric structures via the construction of correspondence spaces and twistor spaces. The structures we deal with are the so-called parabolic geometries, which may be viewed as curved analogs of homogeneous spaces of the form G/P , where G is a semisimple Lie group and $P \subset G$ is a parabolic subgroup. Parabolic geometries form a rather large class of structures, including for example projective, conformal and non-degenerate CR-structures of hypersurface type, as well as certain higher codimension CR structures.

The starting point of twistor theory was R. Penrose's idea to associate to the Grassmannian $Gr_2(\mathbb{C}^4)$ of planes in \mathbb{C}^4 , which is viewed as compactified complexified Minkowski space, the twistor space $\mathbb{C}P^3$, and to study the conformal geometry of $Gr_2(\mathbb{C}^4)$ via the complex geometry of the twistor space. The connection between these two manifolds is the *correspondence space* $F_{1,2}(\mathbb{C}^4)$, the flag manifold of lines in planes in \mathbb{C}^4 , which canonically fibers over $Gr_2(\mathbb{C}^4)$ and over $\mathbb{C}P^3$ and defines a correspondence between the

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two spaces. This correspondence gives rise to the Penrose transform, which is a basic element of twistor theory.

Later on, twistor theory and the Penrose transform have been extended in two directions. On one hand, the original correspondence is homogeneous under the simple Lie group $SL(4, \mathbb{C})$, which governs the projective geometry of $\mathbb{C}P^3$ and (via the isomorphism with $SO(6, \mathbb{C})$) the conformal geometry of $Gr_2(\mathbb{C}^4)$. The subgroups of $SL(4, \mathbb{C})$ leading to these two homogeneous spaces are parabolic subgroups. Now one may replace $SL(4, \mathbb{C})$ by an arbitrary semisimple Lie group G and fix two parabolic subgroups $P_1, P_2 \subset G$ such that $P_1 \cap P_2 \subset G$ is parabolic. Then the natural fibrations from $G/(P_1 \cap P_2)$ onto G/P_1 and G/P_2 define a correspondence. This is the subject of the book [4], in which these correspondences and the resulting Penrose transforms are studied, and applications to differential geometry and representation theory are described.

On the other hand, it has been known for quite some time that certain geometric structures, like conformal, projective, or almost quaternionic structures can be viewed as “curved analogs” of homogeneous spaces of the form G/P as above. In the case of four-dimensional conformal structures, twistor theory has been first extended to curved situations in [1] and [19]. Later on, the techniques were generalized to further geometries and they have found many applications, see for example [16] and the collection [2]. There are two important remarks to be made at this point. One is that twistor theory in many cases requires a restriction on the geometric structure, like self duality in four-dimensional conformal geometry, or torsion freeness in quaternionic geometry. The second important point is that compared to the flat versions of the correspondence discussed above, the situation loses its symmetry. While the passage “up” from the original manifold to the correspondence space is very similar to the flat case, the passage “down” is given by passing to the leaf space of a certain foliation, so usually this is only possible locally. More drastically, usually there is no local geometric structure on the twistor space, but it is only a smooth or complex manifold.

Using E. Cartan’s concept of a generalized space, one may associate to any homogeneous space a geometric structure defined via Cartan connections on suitable principal bundles. In the case of homogeneous spaces of the form G/P as above the corresponding geometric structures are called *parabolic geometries*. A general construction of Cartan connections of this type (under small technical restrictions) was given in the pioneering work of N. Tanaka, see [22]. In [17], these results were embedded into the more general theory of Cartan connections associated to geometric structures on *filtered manifolds*, i.e. manifolds endowed with a filtration of the tangent bundle by subbundles.

The main emphasis in these considerations was the solution of the equivalence problem, while the geometry of the structures in question was only a secondary issue. Although there were some applications of Tanaka theory to geometrical problems which are quite close to twistor theory, see e.g. [20], these works unfortunately never became well known to people working in twistor theory, which may also be due to their rather complicated and technical nature.

During the last years there was a renewed interest in this class of structures and it turned out that apart from containing interesting examples it can be studied in a remarkably uniform way. Besides new constructions of the Cartan connections and descriptions of the underlying structures (see [7]), there is a general theory of classes of preferred connections for parabolic geometries (see [9]) which generalizes Weyl connections in conformal geometry. This leads to a systematic way of expressing the curvature of the Cartan connection (which is the essential invariant of any parabolic geometry) in terms of underlying data and to powerful tools like normal coordinates for arbitrary parabolic geometries. Finally, there are strong general results on invariant differential operators, i.e. differential operators intrinsic to a parabolic geometry (see [11] and [5]).

The impact of this in the direction of twistor theory is that in many cases the geometric structures showing up “on top” of the correspondence (i.e. in the place of the correspondence space) are highly interesting and much more subtle than the geometric structures showing up “downstairs”. In this paper, we will show that the construction of a correspondence space works in the curved setting without restrictions, so one may always pass “up”. Then we will give a complete local characterization of the geometries obtained in that way. This is done in two steps: first we find conditions for the existence of a twistor space, which is a candidate for a space carrying the “downstairs” structure; secondly, we derive the conditions which ensure the existence of this geometric structure. Combining the results for going up and going down, we obtain the precise conditions on the existence of twistor correspondences in the classical sense.

Let us describe the contents of the paper in a little more detail. Technically speaking, we have to deal only with one side of the correspondence. Thus, we consider a semisimple Lie group G and two parabolic subgroups $Q \subset P \subset G$. Starting with any parabolic geometry of type (G, P) on a manifold N , a simple construction leads to the *correspondence space* \mathcal{CN} , which is the total space of a natural fiber bundle over N with fiber a generalized flag manifold, and canonically carries a parabolic geometry of type (G, Q) . In fact, the Cartan connections (and thus also their curvatures) are the same for both structures, which implies some simple restrictions on the curvature of correspondence spaces. It is less obvious but still rather simple, that the normalization conditions for Cartan connections of the two different types are compatible, so for a normal parabolic geometry the correspondence space is normal, too.

The main result of section 2 is that the simple curvature restrictions from above actually characterize correspondence spaces locally. If M is a smooth manifold equipped with a parabolic geometry of type (G, Q) , then the subalgebra $\mathfrak{p} \subset \mathfrak{g}$ gives rise to a distribution on M . Under a weak torsion condition, this distribution is integrable, so we can consider a local leaf-space for the corresponding foliation, which is then called the *twistor space* N of M . The main result is then that if M satisfies the curvature restriction for correspondence spaces from above, then there is a parabolic geometry of type (G, P) on the twistor space N (which is uniquely determined provided that P/Q is connected) such that M is locally isomorphic to \mathcal{CN} .

To get to the classical form of the twistor correspondence, one starts with two parabolics $P_1, P_2 \subset G$ which contain the same Borel subgroup, takes a parabolic geometry of type (G, P_1) , constructs the correspondence spaces associated to $P_1 \cap P_2 \subset P_1$ and then the twistor space corresponding to $P_2 \supset P_1 \cap P_2$. It should be emphasized that for the existence of the Penrose transform, a parabolic geometry on the twistor space is not needed, since one side of the transform (the pull back part) does not need any geometric structure. On the other side one then has a fiber bundle with fiber a generalized flag manifold, and thus results from representation theory apply to the push down part, which is much more subtle.

The curvature of the canonical Cartan connection is a rather complicated object in general. A well known nice feature of regular normal parabolic geometries is that one can pass from this curvature to the harmonic curvature, which is a much simpler object. In section 3, we show how rather deep general results on parabolic geometries can be used to prove that for regular normal parabolic geometries, the curvature restrictions from above are equivalent to the analogous restrictions for the harmonic curvature, which are much easier to verify. Hence we arrive at effective conditions for the existence of a twistor space, as well as for the existence of a parabolic geometry on this twistor space. We show that these results also work in the holomorphic category.

In section 4, we discuss several examples and outline some applications. We first discuss in detail the example of Lagrangian contact structures, which leads to an interesting geometric interpretation of the projective curvature of a linear connection, as well as to results on contact structures and partial connections on projectivized tangent bundles. Next, we briefly outline the case of elliptic partially integrable almost CR structures of CR dimension and codimension two, in which even the construction of correspondence spaces leads to unexpected results. Finally, we briefly discuss the example of almost Grassmannian structures which leads to an interpretation of path geometries as parabolic geometries and contains the classical twistor theory for split signature conformal four-manifolds.

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2. Correspondence spaces and twistor spaces

We start by briefly reviewing some general facts about parabolic geometries, see [7], [11], and [9] for more information.

2.1. $|k|$ -graded Lie algebras and parabolic geometries. Let \mathfrak{g} be a (real or complex) semisimple Lie algebra endowed with a $|k|$ -grading, i.e. a grading of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ such that \mathfrak{g}_1 generates the subalgebra $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, and such that none of the simple ideals of \mathfrak{g} is contained in \mathfrak{g}_0 . Define \mathfrak{p} to be the subalgebra $\mathfrak{g}_0 \oplus \mathfrak{p}_+$. Since we will deal with different parabolics at the same time, we will write \mathfrak{p}_0 for \mathfrak{g}_0 and

we will write \mathfrak{p}_- for the subalgebra $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$, which is usually called \mathfrak{g}_- . Be aware of the fact that \mathfrak{p}_- is not contained in \mathfrak{p} . While the grading of \mathfrak{g} is not \mathfrak{p} -invariant, it gives rise to an invariant decreasing filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k$ defined by $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ for all $i = -k, \dots, k$. It turns out (see [23]), that this filtration is completely determined by the subalgebra \mathfrak{p} which is parabolic, and conversely any parabolic subalgebra gives rise to a $|k|$ -grading.

Next, let G be a Lie group with Lie algebra \mathfrak{g} , and define subgroups $P_0 \subset P \subset G$ (usually P_0 is denoted by G_0) as the subgroups of those elements, whose adjoint actions on \mathfrak{g} preserve the grading respectively the filtration of \mathfrak{g} . One shows that Lie algebras of P and P_0 are \mathfrak{p} and \mathfrak{p}_0 , the exponential map restricts to a diffeomorphism from \mathfrak{p}_+ onto a normal subgroup $P_+ \subset P$, and that P is the semidirect product of P_0 and P_+ .

A *parabolic geometry* of type (G, P) on a smooth manifold M (having the same dimension as the homogeneous space G/P) is given by a principal P -bundle $p : \mathcal{G} \rightarrow M$ and a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e. a smooth one-form with values in \mathfrak{g} such that

- (1) $\omega(\zeta_A) = A$ for all fundamental fields ζ_A , $A \in \mathfrak{p}$
- (2) $(r^g)^*\omega = \text{Ad}(g^{-1}) \circ \omega$ for all $g \in P$
- (3) $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \rightarrow \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

The homogeneous model for this parabolic geometry is the principal P -bundle $p : G \rightarrow G/P$ together with the left Maurer–Cartan form as a Cartan connection. A *morphism* between parabolic geometries $(p : \mathcal{G} \rightarrow M, \omega)$ and $(p' : \mathcal{G}' \rightarrow M', \omega')$ of the same type is a principal bundle map $F : \mathcal{G} \rightarrow \mathcal{G}'$ such that $F^*\omega' = \omega$. This compatibility of F with the Cartan connections implies that it is a local diffeomorphism.

The *curvature-function* $\kappa : \mathcal{G} \rightarrow L(\Lambda^2\mathfrak{g}, \mathfrak{g})$ of a parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$ is defined by $\kappa(u)(X, Y) := d\omega(\omega_u^{-1}(X), \omega_u^{-1}(Y)) + [X, Y]$ for $u \in \mathcal{G}$, so this exactly measures to what extent the Maurer–Cartan equation fails to hold. The defining properties of ω imply that κ is P -equivariant and it vanishes if one of its entries lies in $\mathfrak{p} \subset \mathfrak{g}$. Hence we will view κ as an equivariant smooth function $\mathcal{G} \rightarrow L(\Lambda^2\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$. The values of the curvature function can be used to define various subcategories of parabolic geometries:

First, if κ is identically zero, then the corresponding parabolic geometry is called *(locally) flat*. General results on Cartan connections imply that flat parabolic geometries are locally isomorphic (as parabolic geometries) to the homogeneous model G/P , see Proposition 4.12 in [7] for a proof in the realm of parabolic geometries.

Second, the parabolic geometry is called *torsion-free* if the curvature function has the property that $\kappa(u)(X, Y) \in \mathfrak{p}$ for all $u \in \mathcal{G}$ and $X, Y \in \mathfrak{g}$.

Finally, the parabolic geometry is called *regular* if for all $u \in \mathcal{G}$, all $i, j < 0$ and all $X \in \mathfrak{g}^i$ and $Y \in \mathfrak{g}^j$ one has $\kappa(u)(X, Y) \in \mathfrak{g}^{i+j+1}$. This means that with respect to the grading on \mathfrak{g} , all nonzero homogeneous components of κ are of strictly positive degree. Notice that torsion free parabolic geometries are automatically regular, so regularity should be viewed as a condition avoiding particularly bad types of torsion.

2.2. Normal parabolic geometries. Parabolic geometries are mainly studied because they provide a conceptual way to describe certain underlying geometric structures, for example conformal, almost quaternionic, or CR structures of hypersurface type. This underlying structure easily leads to a principal P_0 -bundle $\mathcal{G}_0 \rightarrow M$ for an appropriate choice of G and P . Using quite sophisticated procedures, one extends this bundle to a principal P -bundle $\mathcal{G} \rightarrow M$ and constructs a canonical Cartan connection on \mathcal{G} . To make this Cartan connection unique (up to isomorphism), one has to impose an additional normalization condition on the curvature. One then arrives at an equivalence of categories between the underlying geometric structures and regular normal parabolic geometries. Different versions of such prolongation procedures can be found in [22], [17], and [7].

