Two tales in tractable robust portfolio optimisation

New perspective on fractional Kelly strategies

Jan Obłój

University of Oxford

based on joint works with

Constantinos Kardaras and Eckhard Platen, and with Sigrid Källblad and Thaleia Zariphopoulou

6$^{\text{TH}}$ European Summer School in Financial Mathematics
Vienna, 26–30 August 2013

Motivating questions

- How to develop a robust approach to optimal investment?
- A long run investor will see one path... can we make sense of optimal investment questions pathwise?
- Can we justify fractional Kelly strategies used by large diversified funds?
- The usual criterion $\sup \mathbb{E}[U(X_T)]$ involves (at least) two arbitrary choices: model $\mathbb{P}$ and utility U. The resulting optimal investment strategy in an entangled result of these two choices. Can we disentangle their influence?

Motivating questions

- How to develop a robust approach to optimal investment?
- A long run investor will see one path... can we make sense of optimal investment questions pathwise?
- Can we justify fractional Kelly strategies used by large diversified funds?
- The usual criterion $\sup \mathbb{E}[U(X_T)]$ involves (at least) two arbitrary choices: model $\mathbb{P}$ and utility U. The resulting optimal investment strategy in an entangled result of these two choices. Can we disentangle their influence?

Motivating questions

- How to develop a robust approach to optimal investment?
- A long run investor will see one path... can we make sense of optimal investment questions pathwise?
- Can we justify fractional Kelly strategies used by large diversified funds?
- The usual criterion $\sup \mathbb{E}[U(X_T)]$ involves (at least) two arbitrary choices: model $\mathbb{P}$ and utility U. The resulting optimal investment strategy in an entangled result of these two choices. Can we disentangle their influence?

Motivating questions

- How to develop a robust approach to optimal investment?
- A long run investor will see one path... can we make sense of optimal investment questions pathwise?
- Can we justify fractional Kelly strategies used by large diversified funds?
- The usual criterion $\sup \mathbb{E}[U(X_T)]$ involves (at least) two arbitrary choices: model $\mathbb{P}$ and utility U. The resulting optimal investment strategy in an entangled result of these two choices. Can we disentangle their influence?

Outline

1. Long-run investment, risk attitudes via drawdown constraints
   - Kelly's long-run investor and the numéraire property
   - Numéraire under drawdown – finite horizon
   - Numéraire under drawdown – asymptotics
2. Robust forward performance criteria
   - Model uncertainty, variational preferences and time homogeneity
   - Logarithmic preferences and fractional Kelly
   - Duality and (S)PDEs
3. Conclusions

1. Long-run investment, risk attitudes via drawdown constraints
   - Kelly's long-run investor and the numéraire property
   - Numéraire under drawdown – finite horizon
   - Numéraire under drawdown – asymptotics
2. Robust forward performance criteria
   - Model uncertainty, variational preferences and time homogeneity
   - Logarithmic preferences and fractional Kelly
   - Duality and (S)PDEs
3. Conclusions

Setup

Consider

- a general *continuous* semimartingale market $(S^1_t, \dots, S^d_t)$ denominated in units of a baseline asset

(i) which admits no opportunity for arbitrage of the first kind;

(ii) and there exists $X \in \mathcal{A}$ such that $X_t \to \infty$ a.s.

where $\mathcal{A} = \left\{X : X = 1 + \int_0^{\cdot} \left(\sum_{i=1}^d H^i_t \mathrm{d}S^i_t\right) \geq 0\right\}$.

Theorem

(i) is equivalent to existence of $\hat{X} \in \mathcal{A}$ such that $X/\hat{X}$ is a supermartingale $\forall\ X \in \mathcal{A}$.
Then (ii) is equivalent to $\lim_{t\to\infty} \hat{X}_t = \infty$ a.s.

Note that $\hat{X}$ solves the log-utility problem on $[0, T]$:

$$\mathbb{E}\left[\log\left(\frac{X_T}{\hat{X}_T}\right)\right] \leq \mathbb{E}\left[\frac{X_T}{\hat{X}_T} - 1\right] \leq 0.$$

Setup

Consider

- a general *continuous* semimartingale market $(S^1_t, \dots, S^d_t)$ denominated in units of a baseline asset

(i) which admits no opportunity for arbitrage of the first kind;

(ii) and there exists $X \in \mathcal{A}$ such that $X_t \to \infty$ a.s.

where $\mathcal{A} = \left\{ X : X = 1 + \int_0^\cdot \left(\sum_{i=1}^d H^i_t \mathrm{d} S^i_t \right) \geq 0 \right\}$.

Theorem

(i) is equivalent to existence of $\hat{X} \in \mathcal{A}$ such that $X/\hat{X}$ is a supermartingale $\forall\ X \in \mathcal{A}$.
Then (ii) is equivalent to $\lim_{t\to\infty} \hat{X}_t = \infty$ a.s.

Note that $\hat{X}$ solves the log-utility problem on $[0, T]$:

$$\mathbb{E}\left[\log\left(\frac{X_T}{\hat{X}_T}\right)\right] \leq \mathbb{E}\left[\frac{X_T}{\hat{X}_T} - 1\right] \leq 0.$$

Background: Kelly's strategy

- Kelly argued that a long-run investor should chose $\hat{X}$ – the growth optimal portoflio (numéraire, benchmark). It has a very attractive pathwise property that

$$\lim_{t\to\infty} \frac{X_t}{\hat{X}_t} \leq 1 \quad \text{a.s., for any investment } X, \; X_0 = \hat{X}_0.$$

  Many, including Markowitz, found this appealing.

- Samuelson argued (in words of one syllable) that $\hat{X}$ does not take into account risk preferences and one should look at general utility maximisation instead. But this requires arbitrary choices of model and preferences.

- Practically both are troublesome: estimating drift is hard and utility elucidation often yields different and contradictory outcomes.

Background: Kelly's strategy

- Kelly argued that a long-run investor should chose $\hat{X}$ – the growth optimal portoflio (numéraire, benchmark). It has a very attractive pathwise property that

$$\lim_{t\to\infty} \frac{X_t}{\hat{X}_t} \leq 1 \quad \text{a.s., for any investment } X,\ X_0 = \hat{X}_0.$$

  Many, including Markowitz, found this appealing.

- Samuelson argued (in words of one syllable) that $\hat{X}$ does not take into account risk preferences and one should look at general utility maximisation instead. But this requires arbitrary choices of model and preferences.

- Practically both are troublesome: estimating drift is hard and utility elucidation often yields different and contradictory outcomes.

Background: Kelly's strategy

- Kelly argued that a long-run investor should chose $\hat{X}$ – the growth optimal portoflio (numéraire, benchmark). It has a very attractive pathwise property that

$$\lim_{t\to\infty} \frac{X_t}{\hat{X}_t} \leq 1 \quad \text{a.s., for any investment } X,\ X_0 = \hat{X}_0.$$

  Many, including Markowitz, found this appealing.

- Samuelson argued (in words of one syllable) that $\hat{X}$ does not take into account risk preferences and one should look at general utility maximisation instead. But this requires arbitrary choices of model and preferences.
- Practically both are troublesome: estimating drift is hard and utility elucidation often yields different and contradictory outcomes.

Background: drawdown constraints

Consider an increasing function $w : \mathbb{R} \to \mathbb{R}$ with $w(x)/x \le \alpha < 1$.
Let $\mathcal{A}^w := \{X \in \mathcal{A} : X_t \ge w(\sup_{u \le t} X_u),\ t \ge 0\}$.

Theorem (Cherny & O. (2013))

Let $\tilde{\log}(-x) = -\log(x)$, $x > 0$. *Under v mild assumptions on* U*:*

$$\sup_{X \in \mathcal{A}^w} \mathcal{R}_U(X) = \sup_{X \in \mathcal{A}} \mathcal{R}_{U \circ F_w}(X),$$

$$\text{where } \mathcal{R}_U(X) := \limsup_{T \to \infty} \frac{1}{T} \tilde{\log} \mathbb{E}\left[U(X_T)\right],$$

and F_w *depends only on* w*. Further if* Y *solves the RHS then* $V = M^{F_w}(Y)$*, the Azéma–Yor transform of* Y

$$\mathrm{d}V_t = (V_t - w(\sup_{u \le t} V_u)) \frac{\mathrm{d}Y_t}{Y_t} = F_w' \left(\sup_{u \le t} Y_u \right) \mathrm{d}Y_t,$$

solves the LHS.

Background: drawdown constraints

Consider an increasing function $w : \mathbb{R} \to \mathbb{R}$ with $w(x)/x \leq \alpha < 1$.
Let $\mathcal{A}^w := \{X \in \mathcal{A} : X_t \geq w(\sup_{u\leq t} X_u),\ t \geq 0\}$.

