Geometric and Asymptotic Group Theory I

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Blatt 1
Free groups

An *action* of a group G on a set X is a homomorphism $G \to \mathrm{Sym}(X)$.

(1) Prove the following Ping-pong Lemma.

Let G be a group acting on a set X. Suppose there exist disjoint nonempty subsets $A^+, A^-, B^+, B^- \subset X$, and two elements a, b of G with the following properties:

a) $A^+ \cup A^- \cup B^+ \cup B^- \subsetneq X$;
b) $a(X - A^-) \subseteq A^+$, $a^{-1}(X - A^+) \subseteq A^-$;
b) $b(X - B^-) \subseteq B^+$, $b^{-1}(X - B^+) \subseteq B^-$.

Then $\langle a, b\rangle \leq G$ is a free subgroup generated by a and b.

(2) Let Γ be a graph with the vertex set V and the set of directed edges E. A *loop based at* $v \in V$ is an edge path starting and ending in v. We say that two loops p_1 and p_2 are *equivalent*, denoted $p_1 \sim p_2$, if one can obtain p_2 from p_1 by a finite sequence of inserting/removing of subpaths of the form ee^{-1}, where $e, e^{-1} \in E$ and e^{-1} is the edge with the same endpoints as e and with the opposite orientation. Define a *product* $[p_1] \cdot [p_2]$ of two $\sim$-equivalence classes of loops p_1 and p_2 based at $v \in V$ as $[p_1] \cdot [p_2] = [p_1 \cdot p_2]$, where $p_1 \cdot p_2$ is the concatenation of the paths p_1 and p_2.

(a) Show that the set of equivalence classes of loops based at a given vertex v, equipped with the product defined above forms a group. This group is called the *fundamental group of* Γ *with the basepoint* v and is denoted by $\pi_1(\Gamma, v)$.
(b) Prove that $\pi_1(\Gamma, v)$ is a free group. What is its rank?
(c) Show that if Γ is connected, then $\pi_1(\Gamma, v) \cong \pi_1(\Gamma, v')$, for any two vertices v and v'.

(3) Using the previous exercise show that for every $n \geq 1$ the free group F_n is a subgroup of F_2.

Hint: Show that a *covering* (i.e. a surjective map that is a local isomorphism) of graphs induces a monomorphism of their fundamental groups.