For general Cartan connections the problem of finding an appropriate normalization condition is very subtle, but for parabolic geometries Lie theory offers a uniform approach. The Killing form of \mathfrak{g} defines a duality between $\mathfrak{g}/\mathfrak{p}$ and \mathfrak{p}_+ which is compatible with the natural P -actions on both spaces (which both come from the restriction of the adjoint action of G). Hence for each k we get an isomorphism $L(\Lambda^k \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \cong \Lambda^k \mathfrak{p}_+ \otimes \mathfrak{g}$ of P -modules. The latter spaces are the groups in the standard complex computing the Lie algebra homology of \mathfrak{p}_+ with coefficients in \mathfrak{g} . The differentials in this standard complex define linear maps

$$\partial^* : L(\Lambda^k \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \rightarrow L(\Lambda^{k-1} \mathfrak{g}/\mathfrak{p}, \mathfrak{g}),$$

which are traditionally referred to as the *codifferential*. From the explicit formula for these differentials one immediately reads off that they are P -homomorphisms. For the curvature function $\kappa : \mathcal{G} \rightarrow L(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ of a parabolic geometry, we can form $\partial^* \circ \kappa : \mathcal{G} \rightarrow L(\mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ and the geometry is called *normal* if this composition vanishes identically.

By construction, $\partial^* \circ \partial^* = 0$ and the resulting complex computes the Lie algebra homology of \mathfrak{p}_+ with coefficients in \mathfrak{g} . Let us denote the k th homology group (which is a P -module by construction) by $\mathbb{H}_{\mathfrak{g}}^k$. One easily shows that P_+ acts trivially on $\mathbb{H}_{\mathfrak{g}}^k$, so the P -action is determined by the action of P_0 . Kostant's version of the Bott–Borel–Weyl theorem in [15] can be used to algorithmically compute the cohomology groups $H^*(\mathfrak{p}_+, \mathfrak{g})$, which are well known to be dual to the homology groups, as representations of P_0 .

The curvature function $\kappa : \mathcal{G} \rightarrow L(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ of any parabolic geometry is by construction P -equivariant. For a normal parabolic geometry, we get $\partial^* \circ \kappa = 0$ and thus we can consider the induced function $\kappa_H : \mathcal{G} \rightarrow \mathbb{H}_{\mathfrak{g}}^2$, which is again P -equivariant. This is called the *harmonic curvature* of the parabolic geometry. For a regular normal parabolic geometry the Bianchi identity implies that κ vanishes identically provided that κ_H vanishes identically. This reduction to the harmonic curvature is a major simplification. Equivariance of κ_H implies that it determines a section of the bundle $\mathcal{G} \times_P \mathbb{H}_{\mathfrak{g}}^2$. But since P_+ acts trivially on $\mathbb{H}_{\mathfrak{g}}^2$ we may identify this associated bundle with $\mathcal{G}_0 \times_{P_0} \mathbb{H}_{\mathfrak{g}}^2$, where $\mathcal{G}_0 := \mathcal{G}/P_+$. This is exactly the bundle encoding the underlying geometric structure, so the harmonic curvature can be directly interpreted in terms of this underlying structure, while to understand the Cartan curvature κ one needs the full Cartan bundle \mathcal{G} .

A conceptual approach to the computation and geometric interpretation of the harmonic curvature is offered by the so-called Weyl-structures, see [9]. These are global smooth G_0 -equivariant sections $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}$ of the natural projection $\mathcal{G} \rightarrow \mathcal{G}/P_+ = \mathcal{G}_0$. In [9] it is shown that such sections always exist, and how the pull-back $\sigma^*\kappa$ can be computed in terms of tensors and connections naturally associated to σ . Now $\sigma^*\kappa$ corresponds to a P_0 -equivariant function $\mathcal{G}_0 \rightarrow L(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ and on the latter space one has a P_0 -equivariant algebraic Hodge decomposition, see 3.1 below. The harmonic part of this function can be interpreted as a P_0 -equivariant function $\mathcal{G}_0 \rightarrow \mathbb{H}_{\mathfrak{g}}^2$, which exactly represents the harmonic curvature κ_H . It should also be noted that usually there are more direct interpretations for the components of κ_H of lowest homogeneity, in particular if they are of torsion type, i.e. have values in \mathfrak{p}_- . We will discuss this in some examples in section 4.

2.3. Suppose that \mathfrak{g} is complex, and we have fixed a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a choice of positive roots. Then any parabolic subalgebra of \mathfrak{g} is conjugate to a standard parabolic subalgebra, i.e. one that contains \mathfrak{h} and all positive root spaces. Standard parabolic subalgebras in \mathfrak{g} are in bijective correspondence with subsets $\Sigma \subset \Delta_0$ of the set of simple roots. The corresponding $|k|$ -grading of \mathfrak{g} is then given by the Σ -height, i.e. the root space corresponding to a root α lies in \mathfrak{g}_j , where j is the sum of the coefficients of all elements of Σ in the (unique) expansion of α as a linear combination of simple roots. In particular, Σ consists of those simple roots α , for which the $(-\alpha)$ -root space is *not* contained in the parabolic.

Let us now consider the case of two nested (standard) parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$. By construction, the subset $\tilde{\Sigma}$ associated to \mathfrak{q} has to contain the subset Σ associated to \mathfrak{p} . In the language of Dynkin diagrams with crossed nodes (see chapter 2 of [4] and [7]) this just means that in the complex case one passes from \mathfrak{p} to \mathfrak{q} by changing any number of uncrossed nodes to crossed nodes. In the real case one in addition has to take care that the new parabolic subalgebra of the complexification comes from the given real form, which can be read off the Satake diagram.

In any case, we get the two decompositions $\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{g} = \mathfrak{q}_- \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_+$. Since $\Sigma \subset \tilde{\Sigma}$ we conclude that $\mathfrak{p}_{\pm} \subset \mathfrak{q}_{\pm}$, $\mathfrak{q}_0 \subset \mathfrak{p}_0$, and that $\mathfrak{q}_- = \mathfrak{p}_- \oplus (\mathfrak{p} \cap \mathfrak{q}_-)$ and $\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{q}_-) \oplus \mathfrak{q}$. Finally note that $\mathfrak{p} \cap \mathfrak{q}_- = \mathfrak{p}_0 \cap \mathfrak{q}_-$.

The first crucial observation is that the codifferentials $\partial_{\mathfrak{p}}^*$ and $\partial_{\mathfrak{q}}^*$ corresponding to the two parabolics are compatible. The inclusion $\mathfrak{p}_+ \hookrightarrow \mathfrak{q}_+$ induces an inclusion

$$j : L(\Lambda^k \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \cong \Lambda^k \mathfrak{p}_+ \otimes \mathfrak{g} \rightarrow \Lambda^k \mathfrak{q}_+ \otimes \mathfrak{g} \cong L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathfrak{g}).$$

Since the standard differentials for Lie algebra homology of \mathfrak{p}_+ and \mathfrak{q}_+ are both restrictions of the standard differential for Lie algebra homology of \mathfrak{g} we conclude

Proposition. *For the natural inclusion $j : L(\Lambda^k \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \rightarrow L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$ we have $\partial_{\mathfrak{q}}^* \circ j = j \circ \partial_{\mathfrak{p}}^*$.*

2.4. Let G be a Lie group with Lie algebra \mathfrak{g} and let $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ be parabolic subalgebras as before. The explicit descriptions of \mathfrak{p} and \mathfrak{q} , the corresponding gradings of \mathfrak{g} in terms of root spaces, and the definition of the subgroups

$P, Q \subset G$ imply that Q is a closed subgroup of P , and $P_+ \subset Q_+$. It can be shown that P/Q is the quotient of the semisimple part of P_0 by some parabolic subgroup and thus a generalized flag manifold, see section 2.4 of [4].

Now suppose that $(p : \mathcal{G} \rightarrow N, \omega)$ is a parabolic geometry of type (G, P) . Since \mathcal{G} is a principal P -bundle, we may restrict the principal action to the closed subgroup $Q \subset P$, which still acts freely on \mathcal{G} . Thus, the orbit space $\mathcal{CN} := \mathcal{G}/Q$ is a smooth manifold.

Definition. The *correspondence space* \mathcal{CN} of the parabolic geometry $(p : \mathcal{G} \rightarrow N, \omega)$ of type (G, P) is the orbit space \mathcal{G}/Q .

Proposition. *The correspondence space \mathcal{CN} is the total space of a natural fiber bundle over N with fiber the generalized flag manifold P/Q . It carries a natural parabolic geometry of type (G, Q) . The curvature function $\kappa^{\mathcal{CN}}$ of this geometry is given by $j \circ \kappa^N$, where κ^N denotes the curvature function of N and $j : L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g}) \rightarrow L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$ is the natural inclusion. In particular:*

- (1) $\kappa^{\mathcal{CN}}(X, Y) = 0$ if $X \in \mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$.
- (2) *If one starts from a normal parabolic geometry on N , then the induced parabolic geometry on \mathcal{CN} is normal, too.*

Proof. The orbit space $\mathcal{CN} = \mathcal{G}/Q$ is naturally isomorphic to the associated bundle $\mathcal{G} \times_P P/Q$, so it is the total space of a natural fiber bundle with fiber P/Q . The natural projection $\pi : \mathcal{G} \rightarrow \mathcal{G}/Q$ which maps any point to its orbit is a principal Q -bundle. The Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ defining the parabolic geometry on N is equivariant for the action of the group $P \supset Q$ and reproduces the generators in $\mathfrak{p} \supset \mathfrak{q}$ of fundamental vector fields. Hence it is also a Cartan connection on the principal Q -bundle $\mathcal{G} \rightarrow \mathcal{CN}$ and we have obtained a natural parabolic geometry of type (G, Q) on \mathcal{CN} .

Since the parabolic geometries on N and \mathcal{CN} are given by the same Cartan connection, $\kappa^{\mathcal{CN}}$ and κ^N coincide as functions with values in $L(\Lambda^2 \mathfrak{g}, \mathfrak{g})$ so we clearly get $\kappa^{\mathcal{CN}} = j \circ \kappa^N$ as a function with values in $L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$. From this, (1) is obvious, while (2) follows from Proposition 2.3. \square

Remark. For parabolic geometries $(p : \mathcal{G} \rightarrow N, \omega)$ and $(p' : \mathcal{G}' \rightarrow N', \omega')$, a morphism by definition is a homomorphism $\Phi : \mathcal{G} \rightarrow \mathcal{G}'$ of principal P -bundles such that $\Phi^* \omega' = \omega$. Of course, this implies that Φ is equivariant for the action of the subgroup $Q \subset P$, so it also defines a morphism of parabolic geometries of type (G, Q) from $(\mathcal{G} \rightarrow \mathcal{CN}, \omega)$ to $(\mathcal{G}' \rightarrow \mathcal{CN}', \omega')$. Thus forming the correspondence space is a functorial construction.

Conversely, a morphism between the two correspondence spaces is a Q -equivariant map $\Phi : \mathcal{G} \rightarrow \mathcal{G}'$ such that $\Phi^* \omega' = \omega$. To understand the behavior of Φ with respect to the action of the group $P \supset Q$, observe that ω and ω' both reproduce the generators of fundamental vector fields corresponding to the P -action. From this one easily concludes that Φ is equivariant for the action of elements of the form $\exp(A)$ for $A \in \mathfrak{p}$. If P/Q is connected, then these elements together with Q generate all of P , so Φ is automatically P -equivariant. In particular we see that if P/Q is connected, then isomorphism of the correspondence spaces \mathcal{CN} and \mathcal{CN}' implies isomorphism of N and N' , and in particular for any N the automorphism groups of the geometries on N and \mathcal{CN} coincide.

2.5. The first step towards a converse of the construction of the correspondence space is to observe that the subalgebra \mathfrak{p} gives rise to a distribution on the base space of any parabolic geometry $(\pi : \mathcal{G} \rightarrow M, \omega)$ of type (G, Q) . Since ω is a Cartan connection on \mathcal{G} , the tangent bundle of M is the associated bundle $\mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$. Since $Q \subset P$, the subspace $\mathfrak{p} \subset \mathfrak{g}$ is Q -invariant, and $\mathfrak{q} \subset \mathfrak{p}$. Thus, $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is a Q -invariant subspace which gives rise to a smooth subbundle $E := \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q})$ of TM , i.e. a smooth distribution of constant rank on M . Explicitly, the subspace $E_x \subset T_x M$ can be described as the space of those tangent vectors such that for one (or equivalently any) point $u \in \mathcal{G}$ such that $\pi(u) = x$ and one (or equivalently any) lift $\tilde{\xi}$ of ξ to $T_u \mathcal{G}$ we have $\omega(\tilde{\xi}) \in \mathfrak{p}$. From the construction of the correspondence space it follows immediately that for $M = \mathcal{C}N$ the distribution E is exactly the vertical subbundle with respect to the projection $\pi : \mathcal{C}N \rightarrow N$, so this distribution is globally integrable with leaf-space N . Integrability of the distribution E is a rather weak condition in general:

Proposition. *Let $(\pi : \mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, Q) with curvature function κ and let $E \subset TM$ be the smooth distribution corresponding to the subalgebra $\mathfrak{p} \supset \mathfrak{q}$. Then the distribution E is integrable if and only if for all $u \in \mathcal{G}$ and $X, Y \in \mathfrak{p}/\mathfrak{q}$, we have $\kappa(u)(X, Y) \in \mathfrak{p} \subset \mathfrak{g}$.*

Proof. Locally, any vector field on M can be lifted to a vector field on \mathcal{G} . Sections of the subbundle $E \subset TM$ correspond exactly to sections of the subbundle $\omega^{-1}(\mathfrak{p}) \subset T\mathcal{G}$. Since the Lie bracket of two lifts is a lift of the Lie bracket of the original fields, we see that E is integrable provided that the space of sections of $\omega^{-1}(\mathfrak{p}) \subset T\mathcal{G}$ is closed under the Lie bracket. By definition of the exterior derivative, this is equivalent to $d\omega(\xi, \eta) \in \mathfrak{p}$ for all $\xi, \eta \in \Gamma(\omega^{-1}(\mathfrak{p}))$. By definition of the curvature function,

$$d\omega(\xi, \eta) = \kappa(\omega(\xi), \omega(\eta)) - [\omega(\xi), \omega(\eta)].$$

Since $\mathfrak{p} \subset \mathfrak{g}$ is a Lie subalgebra, we see that $\kappa(X, Y) \in \mathfrak{p}$ for all $X, Y \in \mathfrak{p}/\mathfrak{q}$ implies integrability of E .

