Geometric and Asymptotic Group Theory II

Damian Osajda
damian.osajda@univie.ac.at
http://www.mat.univie.ac.at/∼dosaj/GGTWien/Course.html
Dienstag, 11:00–12:00, Raum D1.07 UZA 4

Blatt 4
Seifert-van Kampen Theorem

Let $X = A \cup B$, where A, B are open and $x_0 \in A \cap B$. Let $j_A \colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ and $j_B \colon \pi_1(Bx_0) \to \pi_1(X, x_0)$ be homomorphism induced by inclusions, and let $\Phi \colon \pi(a) * \pi(B) \to \pi(X)$ be their extension. Let moreover $i_A \colon \pi_1(A \cap B) \to \pi_1(A)$ and $i_B \colon \pi_1(A \cap B) \to \pi_1(B)$ be induced by inclusions, and let N be the normal subgroup of $\pi_1(X)$ generated by all elements of the form $i_A(g) i_B^{-1}(g)$, for $g \in \pi_1(A \cap B)$.

Seifert-van Kampen Theorem. *If A, B and $A \cap B$ are all path-connected then Φ gives an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(A, x_0) * \pi_1(B, x_0)/N$.*

The following exercises present a proof of the Seifert-van Kampen Theorem.

(1) Show that Φ is surjective.

Hint: Decompose every loop in a product of loops inside pieces (A and B).

(2) For $[f] \in \pi_1(X, x_0)$, a *factorization of* $[f]$ is a product $[f_1] \cdot [f_2] \cdot \ldots \cdot [f_n]$, such that f_i is a loop in a piece based at x_0 and $f_1 \cdot f_2 \cdot \ldots \cdot f_n$ is homotopic with f in X. Suggest (two, obvious) "elementary moves" transforming one factorization into another one.

Hint: Proceed similarly as for the normal form of an element of a free product with amalgamation.

(3) Show that any two factorizations of an element of $\pi_1(X)$ are equivalent wrt. a sequence of elementary moves defined as above.

Hint: For given two factorizations consider a homotopy joining the representing loops. Subdivide the domain of the homotopy into small "rectangles". Modify the homotopy so that the vertices of the subdivision "are mapped to x_0". Observe that passing from one "side" of a rectangle to another one corresponds to an elementary move.

(4) Conclude that point (3) implies injectivity of Φ.