Theorem (Cherny & O. (2013))

Let $\tilde{\log}(-x) = -\log(x)$, $x > 0$. *Under v mild assumptions on* U*:*

$$\sup_{X\in A^w} \mathcal{R}_U(X) = \sup_{X\in\mathcal{A}} \mathcal{R}_{U\circ F_w}(X),$$

$$\textit{where } \mathcal{R}_U(X) := \limsup_{T\to\infty} \frac{1}{T}\tilde{\log}\mathbb{E}\left[U(X_T)\right],$$

and F_w *depends only on* w. *Further if* Y *solves the RHS then* $V = M^{F_w}(Y)$, *the Azéma–Yor transform of* Y

$$\mathrm{d}V_t = \left(V_t - w(\sup_{u\leq t} V_u)\right)\frac{\mathrm{d}Y_t}{Y_t} = F_w'\left(\sup_{u\leq t} Y_u\right)\mathrm{d}Y_t,$$

solves the LHS.

Resulting ideas

- Kelly's pathwise outperformance is an attractive investment criteria.
- Drawdown constraint are an effective way of encoding preferences, and are used in practice.

⇒ Seek pathwise outperformance and encode preferences via pathwise constraints.

Specifically, we consider linear drawdown: $w(x) = \alpha x$, $\alpha \in (0, 1)$ and $\mathcal{A}^\alpha = \{X \in \mathcal{A} : X_t \geq \alpha \sup_{u \leq t} X_u, t \geq 0\}$.
In the unconstrained case $X_t/\hat{X}_t$ is always a supermartingale.
However such process in general fails to exist within the class $\mathcal{A}^\alpha$.
A new criterion is needed!

Resulting ideas

- Kelly's pathwise outperformance is an attractive investment criteria.
- Drawdown constraint are an effective way of encoding preferences, and are used in practice.

$\Rightarrow$ Seek pathwise outperformance and encode preferences via pathwise constraints.

Specifically, we consider linear drawdown: $w(x) = \alpha x$, $\alpha \in (0,1)$ and $\mathcal{A}^\alpha = \{X \in \mathcal{A} : X_t \geq \alpha \sup_{u \leq t} X_u, t \geq 0\}$.

In the unconstrained case $X_t/\hat{X}_t$ is always a supermartingale.

However such process in general fails to exist within the class $\mathcal{A}^\alpha$.

A new criterion is needed!

Resulting ideas

- Kelly's pathwise outperformance is an attractive investment criteria.
- Drawdown constraint are an effective way of encoding preferences, and are used in practice.

$\Rightarrow$ Seek pathwise outperformance and encode preferences via pathwise constraints.

Specifically, we consider linear drawdown: $w(x) = \alpha x$, $\alpha \in (0, 1)$ and $\mathcal{A}^\alpha = \{X \in \mathcal{A} : X_t \geq \alpha \sup_{u \leq t} X_u, t \geq 0\}$.

In the unconstrained case $X_t/\hat{X}_t$ is always a supermartingale.
However such process in general fails to exist within the class $\mathcal{A}^\alpha$.
A new criterion is needed!

Resulting ideas

- Kelly's pathwise outperformance is an attractive investment criteria.
- Drawdown constraint are an effective way of encoding preferences, and are used in practice.

$\Rightarrow$ Seek pathwise outperformance and encode preferences via pathwise constraints.

Specifically, we consider linear drawdown: $w(x) = \alpha x$, $\alpha \in (0, 1)$ and $\mathcal{A}^\alpha = \{X \in \mathcal{A} : X_t \geq \alpha \sup_{u \leq t} X_u, t \geq 0\}$.
In the unconstrained case $X_t / \hat{X}_t$ is always a supermartingale.
However such process in general fails to exist within the class $\mathcal{A}^\alpha$.
A new criterion is needed!

The Numéraire (benchmark) property

For a stopping time T and $X, X' \in \mathcal{A}$, define

$$\mathrm{rr}_T(X|X') := \limsup_{t\to\infty}\left(\frac{X_{T\wedge t} - X'_{T\wedge t}}{X'_{T\wedge t}}\right) = \limsup_{t\to\infty}\left(\frac{X_{T\wedge t}}{X'_{T\wedge t}}\right) - 1,$$

the *return of X relative to X' over the period* $[0, T]$.
Note that we may have $\mathbb{E}\mathrm{rr}_T(X|X') \geq 0$ and $\mathbb{E}\mathrm{rr}_T(X'|X) \geq 0$ however $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ implies $\mathbb{E}\mathrm{rr}_T(X'|X) \geq 0$.

Definition

We say that X' has the *numéraire property* in a certain class of wealth processes for investment over the period $[0, T]$ if and only if $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ holds for all other X in the same class.

The Numéraire (benchmark) property

For a stopping time T and $X, X' \in \mathcal{A}$, define

$$\mathrm{rr}_T(X|X') := \limsup_{t\to\infty}\left(\frac{X_{T\wedge t} - X'_{T\wedge t}}{X'_{T\wedge t}}\right) = \limsup_{t\to\infty}\left(\frac{X_{T\wedge t}}{X'_{T\wedge t}}\right) - 1,$$

the *return of X relative to X' over the period* $[0, T]$.
Note that we may have $\mathbb{E}\mathrm{rr}_T(X|X') \geq 0$ and $\mathbb{E}\mathrm{rr}_T(X'|X) \geq 0$ however $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ implies $\mathbb{E}\mathrm{rr}_T(X'|X) \geq 0$.

Definition

We say that X' has the *numéraire property* in a certain class of wealth processes for investment over the period $[0, T]$ if and only if $\mathbb{E}\mathrm{rr}_T(X|X') \leq 0$ holds for all other X in the same class.

Finite horizon – existence and uniqueness

Theorem

Let T be a finite stopping time. There exists a unique $\tilde{Z} \in \mathcal{A}^\alpha$ such that $\mathbb{E}\mathrm{rr}_T(Z|\tilde{Z}) \leq 0$ holds or all $Z \in \mathcal{A}^\alpha$.

Finite horizon – existence and uniqueness

Theorem

Let T be a finite stopping time. There exists a unique $\tilde{Z} \in \mathcal{A}^\alpha$ such that $\mathbb{E}\mathrm{rr}_T(Z|\tilde{Z}) \leq 0$ holds or all $Z \in \mathcal{A}^\alpha$.

Proof:

Existence via Optional Decomposition + Convexity and boundedness in proba of $\mathcal{A}^\alpha$ + Kardaras (2010) + limiting passages + drawdown specific.

Uniqueness via strategy switching at times of maximum.

Finite horizon – existence and uniqueness

Theorem

Let T be a finite stopping time. There exists a unique $\tilde{Z} \in \mathcal{A}^\alpha$ such that $\mathbb{E}\mathrm{rr}_T(Z|\tilde{Z}) \leq 0$ holds or all $Z \in \mathcal{A}^\alpha$.

Rk: $\tilde{Z}$ solves the log-utility problem on $[0, T]$:

$$\mathbb{E}\left[\log\left(\frac{Z_T}{\tilde{Z}_T}\right)\right] \leq \mathbb{E}\left[\frac{Z_T}{\tilde{Z}_T} - 1\right] = \mathbb{E}\mathrm{rr}_T(Z|\tilde{Z}) \leq 0.$$

Rk2: However, in general $\tilde{Z}$ depends on T. In particular, the global numéraire $\hat{X}$ solves the problem up to the first time it violates the α-DD constraint.

Finite horizon – existence and uniqueness

Theorem

Let T be a finite stopping time. There exists a unique $\tilde{Z} \in \mathcal{A}^\alpha$ such that $\mathbb{E}\text{rr}_T(Z|\tilde{Z}) \leq 0$ holds or all $Z \in \mathcal{A}^\alpha$.

Rk: $\tilde{Z}$ solves the log-utility problem on $[0, T]$:

$$\mathbb{E}\left[\log\left(\frac{Z_T}{\tilde{Z}_T}\right)\right] \leq \mathbb{E}\left[\frac{Z_T}{\tilde{Z}_T} - 1\right] = \mathbb{E}\text{rr}_T(Z|\tilde{Z}) \leq 0.$$

Rk2: However, in general $\tilde{Z}$ depends on T. In particular, the global numéraire $\hat{X}$ solves the problem up to the first time it violates the α-DD constraint.

