Geometric and Asymptotic Group Theory II

Damian Osajda
damian.osajda@univie.ac.at
http://www.mat.univie.ac.at/∼dosaj/GGTWien/Course.html
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Blatt 1

Free groups

(1) Basing only on the "universal" definition of a free group, prove its uniqueness.

(2) An *action* of a group G on a set X is a homomorphism $G \to \mathrm{Sym}(X)$.

Prove the following Ping-pong Lemma.

Let G be a group acting on a set X. Suppose there exist disjoint nonempty subsets $A^+, A^-, B^+, B^- \subset X$, and two elements a, b of G with the following properties:

a) $A^+ \cup A^- \cup B^+ \cup B^- \subsetneq X$;
b) $a(X - A^-) \subseteq A^+$, $a^{-1}(X - A^+) \subseteq A^-$;
b) $b(X - B^-) \subseteq B^+$, $b^{-1}(X - B^+) \subseteq B^-$.

Then $\langle a, b \rangle \leq G$ is a free subgroup generated by a and b.

(3) Prove that the fundamental group of any graph is free.

(4) Show that the fundamental group of any surface with boundary is free.

(5) Show that a group is free (possibly on an infinite set) iff it acts freely by graph automorphisms on a tree.

(6) Show that any subgroup of a free group is free.

(7) Show that the subgroup of $GL_2(\mathbb{R})$ generated by two matrices: $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, with $|a|, |b| > 2$, is free.

(8) For a ring R, let $R[[x]] = R[[x_1, x_2, \ldots, x_k]]$ be a ring of formal power series in x_i's. Prove that the subgroup of $R[[x]]$ generated by all elements of type $1 + x_i$ is free.