Representation theory of groups — Prof. G. Arzhantseva
Winter semester — 2016 — Exam questions

(1) Linear representations of groups: definition, (non)-examples, intertwines, their properties. Regular, faithful, trivial, 1- and -infinite- dimensional representations.

(2) Subrepresentations and induced representations. Characters.

(3) Operations over representations: quotients, direct sums, tensor products, intersection, exact sequences. Contragredient representations and transpose intertwiners. Invariant subspaces.

(4) Irreducible, semi-simple, and completely reducible representations: definitions, (non)-examples. A completely reducible representation contains an irreducible subrepresentation.

(5) Semisimplicity criterion.

(6) Schur's lemma. Description of finite-dimensional irreducible representations of an abelian group over an algebraically closed field.

(7) Maschke's theorem: direct implication.

(8) Maschke's theorem: converse implication.

(9) Matrix coefficients. Isotypic components of finite-dimensional irreducible representations over an algebraically closed field. Decomposition of the regular representation of a finite group.

(10) On the number of irreducible representations over an algebraically closed field.

(11) Representations of groups as modules over the group algebra. Submodules, simple, and semisimple modules. Reformulations of Schur's lemma and Maschke's theorem.

(12) Permutation representations: definition and (non)-examples. The permutation characters and modules.

(13) Topological groups and their representations: Banach space representations, sub representations, intertwines, quotient representations, irreducible representations.

(14) Unitary representations: definition, (non)-examples. Operations over unitary representations. Unrelated unitary representations are orthogonal.

(15) Reducibility of unitary representations. Schur's lemma for unitary representations: 1st statement.

(16) Schur's lemma for unitary representations: 2nd statement.

(17) Various topologies on the space of bounded linear operators on a Hilbert space. Unitary representations relative to these topologies.

(18) Haar measure: definition, existence, uniqueness, (non)-examples.

(19) Unimodular groups : definition of the modular function, examples and non-examples.

(20) Regular representations of locally compact groups on L^2(G, \mu)

(21) Unitarizable representations. Unitarizability of representations of finite and compact groups.

(22) Characterization of compact groups within locally compact groups, using the strong containment property and the Haar measure. The Peter-Weyl theorem (formulation).

(23) Weak containment of unitary representations: definition, properties, (non)-examples.

(24) Representations with invariant vectors and representations almost having invariant vectors. The terminology of fixed points and almost fixed points respectively. Characterizations using strong and weak containment of representations (case of the trivial representation).

(25) Compactly generated locally compact groups. Characterization of unitary representations of such a group weakly containing the trivial representation.

(26) Amenable locally compact groups: definition, (non)-examples, characterization using the concept of weak containment.

(27) Locally compact groups with Kazhdan's Property (T). Characterization of Kazhdan's locally compact groups which are amenable.

(28) Expanders : définition and (non)-examples. Construction of expanders using groups with Kazhdan's Property (T).

(29) Gelfand-Naimark-Segal construction for unitary representations : statement and consequences.

(30) C*-algebras: definition and (non)-examples.

(31) Lance theorem.