Recall that $\widehat{X}$ is the global numéraire (growth optimal) portfolio. From Cherny & O. ('13) we know that

- ${}^{\alpha}\widehat{X} := M_t^{F_\alpha}(\widehat{X})$ solves the long-run log-utility maximisation in $\mathcal{A}^{\alpha}$,
- the mapping $X \to {}^{\alpha}X := M_t^{F_\alpha}(X)$ is a bijection between $\mathcal{A}$ and $\mathcal{A}^{\alpha}$.

Theorem

For any $\alpha \in [0,1)$ and $X \in \mathcal{A}$, we have:

1. $\lim_{t\to\infty}({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t)$ *exists in* $\mathbb{R}_+$ *a.s. Moreover,*

$$\mathrm{rr}_\infty({}^{\alpha}X|{}^{\alpha}\widehat{X}) = \left(\lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right)\right)^{1-\alpha} - 1 = \left(1 + \mathrm{rr}_\infty(X|\widehat{X})\right)^{1-\alpha} - 1.$$

2. *for σ and τ two times of maximum of $\widehat{X}$ with $\sigma \le \tau$ we have*

$$\mathbb{E}\left[\mathrm{rr}_\tau({}^{\alpha}X|{}^{\alpha}\widehat{X}) \;\middle|\; \mathcal{F}_\sigma\right] \le \mathrm{rr}_\sigma({}^{\alpha}X|{}^{\alpha}\widehat{X}) \quad a.s.$$

In particular, letting $\sigma = 0$, $\mathbb{E}\mathrm{rr}_\tau(Z|{}^{\alpha}\widehat{X}) \le 0$ holds for any $\alpha \in [0,1)$ and $Z \in \mathcal{A}^{\alpha}$.

Recall that $\widehat{X}$ is the global numéraire (growth optimal) portfolio. From Cherny & O. ('13) we know that

- ${}^{\alpha}\widehat{X} := M_t^{F_\alpha}(\widehat{X})$ solves the long-run log-utility maximisation in $\mathcal{A}^\alpha$,
- the mapping $X \to {}^{\alpha}X := M_t^{F_\alpha}(X)$ is a bijection between $\mathcal{A}$ and $\mathcal{A}^\alpha$.

Theorem

For any $\alpha \in [0,1)$ and $X \in \mathcal{A}$, we have:

1. $\lim_{t\to\infty}({}^{\alpha}X_t/{}^{\alpha}\widehat{X}_t)$ *exists in* $\mathbb{R}_+$ *a.s. Moreover,*

$$\mathrm{rr}_\infty({}^{\alpha}X|{}^{\alpha}\widehat{X}) = \left(\lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right)\right)^{1-\alpha} - 1 = \left(1 + \mathrm{rr}_\infty(X|\widehat{X})\right)^{1-\alpha} - 1.$$

2. *for σ and τ two times of maximum of $\widehat{X}$ with $\sigma \le \tau$ we have*

$$\mathbb{E}\left[\mathrm{rr}_\tau({}^{\alpha}X|{}^{\alpha}\widehat{X}) \mid \mathcal{F}_\sigma\right] \le \mathrm{rr}_\sigma({}^{\alpha}X|{}^{\alpha}\widehat{X}) \quad a.s.$$

In particular, letting $\sigma = 0$, $\mathbb{E}\mathrm{rr}_\tau(Z|{}^{\alpha}\widehat{X}) \le 0$ holds for any $\alpha \in [0,1)$ and $Z \in \mathcal{A}^\alpha$.

Recall that $\widehat{X}$ is the global numéraire (growth optimal) portfolio. From Cherny & O. ('13) we know that

- ${}^{\alpha}\widehat{X} := M_t^{F_\alpha}(\widehat{X})$ solves the long-run log-utility maximisation in $\mathcal{A}^\alpha$,
- the mapping $X \to {}^{\alpha}X := M_t^{F_\alpha}(X)$ is a bijection between $\mathcal{A}$ and $\mathcal{A}^\alpha$.

Theorem

For any $\alpha \in [0,1)$ and $X \in \mathcal{A}$, we have:

1. $\lim_{t\to\infty}({}^{\alpha}X_t / {}^{\alpha}\widehat{X}_t)$ *exists in $\mathbb{R}_+$ a.s. Moreover,*

$$\mathrm{rr}_\infty({}^{\alpha}X | {}^{\alpha}\widehat{X}) = \left(\lim_{t\to\infty}\left(\frac{X_t}{\widehat{X}_t}\right)\right)^{1-\alpha} - 1 = \left(1 + \mathrm{rr}_\infty(X|\widehat{X})\right)^{1-\alpha} - 1.$$

2. *for σ and τ two times of maximum of $\widehat{X}$ with $\sigma \le \tau$ we have*

$$\mathbb{E}\left[\mathrm{rr}_\tau({}^{\alpha}X | {}^{\alpha}\widehat{X}) \;\middle|\; \mathcal{F}_\sigma\right] \le \mathrm{rr}_\sigma({}^{\alpha}X | {}^{\alpha}\widehat{X}) \quad a.s.$$

In particular, letting $\sigma = 0$, $\mathbb{E}\mathrm{rr}_\tau(Z | {}^{\alpha}\widehat{X}) \le 0$ holds for any $\alpha \in [0,1)$ and $Z \in \mathcal{A}^\alpha$.

More on asymptotic optimality

- The previous result allows to show easily that ${}^{\alpha}\widehat{X}$ maximises the growth rate in $\mathcal{A}^{\alpha}$, extending Cvitanic and Karatzas '94.
- We also show that ${}^{\alpha}\widehat{X}$ is the only process with the numéraire property along a sequence $T_n \to \infty$ a.s.
- Further, when T is large the numéraire over $[0, T]$ will be close (initially in time) to ${}^{\alpha}\widehat{X}$:

Theorem

Consider a sequence of stopping times $T_n \to \infty$ a.s. and let ${}^{\alpha}\widetilde{X}^n \in \mathcal{A}^{\alpha}$ have the numéraire property in $\mathcal{A}^{\alpha}$ over $[0, T_n]$. Then ${}^{\alpha}\widetilde{X}^n \to {}^{\alpha}\widehat{X}$ (locally) in Emery's topology.

Rk. This implies that *both* the wealth processes *and* the investment strategies converge.

More on asymptotic optimality

- The previous result allows to show easily that ${}^{\alpha}\widehat{X}$ maximises the growth rate in $\mathcal{A}^{\alpha}$, extending Cvitanic and Karatzas '94.
- We also show that ${}^{\alpha}\widehat{X}$ is the only process with the numéraire property along a sequence $T_n \to \infty$ a.s.
- Further, when T is large the numéraire over $[0, T]$ will be close (initially in time) to ${}^{\alpha}\widehat{X}$:

Theorem

Consider a sequence of stopping times $T_n \to \infty$ a.s. and let ${}^{\alpha}\widetilde{X}^n \in \mathcal{A}^{\alpha}$ have the numéraire property in $\mathcal{A}^{\alpha}$ over $[0, T_n]$. Then ${}^{\alpha}\widetilde{X}^n \to {}^{\alpha}\widehat{X}$ (locally) in Emery's topology.

Rk. This implies that *both* the wealth processes *and* the investment strategies converge.

More on asymptotic optimality

- The previous result allows to show easily that ${}^{\alpha}\widehat{X}$ maximises the growth rate in $\mathcal{A}^{\alpha}$, extending Cvitanic and Karatzas '94.
- We also show that ${}^{\alpha}\widehat{X}$ is the only process with the numéraire property along a sequence $T_n \to \infty$ a.s.
- Further, when T is large the numéraire over $[0, T]$ will be close (initially in time) to ${}^{\alpha}\widehat{X}$:

Theorem

Consider a sequence of stopping times $T_n \to \infty$ a.s. and let ${}^{\alpha}\widetilde{X}^n \in \mathcal{A}^{\alpha}$ have the numéraire property in $\mathcal{A}^{\alpha}$ over $[0, T_n]$. Then ${}^{\alpha}\widetilde{X}^n \to {}^{\alpha}\widehat{X}$ (locally) in Emery's topology.

Rk. This implies that *both* the wealth processes *and* the investment strategies converge.

More on asymptotic optimality

- The previous result allows to show easily that ${}^{\alpha}\widehat{X}$ maximises the growth rate in $\mathcal{A}^{\alpha}$, extending Cvitanic and Karatzas '94.
- We also show that ${}^{\alpha}\widehat{X}$ is the only process with the numéraire property along a sequence $T_n \to \infty$ a.s.
- Further, when T is large the numéraire over $[0, T]$ will be close (initially in time) to ${}^{\alpha}\widehat{X}$:

Theorem

Consider a sequence of stopping times $T_n \to \infty$ a.s. and let ${}^{\alpha}\widetilde{X}^n \in \mathcal{A}^{\alpha}$ have the numéraire property in $\mathcal{A}^{\alpha}$ over $[0, T_n]$. Then ${}^{\alpha}\widetilde{X}^n \to {}^{\alpha}\widehat{X}$ (locally) in Emery's topology.

