Geometric and Asymptotic Group Theory II

Damian Osajda
damian.osajda@univie.ac.at
http://www.mat.univie.ac.at/∼dosaj/GGTWien/Course.html
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Blatt 1

The Fundamental Group

Let X be a topological space and let $x_0 \in X$. We define the set $\pi_1(X, x_0) = \{[f] \mid f\colon [0,1] \to X,\ f(0) = f(1) = x_0\}$—the set of homotopy classes of loops based at x_0. The *fundamental group* (or the *first homotopy group*) of X at the basepoint x_0 is the set $\pi_1(X, x_0)$ together with the product defined as $[f] \cdot [g] = [f \cdot g]$, where

$$f \cdot g = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} .$$

(1) Show that $\pi_1(X, x_0)$ (with the product defined above) is a group.

(2) Show that the fundamental group of the point (i.e. of the one-element space) is trivial.

(3) Show that if X is path-connected then $\pi_1(X, x_0) = \pi_1(X, x_1)$ for any points $x_0, x_1 \in X$.

(4) Show that if $F\colon (X, x_0) \to (Y, y_0)$ is a homotopy equivalence between two based spaces then $\pi_1(X, x_0) = \pi_1(Y, y_0)$. Conclude then that a contractible space is *simply connected*, i.e. its fundamental group is trivial.

    Hint: Show that the map $F_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ *induced* by F is an isomorphism.

(5) Show that the fundamental group of a finite graph is the same as the fundamental group of some finite bouquet of circles.

    Hint: Look at a maximal subtree of the graph.

(6) *The fundamental group of the circle.* Let $p\colon \mathbb{R} \to S^1$ be the covering map given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$.

    (a) Show that for each path $f\colon [0,1] \to S^1$ starting at a point $x_0 \in S^1$ and each $\widetilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\widetilde{f}\colon [0,1] \to \mathbb{R}$ (i.e. $p \circ \widetilde{f} = f$) starting at $\widetilde{x}_0$.

    Hint: Use the properties of a covering space, i.e. decompose $\mathbb{R}$ into (nice) disjoint subsets which are mapped homeomorphically onto their images by p.

    (b) Show that for each homotopy $H_t\colon [0,1] \to S^1$ of paths starting at x_0 and each $\widetilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\widetilde{H}_t\colon [0,1] \to \mathbb{R}$ of paths starting at $\widetilde{x}_0$.

    Hint: Proceed similarly as in a) above.

    (c) Let $\psi\colon \mathbb{Z} \to \pi_1(S^1, x_0)$ be the map defined as follows: $\psi(n)$ is the homotopy class of the loop f given by $f(t) = p(nt)$, where $t \in [0,1]$. Prove that a) and b) imply that the map ψ is an isomorphism.

(7) Show that $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(8) Find the fundamental group of the following spaces.
   (a) The sphere S^n.
   (b) The torus.
   (c) The projective plane

   Hint: For c), find the corresponding covering of the projective plane and proceed similarly as in Exercise 6.