Universität Wien SS 2013
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Mathematical Finance 2
Exercise sheet 9/10

1. Let $T > 0$ and W be a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$. Consider the following process

$$X_t = \mu t + \sigma W_t, \quad t \in [0, T],$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Construct an equivalent probability measure $\mathbb{Q}$ such that X is a $\mathbb{Q}$-martingale.

2. Let $T > 0$, $\mu \in \mathbb{R}$ and let W be a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$. Define a probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\mu W_T - \frac{1}{2}\mu^2 T}$$

and $\widetilde{W}$ by $\widetilde{W}_t = W_t - \mu t$. Prove Girsanov's theorem, i.e., show that $\widetilde{W}$ is a $\mathbb{Q}$ Brownian motion, by using the moment generating function.

3. (General Bayes formula) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Let X and Z be $\mathbb{R}$-valued random variables, where Z is supposed to be nonnegative and $\mathbb{E}[Z] = 1$. Define a probability measure $\mathbb{Q}$ via $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$. Show that X is $\mathbb{Q}$-integrable if and only if XZ is $\mathbb{P}$-integrable and

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{G}] = \frac{\mathbb{E}_{\mathbb{P}}[XZ|\mathcal{G}]}{\mathbb{E}_{\mathbb{P}}[Z|\mathcal{G}]}.$$

4. Let X be a stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F})_{t>0}, \mathbb{P})$ and $\mathbb{Q} \sim \mathbb{P}$ an equivalent probability measure. Define $Z_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right]$. Prove XZ is a $\mathbb{P}$-martingale if and only if X is a $\mathbb{Q}$-martingale.

5. Solve Exercise 5.3 in Shreve's book.

6. Solve Exercise 5.5 in Shreve's book.