Universität Wien SS 2013
Fakultät für Mathematik
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Mathematical Finance 2
Exercise sheet 3

1. **a)** Define for $\mu \in \mathbb{R}$, $x, y \in \mathbb{R}$ and $\tau > 0$

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(y - x - \mu\tau)^2}{2\tau}\right).$$

Show that p is the transition probability density function of a Brownian motion with drift

$$X_t = \mu t + W_t.$$

This means to show that for bounded measurable functions f and $t > s$, we have

$$E[f(X_t) \mid \mathcal{F}_s] = g(X_s)$$

with $g(x) = \int_{-\infty}^{\infty} f(y)p(t - s, x, y)dy$.

b) Show that the transition probability density function of a geometric Brownian motion

$$S_t = S_0 e^{\sigma W_t + \nu t}$$

is given by

$$p(\tau, x, y) = \frac{1}{\sigma y\sqrt{2\pi\tau}} \exp\left(-\frac{(\log(y/x) - \nu\tau)^2}{2\sigma^2\tau}\right),$$

where $x, y, \tau > 0$.

2. **a)** Solve Exercise 3.2 of Shreve's book.

b) Prove also that

$$\exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right), \quad t \geq 0$$

is a martingale.

c) Show that $X_t = W_t^3$ has constant expectation. Ist X_t a martingale?

Please turn over!

3. Let S_t be a geometric Brownian motion on $[0, T]$, that is,

$$S_t = S_0 e^{\sigma W_t + \nu t}$$

where $\nu \in \mathbb{R}$ and $\sigma > 0$. Let

$$0 = t_0 < t_1 < \cdots < t_m = T$$

be a partition of $[0, T]$. Show that

$$\frac{1}{T} \lim \sum_{j=0}^{m-1} \left(\log \frac{S_{t_{j+1}}}{S_{t_j}} \right)^2 = \sigma^2, \tag{1}$$

as the number m of partitions points approaches infinity and the length of the longest subinterval approaches 0. In which sense does (1) converge?

Website: http://www.mat.univie.ac.at/~finance_hp/exercisesSS13_MF.html