Geometric and Asymptotic Group Theory II

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Dienstag, 11:00–12:00, Raum C2.07 UZA 4

Blatt 2
Residual finiteness of free groups

(1) Show that $\mathbb{Z}^n$ is residually finite.

(2) *Combinatorial proof.* Let $\mathbb{F}_2 = \langle a, b\rangle$. Let $w(a,b) = x_{i_n}^{\epsilon_n} x_{i_{n-1}}^{\epsilon_{n-1}} \cdots x_{i_1}^{\epsilon_1}$ be a reduced word in $a^{\pm}, b^{\pm}$, with $x_j \in \{a,b\}$ and $\epsilon_j = \pm 1$.
 (a) For $k = 1, \ldots, n$, construct permutations σ_{i_k} in S_{n+1} such that the following conditions are satisfied.
$$\sigma_{i_k}(k) = k+1 \text{ if } \epsilon_k = +1, \text{ or}$$
$$\sigma_{i_k}(k+1) = k \text{ if } \epsilon_k = -1.$$
 (b) Define a map $f\colon \mathbb{F}_2 \to S_{n+1}$ such that the image of w is nontrivial.
 (c) Conclude that $\mathbb{F}_2$ is residually finite.

(3) * *Topological proof.* Consider a 2-rose Γ with labels a, b —i.e. a graph with one vertex and two oriented edges (loops) labelled by a, b. Then $\pi_1(\Gamma) = \mathbb{F}_2$.
 (a) For a given reduced word $w(a,b)$ build a path p of (directed) edges labelled by w and consider the obvious (labelled) immersion $i\colon p \to \Gamma$.
 (b) Complete the immersion i to a finite covering of the labelled graph Γ.
 (c) Observe that a loop corresponding to p in Γ does not lift to a loop.
 (d) Conclude that $\mathbb{F}_2$ is residually finite

(4) ** *Probabilistic-topological proof.* Let Γ be a finite graph. Consider its double covering $p\colon \widetilde{\Gamma} \to \Gamma$. It means in particular the following. For each vertex $v \in \Gamma$ there are two vertices $\widetilde{v}_1, \widetilde{v}_2 \in \widetilde{\Gamma}$ with $p(\widetilde{v}_1) = p(\widetilde{v}_2) = v$, and if $\{\widetilde{v}, \widetilde{w}\}$ is an edge in $\widetilde{\Gamma}$ then $\{p(\widetilde{v}), p(\widetilde{w})\}$ is an edge in Γ.
 (a) Observe that $g := \mathrm{girth}(\Gamma) \leq \mathrm{girth}(\widetilde{\Gamma})$.
 (b) Let Z be a random variable counting the number of cycles (i.e. polygonal loops) of length g in a double covering of Γ. Show that EZ (the expected value of Z) equals the number of g-cycles in Γ.
 (c) Conclude that there is a double covering with fewer g-cycles.
 (d) Show that there exists a (not necessarily double) covering $\widetilde{\Gamma}$ with
$$\mathrm{girth}(\widetilde{\Gamma}) > \mathrm{girth}(\Gamma).$$
 (e) Conclude that free groups are residually finite.

(5) *** *Coxeter groups.* Show how the residual finiteness of $\mathbb{F}_n$ follows from Problem (3) on Blatt 1.