Root's and Rost's solution of the SEP

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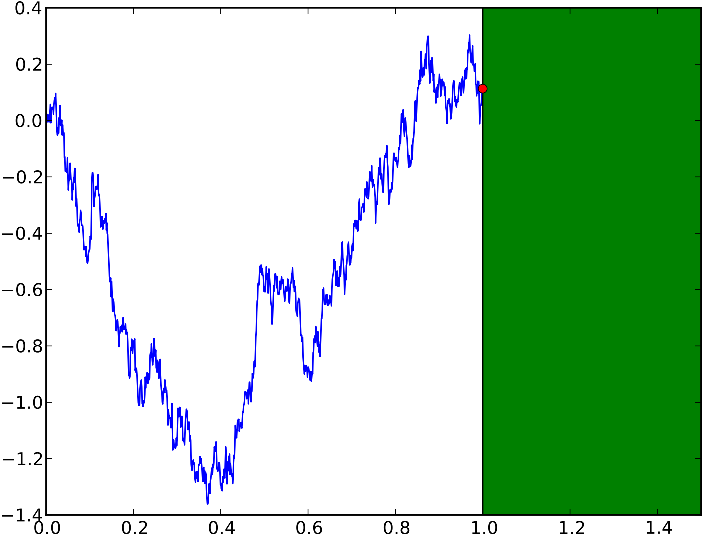
I. An example: generating Brownian increments by Skorokhod stopping times of minimal variance
II. Calculating Root barriers with integral equations
III. Root/Rost barriers, viscosity solutions of obstacle problems and FBSDEs

I. Generating Brownian increments by Skorokhod stopping times

Simulating Brownian motion $B = (B_t)_{t \geq 0}$



Simulating Brownian motion $B = (B_t)_{t \geq 0}$

Algorithm 1. (Euler)

- $(\tau_0, X_0) = (0, 0)$
- Draw

$$\begin{cases} X_{k+1} & = & X_k + N_k \text{ with } N_k \sim \mathcal{N}(0,1) \\ \tau_{k+1} & = & \tau_k + 1 \end{cases}$$

- Gives $(\tau_k, X_k)_{k \geq 0} =^{\mathcal{L}} (\tau_k, B_{\tau_k})_{k \geq 0}$

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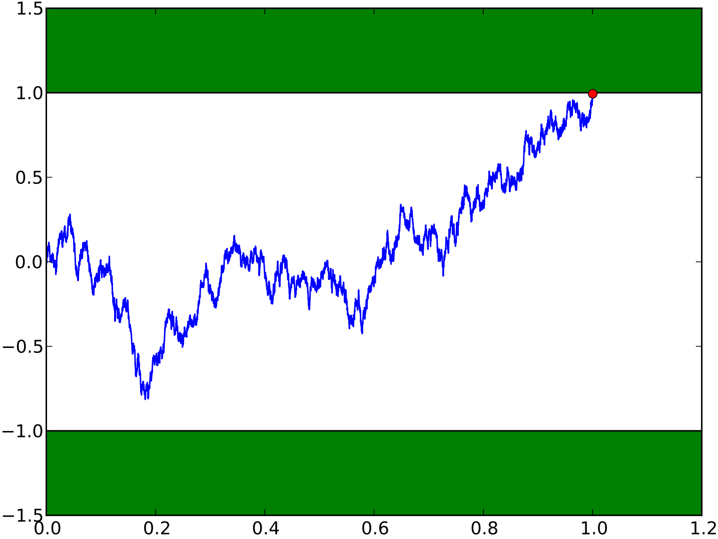
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0.0
0.2
0.4
0.6
0.8
1.0
1.2

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$$\tau_1 \textbf{ solves SEP } B_\tau \sim \mathcal{N}(0,1)$$

Algorithm 2 (Bichteler–Karandhikar)

- $(\tau_0, X_0) = (0,0)$
- Draw

$$\begin{cases} X_{k+1} & = & X_k + N_k \text{ with } \mathbb{P}(N_k = 1) = \mathbb{P}(N_k = -1) = \frac{1}{2} \\ \tau_{k+1} & = & \tau_k + D_k \text{ with } D_k \text{ s.t. } \mathbb{E}[\exp \lambda D_k] = \frac{1}{\cosh(\sqrt{2\lambda})} \end{cases}$$