Conversely, if we find $X, Y \in \mathfrak{p}/\mathfrak{q}$ and a point $u \in \mathcal{G}$ such that $\kappa(u)(X, Y) \notin \mathfrak{p}$, then choose representatives $\tilde{X}, \tilde{Y} \in \mathfrak{p}$ for X and Y . The tangent vectors $\omega^{-1}(\tilde{X})(u)$ and $\omega^{-1}(\tilde{Y})(u)$ have nonzero projections to TM . Extending them to projectable sections of the subbundle $\omega^{-1}(\mathfrak{p})$, we obtain two sections whose Lie bracket does not lie in $\omega^{-1}(\mathfrak{p})$. Their projections on M are sections of E , whose Lie bracket cannot be contained in E . \square

2.6. Once the distribution $E \subset TM$ from 2.5 is integrable, we can locally define a *twistor space* for M as a local leaf space for the corresponding foliation, i.e. a smooth manifold N together with a surjective submersion ψ from an open subset U of M onto N such that $\ker(T_x \psi) = E_x$ for all $x \in U$. The existence of local leaf spaces for integrable distributions immediately follows from the local version of the Frobenius theorem (see e.g. [14], theorem 3.22) by projecting on one factor of an adapted chart. One easily shows that for two local leaf spaces $\psi : U \rightarrow N$ and $\psi' : U' \rightarrow N'$ there is a unique diffeomorphism $\varphi : \psi(U \cap U') \rightarrow \psi'(U \cap U')$ such that $\varphi \circ \psi = \psi'$.

At this stage the twistor space is just a smooth manifold and only locally defined. However Penrose transforms may be used to interpret geometric

objects on the twistor space on the original manifold M , which makes the construction interesting, even without getting some local geometric structures on the twistor space.

Our aim is to find conditions on the curvature function of the geometry $(\pi : \mathcal{G} \rightarrow M, \omega)$ which enable us to define a parabolic geometry of type (G, P) on a sufficiently small local leaf space $\psi : U \rightarrow N$ such that U is isomorphic (as a parabolic geometry) to an open subset in the correspondence space \mathcal{CN} . This will be done in two steps, which require different conditions on κ . First we have to construct a diffeomorphism from an appropriate open subset of \mathcal{G} to an open subset in a principal P -bundle over N which satisfies a certain equivariancy condition. For $X \in \mathfrak{g}$ let us denote by $X^{\mathcal{G}} \in \mathfrak{X}(\mathcal{G})$ the vector field characterized by $\omega(X^{\mathcal{G}}) = X$.

Proposition. *Let $(\pi : \mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, Q) and let $E \subset TM$ be the subbundle corresponding to \mathfrak{p} as in 2.5. Suppose further that the curvature function κ has the property that $\kappa(u)(X, Y) = 0$ for all $u \in \mathcal{G}$ and $X, Y \in \mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$.*

Then for any sufficiently small local leaf space $\psi : U \rightarrow N$ of the foliation defined by E , there is a Q -equivariant diffeomorphism Φ from a Q -invariant open subset of the trivial principal bundle $N \times P$ to a Q -invariant open subset of \mathcal{G} such that $\psi \circ \pi \circ \Phi = \text{pr}_1 : N \times P \rightarrow N$ and such that for $A \in \mathfrak{p}$ and the fundamental vector field $\zeta_A \in \mathfrak{X}(N \times P)$ we get $T\Phi \circ \zeta_A = A^{\mathcal{G}} \circ \Phi$.

Proof. For $A, B \in \mathfrak{p} \subset \mathfrak{g}$ consider $A^{\mathcal{G}}, B^{\mathcal{G}} \in \mathfrak{X}(\mathcal{G})$ and the Lie bracket $[A^{\mathcal{G}}, B^{\mathcal{G}}]$. Then we compute

$$\omega([A^{\mathcal{G}}, B^{\mathcal{G}}]) = -d\omega(A^{\mathcal{G}}, B^{\mathcal{G}}) = [A, B] - \kappa(A + \mathfrak{q}, B + \mathfrak{q}),$$

so the assumption on κ implies that $[A^{\mathcal{G}}, B^{\mathcal{G}}] = [A, B]^{\mathcal{G}}$. Hence $A \mapsto A^{\mathcal{G}}$ restricts to a Lie algebra homomorphism $\mathfrak{p} \rightarrow \mathfrak{X}(\mathcal{G})$, i.e. an action of the Lie algebra \mathfrak{p} on \mathcal{G} . By Lie's second fundamental theorem, see pp. 47–49 and 58 of [18], this Lie algebra action integrates to a local group action. In particular, for any $u_0 \in \mathcal{G}$ there is an open neighborhood W of (u_0, e) in $\mathcal{G} \times P$ and a smooth map $F : W \rightarrow \mathcal{G}$ such that

- If $(u, e) \in W$, then $F(u, e) = u$ and $\frac{d}{dt}|_{t=0} F(u, \exp(tA)) = A^{\mathcal{G}}(u)$ for all $A \in \mathfrak{p}$.
- $F(F(u, g), h) = F(u, gh)$ provided that (u, g) , (u, gh) and $(F(u, g), h)$ all lie in W .

Now consider a local leaf space $\psi : U \rightarrow N$ which is so small that there is a smooth section $\sigma : N \rightarrow \mathcal{G}$ of $\psi \circ \pi : \pi^{-1}(U) \rightarrow N$. Possibly shrinking the leaf space further, we find an open neighborhood \tilde{V} of e in P such that for some set W as above we have $(\sigma(x), g) \in W$ and $(F(\sigma(x), g), e) \in W$ for all $x \in N$ and all $g \in \tilde{V}$. Then we define $\Phi : N \times \tilde{V} \rightarrow \mathcal{G}$ by $\Phi(x, g) := F(\sigma(x), g)$. For $x \in N$ the tangent map $T_{(x,e)}\Phi : T_x N \times \mathfrak{p} \rightarrow T_{\sigma(x)}\mathcal{G}$ is evidently given by $(\xi, A) \mapsto T_x \sigma \cdot \xi + A^{\mathcal{G}}(\sigma(x))$ so it is a linear isomorphism. Possibly shrinking U and \tilde{V} , we may assume that Φ is a diffeomorphism onto an open subset $\tilde{U} \subset \mathcal{G}$. We may further assume that $\tilde{V} = \{\exp(X)\exp(B) : X \in V_1, B \in V_2\}$ where V_1 is an open neighborhood of zero in $\mathfrak{p} \cap \mathfrak{q}_-$ such that $(X, h) \mapsto \exp(X)h$ is a diffeomorphism from $V_1 \times Q$ onto an open neighborhood V of Q in P and V_2 is a ball around zero in \mathfrak{q} .

For a fixed point $x \in N$, any vector tangent to $\{x\} \times \tilde{V}$ can be written as $\zeta_A(x, g) = \frac{d}{dt}|_{t=0}(x, g \exp(tA))$ for some $g \in \tilde{V}$ and $A \in \mathfrak{p}$. For sufficiently small t , we by construction have $\Phi(x, g \exp(tA)) = F(\Phi(x, g), \exp(tA))$, and thus $T_{(x,g)}\Phi$ maps this tangent vector to $A^{\mathcal{G}}(\Phi(x, g))$. Thus we see that $T\Phi \circ \zeta_A = A^{\mathcal{G}} \circ \Phi$ on $N \times \tilde{V}$. Moreover, $T\Phi \circ \zeta_A$ always lies in $\omega^{-1}(\mathfrak{p}) \subset T\mathcal{G}$, which implies that $\tilde{\Phi}(\{x\} \times \tilde{V})$ is contained in one leaf of the foliation corresponding to the integrable distribution $\omega^{-1}(\mathfrak{p}) \subset T\mathcal{G}$. From 2.5 we conclude that the map $\psi \circ \pi \circ \Phi$ is constant on $\{x\} \times \tilde{V}$, and since $\Phi(x, e) = \sigma(x)$ we conclude that $\psi \circ \pi \circ \Phi = \text{pr}_1 : N \times \tilde{V} \rightarrow N$.

For $X \in V_1$ and $B \in V_2$ we have $\exp(X)\exp(tB) \in \tilde{V}$ for all $t \in [0, 1]$. Since $B^{\mathcal{G}}$ is the fundamental vector field on \mathcal{G} generated by $B \in \mathfrak{q}$, the infinitesimal condition $T\Phi \circ \zeta_B = B^{\mathcal{G}} \circ \Phi$ immediately implies that $\Phi(x, \exp(X)\exp(B)) = \Phi(x, \exp(X)) \cdot \exp(B)$, where in the right hand side we use the principal right action on \mathcal{G} . Since Q acts freely both on $N \times V$ and on \mathcal{G} we can uniquely extend Φ to a Q -equivariant diffeomorphism from $N \times V$ to the Q -invariant open subset $\{u \cdot g : u \in \tilde{U}, g \in Q\} \subset \mathcal{G}$. Since the family of fundamental vector fields on $N \times P$ and the family of the vector fields $A^{\mathcal{G}}$ on \mathcal{G} have the same equivariancy property, this extension still satisfies $T\Phi \circ \zeta_A = A^{\mathcal{G}} \circ \Phi$ for all $A \in \mathfrak{p}$. \square

2.7. The second step in the construction is to use the diffeomorphism from Proposition 2.6 to carry over the Cartan connection to the principal bundle $N \times P$, which needs an additional condition on the curvature.

Theorem. *Let $(\pi : \mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, Q) whose curvature κ satisfies $\kappa(u)(X, Y) = 0$ for all $u \in \mathcal{G}$, $X \in \mathfrak{p}$ and all $Y \in \mathfrak{g}$. Let $\psi : U \rightarrow N$ be a sufficiently small local leaf space for the integrable distribution $E \subset TM$ corresponding to $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. Then:*

- (1) ω induces a Cartan connection $\tilde{\omega}$ on the trivial principal bundle $N \times P$, which is normal if $(\pi : \mathcal{G} \rightarrow M, \omega)$ is normal.
- (2) The parabolic geometry $(\pi^{-1}(U), \omega|_{\pi^{-1}(U)})$ is isomorphic to an open subset of the correspondence space $(N \times P \rightarrow N \times (P/Q), \tilde{\omega})$. If P/Q is connected, this condition determines the parabolic geometry on N up to isomorphism.

Proof. (1) By Proposition 2.6 there is a open neighborhood V of e in P and a diffeomorphism Φ from $N \times V$ onto an open subset of \mathcal{G} such that $\psi \circ \pi \circ \Phi = \text{pr}_1 : N \times V \rightarrow N$ and such that $T\Phi \circ \zeta_A = A^{\mathcal{G}} \circ \Phi$ for all $A \in \mathfrak{p}$. Hence we can form $\Phi^*\omega \in \Omega^1(N \times V, \mathfrak{g})$, which restricts to a linear isomorphism on each tangent space. Let us denote by ρ the principal right action of P on $N \times P$. We can extend the values of this form in $N \times \{e\}$ equivariantly to a \mathfrak{g} -valued one-form $\tilde{\omega}$ on $N \times P$ by putting

$$\tilde{\omega}(x, g) := \text{Ad}(g^{-1}) \circ (\Phi^*\omega)(x, e) \circ T\rho^{g^{-1}}.$$

By construction, this form restricts to an isomorphism on each tangent space and satisfies $(\rho^g)^*\tilde{\omega} = \text{Ad}(g)^{-1} \circ \tilde{\omega}$ for all $g \in P$. Since the vector fields ζ_A and $A^{\mathcal{G}}$ are Φ -related, their flows are Φ related. Thus $\Phi \circ \rho^{\exp(tA)} = \text{Fl}_t^{A^{\mathcal{G}}} \circ \Phi$, whenever the left hand side is defined. The curvature condition $\kappa(A, Y) = 0$ for all $Y \in \mathfrak{g}$ and $A \in \mathfrak{p}$ reads as $-\text{ad}(A) \circ \omega = i_{A^{\mathcal{G}}}d\omega = \mathcal{L}_{A^{\mathcal{G}}}\omega$, where

in the last step we have used that $\omega(A^{\mathcal{G}})$ is constant. This infinitesimal equivariancy condition easily implies local equivariancy, i.e. that $(\text{Fl}_t^{A^{\mathcal{G}}})^*\omega = \text{Ad}(\exp(-tA)) \circ \omega$, whenever the flow is defined. Hence for $A \in \mathfrak{p}$ such that $\exp(tA) \in V$ for all $t \in [0, 1]$ we have $(\rho^{\exp(A)})^*\Phi^*\omega(x, e) = \text{Ad}(\exp(A)^{-1}) \circ \omega(x, e)$ for all $x \in N$. This shows that $\tilde{\omega}$ coincides with $\Phi^*\omega$ on an open neighborhood of N in $N \times P$, so $\tilde{\omega}$ is smooth on this neighborhood and hence by equivariancy on all of $N \times P$.

By construction, $\Phi^*\omega$ reproduces the generators of fundamental vector fields. Hence $\tilde{\omega}$ has the same property in points of the form (x, e) with $x \in N$, and thus by equivariancy in all points of $N \times P$. We have therefore verified that $\tilde{\omega} \in \Omega^1(N \times P, \mathfrak{g})$ is a Cartan connection, and thus defines a parabolic geometry of type (G, P) on N . Concerning normality, we have observed above that $\tilde{\omega}$ coincides with $\Phi^*\omega$ on an open neighborhood of $N \times \{e\}$. On this neighborhood, the curvature function $\tilde{\kappa}$ of $\tilde{\omega}$ is given by $j \circ \tilde{\kappa} = \kappa \circ \Phi$, where j is the inclusion from 2.3. Now the claim on normality follows from Proposition 2.3, equivariancy of $\tilde{\kappa}$, and the fact that $\partial_{\mathfrak{p}}^*$ is P -equivariant.