Rk. This implies that *both* the wealth processes *and* the investment strategies converge.

The End: Moral

We obtained a framework where a long term investor's optimal strategy was

$$d\,{}^{\alpha}\widehat{X}_t = \left({}^{\alpha}\widehat{X}_t - \alpha \sup_{u \le t} {}^{\alpha}\widehat{X}_u\right) \frac{d\hat{X}_t}{\hat{X}_t}.$$

Preferences (α) and model ($\hat{X}$) are decoupled.

1. Long-run investment, risk attitudes via drawdown constraints
   - Kelly's long-run investor and the numéraire property
   - Numéraire under drawdown – finite horizon
   - Numéraire under drawdown – asymptotics
2. Robust forward performance criteria
   - Model uncertainty, variational preferences and time homogeneity
   - Logarithmic preferences and fractional Kelly
   - Duality and (S)PDEs
3. Conclusions

So far we obtained a framework where a long term investor's optimal strategy was

$$\mathrm{d}{}^{\alpha}\widehat{X}_t = \Big({}^{\alpha}\widehat{X}_t - \alpha \sup_{u\le t} {}^{\alpha}\widehat{X}_u\Big)\frac{\mathrm{d}\hat{X}_t}{\hat{X}_t}.$$

Preferences (α) and model ($\hat{X}$) are decoupled but still a perfect knowledge of $\hat{X}$ is assumed.

We would like to advance a framework where

- time horizon is arbitrary (neither fixed nor $+\infty$)
- $\hat{X}$ is dynamically estimated and investor's trust in the estimate is reflected in the strategy.

So far we obtained a framework where a long term investor's optimal strategy was

$$\mathrm{d}{}^{\alpha}\widehat{X}_t = \left({}^{\alpha}\widehat{X}_t - \alpha \sup_{u \le t} {}^{\alpha}\widehat{X}_u\right) \frac{\mathrm{d}\hat{X}_t}{\hat{X}_t}.$$

Preferences (α) and model ($\hat{X}$) are decoupled but still a perfect knowledge of $\hat{X}$ is assumed.

We would like to advance a framework where

- time horizon is arbitrary (neither fixed nor $+\infty$)
- $\hat{X}$ is dynamically estimated and investor's trust in the estimate is reflected in the strategy.

We now combine the idea of forward performance/horizon unbiased with variational preferences under model uncertainty (Musiela & Zariphopoulou '09; Henderson & Hobson '07; Gilboa & Schmeidler '89; Maccheroni, Marinacci & Rustichini '06; Schied '07).

Definition (Protagonists:)

A utility random field $U : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is $(\mathcal{F}_t)$–prog. measurable and

- $\forall\ (\omega, t) \in \Omega \times [0, \infty)$, $U(\omega, \cdot, t)$ is a (nice) utility function
- $U(\omega, x, \cdot)$ is càdlàg and $U(\cdot, x, t) \in L^1(\mathcal{F}_t)$.

A family of penalty functions

$$\gamma_{t,T} : \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{F}_T\} \to [0, \infty]$$

convex, l.s.c., finite on a weakly compact set.

We now combine the idea of forward performance/horizon unbiased with variational preferences under model uncertainty (Musiela & Zariphopoulou '09; Henderson & Hobson '07; Gilboa & Schmeidler '89; Maccheroni, Marinacci & Rustichini '06; Schied '07).

Definition (Protagonists:)

A utility random field $U : \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is $(\mathcal{F}_t)$–prog. measurable and

- $\forall\ (\omega, t) \in \Omega \times [0, \infty)$, $U(\omega, \cdot, t)$ is a (nice) utility function
- $U(\omega, x, \cdot)$ is càdlàg and $U(\cdot, x, t) \in L^1(\mathcal{F}_t)$.

A family of penalty functions

$$\gamma_{t,T} : \{\mathbb{Q} : \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{F}_T\} \to [0, \infty]$$

convex, l.s.c., finite on a weakly compact set.

Definition (Dynamic consistency:)

The pair U and $\gamma_{t,T}$ is a robust forward performance (or is time consistent) if

- $\mathbb{E}^{\mathbb{Q}}[U(T,x)]$ is well defined in $(-\infty,\infty]$ for all T,x for $\mathbb{Q}$ with $\gamma_{t,T}(\mathbb{Q})<\infty$,
- $$U(\xi,t)=u(\xi;t,T)\ a.s.\quad \forall 0\le t\le T<\infty,\ \xi\in L^{\infty}(\mathcal{F}_t),$$
  where u is the value function

$$u(\xi;t,T):=\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{bd}}\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_T}\left\{\mathbb{E}^{\mathbb{Q}}\left[U\left(\xi+\int_t^T\pi_u\mathrm{d}S_u,\,T\right)\middle|\mathcal{F}_t\right]+\gamma_{t,T}(\mathbb{Q})\right\}.$$

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $(\mathcal{F}_t)$ generated by a (1 or d-dim) Brownian motion W, prog. measurable λ, σ and

$$\mathrm{d}S_t = S_t\sigma_t(\lambda_t \mathrm{d}t + \mathrm{d}W_t), \quad t \geq 0.$$

This is "true" model, unknown. Instead agent builds her "best prediction" or most likely model described by $\hat{\lambda}$ with $\hat{\mathbb{P}} \sim \mathbb{P}$ on $\mathcal{F}_T$, for all $T > 0$, where

$$\left.\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^{\cdot} (\hat{\lambda}_s - \lambda_s)\mathrm{d}W_s\right)_T.$$

Observe that

$$\mathrm{d}S_t = S_t\sigma_t(\hat{\lambda}_t \mathrm{d}t + \mathrm{d}\hat{W}_t), \qquad \text{for a } \hat{\mathbb{P}} \text{ Brownian motion } \hat{W}.$$

$\hat{\mathbb{P}}$ is "reasonable" in that $\hat{\mathbb{E}}[\int_0^T \hat{\lambda}_s^2 \mathrm{d}s] < \infty$, $T > 0$.
Given $\mathbb{Q} \ll \hat{\mathbb{P}}$ on $\mathcal{F}_T$ we write $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ where

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\hat{\mathbb{P}}}\right|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^{\cdot} \hat{\eta}_s \mathrm{d}\hat{W}_s\right)_T.$$

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $(\mathcal{F}_t)$ generated by a (1 or d-dim) Brownian motion W, prog. measurable λ, σ and

$$\mathrm{d}S_t = S_t\sigma_t(\lambda_t \mathrm{d}t + \mathrm{d}W_t), \quad t \geq 0.$$

This is "true" model, unknown. Instead agent builds her "best prediction" or most likely model described by $\hat{\lambda}$ with $\hat{\mathbb{P}} \sim \mathbb{P}$ on $\mathcal{F}_T$, for all $T > 0$, where

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^{\cdot} (\hat{\lambda}_s - \lambda_s)\mathrm{d}W_s\right)_T.$$

Observe that

$$\mathrm{d}S_t = S_t\sigma_t(\hat{\lambda}_t \mathrm{d}t + \mathrm{d}\hat{W}_t), \qquad \text{for a } \hat{\mathbb{P}} \text{ Brownian motion } \hat{W}.$$

$\hat{\mathbb{P}}$ is "reasonable" in that $\hat{\mathbb{E}}[\int_0^T \hat{\lambda}_s^2 \mathrm{d}s] < \infty$, $T > 0$.