- Gives $(\tau_k, X_k)_{k\geq 0} =^{\mathcal{L}} (\tau_k, B_{\tau_k})_{k\geq 0}$

$$\tau_1 \textbf{ solves SEP } B_\tau \sim \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

Root's barrier

- τ_1 solution of SEP $B_{\tau_1} \sim \mu$
  - in algorithm 1 $\mu = \mathcal{N}(0,1)$
  - in algorithm 2 $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$

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- τ_1 hitting time of time-space process $t \mapsto (t, B_t)$

$$\tau_1 = \inf\{t > 0 : (t, B_t) \in R\}$$

  - in algorithm 1 $R = \{(s,x) : s \geq 1, x \in \mathbb{R}\}$
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Theorem (Root's barrier, 1968)

Let μ be centered and have finite second moment. Then there exists a closed set

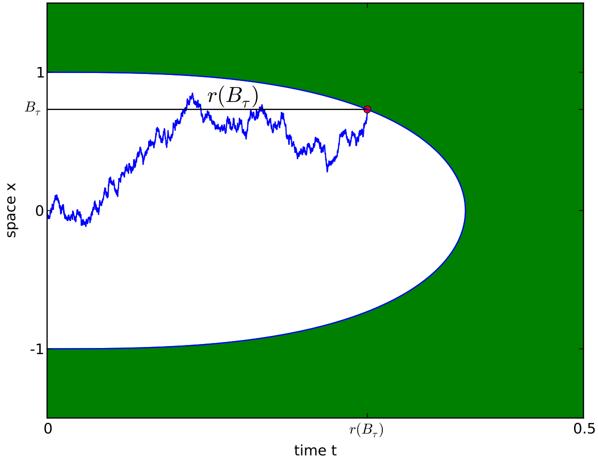
$$R \subset [0,\infty] \times [-\infty,\infty]$$

such that $\tau_R := \inf\{t \geq 0 : (t, B_t) \in R\}$ solves the Skorokhod embedding $B_\tau \sim \mu$, $B^\tau = (B_{t\wedge\tau})_{t\geq 0}$ u.i.

Better: solve SEP with $\mu = \mathcal{U}[-1,1]$

Corollary

$\exists r \in C(\mathbb{R}, [0,\infty))$ s.t. $R = \{(t,x) : t \geq r(x)\}$ is the Root barrier for the SEP $B_\tau \sim \mathcal{U}[-1,1]$.



Algorithm 3

- $(\tau_0, X_0) = (0, 0)$
- Draw

$$\begin{cases} X_{k+1} &= X_k + U_k \text{ with } U_k \sim \mathcal{U}[-1, 1] \\ \tau_{k+1} &= \tau_k + r(U_k) \end{cases}$$

- Then $(\tau_k, X_k)_{k\geq 0} =^{\mathcal{L}} (\tau_k, B_{\tau_k})_{k\geq 0}$.
  - Trivial to simulate (once you know r)
  - Increments bounded in space AND time (scaled Monte-Carlo; example knock-out options)
  - Similar schemes without SEP (Milstein–Tretyakov, Lejay, Deaconu–Hermann, etc.)
  - For more applications see Gassiat&Mijatovic&O13

Problem: Find Root barrier R for any given distribution μ

Unfortunately:

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Rest of talk:

- 1-1 correspondence of Root barrier and solutions of nonlinear integral equation
- 1-1 correspondence between Root barrier and viscosity solution of parabolic obstacle problem (Dupire, Cox–Wang)
- 1-1 correspondence of R with solution of reflected FBSDE
- Barles–Souganidis for numerical schemes parabolic obstacle problem

II. Root barrier and integral equations

Set

$$g(t,x) = \mathbb{E}L_t^x = \sqrt{\frac{2}{\pi}}\sqrt{t}e^{-\frac{x^2}{2t}} - x\operatorname{erfc}\left(\frac{x}{\sqrt{2}\sqrt{t}}\right)$$

$$K(r,\overline{r},x,y) = \frac{1}{2}\left(g(r-\overline{r},x-y)+g(r-\overline{r},x+y)\right)$$

Theorem (Gassiat&Mijatovic&O13)

$\exists! r \in C_b([-1,1],\mathbb{R}_{\geq 0})$ *which solves the integral equation*

$$\frac{x^2+1}{2} - x = g(r(x),x) - \int_x^1 K(r(x),r(y),x,y)\,dy$$

Moreover, if we extend r *to* $\mathbb{R}$ *by* $r(x)=0$ *for* $x \in \mathbb{R}\setminus[-1,1]$ *then*

$$R = \{(t,x) : t \geq r(x)\}$$

is the Root barrier for the SEP $B_\tau \sim \mathcal{U}[-1,1]$.