(2) Take the map $\Phi : N \times V \rightarrow \mathcal{G}$ from (1). By Proposition 2.6 we may assume $N \times V$ to be a Q -invariant subset of $N \times P$ and Φ to be a Q -equivariant diffeomorphism onto a Q -invariant open subset $\tilde{U} \subset \mathcal{G}$. From (1) we know that $\Phi^*\omega = \tilde{\omega}$ on an open subset of N , which has to be Q -invariant since both $\Phi^*\omega$ and $\tilde{\omega}$ are Q -equivariant. But this exactly means that Φ defines an isomorphism of parabolic geometries of type (G, Q) from an open subset in $\mathcal{CN} = N \times V/Q$ to the open subset $\pi(\tilde{U}) \subset M$. Finally, we have seen in Remark 2.4 that isomorphism of the correspondence spaces implies isomorphism of the underlying spaces provided that P/Q is connected. \square

Corollary. *Suppose that P/Q is connected and let $(\mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, Q) satisfying the curvature restriction of the theorem. Suppose that $\psi : U \rightarrow N$ is any local leaf space for the foliation corresponding to E . Then N carries a canonical parabolic geometry of type (G, P) , which is normal if (\mathcal{G}, ω) is normal and such that U is isomorphic to an open subset of \mathcal{CN} as a parabolic geometry.*

Proof. By part (1) of the theorem we get appropriate parabolic geometries on sufficiently small open subsets of N . By part (2) these locally defined structures fit together to define a principal bundle and a Cartan connection on N . Also the isomorphisms between open subsets of the correspondence spaces and appropriate subsets of U piece together smoothly by part (2) and Remark 2.4. \square

Remarks. (1) For many structures, there are intermediate curvature conditions lying between the one in Proposition 2.5, which ensures existence of a twistor space, and the one in the theorem above, which ensures existence of a parabolic geometry on the twistor space. In some cases, these conditions imply the existence of geometric structures on the twistor space which are weaker than a parabolic geometry, see 4.6 for an example.

(2) To get the classical forms of the twistor correspondence, one has to combine the constructions of correspondence and twistor spaces. One starts with two parabolic subalgebras $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathfrak{g}$ which contain the same Borel subalgebra. Then $\mathfrak{q} = \mathfrak{p}_1 \cap \mathfrak{p}_2$ is a parabolic subalgebra, too. Starting from

a parabolic geometry of type (G, P_1) on a manifold M one forms the correspondence space $\mathcal{C}M$, which carries a parabolic geometry of type (G, Q) . The parabolic $\mathfrak{p}_2 \supset \mathfrak{q}$ gives rise to a distribution E on $\mathcal{C}M$. Since the curvature function of $\mathcal{C}M$ coincides with the curvature function of M , the condition for integrability of this distribution from Proposition 2.5 can be expressed in terms of the curvature of M (and with the help of the results in the next section even in terms of the harmonic curvature). Given the corresponding curvature restrictions, one can form local twistor spaces. A priori, they are only defined locally on $\mathcal{C}M$, but using the fact that the fibers P_1/Q of $\mathcal{C}M \rightarrow M$ are compact, one obtains a full correspondence over sufficiently small open subsets of M . The fibers of $\mathcal{C}M \rightarrow M$ then descend to submanifolds in such twistor space, and the global structure of these submanifolds encodes the local geometry of M . On the other hand, under further restrictions on the curvature, the local twistor spaces inherit local geometric structures. Again, this will be discussed in the examples in section 4.

3. Reduction to harmonic curvature components

The curvature of a Cartan connection is a rather complicated object, and one needs the full Cartan bundle to interpret it geometrically. As we have noted in 2.2 in the case of regular normal parabolic geometries there is the harmonic curvature, which can be directly interpreted in terms of the underlying structure determining the parabolic geometry. As noted in 2.2 the harmonic curvature is still a complete obstruction to local flatness in the regular normal case. Our aim in this section is to express (in the regular normal case) the curvature condition from 2.5–2.7 equivalently in terms of the harmonic curvature, which leads to much more effective conditions.

The key to this is that one may recover the full curvature from the harmonic curvature by applying an invariant differential operator, which was first shown by D. Calderbank and T. Diemer in [5]. The authors suggested to use this as a strengthening of the Bianchi identity. The relation between curvature and harmonic curvature follows from the theory of Bernstein–Gelfand–Gelfand (BGG) sequences. The main result of this section is Theorem 3.2, which partly works in the realm of general BGG sequences.

Throughout this section, we fix two parabolic subgroups $Q \subset P$ of a semisimple Lie group G corresponding to subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ as in 2.3.

3.1. BGG sequences. Any representation of the Lie group Q gives rise to a natural vector bundle on the category of parabolic geometries of type (G, Q) by forming associated bundles to the principal Cartan bundles. The first step towards BGG sequences is the observation that restrictions of representations of G play a special role. If \mathbb{V} is representation of G , then we can also view it as a representation of $Q \subset G$, and the corresponding natural vector bundle is called a *tractor bundle*. The main feature of these bundles is that for each parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$, the Cartan connection ω induces a linear connection on the tractor bundle $VM := \mathcal{G} \times_Q \mathbb{V}$, see [6]. To deal with the curvature, we have to consider $\mathbb{V} = \mathfrak{g}$, and the corresponding tractor bundle $\mathcal{A}M := \mathcal{G} \times_Q \mathfrak{g}$ is called the *adjoint tractor bundle*.

There are two ways to extend the natural linear connection on a tractor bundle VM to an operator on VM -valued differential forms, see section 2 of [11]. First, there is the usual *covariant exterior derivative*, which we denote by $d^\omega : \Omega^k(M, VM) \rightarrow \Omega^{k+1}(M, VM)$. Second, one may start from the covariant exterior derivative on the homogeneous model G/Q , and extend this in a different way to all parabolic geometries of type (G, Q) . This leads to the *twisted exterior derivative* $d_{\mathbb{V}}$. The relation between these two operators can be easily described explicitly. The curvature function κ has values in $L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$. Projecting in the last factor to $\mathfrak{g}/\mathfrak{q}$ we obtain a Q -equivariant map $\mathcal{G} \rightarrow L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g}/\mathfrak{q})$, which corresponds to a section κ_- of the associated bundle $\Lambda^2 T^*M \otimes TM$. This is exactly the torsion of the Cartan connection ω . This leads to an insertion operator i_κ , i.e. for $\varphi \in \Omega^k(M, VM)$, the form $i_\kappa \varphi \in \Omega^{k+1}(M, VM)$ is the alternation of $(\xi_0, \dots, \xi_k) \mapsto \varphi(\kappa_-(\xi_0, \xi_1), \xi_2, \dots, \xi_k)$. Then $d^\omega \varphi = d_{\mathbb{V}} \varphi + i_\kappa \varphi$, so in particular, the two operators coincide for torsion free parabolic geometries. To unify the presentation, let us write d for either d^ω or $d_{\mathbb{V}}$ in the sequel.

The second step is to compress either of these sequences of first order operators defined on VM -valued differential forms to a sequence of higher order operators defined on smaller bundles. The bundle $\Lambda^k T^*M \otimes VM$ corresponds to the representation $L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V})$ of Q . Similarly as in 2.3, via the duality $(\mathfrak{g}/\mathfrak{q})^* \cong \mathfrak{q}_+$ the differentials in the standard complex computing Lie algebra homology of \mathfrak{q}_+ with coefficients in \mathbb{V} define Q -homomorphisms $\partial^* : L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V}) \rightarrow L(\Lambda^{k-1} \mathfrak{g}/\mathfrak{q}, \mathbb{V})$ such that $\partial^* \circ \partial^* = 0$. The quotient $\ker(\partial^*)/\text{im}(\partial^*)$ is a representation of Q , which we denote by $\mathbb{H}_{\mathbb{V}}^k$. It turns out that $Q_+ \subset Q$ acts trivially on this quotient, so this representation is completely reducible and determined by the Q_0 -action.

The Q -homomorphisms ∂^* induce vector bundle maps $\Lambda^k T^*M \otimes VM \rightarrow \Lambda^{k-1} T^*M \otimes VM$ and correspondingly algebraic operators on VM -valued forms, which we all denote by the same symbol. The kernel and image of ∂^* are natural subbundles in $\Lambda^k T^*M \otimes VM$ and their quotient can be naturally identified with $H_{\mathbb{V}}^k M := \mathcal{G} \times_Q \mathbb{H}_{\mathbb{V}}^k$. Since Q_+ acts trivially on $\mathbb{H}_{\mathbb{V}}^k$ we can also view $H_{\mathbb{V}}^k M$ as $\mathcal{G}_0 \times_{Q_0} \mathbb{H}_{\mathbb{V}}^k$, so this admits a direct interpretation in terms of underlying structures.

The key step for the construction of BGG sequence is to construct an invariant differential operator $S : \Gamma(H_{\mathbb{V}}^k M) \rightarrow \Omega^k(M, VM)$, called the splitting operator, such that for all $s \in \Gamma(H_{\mathbb{V}}^k M)$ we have

- $\partial^*(S(s)) = 0$ and $\pi_H(S(s)) = s$.
- $\partial^*(dS(s)) = 0$.

Here π_H denotes the natural algebraic projection from sections of $\ker(\partial^*) \subset \Lambda^k T^*M \otimes VM$ to sections of $H_{\mathbb{V}}^k M$.

These operators were first constructed in [11] in terms of homomorphisms of semi-holonomic jet modules. In [5] the authors gave a simpler construction and extended them to operators defined on $\Omega^k(M, VM)$. We will sketch this construction next. The operator $\square^R := d \circ \partial^* + \partial^* \circ d$ is an invariant operator on $\Omega^k(M, VM)$ for each k . Since $\partial^* \circ \partial^* = 0$, this operator preserves the subspace of sections of the bundle $\text{im}(\partial^*)$. One shows that the restriction of \square^R to sections of this subbundle is invertible and

the inverse is again a differential operator, which then is natural by construction. Let $L : \Omega^k(M, VM) \rightarrow \Omega^{k-1}(M, VM)$ be the composition of this inverse with ∂^* . Given a section $\sigma \in \Gamma(H_{\mathbb{V}}^k M)$ one chooses a section φ of $\ker(\partial^*) \subset \Lambda^k T^*M \otimes VM$ such that $\pi_H(\varphi) = \sigma$ and shows that $S(\sigma) := \varphi - Ld\varphi$ is independent of the choice of φ and this recovers the splitting operator S . From this point of view it is also easy to see that S is uniquely determined by the two properties listed above. Suppose that $\varphi \in \Omega^k(M, VM)$ is such that $\partial^*(\varphi) = 0$ and $\partial^*d\varphi = 0$. Then one easily shows that $\varphi - S(\pi_H(\varphi))$ must be a section of $\text{im}(\partial^*)$ which lies in the kernel of \square^R , and thus vanishes identically.

In particular, we may apply this to the special case $\mathbb{V} = \mathfrak{g}$, $k = 2$ and the curvature κ . Equivariance of the curvature function implies that κ can be viewed as an element of $\Omega^2(M, \mathcal{A}M)$. Normality of the parabolic geometry exactly means $\partial^*\kappa = 0$. Moreover, κ coincides with the curvature of the natural linear connection on $\mathcal{A}M$, so the Bianchi identity for linear connections implies $d^\omega(\kappa) = 0$. In particular, $\partial^*(d^\omega(\kappa)) = 0$, which implies that $\kappa = S(\kappa_H)$, where S is the splitting operator for $d = d^\omega$.

One of the tricky points of the theory is that there seems to be no way to write down the operator L (or S) in a manifestly invariant way. To get a formula, one first has to choose a Weyl structure σ , see 2.2. This reduces the structure group of all bundles in question to Q_0 . Now the Q_0 -submodule $\mathfrak{q}_- \subset \mathfrak{g}$ is complementary to \mathfrak{q} , so $\mathfrak{g}/\mathfrak{q} \cong \mathfrak{q}_-$ as a Q_0 -module. Consequently, on the level of Q_0 , we can view the spaces $L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V})$ as the chain groups in the standard complex computing the Lie algebra cohomology $H^*(\mathfrak{q}_-, \mathbb{V})$. The differential ∂ in this complex is a Q_0 -homomorphism (but not a Q -homomorphism). Then $\square := \partial \circ \partial^* + \partial^* \circ \partial$ is a Q_0 -homomorphism and thus, after a choice of a Weyl structure, gives rise to a bundle map on each $\Lambda^k T^*M \otimes VM$. It turns out ∂ and ∂^* are adjoint with respect to a certain inner product. The resulting algebraic Hodge decomposition shows that as a Q_0 -representation we have $\mathbb{H}_{\mathbb{V}}^k \cong \ker(\square) \subset L(\Lambda^k \mathfrak{q}_-, \mathbb{V})$, and this representation is computable via Kostant's version of the Bott–Borel–Weil theorem, see [15]. The Hodge decomposition also implies that \square is invertible on $\text{im}(\partial^*)$, and we can finally state the formula:

$$L = \left(\sum_{i=0}^{\infty} (-1)^i (\square^{-1} \square^R - \text{id})^i \right) \square^{-1} \partial^*.$$

The sum in this formula is actually finite since $\square^{-1} \square^R - \text{id}$ increases homogeneous degrees with respect to a natural grading on \mathbb{V} .

3.2. We next have to study algebraic properties of the operators L and S . Consider a Q -submodule $\mathbb{E} \subset \ker(\partial^*) \subset L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V})$. Then we get a Q_0 -submodule $\mathbb{E}_0 := \mathbb{E} \cap \ker(\square) \subset \mathbb{H}_{\mathbb{V}}^k$ by identifying $L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V})$ with $L(\Lambda^k \mathfrak{q}_-, \mathbb{V})$ as above. We want to find conditions which make sure that the splitting operator S maps sections of $\mathcal{G}_0 \times_{Q_0} \mathbb{E}_0$ to sections of $\mathcal{G} \times_Q \mathbb{E}$. For the operator corresponding to $d = d_{\mathbb{V}}$ this is rather easy and has been discussed in [11], but for $d = d^\omega$, the problem is much more subtle.

Definition. Let $\mathbb{E} \subset L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V})$ and $\mathbb{F} \subset L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$ be Q -submodules. We say that \mathbb{E} is *stable under \mathbb{F} -insertions* if for $\varphi \in \mathbb{E}$ and $\psi \in \mathbb{F}$ we

have $\partial^*(i_\psi\varphi) \in \mathbb{E}$, where $i_\psi\varphi$ is the alternation of the map $(X_0, \dots, X_k) \mapsto \varphi(\psi(X_0, X_1) + \mathfrak{q}, X_2, \dots, X_k)$ for $X_i \in \mathfrak{g}/\mathfrak{q}$.