Given $\mathbb{Q} \ll \hat{\mathbb{P}}$ on $\mathcal{F}_T$ we write $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ where

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\hat{\mathbb{P}}}\Big|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^{\cdot} \hat{\eta}_s \mathrm{d}\hat{W}_s\right)_T.$$

Consider $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $(\mathcal{F}_t)$ generated by a (1 or d-dim) Brownian motion W, prog. measurable λ, σ and

$$\mathrm{d}S_t = S_t\sigma_t(\lambda_t \mathrm{d}t + \mathrm{d}W_t), \quad t \geq 0.$$

This is "true" model, unknown. Instead agent builds her "best prediction" or most likely model described by $\hat{\lambda}$ with $\hat{\mathbb{P}} \sim \mathbb{P}$ on $\mathcal{F}_T$, for all $T > 0$, where

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^{\cdot} (\hat{\lambda}_s - \lambda_s)\mathrm{d}W_s\right)_T.$$

Observe that

$$\mathrm{d}S_t = S_t\sigma_t(\hat{\lambda}_t \mathrm{d}t + \mathrm{d}\hat{W}_t), \quad \text{for a } \hat{\mathbb{P}} \text{ Brownian motion } \hat{W}.$$

$\hat{\mathbb{P}}$ is "reasonable" in that $\hat{\mathbb{E}}[\int_0^T \hat{\lambda}_s^2 \mathrm{d}s] < \infty$, $T > 0$.
Given $\mathbb{Q} \ll \hat{\mathbb{P}}$ on $\mathcal{F}_T$ we write $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ where

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\hat{\mathbb{P}}}\Big|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^{\cdot} \hat{\eta}_s \mathrm{d}\hat{W}_s\right)_T.$$

Logarithmic preferences

Propositon

Let $\hat{\lambda}$ as above and $\delta \geq 0$ prog. measurable. The utility field

$$U(x,t) := \ln x - \frac{1}{2}\frac{\delta_t}{1+\delta_t}\int_0^t \hat{\lambda}_s^2 \mathrm{d}s$$

and the penalty function

$$\gamma_{t,T}(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{\delta_s}{2}\hat{\eta}_s^2 \mathrm{d}s \middle| \mathcal{F}_t\right] \quad \text{if } \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{\delta_s}{1+\delta_s}\hat{\lambda}_s^2 \mathrm{d}s\right] < \infty$$

and $+\infty$ elsewhere, form a robust forward criteria.
Investor's optimal wealth process evolves as

$$\mathrm{d}X_t^{\bar{\pi}} = \frac{\delta_t}{1+\delta_t}\frac{\hat{\lambda}_t}{\sigma_t}X_t^{\bar{\pi}}\frac{\mathrm{d}S_t}{S_t}$$

Remarks

- The choice of learning and investor's confidence, i.e. choice of $\hat{\lambda}$ and δ, arbitrary!
- Under $\hat{\mathbb{P}}$ the Kelly/growth optimal portfolio is

$$\mathrm{d}\hat{X}_t = \frac{\hat{\lambda}_t}{\sigma_t}\hat{X}_t\frac{\mathrm{d}S_t}{S_t}$$

- The investor follows a fractional Kelly strategy, investing a fraction $\frac{\delta_t}{1+\delta_t}$ of her wealth

$$\mathrm{d}X_t^{\bar{\pi}} = \frac{\delta_t}{1+\delta_t}X_t^{\bar{\pi}}\frac{\mathrm{d}\hat{X}_t}{\hat{X}_t} = \frac{\delta_t}{1+\delta_t}\frac{\hat{\lambda}_t}{\lambda_t}\frac{\lambda_t}{\sigma_t}X_t^{\bar{\pi}}\frac{\mathrm{d}S_t}{S_t} = \frac{\delta_t\hat{\lambda}_t}{1+\delta_t}\frac{1}{\sigma_t}X_t^{\bar{\pi}}\frac{\mathrm{d}S_t}{S_t}.$$

which is the Kelly strategy under $\bar{\mathbb{P}} := \mathbb{Q}^{\bar{\eta}}$ for $\bar{\eta}_t := \frac{-\hat{\lambda}_t}{1+\delta_t}$.

Remarks

- The choice of learning and investor's confidence, i.e. choice of $\hat{\lambda}$ and δ, arbitrary!
- Under $\hat{\mathbb{P}}$ the Kelly/growth optimal portfolio is

$$\mathrm{d}\hat{X}_t = \frac{\hat{\lambda}_t}{\sigma_t}\hat{X}_t\frac{\mathrm{d}S_t}{S_t}$$

- The investor follows a fractional Kelly strategy, investing a fraction $\frac{\delta_t}{1+\delta_t}$ of her wealth

$$\mathrm{d}X_t^{\bar{\pi}} = \frac{\delta_t}{1+\delta_t}X_t^{\bar{\pi}}\frac{\mathrm{d}\hat{X}_t}{\hat{X}_t} = \frac{\delta_t}{1+\delta_t}\frac{\hat{\lambda}_t}{\lambda_t}\frac{\lambda_t}{\sigma_t}X_t^{\bar{\pi}}\frac{\mathrm{d}S_t}{S_t} = \frac{\delta_t\hat{\lambda}_t}{1+\delta_t}\frac{1}{\sigma_t}X_t^{\bar{\pi}}\frac{\mathrm{d}S_t}{S_t}.$$

which is the Kelly strategy under $\bar{\mathbb{P}} := \mathbb{Q}^{\bar{\eta}}$ for $\bar{\eta}_t := \frac{-\lambda_t}{1+\delta_t}$.

Remarks

- The choice of learning and investor's confidence, i.e. choice of $\hat{\lambda}$ and δ, arbitrary!
- Under $\hat{\mathbb{P}}$ the Kelly/growth optimal portfolio is

$$\mathrm{d}\hat{X}_t = \frac{\hat{\lambda}_t}{\sigma_t}\hat{X}_t\frac{\mathrm{d}S_t}{S_t}$$

- The investor follows a fractional Kelly strategy, investing a fraction $\frac{\delta_t}{1+\delta_t}$ of her wealth

$$\mathrm{d}X^{\bar{\pi}}_t = \frac{\delta_t}{1+\delta_t}X^{\bar{\pi}}_t\frac{\mathrm{d}\hat{X}_t}{\hat{X}_t} = \frac{\delta_t}{1+\delta_t}\frac{\hat{\lambda}_t}{\lambda_t}\frac{\lambda_t}{\sigma_t}X^{\bar{\pi}}_t\frac{\mathrm{d}S_t}{S_t} = \frac{\delta_t\hat{\lambda}_t}{1+\delta_t}\frac{1}{\sigma_t}X^{\bar{\pi}}_t\frac{\mathrm{d}S_t}{S_t}.$$

which is the Kelly strategy under $\bar{\mathbb{P}} := \mathbb{Q}^{\bar{\eta}}$ for $\bar{\eta}_t := \frac{-\hat{\lambda}_t}{1+\delta_t}$.

Remarks

- The choice of learning and investor's confidence, i.e. choice of $\hat{\lambda}$ and δ, arbitrary!
- Under $\hat{\mathbb{P}}$ the Kelly/growth optimal portfolio is

$$\mathrm{d}\hat{X}_t = \frac{\hat{\lambda}_t}{\sigma_t}\hat{X}_t\frac{\mathrm{d}S_t}{S_t}$$

- The investor follows a fractional Kelly strategy, investing a fraction $\frac{\delta_t}{1+\delta_t}$ of her wealth

$$\mathrm{d}X_t^{\bar{\pi}} = \frac{\delta_t}{1+\delta_t}X_t^{\bar{\pi}}\frac{\mathrm{d}\hat{X}_t}{\hat{X}_t} = \frac{\delta_t}{1+\delta_t}\frac{\hat{\lambda}_t}{\lambda_t}\frac{\lambda_t}{\sigma_t}X_t^{\bar{\pi}}\frac{\mathrm{d}S_t}{S_t} = \frac{\delta_t\hat{\lambda}_t}{1+\delta_t}\frac{1}{\sigma_t}X_t^{\bar{\pi}}\frac{\mathrm{d}S_t}{S_t}.$$

which is the Kelly strategy under $\bar{\mathbb{P}} := \mathbb{Q}^{\bar{\eta}}$ for $\bar{\eta}_t := \frac{-\hat{\lambda}_t}{1+\delta_t}$.

Remarks

- The choice of learning and investor's confidence, i.e. choice of $\hat{\lambda}$ and δ, arbitrary!
- Under $\hat{\mathbb{P}}$ the Kelly/growth optimal portfolio is

$$\mathrm{d}\hat{X}_t = \frac{\hat{\lambda}_t}{\sigma_t}\hat{X}_t \frac{\mathrm{d}S_t}{S_t}$$

- The investor follows a fractional Kelly strategy, investing a fraction $\frac{\delta_t}{1+\delta_t}$ of her wealth

$$\mathrm{d}X_t^{\bar{\pi}} = \frac{\delta_t}{1+\delta_t} X_t^{\bar{\pi}} \frac{\mathrm{d}\hat{X}_t}{\hat{X}_t} = \frac{\delta_t}{1+\delta_t}\frac{\hat{\lambda}_t}{\lambda_t}\frac{\lambda_t}{\sigma_t} X_t^{\bar{\pi}} \frac{\mathrm{d}S_t}{S_t} = \frac{\delta_t \hat{\lambda}_t}{1+\delta_t}\frac{1}{\sigma_t} X_t^{\bar{\pi}} \frac{\mathrm{d}S_t}{S_t}.$$

which is the Kelly strategy under $\bar{\mathbb{P}} := \mathbb{Q}^{\bar{\eta}}$ for $\bar{\eta}_t := \frac{-\hat{\lambda}_t}{1+\delta_t}$.