Potential functions

B one-dimensional Brownian motion; denote semigroup (of transformations on measures) $\left(P_t^B\right)$. Define operator U^B

$$\mu \mapsto U^B \mu := \int_0^\infty P_t^B \mu dt$$

If μ is a signed measure with $\mu(\mathbb{R}) = 0$ then

$$\frac{dU^B \mu}{dx} = -\int_{\mathbb{R}} |x - y| \, \mu(dy) =: u_\mu(x)$$

(rhs well-defined also for positive measures with finite moment).

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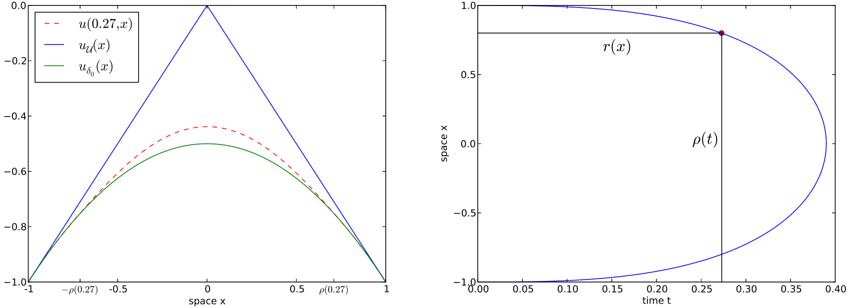
Definition

For probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite first moment, associate with it a function $u_\mu \in C(\mathbb{R}, \mathbb{R}_{\leq 0})$

$$u_\mu(x) = -\int_{\mathbb{R}} |x-y| \, \mu(dy)$$

We call u_μ the **potential function** of μ.

$u(t,x) = -\mathbb{E}\left[|B_{t\wedge\tau} - x|\right]$



- $u(0,x) = u_{\delta_0}(x) = -x,\ u(\infty,x) = u_{\mathcal{U}}(x) = \frac{x^2+1}{2}$
- $R = \{(t,x) : u(t,x) = u_{\mathcal{U}}(x)\}$

- Let R be the Root barrier for $B_\tau \sim \mathcal{U}[-1,1]$, $r(x)$ s.t. $R = \{(t,x) : t \geq r(x)\}$ and $\rho(t) = r^{-1}(t)$ (positive)
- Tanaka: $\forall (t,x)$

$$\begin{aligned} u(t,x) &= u_\delta(x) - \mathbb{E}\left[L^x_{t\wedge\tau}\right] = -|x| - \mathbb{E}\left[L^x_t + 1_{t>\tau}\left(L^x_\tau - L^x_t\right)\right] \\ &= -|x| - g(t,x) - \mathbb{E}\left[1_{t>\tau}\left(L^x_\tau - L^x_t\right)\right] \end{aligned}$$

- At $x = \rho_t$, $u(t,\rho_t) = u_{\mathcal{U}}(x)$ above becomes

$$u_{\mathcal{U}}(\rho_t) = -|\rho_t| - g(t,\rho_t) - \mathbb{E}\left[1_{t>\tau}\left(L^{\rho_t}_\tau - L^{\rho_t}_t\right)\right]$$

- Finished if we can write as explicit $\mathbb{E}\left[1_{t>\tau}\left(L^{\rho_t}_\tau - L^{\rho_t}_t\right)\right]$ functional of ρ.

- Note $\mathbb{P}(\tau < t) = \mathbb{P}(\mathcal{U} \notin [-\rho_t, \rho_t]) = 1 - \rho_t$ hence

$$\mathbb{P}(\tau \in dt) = -d\rho_t$$

- Using Markovianity and symmetry

$$\begin{aligned}
\mathbb{E}\left[(L_t^x - L_\tau^x) 1_{t>\tau}\right] &= \int_0^t \mathbb{E}\left[(L_t^x - L_s^x) \,|\tau = s\right] \mathbb{P}(\tau \in ds) \\
&= -\int_0^t \mathbb{E}\left[(L_t^x - L_s^x) \,|\tau = s\right] d\rho_s \\
&= -\int_0^t \frac{1}{2}\left(\mathbb{E}\left[L_{t-s}^{x-\rho_s}\right] + \mathbb{E}\left[L_{t-s}^{x+\rho_s}\right]\right) d\rho_s .
\end{aligned}$$