Theorem. *Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a regular normal parabolic geometry, with curvature κ , and harmonic curvature κ_H . Let \mathbb{V} be a representation of G , and let $\mathbb{E} \subset \ker(\partial^*) \subset L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V})$ and $\mathbb{F} \subset \ker(\partial^*) \subset L(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ be Q -submodules. Put $\mathbb{E}_0 := \mathbb{E} \cap \ker(\square) \subset \mathbb{H}_{\mathbb{V}}^k$ and $\mathbb{F}_0 := \mathbb{E} \cap \ker(\square) \subset \mathbb{H}_{\mathfrak{g}}^2$. Let $E_0M = \mathcal{G}_0 \times_{Q_0} \mathbb{E}_0 \subset H_{\mathbb{V}}^k M$ and $EM = \mathcal{G} \times_Q \mathbb{E} \subset \Lambda^k T^*M \otimes VM$ be the corresponding subbundles, and similarly for F_0M and FM .*

(1) *The splitting operator $S : \Gamma(H_{\mathbb{V}}^k M) \rightarrow \Omega^k(M, VM)$ for $d = d_{\mathbb{V}}$ maps sections of E_0M to sections of EM .*

(2) *Suppose that $\square(\mathbb{E}) \subset \mathbb{E}$, \mathbb{E} is stable under \mathbb{F} insertions and κ has values in FM . Then the splitting operator for $d = d^\omega$ maps sections of E_0M to sections of EM .*

(3) *If $\square(\mathbb{F}) \subset \mathbb{F}$, \mathbb{F} is stable under \mathbb{F} insertions and κ_H is a section of F_0M , then κ is a section of FM .*

Proof. We first claim that if $\square(\mathbb{E}) \subset \mathbb{E}$, then $\partial^* \circ d_{\mathbb{V}}$ maps sections of EM to sections of EM : Let us compute in jet-modules as it is done in [11]. So consider the first jet prolongation $\mathcal{J}^1(\Lambda^k(\mathfrak{g}/\mathfrak{q})^* \otimes \mathbb{V})$, which as a Q_0 -module is isomorphic to $(\Lambda^k(\mathfrak{q}_-)^* \otimes \mathbb{V}) \oplus (\mathfrak{q}_-^* \otimes \Lambda^k(\mathfrak{q}_-)^* \otimes \mathbb{V})$. By lemma 2.1 of [11] $d_{\mathbb{V}}$ is induced by the homomorphism $\mathcal{J}^1(\Lambda^k(\mathfrak{q}_-)^* \otimes \mathbb{V}) \rightarrow \Lambda^{k+1}(\mathfrak{q}_-)^* \otimes \mathbb{V}$, which is given by $(e, Z \otimes f) \mapsto \partial(e) + (n+1)Z \wedge f$ for $e, f \in \Lambda^k(\mathfrak{q}_-)^* \otimes \mathbb{V}$ and $Z \in \mathfrak{q}_-^*$. Thus, $\partial^* \circ d_{\mathbb{V}}$ corresponds to the homomorphism $(e, Z \otimes f) \mapsto \partial^*\partial(e) + (n+1)\partial^*(Z \wedge f)$. Since $\mathbb{E} \subset \ker(\partial^*)$ the first summand coincides with $\square(e)$ and by formula (1.2) of [11] the second summand gives $-(n+1)Z \cdot f$, so the claim follows.

(1) Let $\tilde{\mathbb{E}}$ be the P -submodule of $L(\Lambda^k \mathfrak{g}/\mathfrak{q}, \mathbb{V})$ generated by \mathbb{E}_0 , and let $\tilde{E}M$ be the corresponding bundle, so $\tilde{E}M \subset EM$. By claim 2.2 of [11] we get $\square(\tilde{\mathbb{E}}) \subset \tilde{\mathbb{E}}$, so by the above claim $\partial^* \circ d_{\mathbb{V}}$ maps sections of E_0M to sections of $\tilde{E}M$. By definition

$$L \circ d_{\mathbb{V}} = \left(\sum_{i=0}^{\infty} (-1)^i (\square^{-1} \square^R - \text{id})^i \right) \square^{-1} \partial^* d_{\mathbb{V}}.$$

Since \square preserves $\tilde{\mathbb{E}}$, the corresponding algebraic operator preserves sections of $\tilde{E}M$ for any choice of Weyl structure. Hence the operator \square^{-1} preserves sections of the subbundle $\text{im}(\partial^*) \cap \tilde{E}M$. Finally, on the image of ∂^* the operator \square^R by definition coincides with $\partial^* \circ d_{\mathbb{V}}$, so it preserves sections of $\text{im}(\partial^*) \cap \tilde{E}M$. Hence the splitting operator maps sections of E_0M to sections of $\tilde{E}M \subset EM$.

(2) Since \mathbb{E} is stable under \mathbb{F} insertions and κ has values in FM we see that $\partial^* \circ i_\kappa$ maps sections of EM to sections of EM . Since $\square(\mathbb{E}) \subset \mathbb{E}$, this together with the claim implies that $\partial^* \circ d^\omega$ maps sections of EM to sections of EM . Now the result follows exactly as in (1).

(3) Let us denote by L the operator corresponding to d^ω . Further, put $\square^R = \partial^* \circ d_{\mathbb{V}} + d_{\mathbb{V}} \circ \partial^*$ and $\tilde{\square}^R = \partial^* \circ d^\omega + d^\omega \circ \partial^*$. Since $d^\omega = d_{\mathbb{V}} + i_\kappa$, we conclude that on the image of ∂^* , we have

$$(*) \quad \square^{-1} \tilde{\square}^R - \text{id} = \square^{-1} \square^R - \text{id} + \square^{-1} \partial^* i_\kappa.$$

From 3.1 we know that $\kappa = \kappa_H - L(d^\omega \kappa_H)$. Since we deal with a regular parabolic geometry, all nonzero homogeneous components of κ have degree bounded from below by some $\ell > 0$, and by the Bianchi identity (see [7], 4.9) the lowest nonzero homogeneous component of κ is harmonic. Hence κ is congruent to $\kappa_H \in \Gamma(FM)$ modulo elements of homogeneous degree $\geq \ell + 1$. Hence $\partial^*(i_\kappa \kappa_H)$ is congruent modulo elements homogeneous of degree $\geq 2\ell + 1$ to $\partial^*(i_{\kappa_H} \kappa_H)$, and the latter element lies in $\Gamma(FM)$, since \mathbb{F} is stable under \mathbb{F} -insertions. Hence we conclude that $\partial^* d^\omega \kappa_H$ is congruent to a section of FM modulo elements of homogeneous degree $\geq 2\ell + 1$. From the definition of L and formula (*) above, we then conclude that $\kappa = \kappa_H - L(d^\omega \kappa_H)$ is congruent to a section of FM modulo elements of that homogeneity.

As above, this implies that $\partial^*(i_\kappa \kappa_H)$ is congruent to a section of FM modulo elements homogeneous of degree $\geq 3\ell + 1$. Hence $\partial^* d^\omega \kappa_H$ and therefore κ are congruent to sections of FM modulo elements of that homogeneity, and iterating this argument, the result follows. \square

Corollary. *Let $\mathbb{E} \subset \ker(\partial^*) \subset L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$ be a Q -submodule and put $\mathbb{E}_0 := \mathbb{E} \cap \ker(\square) \subset \mathbb{H}_{\mathfrak{g}}^2$. Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a regular normal parabolic geometry such that the harmonic curvature κ_H has values in \mathbb{E}_0 . If either (\mathcal{G}, ω) is torsion free or $\square(\mathbb{E}) \subset \mathbb{E}$ and \mathbb{E} is stable under \mathbb{E} -insertions, then the curvature function κ has values in \mathbb{E} .*

3.3. Suppose that G is a semisimple Lie group, $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ are two parabolic subalgebras as in 2.3 and $Q \subset P \subset G$ are the corresponding subgroups. Consider a regular normal parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$ of type (G, Q) . Let κ be the curvature function, κ_H the harmonic curvature and κ_- the torsion, which we consider as a section of the bundle $L(\Lambda^2 TM, TM)$. Let $E \subset TM$ be the distribution corresponding to $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. Then we have:

Theorem. (1) *The distribution E is integrable if and only if $\kappa_-(\xi, \eta) \in E$ for all $\xi, \eta \in E$.*

(2) *The parabolic geometry (\mathcal{G}, ω) satisfies the curvature condition of Theorem 2.7 if and only if κ_H has values in the space of those maps which vanish if one of their entries is from $\mathfrak{p} \cap \mathfrak{q}_-$.*

Proof. (1) is only a straightforward reformulation of Proposition 2.5.

(2) Consider the submodule $\mathbb{E} \subset L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$ of those maps which vanish if one of their entries lies in $\mathfrak{p}/\mathfrak{q}$. To prove that κ has values in \mathbb{E} , by Corollary 3.2 we have to show that $\square(\mathbb{E}) \subset \mathbb{E}$ and \mathbb{E} is stable under \mathbb{E} -insertions. By definition \mathbb{E} is exactly the image of the natural inclusion

$$j : L(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g}) \cong \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{q}_+ \otimes \mathfrak{g} \cong L(\Lambda^2 \mathfrak{g}/\mathfrak{q}, \mathfrak{g})$$

from 2.3. By definition of the insertion operator, for $j(\varphi), j(\psi) \in \mathbb{E}$, we get $i_{j(\varphi)} j(\psi) = j(i_\varphi \psi) \in \mathbb{E}$. But by Proposition 2.3 we get $\partial_{\mathfrak{q}}^*(j(i_\varphi \psi)) = j(\partial_{\mathfrak{p}}^*(i_\varphi \psi)) \in \mathbb{E}$, so \mathbb{E} is stable under \mathbb{E} -insertions.

To prove that $\square(\mathbb{E}) \subset \mathbb{E}$, we use Kostant's algebraic formula for \square . Let $\tilde{j} : \Lambda^2 \mathfrak{q}_+ \otimes \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{g} \otimes \mathfrak{g}$ be the inclusion. Recall from 2.3 that we have the decomposition $\mathfrak{g} = \mathfrak{p}_- \oplus (\mathfrak{p} \cap \mathfrak{q}_-) \oplus \mathfrak{q}_0 \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_+) \oplus \mathfrak{p}_+$. The Killing form of \mathfrak{g} induces dualities between the first and last, and the second and fourth summands and its restriction to the third summand is non-degenerate. Now

we choose a basis $\{X_\alpha\}$ of \mathfrak{g} consisting of homogeneous elements, which starts with bases of the first three summands and has the duals of the first two bases in the last two summands. Denoting by $\{Y_\alpha\}$ the dual basis with respect to the Killing form, we by construction have $[X_\alpha, Y_\alpha] \in \mathfrak{p}_0$ for all α and the first elements of $\{Y_\alpha\}$ (which lie in \mathfrak{p}_+) coincide with the last elements of $\{X_\alpha\}$ and vice versa. According to Theorem 4.4 of [15], one can write $\tilde{j} \circ \square$ as

$$\frac{1}{2} \left(\sum_{\alpha} \text{id} \otimes (\text{ad}_{Y_\alpha} \circ \text{ad}_{X_\alpha}) + \sum_{\alpha: X_\alpha \in \mathfrak{q}_-} \rho_{Y_\alpha} \circ \rho_{X_\alpha} - \sum_{\alpha: X_\alpha \in \mathfrak{q}} \rho_{Y_\alpha} \circ \rho_{X_\alpha} \right) \circ \tilde{j}.$$

Here ad denotes the adjoint action of \mathfrak{g} on \mathfrak{g} , while ρ denotes the natural action of \mathfrak{g} on $\Lambda^2 \mathfrak{g} \otimes \mathfrak{g}$. The subspace $\text{im}(\tilde{j} \circ j)$ is invariant under any map which acts only on the \mathfrak{g} part, as well as under ρ_A for each $A \in \mathfrak{p}$. In particular, any summand in the first sum preserves this subspace, while for the other two sums the same holds for summands in which X_α lies in $(\mathfrak{p} \cap \mathfrak{q}_-) \oplus \mathfrak{q}_0 \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_+) \subset \mathfrak{p}$, since then also Y_α lies in this subspace. It remains to show that $\sum_{\alpha: X_\alpha \in \mathfrak{p}_-} \rho_{Y_\alpha} \circ \rho_{X_\alpha} - \sum_{\alpha: X_\alpha \in \mathfrak{p}_+} \rho_{Y_\alpha} \circ \rho_{X_\alpha}$ also preserves this subspace. But by construction, we can rewrite this part as

$$\sum_{\alpha: X_\alpha \in \mathfrak{p}_-} (\rho_{Y_\alpha} \circ \rho_{X_\alpha} - \rho_{X_\alpha} \circ \rho_{Y_\alpha}) = \sum_{\alpha: X_\alpha \in \mathfrak{p}_-} \rho_{[Y_\alpha, X_\alpha]},$$

and since $[Y_\alpha, X_\alpha] \in \mathfrak{p}$, the result follows. \square

3.4. The relation between real and complex parabolic geometries.

Let us demonstrate the power of the theory developed so far by analyzing the relation between real and complex parabolic geometries. In particular, we will show that the theory of twistor spaces also works in the holomorphic category. This is of considerable interest, since twistor correspondences usually only work directly for some real forms of a complex semisimple Lie algebra. For other real forms of interest, one has to restrict to real analytic structures, pass to complexifications, and interpret the final result on the original manifold.

Let \mathfrak{g} be a complex semisimple Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra, G a Lie group with Lie algebra \mathfrak{g} and $P \subset G$ the parabolic subgroup corresponding to \mathfrak{p} . Then one can consider *complex parabolic geometries* of type (G, P) , defined as holomorphic principal P -bundles $\mathcal{G} \rightarrow M$ over complex manifolds M which are endowed with holomorphic Cartan connections $\omega \in \Omega^{1,0}(\mathcal{G}, \mathfrak{g})$. On the other hand, we may also consider \mathfrak{g} as a real Lie algebra endowed with a parabolic subalgebra \mathfrak{p} , and thus consider real parabolic geometries of type (G, P) , which are given by smooth principal bundles endowed with smooth Cartan connections. See 4.6 for the discussion of an interesting example.