First Proof (direct)

W.l.o.g. $t = 0$. Given π and $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ define

$$N_t^{\pi,\hat{\eta}} := U(X_t^\pi, t) + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u = \ln X_t^\pi - \int_0^t \hat{\lambda}_u \mathrm{d}u + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u$$

Then

$$u(x_0; t, T) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_T} \left\{ \mathbb{E}^{\mathbb{Q}}\left[U\left(X_T^\pi, T\right)\right] + \gamma_{0,T}(\mathbb{Q}) \right\}$$

First Proof (direct)

W.l.o.g. $t = 0$. Given π and $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ define

$$N_t^{\pi,\hat{\eta}} := U(X_t^{\pi}, t) + \int_0^t \frac{\delta_u}{2} \hat{\eta}_u^2 \mathrm{d}u = \ln X_t^{\pi} - \int_0^t \hat{\lambda}_u \mathrm{d}u + \int_0^t \frac{\delta_u}{2} \hat{\eta}_u^2 \mathrm{d}u$$

Then

$$u(x_0; t, T) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}_T} \mathbb{E}^{\mathbb{Q}} \left[U\left(X_T^{\pi}, T\right) + \int_0^T \frac{\delta_u}{2} \hat{\eta}_u^2 \mathrm{d}u \right]$$

First Proof (direct)

W.l.o.g. $t = 0$. Given π and $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ define

$$N_t^{\pi,\hat{\eta}} := U(X_t^{\pi}, t) + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u = \ln X_t^{\pi} - \int_0^t \hat{\lambda}_u \mathrm{d}u + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u$$

Then

$$u(x_0; t, T) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}_T} \mathbb{E}^{\mathbb{Q}}\left[N_T^{\pi,\hat{\eta}}\right]$$

First Proof (direct)

W.l.o.g. $t = 0$. Given π and $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ define

$$N_t^{\pi,\hat{\eta}} := U(X_t^{\pi}, t) + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u = \ln X_t^{\pi} - \int_0^t \hat{\lambda}_u \mathrm{d}u + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u$$

Then

$$u(x_0; t, T) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}_T} \mathbb{E}^{\mathbb{Q}}\left[N_T^{\pi,\hat{\eta}}\right]$$

First Proof (direct)

W.l.o.g. $t = 0$. Given π and $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ define

$$N_t^{\pi,\hat{\eta}} := U(X_t^{\pi}, t) + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u = \ln X_t^{\pi} - \int_0^t \hat{\lambda}_u \mathrm{d}u + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u$$

Then

$$u(x_0; t, T) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}_T} \mathbb{E}^{\mathbb{Q}} \left[N_T^{\pi,\hat{\eta}} \right]$$

A direct computation gives $\forall \pi \in \mathcal{A}$, $N_t^{\pi,\bar{\eta}}$ is a supermartingale

$$u(x_0; 0, T) \leq \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \mathbb{E}^{\bar{\mathbb{P}}} \left[N_T^{\pi,\bar{\hat{\eta}}} \right] \leq N_0^{\pi,\bar{\hat{\eta}}} = U(x_0, 0).$$

First Proof (direct)

W.l.o.g. $t = 0$. Given π and $\mathbb{Q} = \mathbb{Q}^{\hat{\eta}}$ define

$$N_t^{\pi,\hat{\eta}} := U(X_t^\pi, t) + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u = \ln X_t^\pi - \int_0^t \hat{\lambda}_u \mathrm{d}u + \int_0^t \frac{\delta_u}{2}\hat{\eta}_u^2 \mathrm{d}u$$

Then

$$u(x_0; t, T) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_T} \mathbb{E}^{\mathbb{Q}}\left[N_T^{\pi,\hat{\eta}}\right]$$

A direct computation gives $\forall \mathbb{Q} \in \mathcal{Q}_T$, $N_t^{\bar{\pi},\hat{\eta}}$ is a submartingale

$$u(x_0; 0, T) \geq \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_T} \mathbb{E}^{\mathbb{Q}}\left[N_T^{\bar{\pi},\hat{\eta}}\right] \geq N_0^{\pi,\hat{\eta}} = U(x_0, 0).$$

Dual field

Consider now a general semimartingale setup, U on $\mathbb{R}$ and $\mathcal{A} = \mathcal{A}_{bd}$.

Let V be the Fenchel transform of U: $V(t, y) = \sup_{x\in\mathbb{R}}(U(t, x) - xy)$.

Definition (3rd protagonist: the Dual field)

Given a utility field U and a penalty function γ, the dual field v is

$$v(\eta; t, T) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_T} \operatorname*{ess\,inf}_{\mathbb{M}\in\mathcal{M}_T^{\mathbb{Q}}} \left\{ \mathbb{E}^{\mathbb{Q}} \left[V\left(\eta Z_{t,T}^{\mathbb{MQ}}, T\right) \middle| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}) \right\},$$

for $\eta \in L^0_+(\mathcal{F}_t)$ and where $Z_{t,T}^{\mathbb{MQ}} = \frac{d\mathbb{M}}{d\mathbb{Q}}\Big|\mathcal{F}_T \cdot \frac{d\mathbb{Q}}{d\mathbb{M}}\Big|\mathcal{F}_t$ and $\mathcal{M}_T^{\mathbb{Q}}$ are $\mathbb{Q}$–abs. cont. local martingale measures.

The pair of dual field V and the family of penalty functions $\gamma_{t,T}$ is time-homogeneous if

$$V(\eta, t) = v(\eta; t, T) \text{ a.s.}$$

for all $0 \le t \le T$ and $\eta \in L^0_+(\mathcal{F}_t)$.

Rk: Global inf instead of a saddle point!

Dual field

Consider now a general semimartingale setup, U on $\mathbb{R}$ and $\mathcal{A} = \mathcal{A}_{bd}$.
Let V be the Fenchel transform of U: $V(t, y) = \sup_{x\in\mathbb{R}}(U(t, x) - xy)$.

Definition (3rd protagonist: the Dual field)

Given a utility field U and a penalty function γ, the dual field v is

$$v(\eta; t, T) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_T} \operatorname*{ess\,inf}_{\mathbb{M}\in\mathcal{M}_T^{\mathbb{Q}}} \left\{ \mathbb{E}^{\mathbb{Q}} \left[V\left(\eta Z_{t,T}^{\mathbb{M}\mathbb{Q}}, T\right) \middle| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}) \right\},$$

for $\eta \in L^0_+(\mathcal{F}_t)$ and where $Z_{t,T}^{\mathbb{M}\mathbb{Q}} = \frac{\mathrm{d}\mathbb{M}}{\mathrm{d}\mathbb{Q}}\Big|\mathcal{F}_T \cdot \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{M}}\Big|\mathcal{F}_t$ and $\mathcal{M}_T^{\mathbb{Q}}$ are $\mathbb{Q}$–abs. cont. local martingale measures.
The pair of dual field V and the family of penalty functions $\gamma_{t,T}$ is time-homogeneous if

$$V(\eta, t) = v(\eta; t, T) \ a.s.$$

for all $0 \le t \le T$ and $\eta \in L^0_+(\mathcal{F}_t)$.

Rk: Global inf instead of a saddle point!

Dual field

Consider now a general semimartingale setup, U on $\mathbb{R}$ and $\mathcal{A} = \mathcal{A}_{bd}$.
Let V be the Fenchel transform of U: $V(t, y) = \sup_{x\in\mathbb{R}}(U(t, x) - xy)$.

Definition (3rd protagonist: the Dual field)

Given a utility field U and a penalty function γ, the dual field v is

$$v(\eta; t, T) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_T} \operatorname*{ess\,inf}_{\mathbb{M}\in\mathcal{M}_T^{\mathbb{Q}}} \left\{ \mathbb{E}^{\mathbb{Q}} \left[V\left(\eta Z_{t,T}^{\mathbb{MQ}}, T\right) \Big| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}) \right\},$$

for $\eta \in L^0_+(\mathcal{F}_t)$ and where $Z_{t,T}^{\mathbb{MQ}} = \frac{d\mathbb{M}}{d\mathbb{Q}}\Big|\mathcal{F}_T \cdot \frac{d\mathbb{Q}}{d\mathbb{M}}\Big|\mathcal{F}_t$ and $\mathcal{M}_T^{\mathbb{Q}}$ are $\mathbb{Q}$–abs. cont. local martingale measures.
The pair of dual field V and the family of penalty functions $\gamma_{t,T}$ is time-homogeneous if

$$V(\eta, t) = v(\eta; t, T) \ a.s.$$

for all $0 \leq t \leq T$ and $\eta \in L^0_+(\mathcal{F}_t)$.