- Putting this into above

$$u_{\mathcal{U}}(\rho_t) = u_\delta(\rho_t) - g(t, \rho_t) + \frac{1}{2}\int_0^t \overbrace{\mathbb{E}\left[L_{t-s}^{\rho_t-\rho_s}\right] + \mathbb{E}\left[L_{t-s}^{\rho_t+\rho_s}\right]}^{=g(t-s,\rho_t-\rho_s)+g(t-s,\rho_t+\rho_s)} d\rho_s$$

- Finish by change of variable $dy = d\rho(s)$:

$$u_{\mathcal{U}}(x) = u_{\delta,}(x) - g(r(x), x) + \frac{1}{2}\int_1^x K(r(x), r(y), x, y)\, dy$$

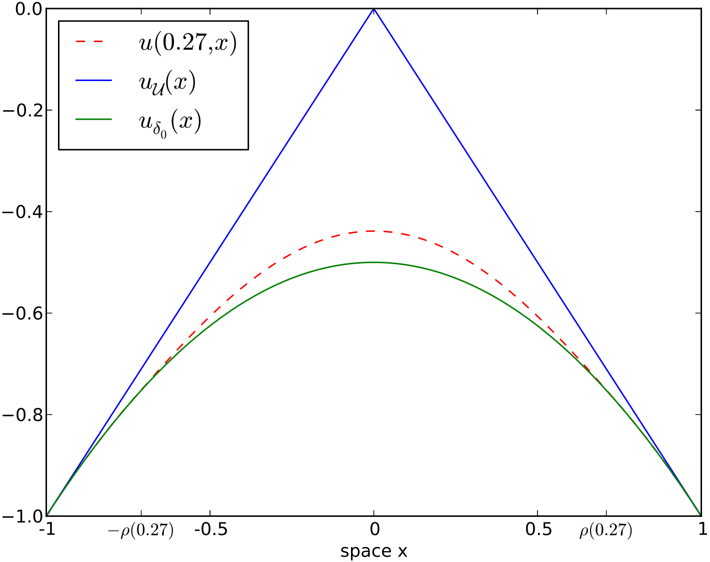
- Derivation purely probabilistic...no PDE techniques
- Extends to other target distributions (but there are limits)
- Uniqueness of solutions is hard (without using PDE uniqueness)! see Gassiat&Mijatovich&O13
- Useful? Solving this integral equation is numerically MUCH MUCH better than solving for free boundary via PDE
- The integral term in

$$\frac{x^2+1}{2} - x = g\left(r\left(x\right), x\right) - \int_x^1 K\left(r\left(x\right), r\left(y\right), x, y\right) dy$$

  is very small. Hence applying $\frac{d}{dx}$ to both both sides of $\frac{x^2+1}{2} - x = g\left(r\left(x\right), x\right)$ gives ODE for r which is a very good approximation.

Root/Rost barriers, viscosity solutions of obstacle problems and FBSDEs

$$u(t,x) = -\mathbb{E}\left[|B_{t\wedge\tau} - x|\right]$$



Recall viscosity theory

Definition

$\mathcal{O}$ a locally compact subset of $\mathbb{R}$, $\mathcal{O}_T = (0, T) \times \mathcal{O}$ for $T \in (0, \infty]$.
Let $u : \mathcal{O}_T \to \mathbb{R}$ and define for $(s, z) \in \mathcal{O}_T$ the parabolic superjet $\mathcal{P}^{2,+}_{\mathcal{O}} u(s, z)$ as the set of triples $(a, p, m) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ which fulfill

$$\begin{aligned} u(t, x) \leq & \; u(s, z) + a(t - s) + p(x - z) \\ & + m \frac{(x - z)^2}{2} + o\left(|t - s| + |x - z|^2\right) \end{aligned}$$

as $\mathcal{O}_T \ni (t, x) \to (s, z)$.
Similarly we define the parabolic subjet $\mathcal{P}^{2,-}_{\mathcal{O}} u(s, z)$ such that $\mathcal{P}^{2,-}_{\mathcal{O}} u = -\mathcal{P}^{2,+}(-u)$.

