Universität Wien SS 2013
Fakultät für Mathematik
Christa Cuchiero

Mathematical Finance 2
Exercise sheet 8

1. For a European call option expiring at time T with strike price K the Black-Scholes price at time t, if the time-t stock price is x, is given by

$$c(t,x) = xN\left(d_1(T-t,x)\right) - Ke^{-r(T-t)}N\left(d_2(T-t,x)\right), \tag{1}$$

where

$$d_1(\tau,x) = \frac{\ln\left(\frac{x}{K}\right) + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \qquad d_2(\tau,x) = d_1(\tau,x) - \sigma\sqrt{\tau},$$

and N is the standard Gaussian distribution function. Moreover, in the sequel $n(y)$ denotes the standard Gaussian probability density $n(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$.

a) Verify
$$Ke^{-r(T-t)}n\left(d_2(T-t,x)\right) = xn\left(d_1(T-t,x)\right).$$

b) Show that
$$\partial_x c(t,x) = N(d_1(T-t,x)).$$
This is the so-called *delta* of the option.

c) Show that
$$\partial_{xx} c(t,x) = \frac{n(d_1(T-t,x))}{x\sigma\sqrt{T-t}},$$
which is the so-called *gamma* of the option.

d) Show that
$$\partial_t c(t,x) = -rKe^{-r(T-t)}N(d_2(T-t,x)) - \frac{x\sigma}{2\sqrt{T-t}}n(d_1(T-t,x)),$$
which is the so-called *theta* of the option.

e) Use these formulas to show that $c(t,x)$ satisfies the Black-Scholes partial differential equation:
$$\partial_t c(t,x) + rx\partial_x c(t,x) + \frac{1}{2}\sigma^2x^2\partial_{xx}c(t,x) = rc(t,x), \quad 0 \le t < T, \quad x > 0.$$

Please turn over!

2. Consider the Black-Scholes model and let $c(t, x)$ denote the Black-Scholes price at time t of a European call option with maturity T and strike K, when the time-t stock price is x. Prove the following representation of the Black-Scholes formula

$$c(t, x) = (x - K)^+ + x \int_0^a n\left(\frac{\ln(x/K)}{y} + \frac{y}{2}\right) dy,$$

where $a = \sigma\sqrt{T - t}$ and $n(y)$ denotes the standard normal density.

3. Consider the Black-Scholes model and let $c(t, x)$ denote the Black-Scholes price at time t of a European call option with maturity T and strike K, when the time-t stock price equals x. Define the *elasticity*, i.e., the relative change of the option's price as the stock prices moves by

$$\eta_t^c := \frac{\partial_x c(t, S_t) S_t}{c(t, S_t)}.$$

Show that $\eta_t^c > 1$. Moreover, show that, if the stock price changes, the absolute change in the option price is smaller than the absolute change in the stock price.

Website: http://www.mat.univie.ac.at/~finance_hp/exercisesSS13_MF.html