If $(p : \mathcal{G} \rightarrow M, \omega)$ is a real parabolic geometry of type (G, P) , then for each point $u \in \mathcal{G}$, $\omega(u)$ is a linear isomorphism from $T_u \mathcal{G}$ to the complex vector space \mathfrak{g} , so ω defines an almost complex structure $J^{\mathcal{G}}$ on \mathcal{G} . The vertical subspace in $T_u \mathcal{G}$ is just the preimage under $\omega(u)$ of \mathfrak{p} . Since this is a complex subspace, we get a complex structure on the quotient space $T_{p(u)} M$. If we change from u to another preimage of $p(u)$, the resulting isomorphism $T_{p(u)} M \rightarrow \mathfrak{g}/\mathfrak{p}$ changes by the adjoint action of an element of P , which is a

complex linear map. Thus, we also get a well defined almost complex structure J on M , and the tangent map to the projection $p : \mathcal{G} \rightarrow M$ is complex linear. These almost complex structures imply that differential forms on M and \mathcal{G} with values in any complex vector space or vector bundle split according to their complex linearity or anti-linearity properties into (p, q) -types. In particular, the adjoint tractor bundle is a complex vector bundle in this case, so this applies to the curvature and its harmonic part viewed as two-forms on M . (Note that even the real cohomologies canonically are complex vector spaces.)

Theorem. *Let \mathfrak{g} be a complex semisimple Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalgebra, G a complex Lie group with Lie algebra \mathfrak{g} and $P \subset G$ the parabolic subgroup corresponding to \mathfrak{p} . A real parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$ of type (G, P) is actually a complex parabolic geometry (i.e. M is a complex manifold, $p : \mathcal{G} \rightarrow M$ a holomorphic principal bundle and ω a holomorphic Cartan connection) if and only if its curvature function is of type $(2, 0)$.*

If the parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$ is regular and normal, then it suffices that the harmonic curvature is of type $(2, 0)$.

Proof. We can split $L(\Lambda^2 \mathfrak{g}/\mathfrak{p}, \mathfrak{g})$ according to (p, q) -types into $L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$. Since the adjoint action of P is by complex linear maps, this is a splitting of P -modules. Since \mathfrak{g} is a complex Lie algebra, both ∂ and ∂^* preserve complex multilinear maps, so for $\mathbb{E} := L^{2,0}$ we get $\square(\mathbb{E}) \subset \mathbb{E}$. One easily verifies that \mathbb{E} is stable under \mathbb{E} -insertions. Hence by Corollary 3.2, in the regular normal case, the harmonic curvature κ_H is of type $(2, 0)$ if and only if the whole curvature κ is of type $(2, 0)$.

For $X \in \mathfrak{g}$, we have the vector field $X^{\mathcal{G}} = \omega^{-1}(X)$ on \mathcal{G} , and in the proof of Proposition 2.6 we have seen that $\kappa(X, Y) = [X, Y] - \omega([X^{\mathcal{G}}, Y^{\mathcal{G}}])$. Since the bracket in \mathfrak{g} is complex bilinear this formula immediately implies that the $(0, 2)$ -part of κ maps $X, Y \in \mathfrak{g}$ to $-\frac{1}{4}\omega(N(X^{\mathcal{G}}, Y^{\mathcal{G}}))$, where N denotes the Nijenhuis tensor of $J^{\mathcal{G}}$. Consequently, the integrability of the almost complex structure $J^{\mathcal{G}}$ is equivalent to vanishing of the $(0, 2)$ -component of the curvature function, and this vanishing also implies integrability of the almost complex structure J on M . If the almost complex structures are integrable then the projection $p : \mathcal{G} \rightarrow M$ is by construction holomorphic. Equivariancy of ω immediately implies that for $u \in \mathcal{G}$ and $g \in P$ we get $\omega_{u \cdot g} \circ T_u r^g = \text{Ad}(b^{-1}) \circ \omega_u$, which implies that $r^g : \mathcal{G} \rightarrow \mathcal{G}$ is a holomorphic mapping. On the other hand, since ω reproduces the generators of fundamental vector fields, the map $g \mapsto u \cdot g$ is holomorphic, too, and these two facts imply that the principal right action $r : \mathcal{G} \times P \rightarrow \mathcal{G}$ is holomorphic, so $p : \mathcal{G} \rightarrow M$ is a holomorphic principal bundle.

Given that \mathcal{G} is a complex manifold, the Cartan connection ω , which by construction is a $(1, 0)$ -form, is holomorphic, if and only if its exterior derivative is a $(2, 0)$ -form, and since the bracket in \mathfrak{g} is complex bilinear, this is equivalent to the curvature being of type $(2, 0)$, which implies the result. \square

We can deduce from this that our results on correspondence spaces and twistor spaces continue to hold in the realm of complex parabolic geometries. While this is obvious for the correspondence part, it is quite nontrivial for

the twistor part. Assume that \mathfrak{g} is a complex semisimple Lie algebra with standard parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ as in 2.3, G is a complex Lie group with Lie algebra \mathfrak{g} and $Q \subset P \subset G$ are the parabolic subgroups corresponding to \mathfrak{q} and \mathfrak{p} . If $(p : \mathcal{G} \rightarrow N, \omega)$ is a complex parabolic geometry, then clearly $\mathcal{CN} = \mathcal{G} \times_P P/Q$ is a complex manifold, and $(\mathcal{G} \rightarrow \mathcal{CN}, \omega)$ is a complex parabolic geometry of type (G, Q) , which is normal if $(\mathcal{G} \rightarrow N, \omega)$ is normal.

On the other hand, assume that $(p : \mathcal{G} \rightarrow M, \omega)$ is a complex parabolic geometry of type (G, Q) , and suppose that the curvature satisfies the restrictions of Theorem 2.7 or (in the regular normal case) 3.3. Then viewed as a real parabolic geometry, M is locally isomorphic to a correspondence space \mathcal{CN} for a (unique) real parabolic geometry of type (G, P) on a manifold N . But the curvature function for this parabolic geometry is the same as the curvature function of M , so in particular, it has complex bilinear values. By the theorem N is a complex parabolic geometry, and we get

Corollary. *Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a complex parabolic geometry of type (G, Q) , which satisfies the curvature restrictions of Theorem 2.7 respectively 3.3. Then the twistor space N of M is automatically a complex parabolic geometry (and in particular a complex manifold) and the local isomorphism between M and the correspondence space \mathcal{CN} is automatically a biholomorphism.*

4. Examples and applications

In this section, we apply our results to three concrete examples. These relate Lagrangian contact structures to projective structures, certain almost CR structures to an almost complex version of projective structures, and finally projective and almost Grassmannian structures to a generalization of path geometries.

Lagrangian contact structures

In this case, the construction of the correspondence space is described in [20] but the main result obtained there is just that the correspondence space is locally flat if and only if the original geometry is locally flat. Our results in this case go much further. In one direction, we obtain a nice geometric interpretation of the projective curvature of a torsion free affine connection, while in the other direction we get results on contact structures and on partial connections on projectivized cotangent bundles.

4.1. For $n \geq 2$ consider the Lie group $G = PSL(n+1, \mathbb{R})$, the quotient of $SL(n+1, \mathbb{R})$ by its center. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ consists of all tracefree linear endomorphisms of \mathbb{R}^{n+1} . Define $\mathfrak{p} \subset \mathfrak{g}$ to be the stabilizer of the line through the first vector in the standard basis of \mathbb{R}^{n+1} . Let $\mathfrak{q} \subset \mathfrak{p}$ be the subalgebra of those maps which in addition preserve the hyperplane W generated by the first n vectors in the standard basis. Then \mathfrak{p} and \mathfrak{q} give rise to a $|1|$ -grading, respectively a $|2|$ -grading of \mathfrak{g} , defined by

$$\begin{pmatrix} \mathfrak{p}_0 & \mathfrak{p}_1 \\ \mathfrak{p}_{-1} & \mathfrak{p}_0 \end{pmatrix} \quad \begin{pmatrix} \mathfrak{q}_0 & \mathfrak{q}_1^L & \mathfrak{q}_2 \\ \mathfrak{q}_{-1}^L & \mathfrak{q}_0 & \mathfrak{q}_1^R \\ \mathfrak{q}_{-2} & \mathfrak{q}_{-1}^R & \mathfrak{q}_0 \end{pmatrix},$$

where in the first matrix, the blocks are of size 1 and n , while in the second matrix, they are of size 1, $n - 1$ and 1. For later use, we have indicated the finer decomposition $\mathfrak{q}_{\pm 1} = \mathfrak{q}_{\pm 1}^L \oplus \mathfrak{q}_{\pm 1}^R$.

The group G does not act on \mathbb{R}^{n+1} but only on the projective space $\mathbb{R}P^n$. The parabolic subgroups $P, Q \subset G$ are defined as consisting of those elements whose adjoint action preserves the filtration of \mathfrak{g} induced by the corresponding subalgebra. One easily verifies that P is exactly the stabilizer of the point in $\mathbb{R}P^n$ corresponding to the line through the first basis vector, so $G/P \cong \mathbb{R}P^n$, and $Q \subset P$ consists of those elements which in addition stabilize the projective hyperplane corresponding to W .

It is well known, see e.g. [20], that normal parabolic geometries of type (G, P) (which are automatically regular since we deal with a $|1|$ -grading) are exactly the classical projective structures on n -manifolds. Hence specifying such a geometry on an n -dimensional manifold M is equivalent to giving a projective equivalence class $[\nabla]$ of torsion-free connections on the tangent bundle TM . Projective equivalence means that two connections in the class differ by the action of a one-form $\Upsilon \in \Omega^1(M)$, i.e. $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$, and it says precisely, that ∇ and $\hat{\nabla}$ have the same geodesics up to parametrization. Moreover, $P_0 = GL(n, \mathbb{R})$ and the underlying P_0 -bundle $\mathcal{G}_0 \rightarrow M$ is simply the full first order frame bundle in this case.

The parabolic subalgebra \mathfrak{q} is of contact type, i.e. it defines a $|2|$ -grading on \mathfrak{g} , the components $\mathfrak{q}_{\pm 2}$ have dimension one, and the bracket $\mathfrak{q}_{-1} \times \mathfrak{q}_{-1} \rightarrow \mathfrak{q}_{-2}$ is non degenerate. Therefore we get an example of a *parabolic contact structure*, i.e. regular normal parabolic geometries of type (G, Q) have an underlying contact structure. These geometries have been first studied in [20], where they were called Lagrangian contact structures (although Legendrian contact structures would be closer to the usual terminology). To describe them, consider a contact structure $H \subset TM$ on a smooth manifold M of dimension $2n + 1$. This means that H is a subbundle of rank $2n$ which is maximally non-integrable in the sense that the skew symmetric bundle map $\mathcal{L} : H \times H \rightarrow TM/H$ induced by the Lie bracket of vector fields is non-degenerate. A subbundle $E \subset H$ is called *isotropic* if the restriction of \mathcal{L} to $E \times E$ vanishes identically. Elementary linear algebra shows that isotropic subbundles have rank at most n , and isotropic subbundles of maximal rank are usually called *Legendrian* or *Lagrangian*.

Definition. A *Lagrangian contact structure* on a smooth manifold M of dimension $2n + 1$ is a contact structure $H \subset TM$ together with a fixed splitting $H = H^L \oplus H^R$ of the contact subbundle as a direct sum of two rank n isotropic subbundles.

Note that if $(M, H = H^L \oplus H^R)$ is a Lagrangian contact structure, then \mathcal{L} induces an isomorphism $H^R \cong L(H^L, TM/H)$, so the two subbundles are almost dual to each other.

In [20], the author used Tanaka's prolongation procedure from [22] to construct a parabolic geometry from a Lagrangian contact structure. We sketch a construction using the procedure from [7], which uses a simpler description of the underlying structures. We have noted above that the Lie bracket $[,] : \mathfrak{q}_{-1} \times \mathfrak{q}_{-1} \rightarrow \mathfrak{q}_{-2}$ is non-degenerate, and one observes that the

subspaces \mathfrak{q}_{-1}^L and \mathfrak{q}_{-1}^R are isotropic. Next, one verifies that the adjoint action identifies the subgroup $Q_0 \subset Q$ with the group of all linear isomorphisms $\varphi : \mathfrak{q}_{-1} \rightarrow \mathfrak{q}_{-1}$ which preserve the decomposition $\mathfrak{q}_{-1}^L \oplus \mathfrak{q}_{-1}^R$ and have the property that there exists a linear isomorphism $\psi : \mathfrak{q}_{-2} \rightarrow \mathfrak{q}_{-2}$ such that $[\varphi(X), \varphi(Y)] = \psi([X, Y])$ for all $X, Y \in \mathfrak{g}_{-1}$. Of course, the map ψ is then uniquely determined by φ . In the description of [7] for a parabolic geometry of type (G, Q) one first needs a manifold of dimension $2n+1$ together with a subbundle $H \subset TM$ of rank $\dim(\mathfrak{g}_{-1}) = 2n$. Then regular normal parabolic geometries of type (G, Q) are in bijective correspondence with reductions of the associated graded $H \oplus TM/H$ to the structure group Q_0 such that the bundle map $\Lambda^2 H \rightarrow TM/H$ obtained from the Lie bracket of vector fields looks like the bracket $[\cdot, \cdot] : \Lambda^2 \mathfrak{q}_{-1} \rightarrow \mathfrak{q}_{-2}$ in each fiber. The condition on the bracket exactly means that H defines a contact structure on M , and the reduction to Q_0 is exactly equivalent to the decomposition $H = H^L \oplus H^R$ into a sum of Lagrangian subbundles.