Definition

A function $F : \mathcal{O}_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is proper if $\forall (t, x, a, p) \in \mathcal{O}_T \times \mathbb{R} \times \mathbb{R}$

$$F(t, x, r, a, p, m) \leq F(t, x, s, a, p, m') \ \ \forall m \geq m', s \geq r.$$

Denote the real-valued, upper semicontinuous functions on $\mathcal{O}_T$ with $USC(\mathcal{O}_T)$. A subsolution of

$$\left\{ \begin{array}{rcl} F\left(t, x, u, \partial_t u, Du, D^2 u\right) & = & 0 \\ u(0, .) & = & u_0(.) \end{array} \right. \tag{1}$$

is a function $u \in USC(\mathcal{O}_T)$ such that

$$\begin{array}{rcl} F(t, x, a, p, m) & \leq & 0 \text{ for } (t, x) \in \mathcal{O}_T \text{ and } (a, p, m) \in \mathcal{P}^{2,+}_{\mathcal{O}} u(t, x) \\ u(0, .) & \leq & u_0(.) \text{ on } \mathcal{O} \end{array}$$

The definition of a supersolution follows by replacing upper by lower semicontinuous, $\mathcal{P}^{2,+}_{\mathcal{O}}$ by $\mathcal{P}^{2,-}_{\mathcal{O}}$ and $\leq$ by $\geq$.

- If u is a classic $C^{1,2}\left(\left(0,T\right)\times\mathbb{R},\mathbb{R}\right)$ solution of

$$\begin{cases} F\left(t,x,u,\partial_t u,Du,D^2 u\right) & = & 0 \\ u\left(0,.\right) & = & u_0\left(.\right) \end{cases}$$

  then v is also a viscosity solution
- Comparison Theorem (Maximum Principle): $u\left(0,.\right)\leq v\left(0,.\right)$, u sub- and v supersolution implies $u\leq v$

Barles-Perthame's semi-relaxed limits

Proposition

Let $(u^n)_n \subset USC(\mathcal{O}_T)$*,*$\mathcal{O}$ *a locally compact subset of* $\mathbb{R}$*,* (F_n) *a sequence of maps*

$$F_n : \mathcal{O}_T \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

each u^n *a subsolution of* $F_n\left(t, x, v, \partial_t v, D^2 v\right) = 0$*. Assume* $(u^n)_n$ *and* $(F^n)_n$ *are locally uniformly bounded. Then* $\underline{u}(t,x) = \liminf_{(s,y)\to(t,x),n\to\infty} u^n(s,y)$ *is a subsolution of*

$$\underline{F}\left(t, x, \underline{u}, \partial_t \underline{u}, D^2 \underline{u}\right) = 0 \text{ on } \mathcal{O}$$

The analogous statement holds for a sequence of $LSC(\mathcal{O}_T)$ *functions which are supersolutions. Further, if* $\overline{u} = \underline{u}$ *then the convergence of* (u_n) *to* $\overline{u} = \underline{u}$ *is locally uniform.*

Theorem (O&Reis13)

Let $\mu, \nu \in \mathcal{M}^2$, $\mu \leq_{cx} \nu$ *and denote with* R *the Root barrier for the SEP*

$$dX_t = \sigma(t, X_t)\, dB_t,\ X_0 \sim \mu,\ X_{\tau_R} \sim \nu$$

Define

$$u(t,x) := -\mathbb{E}\left[|X_{t\wedge\tau_R} - x|\right].$$

Then u *is the unique viscosity solution of the obstacle problem*

$$\begin{cases} \min\left(u - u_\nu, \partial_t u - \frac{\sigma^2}{2}\Delta u\right) & = & 0, \\ u(0,.) & = & u_\mu(.) . \end{cases} \tag{2}$$

Moreover,

1. $t \mapsto u(t,x)$ *is non-increasing and* $x \mapsto u(t,x)$ *is Lipschitz (uniformly in time)*
2. $u_\nu(x) \leq u(t,x) \leq u_\mu(x)$,
3. $\lim_{t\to\infty} u(t,x) = u_\nu(x)$.

Proof (Sketch).

