Universität Wien SS 2013
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Mathematical Finance 2
Exercise sheet 5

1. Let $W = (W_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion and $\sigma, \mu \in \mathbb{R}$. Prove that

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right), \quad t \geq 0$$

is the unique solution of

$$dS_t = S_t \mu dt + S_t \sigma dW_t, \quad t \geq 0,$$

with deterministic initial condition $S_0 > 0$.

Hint: For uniqueness, let $\widetilde{S}$ denote another solution of the SDE and apply the two-dimensional Itô-formula to the process $X_t = \frac{\widetilde{S}_t}{S_t}$ for $t \geq 0$.

2. Consider a discrete-time model with finite time horizon T defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\{0,\ldots,T\}}, \mathbb{P})$ with stock price process S and bank account B whose initial value satisfies $B_0 = 1$. A strategy $\phi = (\phi^{(0)}, \phi^{(1)})$, that is, a predictable process[1] taking values in $\mathbb{R}^2$ is called *self-financing* if, for every $n \in \{1, \ldots, T\}$, the value of the portfolio given by

$$V_n(\phi) := \phi_n^{(0)} B_n + \phi_n^{(1)} S_n$$

satisfies

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j^{(0)}(B_j - B_{j-1}) + \sum_{j=1}^{n} \phi_j^{(1)}(S_j - S_{j-1}).$$

a) Prove that a strategy ϕ is self-financing if and only if

$$\phi_n^{(0)} B_n + \phi_n^{(1)} S_n = \phi_{n+1}^{(0)} B_n + \phi_{n+1}^{(1)} S_n$$

for all $n \in \{0, \ldots, T-1\}$. Give a verbal interpretation of this property.

[1] In the discrete time setting, this simply means that ϕ_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \{1, \ldots, T\}$.

Please turn over!

b) Show that for any $\mathbb{R}$-valued predictable process $\phi^{(1)}$ and any $\mathcal{F}_0$-measurable random variable V_0, there exists a unique real-valued predictable process $\phi^{(0)}$ such that the strategy $\phi = (\phi^{(0)}, \phi^{(1)})$ is self-financing with $V_0(\phi) = V_0$.

3. Let W be a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let h be a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$. Fix $T > 0$ and let $t \in [0, T]$ and $x \in \mathbb{R}$. Assume that $\mathbb{E}_{\mathbb{P}}[|h(W_T)||W_t = x] < \infty$ for all t and x and define the function

$$u(x,t) := \mathbb{E}_{\mathbb{P}}\left[e^{-r(T-t)}h(W_T)|W_t = x\right],$$

Prove that $u(t, x)$ satisfies the following PDE

$$-\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - ru(x,t), \quad (x,t) \in \mathbb{R} \times [0,T)$$

with terminal condition $u(x, T) = h(x)$ for $x \in \mathbb{R}$.

Website: http://www.mat.univie.ac.at/∼finance_hp/exercisesSS13_MF.html