Rk: Global inf instead of a saddle point!

Duality theorem

Theorem

(under some integrability and compactness assumptions) The primal and dual value functions satisfy

$$u(\xi; t, T) = \operatorname*{ess\,inf}_{\eta \in L^0_+(\mathcal{F}_t)} \left(v(\eta; t, T) + \xi\eta \right) \quad a.s.$$
$$v(\eta; t, T) = \operatorname*{ess\,sup}_{\xi \in L^\infty(\mathcal{F}_t)} \left(u(\xi; t, T) - \xi\eta \right) \quad a.s. \tag{1}$$

for all $0 \le t \le T$, $\xi \in L^\infty(\mathcal{F}_t)$ *and* $\eta \in L^0_+(\mathcal{F}_t)$.

Proof: Follows the ideas in Schied '07 but using duality in Zitkovic '09 instead of Kramkov & Schachermayer '99.

Corollary

U and γ are time-consistent if and only if V and γ are.

Duality theorem

Theorem

(under some integrability and compactness assumptions) The primal and dual value functions satisfy

$$
\begin{aligned}
u(\xi; t, T) &= \operatorname*{ess\,inf}_{\eta \in L^0_+(\mathcal{F}_t)} \left(v(\eta; t, T) + \xi\eta \right) \quad a.s. \\
v(\eta; t, T) &= \operatorname*{ess\,sup}_{\xi \in L^\infty(\mathcal{F}_t)} \left(u(\xi; t, T) - \xi\eta \right) \quad a.s.
\end{aligned}
\tag{1}
$$

for all $0 \le t \le T$, $\xi \in L^\infty(\mathcal{F}_t)$ *and* $\eta \in L^0_+(\mathcal{F}_t)$.

Proof: Follows the ideas in Schied '07 but using duality in Zitkovic '09 instead of Kramkov & Schachermayer '99.

Corollary

U *and* γ *are time-consistent if and only if* V *and* γ *are.*

To model uncertainty and back

Consider again a Brownian filtration, estimated model $\hat{\mathbb{P}}$, and

$$\gamma_{t,T}(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^T g_u(\hat{\eta}_u)\mathrm{d}u \Big| \mathcal{F}_t\right],$$

g_t convex, l.s.c., $g_t(\eta) \geq -a + b|\eta|^2$. If U, γ are time-consistent and a saddle point $(\bar{\pi}, \bar{\eta})$ exists we have

$$U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u = \sup_{\pi \in \mathcal{A}} \mathbb{E}^{\bar{\mathbb{Q}}}\left[U\left(x + \int_t^T \pi_u \mathrm{d}S_u\right) + \int_0^T g_u(\bar{\eta}_u)\mathrm{d}u \Big| \mathcal{F}_t\right]$$

and hence, in the spirit of Skiadas '03, the problem is equivalent to non-robust forward performance criteria $\bar{U}(x,t) = U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u$ under $\bar{\mathbb{Q}}$. Or yet, to the (non-robust) forward problem under $\mathbb{P}$ with

$$\tilde{U}(x,t) := \frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} \cdot \bar{U}(x,t) = \frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}\left(U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u\right).$$

Note that $\tilde{U}$ necessarily has non-trivial volatility.

To model uncertainty and back

Consider again a Brownian filtration, estimated model $\hat{\mathbb{P}}$, and

$$\gamma_{t,T}(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^T g_u(\hat{\eta}_u)\mathrm{d}u \Big| \mathcal{F}_t\right],$$

g_t convex, l.s.c., $g_t(\eta) \geq -a + b|\eta|^2$. If U, γ are time-consistent and a saddle point $(\bar{\pi}, \bar{\eta})$ exists we have

$$U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u = \sup_{\pi\in\mathcal{A}} \mathbb{E}^{\bar{\mathbb{Q}}}\left[U\left(x + \int_t^T \pi_u \mathrm{d}S_u\right) + \int_0^T g_u(\bar{\eta}_u)\mathrm{d}u \Big| \mathcal{F}_t\right]$$

and hence, in the spirit of Skiadas '03, the problem is equivalent to non-robust forward performance criteria $\bar{U}(x,t) = U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u$ under $\bar{\mathbb{Q}}$. Or yet, to the (non-robust) forward problem under $\mathbb{P}$ with

$$\tilde{U}(x,t) := \frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} \cdot \bar{U}(x,t) = \frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}\left(U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u\right).$$

Note that $\tilde{U}$ necessarily has non-trivial volatility.

To model uncertainty and back

Consider again a Brownian filtration, estimated model $\hat{\mathbb{P}}$, and

$$\gamma_{t,T}(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^T g_u(\hat{\eta}_u)\mathrm{d}u \Big| \mathcal{F}_t\right],$$

g_t convex, l.s.c., $g_t(\eta) \geq -a + b|\eta|^2$. If U, γ are time-consistent and a saddle point $(\bar{\pi}, \bar{\eta})$ exists we have

$$U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u = \sup_{\pi \in \mathcal{A}} \mathbb{E}^{\bar{\mathbb{Q}}}\left[U\left(x + \int_t^T \pi_u \mathrm{d}S_u\right) + \int_0^T g_u(\bar{\eta}_u)\mathrm{d}u \Big| \mathcal{F}_t\right]$$

and hence, in the spirit of Skiadas '03, the problem is equivalent to non-robust forward performance criteria $\bar{U}(x,t) = U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u$ under $\bar{\mathbb{Q}}$. Or yet, to the (non-robust) forward problem under $\mathbb{P}$ with

$$\tilde{U}(x,t) := \frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} \cdot \bar{U}(x,t) = \frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}\left(U(x,t) + \int_0^t g_u(\bar{\eta}_u)\mathrm{d}u\right).$$

Note that $\tilde{U}$ necessarily has non-trivial volatility.

Non-volatile criteria

Time consistency of the dual field boils down to

$$V(yZ_t^{\mathbb{M}\mathbb{Q}}, t) \leq \mathbb{E}^{\mathbb{Q}}\left[V\left(yZ_T^{\mathbb{M}\mathbb{Q}}, T\right)\middle|\mathcal{F}_t\right] + \gamma_{t,T}(\mathbb{Q})$$

with equality for some $\bar{\mathbb{M}}$, $\bar{\mathbb{Q}}$.

We expect V to follow

$$\mathrm{d}V(y,t) = b(y,t)\mathrm{d}t + a(y,t)\mathrm{d}W_t$$

which should lead to SPDE for V (or U).

In the non-robust setting ($g \equiv 0$) we recover

$$\mathrm{d}U(x,t) = \frac{1}{2}\frac{|\lambda_t U_x(x,t) + \sigma_t\sigma_t'\tilde{a}_x(x,t)|^2}{U_{xx}(x,t)}\mathrm{d}t + \tilde{a}(x,t)\mathrm{d}W_t$$

Non-volatile criteria

Time consistency of the dual field boils down to

$$V(yZ_t^{\mathbb{M}\mathbb{Q}}, t) \leq \mathbb{E}^{\mathbb{Q}}\left[V\left(yZ_T^{\mathbb{M}\mathbb{Q}}, T\right)\Big|\mathcal{F}_t\right] + \gamma_{t,T}(\mathbb{Q})$$

with equality for some $\bar{\mathbb{M}}$, $\bar{\mathbb{Q}}$.

We expect V to follow

$$\mathrm{d}V(y,t) = b(y,t)\mathrm{d}t + a(y,t)\mathrm{d}W_t$$

which should lead to SPDE for V (or U).

In the non-robust setting ($g \equiv 0$) we recover

$$\mathrm{d}U(x,t) = \frac{1}{2}\frac{|\lambda_t U_x(x,t) + \sigma_t\sigma_t'\tilde{a}_x(x,t)|^2}{U_{xx}(x,t)}\mathrm{d}t + \tilde{a}(x,t)\mathrm{d}W_t$$

Non-volatile criteria

Time consistency of the dual field boils down to

$$V(yZ_t^{\mathbb{M}\mathbb{Q}}, t) \leq \mathbb{E}^{\mathbb{Q}}\left[V\left(yZ_T^{\mathbb{M}\mathbb{Q}}, T\right)\Big|\mathcal{F}_t\right] + \gamma_{t,T}(\mathbb{Q})$$

with equality for some $\bar{\mathbb{M}}$, $\bar{\mathbb{Q}}$.

We expect V to follow

$$\mathrm{d}V(y,t) = b(y,t)\mathrm{d}t + a(y,t)\mathrm{d}W_t$$

which should lead to SPDE for V (or U).