- Show

$$\begin{aligned}
\left(\partial_t - \frac{\sigma^2}{2}\Delta\right) u &\geq 0 \text{ on } [0,\infty)\times\mathbb{R}, \\
u - u_\nu &\geq 0 \text{ on } [0,\infty)\times\mathbb{R}, \\
\left(\partial_t - \frac{\sigma^2}{2}\Delta\right) u &= 0 \text{ on } R^c, \\
u - u_\nu &= 0 \text{ on } R,
\end{aligned}$$

- **Step 1.** $u - U_\nu \geq 0$. By Jensen

$$\begin{aligned}
u(t,x) &= -\mathbb{E}\left[\left|X_t^{\tau_R} - x\right|\right] \geq \mathbb{E}\left[\mathbb{E}\left[-\left|X_{\tau_R} - x\right| \left|\mathcal{F}_{t\wedge\tau_R}\right.\right]\right] \\
&= -\mathbb{E}\left[\left|X_{\tau_R} - x\right|\right] = u_\nu(x)
\end{aligned}$$

- Take $(\psi_n), \psi_n \subset C^2(\mathbb{R}, \mathbb{R})$, $\psi_n \to |.|$ uniformly, $\Delta\psi_n(.)$ is continuous, $\Delta\psi_n \geq 0$ and $supp(\Delta\psi_n) \subset \left[-\frac{1}{n}, \frac{1}{n}\right]$;
- Define

$$
\begin{aligned}
u^n(t,x) &= -\mathbb{E}\left[\psi^n\left(X_t^\tau - x\right)\right] \to_{n\to\infty} u(t,x) \\
u_\mu^n &= -\mathbb{E}\left[\psi^n\left(X_0 - x\right)\right] \to_{n\to\infty} u_\mu(x) \\
u_\nu^n &= -\mathbb{E}\left[\psi^n\left(X_\tau - x\right)\right] \to_{n\to\infty} u_\nu(x)
\end{aligned}
$$

- Apply Ito to $\psi^n\left(X_.^{\tau_R} - x\right)$

$$
u^n(t,x) = u_\mu^n(x) - \int_0^t \mathbb{E}\left[\frac{\sigma^2(r, X_r)}{2} \Delta\psi^n(X_r - x) 1_{r<\tau_R}\right] dr
$$

- **Step 2.** $u - U_\mu = 0$ on R
  - By Ito, applied to $\psi^n \left(X_\cdot^{\tau_R} - x \right)$

$$u^n(t,x) = u^n_\mu(x) - \int_0^t \mathbb{E}\left[\frac{\sigma^2(r, X_r)}{2} \Delta\psi^n (X_r - x) 1_{r<\tau_R} \right] dr$$

  Take $\lim_{t\to\infty}$

$$u^n_\nu(x) = u^n_\mu(x) - \int_0^\infty \mathbb{E}\left[\frac{\sigma^2(r, X_r)}{2} \Delta\psi^n (X_r - x) 1_{r<\tau_R} \right] dr$$

  - Hence

$$u^n(t,x) - u_\nu(x) = \int_t^\infty \mathbb{E}\left[\frac{\sigma^2(r, X_r)}{2} \Delta\psi^n (X_r - x) 1_{r<\tau_R} \right] dr$$

  - Fix $(t,x) \in R^o$, then $(t+r, x) \in R^o$ for $r \geq 0$ hence

$$\begin{aligned} \lim_{n\to\infty} (u^n - u^n_\nu)(t,x) &= \int_t^\infty \mathbb{E}\left[\frac{\sigma^2(r, X_r)}{2} \lim_{n\to\infty} \Delta\psi^n (X_r - x) 1_{r<\tau_R} \right. \\ &= 0 \end{aligned}$$

Step 3. $\left(\partial_t - \frac{\sigma^2}{2}\Delta\right) u \geq 0$ on $[0,\infty) \times \mathbb{R}$.

- First show that u^n solves

$$\begin{cases} \left(\partial_t - \frac{\sigma^2}{2}\Delta\right) u^n - \frac{1}{2} I_n & = & 0 \text{ on } (0,\infty) \times \mathbb{R} \\ u^n(0,.) & = & u^n_\mu(.) \,. \end{cases}$$

  - $u^n(t,x)$ has a right- and left derivative $\forall (t,x) \in [0,\infty) \times \mathbb{R}$ and continuous derivative $\Delta u^n(t,x)$
  - $\lim_{n\to 0} I^n = 0$ loc. uniformly