Now we can apply the construction of correspondence spaces to this case, which recovers all the results of [20]:

Theorem. *Let $[\nabla]$ be a projective structure on a smooth manifold N , and let $(p : \mathcal{G} \rightarrow N, \omega)$ be the associated parabolic geometry of type (G, P) . Then the correspondence space \mathcal{CN} for $\mathfrak{q} \subset \mathfrak{p}$ is the projectivized cotangent bundle $\mathcal{P}(T^*N)$. The induced Lagrangian contact structure $H = H^L \oplus H^R$ on $\mathcal{P}(T^*N)$ has the following form: $H \subset T\mathcal{P}(T^*N)$ is the canonical contact structure, H^R is the vertical subbundle of $\mathcal{CN} \rightarrow N$, and H^L is obtained from the horizontal distributions of the connections in the projective class.*

Proof. Since the Killing form induces a duality between $\mathfrak{g}/\mathfrak{p}$ and \mathfrak{p}_1 , the cotangent bundle T^*N is the associated bundle $\mathcal{G} \times_P \mathfrak{p}_1$. By definition, the subgroup $Q \subset P$ preserves the line $\mathfrak{q}_2 \subset \mathfrak{p}_1$, and one immediately verifies that this property characterizes Q . Passing to the projectivization, we see that $\mathcal{P}(\mathfrak{p}_1) \cong P/Q$, which implies that $\mathcal{CN} = \mathcal{G}/Q = \mathcal{G} \times_P (P/Q)$ is exactly the projectivized cotangent bundle $\mathcal{P}(T^*N)$.

The tangent bundle to $\mathcal{P}(T^*N)$ is $\mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$, while the tangent bundle to N is $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. The tangent map of the projection $\pi : \mathcal{P}(T^*N) \rightarrow N$ corresponds in this picture exactly to the projection $\mathfrak{g}/\mathfrak{q} \rightarrow \mathfrak{g}/\mathfrak{p}$. The contact distribution $H \subset T\mathcal{P}(T^*N)$ is given by $\mathcal{G} \times_Q ((\mathfrak{q}_{-1} \oplus \cdots \oplus \mathfrak{q}_2)/\mathfrak{q})$. Since $\mathfrak{q}_{-1} \oplus \cdots \oplus \mathfrak{q}_2$ is exactly the annihilator of \mathfrak{q}_2 with respect to the Killing form, we conclude that for a point $\ell \in \mathcal{P}(T^*N)$ (i.e. ℓ is a line in $T_{\pi(\ell)}^*M$), we have $H_\ell = \{\xi : \ell(T\pi \cdot \xi) = 0\}$, so we exactly get the canonical contact structure on $\mathcal{P}(T^*N)$. Since $\mathfrak{q}_{-1}^R = \mathfrak{q} \cap \mathfrak{p}$, the subbundle H^R consists of those tangent vectors, which project to zero, so this is exactly the vertical subbundle in $T\mathcal{P}(T^*N)$.

Any connection ∇ in the projective class induces a linear connection on T^*M , which gives rise to a vertical projection from TT^*M onto the vertical subbundle. This vertical projection is characterized by the fact that the covariant derivative of a one form φ is the nontrivial component of the composition of the vertical projection with the tangent map $T\varphi$. Factoring to the projectivization we see, that any linear connection on TN gives rise

to a vertical projection on $T\mathcal{P}(T^*N)$. For projectively equivalent connections ∇ and $\hat{\nabla}$, one easily computes that on the level of one-forms one gets $\hat{\nabla}_\xi\varphi = \nabla_\xi\varphi - \Upsilon(\xi)\varphi - \varphi(\xi)\Upsilon$. Passing from $T_\varphi T^*N$ to $T_{[\varphi]}\mathcal{P}(T^*N)$ means exactly factoring by the line generated by φ in the vertical subspace, and the subbundle H is characterized by $\xi \in H_{[\varphi]}$ iff $\varphi(\xi) = 0$, which shows that the restriction of the vertical projection to $H \subset T\mathcal{P}(T^*N)$ depends only on the projective class. One verifies directly that this construction really describes the bundle H^L . \square

Note that the subbundle $H^L \subset H$ which is complementary to the vertical subbundle, is very similar to a connection on the fiber bundle $\pi : \mathcal{P}(T^*N) \rightarrow N$. The difference to a true connection is that the vertical projection is only defined on the subbundle $H \subset T\mathcal{P}(T^*N)$. Thus, H^L defines a *partial connection* on $\mathcal{P}(T^*N)$.

4.2. Harmonic curvature. To proceed further, we have to describe the harmonic curvature both for projective structures and for Lagrangian contact structures. The structure of this curvature is essentially different for $n = 2$ and $n \geq 3$. We discuss the case $n \geq 3$ in detail and make some remarks on the case $n = 2$ below.

We have to compute the second cohomologies $H^2(\mathfrak{p}_-, \mathfrak{g})$ and $H^2(\mathfrak{q}_-, \mathfrak{g})$, and this is a simple exercise in applying Kostant's version of the Bott–Borel–Weil theorem, see [15]. Since we are dealing with the split real form $\mathfrak{sl}(n+1, \mathbb{R})$ of $\mathfrak{sl}(n+1, \mathbb{C})$ here, the real cohomologies look exactly as the complex cohomologies. Using the Dynkin–diagram notation from chapter 3 of [4], i.e. the numbers over the nodes indicate the coefficient of the corresponding fundamental weight in the highest weight of the dual representation, and the algorithms from section 8.5 of [4] one gets $H^2(\mathfrak{p}_-, \mathfrak{g}) = \overset{-4}{\times} \overset{1}{\bullet} \overset{2}{\bullet}$ respectively $\overset{-4}{\times} \overset{1}{\bullet} \overset{1}{\bullet} \overset{0}{\bullet} \overset{1}{\bullet}$ for the projective case. This is immediately seen to be exactly the irreducible component of highest weight in $\Lambda^2(\mathfrak{p}_{-1})^* \otimes L(\mathfrak{p}_{-1}, \mathfrak{p}_{-1})$. Following the description in 2.2 one shows that the harmonic curvature is represented by trace free part of the curvature of any connection in the projective class.

For the Lagrangian contact structures, the situation is a little more complicated, since there are three irreducible components in the curvature. One obtains the following picture:

$\overset{-4}{\times} \overset{1}{\bullet} \overset{1}{\bullet} \overset{0}{\bullet} \overset{1}{\times}$	torsion $T^R : \Lambda^2 H^R \rightarrow H^L$
$\overset{1}{\times} \overset{0}{\bullet} \overset{1}{\bullet} \overset{1}{\bullet} \overset{-4}{\times}$	torsion $T^L : \Lambda^2 H^L \rightarrow H^R$
$\overset{-3}{\times} \overset{2}{\bullet} \overset{0}{\bullet} \overset{2}{\bullet} \overset{-3}{\times}$	curvature $\rho : H^L \times H^R \rightarrow L(H^L, H^L)$

The first column contains the representations for $n \geq 4$, and the second column contains the corresponding geometric object. For $n = 3$, the representations are slightly different, but the geometric objects remain the same. The representations are computed as before and the geometric objects are determined by identifying them with components in $\Lambda^2(\mathfrak{g}/\mathfrak{q}, \mathfrak{g})$ and passing to the corresponding bundles. The torsions T^L and T^R are easy to interpret geometrically. Since H^L is isotropic, the bracket of two sections of H^L is a section of H and thus can be projected to H^R . Similarly as in Proposition

3.4 of [8] one shows that the resulting tensor field coincides with T^L up to a nonzero multiple. In particular, T^L is exactly the obstruction to integrability of the subbundle $H^L \subset TM$. The interpretation of T^R is completely analogous. To compute the remaining harmonic curvature component ρ , one has to choose a Weyl-structure and compute the appropriate harmonic part of its curvature.

Since $\mathfrak{q}_{-1} \cap \mathfrak{p} = \mathfrak{q}_{-1}^R$, the distribution corresponding to $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is exactly H^R . Integrability of this subbundle is equivalent to vanishing of T^R . Moreover, Theorem 3.3 implies that local leaf spaces carry an induced projective structure (an hence the original structure is locally isomorphic to a correspondence space) if and only also ρ vanishes identically. In particular, for a correspondence space the torsion T^L is a complete obstruction against local flatness, and we obtain:

Theorem. *The projective Weyl curvature of a projective structure $[\nabla]$ on a smooth manifold N of dimension $n \geq 3$ is exactly the obstruction to integrability of the bundle H^L on $\mathcal{P}(T^*N)$, i.e. to flatness of the induced partial connection on $\mathcal{P}(T^*N) \rightarrow N$.*

Having a twistor correspondence in the classical sense would mean that one starts with a projective manifold $(N, [\nabla])$, forms the correspondence space $\mathcal{C}N$ and then a twistor space with respect to the parabolic $\tilde{\mathfrak{p}} \supset \mathfrak{q}$, which is the stabilizer of the line spanned by the last vector in the standard basis of \mathbb{R}^{n+1} . The corresponding distribution is H^L , so the theorem shows that this is possible only in the locally projectively flat case, in which one recovers projective duality.

4.3. Let us next look at a Lagrangian contact structure $(M, H^L \oplus H^R)$ which admits a twistor space but not a projective structure on this twistor space. So assume that H^R is integrable and let $\psi : U \rightarrow N$ be a local leaf space, see 2.6. For any $x \in U$, we have the subspace $H_x \subset T_x U$, and by construction the image of this subspace is a hyperplane in $T_{\psi(x)} N$. The annihilator of this hyperplane is a line in $T_x^* N$, and thus a point $\tilde{\psi}(x) \in \mathcal{P}(T^*N)$. Clearly, this defines a smooth mapping $\tilde{\psi} : U \rightarrow \mathcal{P}(T^*N)$ such that $\pi \circ \tilde{\psi} = \psi$, where $\pi : \mathcal{P}(T^*N) \rightarrow N$ is the projection. To understand $\tilde{\psi}$, it is better to view $\mathcal{P}(T^*N)$ as the space $Gr_n(TN)$ of hyperplanes in the tangent space of N . By construction, the kernel of $T_x \tilde{\psi}$ has to be contained in $\ker(T_x \psi) = H_x^R$. For $\xi \in H_x^R$ the image $T_x \tilde{\psi} \cdot \xi$ is contained in the vertical subspace $V_{\tilde{\psi}(x)} Gr_n(TN)$. This is the tangent space to the fiber at $\tilde{\psi}(x)$, so it can be identified with the space $L(\tilde{\psi}(x), T_{\psi(x)} N / \tilde{\psi}(x))$ of linear maps. Via $T_x \psi$, this space is isomorphic to $L(H_x / H_x^R, T_x M / H_x)$. Going through the identifications one sees that $T_x \tilde{\psi} \cdot \xi(x)(\eta) = \mathcal{L}(\xi(x), \eta)$ for all $\eta \in H_x$, so non degeneracy of \mathcal{L} implies that $T_x \tilde{\psi}$ is injective.

Since M and $\mathcal{P}(T^*N)$ have the same dimension, we conclude that, possibly shrinking U , that $\tilde{\psi}$ is a diffeomorphism from U onto an open subset V of $\mathcal{P}(T^*N)$. Moreover, from the construction it is obvious that this is a

contact diffeomorphism, and the integrable Lagrangian subbundle H^R is exactly mapped to the vertical subbundle of $\mathcal{P}(T^*N)$. Notice that the complementary Lagrangian subbundle H^L was not used at all in the construction. Hence we obtain the following strengthening of the Darboux theorem:

Theorem. *Let (M, H) be a contact manifold of dimension $2n + 1$, and let $H^R \subset H$ be an integrable Lagrangian subbundle. Then locally M is contact diffeomorphic to the projectivized tangent bundle of \mathbb{R}^{n+1} in such a way that H^R is mapped to the vertical subbundle.*

4.4. We now know that a Lagrangian contact manifold $(M, H^L \oplus H^R)$ which has the property that T^R is identically zero is locally contact diffeomorphic to an open subset of the projectivized tangent bundle of its twistor space in such a way that H^R is mapped to the vertical subbundle. On such an open subset the image of H^L defines a complement to the vertical subbundle in the contact subbundle, so we can view this as a locally defined partial connection.

As we have observed, any linear connection ∇ on TN gives rise to such a partial connection (even globally defined), and moreover, this partial connection depends only on the projective class of ∇ . By Theorem 3.3, a locally defined partial connection comes from a linear connection on TN if and only if the curvature $\rho \in \Gamma((H^L)^* \otimes (H^R)^* \otimes L(H^L, H^L))$ of the corresponding Lagrangian contact structure is identically zero, so we get:

Theorem. *Let N be a smooth manifold of dimension $n \geq 3$ and let $U \subset \mathcal{P}(T^*N)$ be an open subset. Let H be the restriction of the contact subbundle to U and let $H^L \subset H$ be a Lagrangian subbundle, which is complementary to the vertical subbundle H^R . Then H^L is obtained from a linear connection on TN as described in the end of 4.1 if and only if the curvature ρ of the Lagrangian contact structure $(U, H^L \oplus H^R)$ vanishes identically.*

4.5. The case $n = 2$. In this case, the structure of the harmonic curvature is completely different. Rather than having two torsions and one curvature, one has two curvatures in this case. The Lagrangian contact structures in this case admit a natural interpretation as path geometries, so this can also be viewed as a special case of the structures discussed in 4.7. Also projective structures in two dimensions behave differently than in general. For dimensional reasons the tracefree part of the curvature of any linear connection in two dimensions vanishes identically. The complete obstruction to local projective flatness is a tensor analogous to the Cotton–York tensor in conformal geometry. Passing to the correspondence space, this corresponds exactly to one of the two curvatures. Since the bundles H^L and H^R have rank one, they are always integrable. In particular, in this special dimension, one can obtain a twistor correspondence in the classical sense in non-flat situations. This looks particularly nice in the picture of path geometries, viewed as encoding second order ODE's, compare with 4.7. In this picture, the twistor space is the space of all solutions of the equation. The holomorphic version of this correspondence was used in [13] to study Schlesinger's equation.