In the non-robust setting ($g \equiv 0$) we recover

$$\mathrm{d}U(x,t) = \frac{1}{2}\frac{|\lambda_t U_x(x,t) + \sigma_t\sigma_t'\tilde{a}_x(x,t)|^2}{U_{xx}(x,t)}\mathrm{d}t + \tilde{a}(x,t)\mathrm{d}W_t$$

Non-volatile criteria (cont.)

Now, if $a \equiv 0$ the submartingale property $\Rightarrow$ a random PDE

$$V_t(y,t) + \inf_{\eta} \left\{ g(\eta) + \frac{y^2 V_{yy}(y,t)}{2} (\eta + \lambda)^2 \right\} = 0, \quad a.s., t \geq 0.$$

Existence? Two difficulties:

- non-linearity: optimal $\bar{\eta}$ in function of V_{yy}
- solving for all $t \geq 0$: even if $g \equiv 0$, changing variables $V_y(y,t) = -h(\ln y + \frac{1}{2}\int_0^t \lambda_u^2 \mathrm{d}u, \int_0^t \lambda_u^2 \mathrm{d}u)$, we obtain

$$h_t(y,t) + \frac{1}{2} h_{yy}(y,t) = 0, \quad a.s., t \geq 0$$

  the backward heat equation. Solutions characterised by Widder's thm.

Taking $V(y,t) = -\ln y + \int_0^t b_u \mathrm{d}u$ and g quadratic leads to the logarithmic example.

Non-volatile criteria (cont.)

Now, if $a \equiv 0$ the submartingale property $\Rightarrow$ a random PDE

$$V_t(y,t) + \inf_\eta \left\{ g(\eta) + \frac{y^2 V_{yy}(y,t)}{2}(\eta+\lambda)^2 \right\} = 0, \quad a.s., t \geq 0.$$

Existence? Two difficulties:

- non-linearity: optimal $\bar{\eta}$ in function of V_{yy}
- solving for all $t \geq 0$: even if $g \equiv 0$, changing variables $V_y(y,t) = -h(\ln y + \frac{1}{2}\int_0^t \lambda_u^2 \mathrm{d}u, \int_0^t \lambda_u^2 \mathrm{d}u)$, we obtain

$$h_t(y,t) + \frac{1}{2} h_{yy}(y,t) = 0, \quad a.s., t \geq 0$$

  the backward heat equation. Solutions characterised by Widder's thm.

Taking $V(y,t) = -\ln y + \int_0^t b_u \mathrm{d}u$ and g quadratic leads to the logarithmic example.

Non-volatile criteria (cont.)

Now, if $a \equiv 0$ the submartingale property $\Rightarrow$ a random PDE

$$V_t(y,t) + \inf_{\eta} \left\{ g(\eta) + \frac{y^2 V_{yy}(y,t)}{2} (\eta + \lambda)^2 \right\} = 0, \quad a.s., t \geq 0.$$

Existence? Two difficulties:

- non-linearity: optimal $\bar{\eta}$ in function of V_{yy}
- solving for all $t \geq 0$: even if $g \equiv 0$, changing variables $V_y(y,t) = -h(\ln y + \frac{1}{2}\int_0^t \lambda_u^2 \mathrm{d}u, \int_0^t \lambda_u^2 \mathrm{d}u)$, we obtain

$$h_t(y,t) + \frac{1}{2} h_{yy}(y,t) = 0, \quad a.s., t \geq 0$$

  the backward heat equation. Solutions characterised by Widder's thm.

Taking $V(y,t) = -\ln y + \int_0^t b_u \mathrm{d}u$ and g quadratic leads to the logarithmic example.

Non-volatile criteria (cont.)

Now, if $a \equiv 0$ the submartingale property $\Rightarrow$ a random PDE

$$V_t(y,t) + \inf_{\eta}\left\{ g(\eta) + \frac{y^2 V_{yy}(y,t)}{2}(\eta+\lambda)^2 \right\} = 0, \quad a.s., t \geq 0.$$

Existence? Two difficulties:

- non-linearity: optimal $\bar{\eta}$ in function of V_{yy}
- solving for all $t \geq 0$: even if $g \equiv 0$, changing variables $V_y(y,t) = -h(\ln y + \frac{1}{2}\int_0^t \lambda_u^2 \mathrm{d}u, \int_0^t \lambda_u^2 \mathrm{d}u)$, we obtain

$$h_t(y,t) + \frac{1}{2}h_{yy}(y,t) = 0, \quad a.s., t \geq 0$$

  the backward heat equation. Solutions characterised by Widder's thm.

Taking $V(y,t) = -\ln y + \int_0^t b_u \mathrm{d}u$ and g quadratic leads to the logarithmic example.

1. Long-run investment, risk attitudes via drawdown constraints
   - Kelly's long-run investor and the numéraire property
   - Numéraire under drawdown – finite horizon
   - Numéraire under drawdown – asymptotics
2. Robust forward performance criteria
   - Model uncertainty, variational preferences and time homogeneity
   - Logarithmic preferences and fractional Kelly
   - Duality and (S)PDEs
3. Conclusions

Conclusions

- We present two portfolio choice problems which avoid the classical pitfalls and produce practically relevant strategies.
- Long run investor can both use pathwise outperformance and encode risk preferences by setting drawdown constraints. This decouples the ambiguity in specification of model (finding growth optimal portfolio) and preferences (setting drawdown level α).
- We consider variational preferences in the setting of model uncertainty and focus on time-consistent (forward) criteria. In particular, we show that fractional Kelly strategies which use a (dynamic) estimate of the true model are optimal.

⇒ It would be interesting to find another instances where preferences are effectively encoded via restrictions on the set of trading strategies.

⇒ Is it true that complexity of decision criteria (e.g. stochastic utilities) can be understood as simpler criteria but under model uncertainty? Can analyse the (S)PDEs which arise?

Conclusions

- We present two portfolio choice problems which avoid the classical pitfalls and produce practically relevant strategies.
- Long run investor can both use pathwise outperformance and encode risk preferences by setting drawdown constraints. This decouples the ambiguity in specification of model (finding growth optimal portfolio) and preferences (setting drawdown level α).
- We consider variational preferences in the setting of model uncertainty and focus on time-consistent (forward) criteria. In particular, we show that fractional Kelly strategies which use a (dynamic) estimate of the true model are optimal.

$\Rightarrow$ It would be interesting to find another instances where preferences are effectively encoded via restrictions on the set of trading strategies.

$\Rightarrow$ Is it true that complexity of decision criteria (e.g. stochastic utilities) can be understood as simpler criteria but under model uncertainty? Can analyse the (S)PDEs which arise?

Conclusions

- We present two portfolio choice problems which avoid the classical pitfalls and produce practically relevant strategies.
- Long run investor can both use pathwise outperformance and encode risk preferences by setting drawdown constraints. This decouples the ambiguity in specification of model (finding growth optimal portfolio) and preferences (setting drawdown level α).
- We consider variational preferences in the setting of model uncertainty and focus on time-consistent (forward) criteria. In particular, we show that fractional Kelly strategies which use a (dynamic) estimate of the true model are optimal.

⇒ It would be interesting to find another instances where preferences are effectively encoded via restrictions on the set of trading strategies.

⇒ Is it true that complexity of decision criteria (e.g. stochastic utilities) can be understood as simpler criteria but under model uncertainty? Can analyse the (S)PDEs which arise?

Conclusions

- We present two portfolio choice problems which avoid the classical pitfalls and produce practically relevant strategies.
- Long run investor can both use pathwise outperformance and encode risk preferences by setting drawdown constraints. This decouples the ambiguity in specification of model (finding growth optimal portfolio) and preferences (setting drawdown level α).
- We consider variational preferences in the setting of model uncertainty and focus on time-consistent (forward) criteria. In particular, we show that fractional Kelly strategies which use a (dynamic) estimate of the true model are optimal.

$\Rightarrow$ It would be interesting to find another instances where preferences are effectively encoded via restrictions on the set of trading strategies.

$\Rightarrow$ Is it true that complexity of decision criteria (e.g. stochastic utilities) can be understood as simpler criteria but under model uncertainty? Can analyse the (S)PDEs which arise?

THANK YOU!

- Källblad, Obłój and Zariphopoulou, *work in progress*
- C. Kardaras, J. Obłój and E. Platen, *The Numéraire property and long-term growth optimality for drawdown-constrained investments*, arXiv:1206.2305.
- V. Cherny and J. Obłój, *Portfolio optimisation under non-linear drawdown constraints in a semimartingale financial model*, Finance & Stochastics (published online), 2013.
- L. Carraro, N. El Karoui and J. Obłój, *On Azéma–Yor processes, their optimal properties and the Bachelier – Drawdown equation*. Annals of Probability **40**(1): 372–400, 2012.