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  - $u^n(t, x)$ has a right- and left derivative $\forall (t, x) \in [0, \infty) \times \mathbb{R}$ and continuous derivative $\Delta u^n(t, x)$
  - $\lim_{n \to 0} I^n = 0$ loc. uniformly

- Apply the method of semi-relaxed limits: $u^n \to u$ and $u^n_\mu(.) \to u_\mu(.)$ uniformly hence u is viscosity supersolution of

$$\left\{ \begin{array}{rcl} \left(\partial_t - \frac{\sigma^2}{2}\Delta\right) u & = & 0 \text{ on } [0, \infty) \times \mathbb{R} \\ u(0, .) & = & u_\mu(.) \end{array} \right.$$

Step 4. $\left(\partial_t - \frac{\sigma^2}{2}\Delta\right) u = 0$ on R^c:

- need to show that u is a subsolution (supersolution follows from above
- R is a Root barrier, hence

$$(\tau_R + r, X_{\tau_R}) \in R \ \ \forall r \geq 0,$$

hence if $(t,x) \in R^c$ and $t \geq \tau_R$ then $X_{\tau_R} \neq x$. Therefore
$\lim_{n\to\infty} \sup_{(t,x)\in K} \Delta\psi^n \left(X_{\tau_R} - x\right) 1_{t\geq\tau_R} = 0$ for every compact $K \subset R^c$

c) From PDE to barrier

Proposition

Under (TC) there exists a $v_\infty \in C\left([0,\infty) \times \mathbb{R}, \mathbb{R}\right)$ *which is a viscosity solution of*

$$\begin{cases} \min\left(v - h, \partial_t v - \frac{\sigma^2}{2}\Delta v\right) & = & 0,\ (t,x) \in (0,\infty) \times \mathbb{R} \\ v(0,x) & = & u_0(x),\ x \in \mathbb{R}. \end{cases}$$

Moreover, for $T < \infty$

1. $\forall (t,x) \in [0,T] \times \mathbb{R}\ \exists!\ \left(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x}\right)_{s\in[t,T]}$ *of* $\{\mathcal{F}_s^t\}$*-progressively measurable processes, solution of the rFBSDE*

$$\begin{cases} X_s^{t,x} & = & x + \int_t^s \sigma\left(T-r, X_r^{t,x}\right) dW_r, \\ Y_s^{t,x} & = & u_0\left(X_T^{t,x}\right) + K_T^{t,x} - K_s^{t,x} - \int_s^T Z_r^{t,x} dW_r, \\ Y_s^{t,x} & \geq & h\left(X_s^{t,x}\right),\ t < s \leq T, \int_t^T \left(Y_s^{t,x} - h\left(X_s^{t,x}\right)\right) dK_s^{t,x} = 0, \end{cases}$$

2. $\left(K_s^{t,x}\right)_{s\in[t,T]}$ *increasing, continuous, and* $K_t^{t,x} = 0$.
3. $v_\infty|_{[0,T]\times\mathbb{R}}(t,x) \equiv Y_{T-t}^{T-t,x}$.

Theorem (O&Reis13)

Let μ, ν have a second moment, $\mu \leq_{cx} \nu$ and σ Lip+LG. Then the the free boundary R

$$R = \{(t,x) : u(t,x) = u_\nu(x)\}$$

of the obstacle problem

$$\begin{cases} \min\left(u - u_\nu, \partial_t u - \frac{\sigma^2}{2}\Delta u\right) = 0, \\ u(0,.) = u_\mu(.). \end{cases} \tag{3}$$

solves the SEP

$$dX_t = \sigma(t, X_t)\, dB_t,\ X_0 \sim \mu,\ X_{\tau_R} \sim \nu.$$

Proof.

(sketch)

- Under above assumptions the PDE has exactly one solution $\mathcal{O}\left(u_{\mu}, u_{\nu}, \sigma\right) = \{u\}$
- $\mathcal{R}\left(\mu, \nu, \sigma\right) = \{R\}$
- From previous Theorem $u\left(t, x\right) = -\mathbb{E}\left[\left|X_{t\wedge\tau_R} - x\right|\right]$
- By proof of previous theorem, hence $R \subset \{\left(t, x\right) : u\left(t, x\right) = u_{\nu}\left(x\right)\}$
- To see $R \supset \{\left(t, x\right) : u\left(t, x\right) = u_{\nu}\left(x\right)\}$ use the representation $u\left(t + r, x\right) - u\left(t, x\right) = \mathbb{E}\left[L_t^x - L_{t+r}^x\right]$