Elliptic partially integrable almost CR manifolds

4.6. This is an almost complex version of the situation discussed in 4.5 above. We consider $G = PSL(3, \mathbb{C})$ as a real Lie group, the Borel subgroup $Q \subset G$ (i.e. Q is the stabilizer of the standard flag $\mathbb{C} \subset \mathbb{C}^2 \subset \mathbb{C}^3$) and the stabilizer P of the complex line $\mathbb{C} \subset \mathbb{C}^3$. This case is a bit involved and will be taken up in detail elsewhere. Here we only give a brief outline. From their definition, real parabolic geometries of type (G, Q) are six dimensional smooth manifolds endowed with an almost complex structure and two complementary complex line bundles $H^+, H^- \subset TM$, such that, with $H = H^+ \oplus H^-$, the tensorial map $\mathcal{L} : H \otimes H \rightarrow TM/H$ induced by the Lie bracket is complex bilinear and non-degenerate. Building on earlier work in [21] it has been shown in [8] that flipping the almost complex structure on the subbundle H^+ leads to an equivalence of categories between the category of regular normal parabolic geometries of type (G, Q) and the category of elliptic partially integrable almost CR manifolds of CR dimension and codimension two. Since we are dealing with the underlying real Lie algebra of a complex Lie algebra, the structure of torsions and curvatures is rather complicated. In the notation of [8] there are the following irreducible components of the harmonic curvature:

S^\pm	$(0, 2)$	$TM/HM \times H^\pm M \rightarrow TM/HM$	almost complex structure
N^\pm	$(1, 1)$	$H^+M \times H^-M \rightarrow H^\pm M$	almost CR structure
T^\pm	$(1, 1)$	$\Lambda^2 H^\pm M \rightarrow H^\mp M$	integrability of $H^\pm M$
ρ^\pm	$(2, 0)$	$TM/HM \times H^\pm \rightarrow (H^\pm)^*$	

The second column contains the (p, q) -types of the components, while the last column indicates the structure whose integrability is obstructed by the component. The first three lines correspond to the components of torsion type, while the two components in the last line are curvatures. From this table and Theorem 3.4 we immediately conclude that complex regular normal parabolic geometries of type (G, Q) are characterized among the real ones by vanishing of S^\pm , N^\pm , and T^\pm , which is easily seen to be equivalent to torsion freeness. For such a complex parabolic geometry the almost CR structure is automatically integrable (since N^\pm vanishes) and real analytic, which implies that torsion free elliptic CR manifolds are always locally embeddable, see [8].

On the other hand, it turns out that normal parabolic geometries of type (G, P) are an almost complex analog of two dimensional projective structures. Given an almost complex manifold (\mathcal{N}, J) we call two linear connections ∇ and $\hat{\nabla}$ *projectively equivalent* if there exists a smooth $(1, 0)$ -form Υ on \mathcal{N} such that $\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. Note that does *not* imply projective equivalence in the real sense, since complex multiples of ξ and η are involved. One easily shows projectively equivalent connections have the same torsion and if $\nabla J = 0$ for some connection ∇ , then the same is true for any projectively equivalent connection. Now we define a *compatible projective structure* on (\mathcal{N}, J) to be a projective class $[\nabla]$ of connections such that $\nabla J = 0$ and the torsion is of type $(0, 2)$. This is the best possible normalization of the torsion and it implies that the torsion is given by $-1/4$ times the Nijenhuis tensor of J . Similar to the case of classical projective structures one proves that normal parabolic geometries of type (G, P)

are exactly compatible projective structures on almost complex manifolds (\mathcal{N}, J) of real dimension four.

The harmonic curvature of these compatible projective structures consists of three irreducible components, one in each of the types $(0, 2)$, $(1, 1)$, and $(2, 0)$. The $(0, 2)$ -part is a multiple of the Nijenhuis tensor of J , the $(1, 1)$ -part is essentially the obstruction against the projective class locally containing holomorphic connections, while the $(2, 0)$ -part is exactly the complex analog of the projective curvature in two dimensions.

Compatible projective structures are by far simpler in nature than partially integrable almost CR structures. In particular, they exist on any almost complex manifold (\mathcal{N}, J) and the set of all such structures is very easy to describe. Hence in this case already the correspondence space construction leads to a very interesting result:

Theorem. *Let (\mathcal{N}, J) be an almost complex manifold of real dimension four. Then any choice of a compatible projective structure $[\nabla]$ on \mathcal{N} endows the space $M := \mathcal{P}_{\mathbb{C}}(T\mathcal{N})$ of complex lines in the tangent spaces of \mathcal{N} with an elliptic CR structure of CR dimension and codimension 2.*

For this CR structure, the components T^- , N^\pm , S^- and ρ^- of the harmonic curvature vanish identically, while the three remaining components S^+ , T^+ , and ρ^+ correspond directly to the three components of the harmonic curvature of $(\mathcal{N}, J, [\nabla])$ of the respective (p, q) -type. The CR automorphism group of this structure coincides with the group of projective automorphisms of $(\mathcal{N}, J, [\nabla])$.

On the other hand, for a general regular normal parabolic geometry of type (G, P) , one can use the tools developed in this paper to give a twistorial interpretation of the components of the harmonic curvature. The subbundle in TM corresponding to $\mathfrak{p} \supset \mathfrak{q}$ is the bundle H^- . Here there are three steps:

existence of a local leaf space \mathcal{N}	$T^- = 0$
almost complex structure J on \mathcal{N}	$N^+ = 0, S^- = 0$
compatible projective structure on (\mathcal{N}, J)	$N^- = 0, \rho^- = 0$

In the second step, one asks whether the almost complex structure on M descends to a local leaf space \mathcal{N} , and verifying that this is equivalent to the vanishing of N^+ and S^- needs a bit of extra work. If this is the case, then M is locally diffeomorphic to the complex projectivization of $T\mathcal{N}$. Under this condition, the geometry on M can then be described locally as an almost complex version of a path geometry, or equivalently as a partial connection on $\mathcal{P}_{\mathbb{C}}(T\mathcal{N})$. The last step directly follows from Theorem 3.3.

Let us finally mention that in this case there is the possibility to form a twistor correspondence in the classical sense. Given a compatible projective structure $(\mathcal{N}, J, [\nabla])$ one can form the correspondence space M , and the harmonic curvature is encoded in S^+ , T^+ , and ρ^+ . If the component corresponding to T^+ vanishes, then the bundle H^+ is integrable, and one can form a local leaf space Z . If one wants this leaf space to carry an induced almost complex structure, then also the component corresponding to S^+ has to vanish, and one is in the holomorphic situation as discussed in [13].

Almost Grassmannian structures

This example, which we only outline briefly, contains the twistor theory for paraconformal manifolds of [3] as well as twistor theory for conformal four manifolds in split signature case. On the other hand, via the notion of path geometries, there is a relation to the geometric theory of systems of second order ODE's, see [12].

4.7. For $n \geq 2$ we consider $G = PSL(n+2, \mathbb{R})$ and the parabolic subgroups P and \hat{P} of G defined as the stabilizer of a point respectively a line containing that point in $\mathbb{R}P^{n+1}$. Then also $Q := P \cap \hat{P}$ is a parabolic subgroup of G . From 4.1 we know that normal parabolic geometries of type (G, P) are classical projective structures on $n+1$ -dimensional manifolds. On the other hand, normal parabolic geometries of type (G, \hat{P}) are exactly almost Grassmannian (also called paraconformal) structures, see [3]. Essentially they are defined as smooth manifolds \hat{N} of dimension $2n$ together with an isomorphism $T\hat{N} \cong E^* \otimes F$, where E and F are auxiliary bundles on \hat{N} of rank 2 and n , respectively. In the special case $n = 2$, one obtains exactly four dimensional split signature conformal manifolds. It should also be noted that for a different real form, one obtains almost quaternionic structures.

We have described the $|1|$ -grading corresponding to $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ in 4.1. For $\hat{\mathfrak{p}}$, one also obtains a $|1|$ -grading, which has the same form, except that the blocks have size 2 and n rather than 1 and $n+1$. For $\mathfrak{q} = \mathfrak{p} \cap \hat{\mathfrak{p}}$, one obtains a $|2|$ -grading on \mathfrak{g} , with $\mathfrak{q}_{\pm 2} = \mathfrak{p}_{\pm} \cap \hat{\mathfrak{p}}_{\pm}$ of dimension n , and $\mathfrak{q}_{\pm 1} = \mathfrak{q}_{\pm 1}^L \oplus \mathfrak{q}_{\pm 1}^R$. Here $\mathfrak{q}_{\pm 1}^L = \mathfrak{q}_{\pm 1} \cap \hat{\mathfrak{p}}_{\pm}$ is one-dimensional while $\mathfrak{q}_{\pm 1}^R = \mathfrak{q}_{\pm 1} \cap \mathfrak{p}_{\pm}$ has dimension n .

Using the main result of [7], one shows that regular normal parabolic geometries of type (G, Q) are $2n+1$ -dimensional manifolds M equipped with complementary subbundles $H^L, H^R \subset TM$ of rank 1 and n , respectively, which have the property that the bracket of two sections of H^R is a section of $H := H^L \oplus H^R$ and the tensorial map $\mathcal{L} : H \times H \rightarrow TM/H$ induced by the Lie bracket is non-degenerate. Note that these assumptions in particular imply that $H^R \cong L(H^L, TM/H)$.

From 4.1 we know that the harmonic curvature for projective structures is the Weyl curvature, i.e. the tracefree part of the curvature of any connection in the projective class. The harmonic curvatures for almost Grassmannian structures are also well known. There are always two irreducible component, but there is an important difference between the cases $n = 2$ and $n \geq 3$. For $n = 2$ both of the two components are curvatures, and they correspond to the self dual and the anti self dual part of the Weyl curvature for four dimensional conformal structures. On the other hand, for $n \geq 3$, one of the two components is a torsion, while the other one is a curvature. The torsion is exactly the obstruction to the existence of a torsion free connection preserving the almost Grassmannian structure, i.e. to the structure being Grassmannian.

It turns out that the harmonic curvature for geometries of type (G, Q) always has three irreducible components and the highest weights of the corresponding representations are exactly the restrictions to $\mathfrak{q}_0 = \mathfrak{p}_0 \cap \hat{\mathfrak{p}}_0$ of

the highest weights of the representations corresponding to the three curvature components discussed above. The component corresponding to the projective Weyl curvature is realized by a torsion $T : H^L \times TM/H \rightarrow H^R$, while the component corresponding to the curvature on the almost Grassmannian side is realized by a curvature $\rho : H^R \times TM/H \rightarrow L(H^R, H^R)$. For $n = 2$ the second curvature on the Grassmannian side is realized by a torsion $\tau : \Lambda^2 H^R \rightarrow H^L$, while for $n \geq 3$ the torsion τ corresponding to the torsion on the Grassmannian side has the form $\tau : \Lambda^2 H^R \rightarrow TM/H$. Hence for $n \geq 3$ the torsion τ is homogeneous of degree zero, so it has to vanish for regular normal parabolic geometries. In particular, correspondence spaces of almost Grassmannian structures with nontrivial torsion are examples of parabolic geometries of type (G, Q) which are normal but not regular.

Similarly as in 4.1 one verifies that for a projective manifold $(N, [\nabla])$ of dimension $n+1$ the correspondence space for $\mathfrak{q} \subset \mathfrak{p}$ is the projectivized tangent bundle $\mathcal{P}(TN)$. The bundle H^R is the vertical subbundle of $\mathcal{P}(TN) \rightarrow N$ and H is the tautological bundle, whose fiber at a point ℓ consists of all tangent vectors whose projection to TN lie in the line ℓ . The line bundle $H^L \subset H$ which is complementary to H^R is constructed from the horizontal lifts of the connections in the projective class as in 4.1. For an almost Grassmannian manifold \hat{N} , the correspondence space for $\mathfrak{q} \subset \hat{\mathfrak{p}}$ is the projectivization of the auxiliary rank two bundle E , H^L is the vertical bundle, and H^R is constructed from the almost Grassmannian structure.

For $n = 2$, Theorem 3.3 implies that integrability of H^R is equivalent to vanishing of τ . For $n \geq 3$ one shows using Lemma 3.2 of [8] that H^R is integrable for any regular normal parabolic geometry of type (G, Q) . If this is satisfied and N is a local leaf space, then similarly as in 4.3 one shows that M is locally diffeomorphic to $\mathcal{P}(TN)$ and this diffeomorphism maps the subbundles $H^R \subset H \subset TM$ to the vertical respectively the tautological subbundle of $T\mathcal{P}(TN)$ as described above. The subbundle H^L is then mapped to a line subbundle in $T\mathcal{P}(TN)$ which is complementary to the vertical subbundle in the tautological subbundle and contains the complete information about the local geometry on M . Conversely, the non-integrability properties of the tautological subbundle in $T\mathcal{P}(TN)$ imply that such a complementary line bundle always gives rise to a regular normal parabolic geometry of type (G, Q) . These complementary line bundles are exactly the *path-geometries* as defined for example in [12] (via differential systems), and using Theorem 3.3 we obtain:

Theorem. (1) *Any path geometry on a smooth manifold N of dimension $n + 1$ gives rise to a regular normal parabolic geometry of type (G, Q) . If $n = 2$ then the torsion τ of this geometry vanishes identically. Conversely, if $n \geq 3$ or $\tau = 0$ any such parabolic geometry on a manifold M locally admits a twistor space N corresponding to $P \supset Q$, and is locally isomorphic to a path geometry on this twistor space.*

(2) *The path geometry in (1) comes from a projective structure on N if and only if ρ vanishes identically, and then T corresponds to the projective Weyl curvature.*

(3) *For any regular normal parabolic geometry of type (G, Q) on M , there exists a local twistor space \hat{N} corresponding to $\hat{P} \supset Q$. The structure on M*

descends to a Grassmannian (respectively anti self dual conformal) structure on \hat{N} if and only if T vanishes identically.

Via the correspondence between path geometries and systems of second order ODE's, part (2) describes when such a system can be written as the geodesic equation for some connection. The definition of torsion free path geometries in [12] is easily seen to be equivalent to vanishing of the torsion T . Parts (1) and (2) imply that a projective structure leading to a torsion free path geometry is locally flat, which is Theorem 1 of [12]. Part (3) implies that for torsion free equations the structure descends to a Grassmannian (respectively anti self dual conformal) structure on \hat{N} (which is constructed in [12] via a Segre structure). Hence the curvature descends to \hat{N} , which generically leads to explicit solutions for equations corresponding to torsion free path geometries.

Applying part (1) to the correspondence space of an almost Grassmannian (respectively split signature conformal) shows that the conditions for existence of a twistor space are vanishing torsion respectively anti self duality. This recovers the standard twistor theory for these structures, see [3] for the Grassmannian case. From (2) we conclude that a local geometric structure on the twistor space is only available in the locally flat case.

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