$\square$

(d) Numerics

Assume μ, ν support in interval $[a, b] \subset \mathbb{R}$

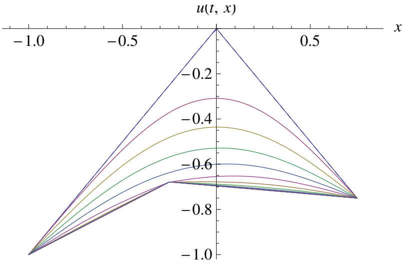
$$S^h\left[u^h\right](t,x) := \begin{cases} u_\mu(x) \text{ in } [0, \Delta t) \times (a, b) \\ u^h(t,x) + \frac{\Delta t \sigma^h(t,x)}{2(\Delta x)^2}\left(u^h(t, x + \Delta x) - 2u^h(t,x) + u^h\right. \\ u_\mu(x) = u_\nu(x) \text{ in } [0, T] \times \{a, b\} \end{cases}$$

Proposition (O&Reis13)

Let $T < \infty$ and let μ, ν have second moments, compact support, $\mu \leq_{cx} \nu$. If $\Delta t \, |\sigma|_{\infty;[a,b]\times[0,T]} < (\Delta x)^2$ Then $u^h \in \mathcal{B}([0, T] \times \mathbb{R}, \mathbb{R})$ and

$$\left|u^h - u\right|_{\infty;[0,T]\times\mathbb{R}} \to 0 \text{ as } h \to (0,0)$$

on $[0, T]$, $\{u\} \in \mathcal{O}(u_\mu, u_\nu, \sigma)$.



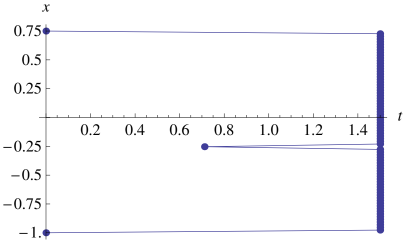


Figure: $\sigma = 1, \mu = \delta_0$ and $\nu = \frac{2}{7}\delta_{-1} + \frac{1}{4}\delta_{-\frac{1}{4}} + \frac{13}{28}\delta_{\frac{3}{4}}$.

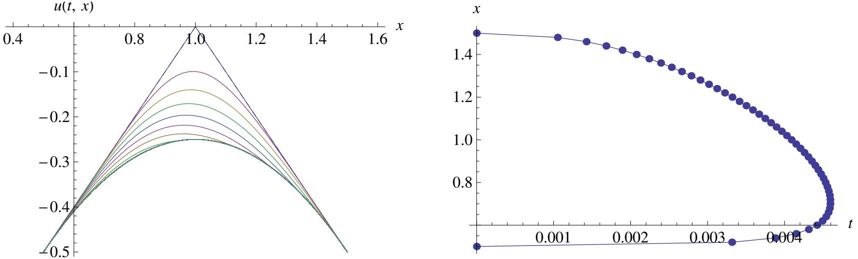


Figure: $\sigma(x) = x, \mu = \delta_1$ and $\nu = \mathcal{U}\left(\left[\frac{1}{2}, \frac{3}{2}\right]\right)$

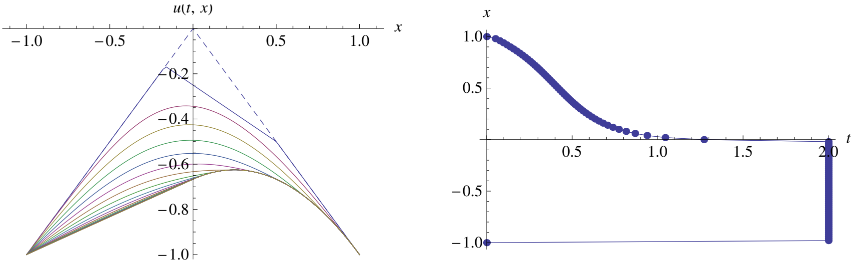


Figure: $\sigma(x) = 1, \mu = \frac{3}{4}\delta_{-\frac{1}{6}} + \frac{1}{4}\delta_{0.5}$, $\nu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\mathcal{U}([0,1])$.

[1, 2, 3, 4, 5, 6, 7]

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THANK YOU FOR YOUR TIME!