Representation Theory of Groups - Blatt 1

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`http://www.mat.univie.ac.at/~gagt/rep_theory2017`

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We begin by recalling a few notions from linear algebra. Let V and W be k-vector spaces.

- The set $\mathrm{Hom}(V, W)$ of all linear maps from V to W is a vector space with pointwise operations;

- The tensor product $V \otimes W$ is the universal vector space for bilinear maps: that is $V \otimes W$ is the quotient of the free vector space on the set $V \times W$ under the equivalence relation generated by:

$$\begin{aligned}(v_1, w) + (v_2, w) &= (v_1 + v_2, w)\\ (v, w_1) + (v, w_2) &= (v, w_1 + w_2)\\ \lambda(v, w) &= (\lambda v, w) = (v, \lambda w).\end{aligned}$$

  We denote elements of $V \otimes W$ by $v \otimes w$, moreover if $\{e_i\}_i$ and $\{f_j\}_j$ are bases for V and W respectively, $\{e_i \otimes f_j\}_{i,j}$ is a basis for $V \otimes W$. Finally, as an example, one should show that $\mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{nm}$ as finite dimensional vector spaces.

- Let $\{e_i\}_{i=1}^{\dim(V)}$ be a basis of V. The dual space to V, denoted V^*, is the vector space of all linear functions $V \to k$ with pointwise operations, which is spanned by the linear extensions of the functions $f_i(e_i) = 1$. If V has an inner product, and $\{e_i\}_i$ is orthonormal with respect to that inner product, then V^* also has an inner product such that $\{f_i\}$ are orthonormal. If V is finite dimensional, then $V \cong V^*$, but otherwise not.

Question 1. Let V and W be finite dimensional k-vector spaces for some field k. Show that $\mathrm{Hom}(V, W) \cong V^* \otimes W$ is an isomorphism of vector spaces, where V^* denotes the dual vector space of V.

The following questions are about the behaviour of *characters*, that is values obtained from the *trace* on the matrices. Let ρ be a representation of G in $GL(V)$, and let χ be the function:

$$\chi(g) = \mathrm{Tr}(\rho(g)).$$

Then χ is the *character* of ρ.

Let G be a finite group and X be a finite set on which G acts. We denote by ρ the the corresponding permutation representation on the free vector space $\mathbb{C}X$ (that is $\rho(g)$ is defined on the basis $\{\delta_x\}_{x\in X}$ by $\rho(g)(\delta_x) = \delta_{g.x}$) and χ the corresponding character.

Question 2. Show that for every $g \in G$ show that $\chi(g) = \mathrm{Tr}(\rho(g))$ is the number of elements of X fixed by g.

We will also consider the dual vector space representation and its character.

Question 3. Let V be a finite dimensional vector space with a representation $\rho : G \to GL(V)$ and let V^* be the vector space dual of V. Let $\langle v, \phi \rangle = \phi(v)$ be the natural pairing between the dual space V^* and V.

1. Show that there is a unique representation $\rho^* : G \to GL(V^*)$ satisfying:

$$\langle \rho(g)(v), \rho^*(g)(\phi) \rangle = \langle v, \phi \rangle$$

   for each $g \in G$.

2. By fixing a basis for V and using the dual basis for V^*, show that $\rho^*(g) = \rho(g^{-1})^T$.

We will consider what happens when we tensor two representations together and the effect of this on the corresponding characters.

Question 4. Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be two representations with characters χ_1 and χ_2 respectively, and let $W := \mathrm{Hom}(V_1, V_2)$. For every $g \in G$ and $f \in W$, denote by $\rho(g)f$ the element of W given by:

$$\rho(g)f = \rho_2(g) \circ f \circ \rho_1^{-1}(g).$$

a) Show that $\rho : G \to GL(W)$ is a representation of G;

b) Show that the character of ρ is $\chi_1^*\chi_2$, where χ_1^* denotes the character of the dual representation of ρ_1 defined in Question 3;

c) Show that ρ is isomorphic as a representation to the representation $\rho_1^* \otimes \rho_2$.

(Note that here, we are considering the tensor product representation of G, not $G \times G$. This is obtained by composing with the map: $G \to G \times G$, and using the tensor product of the representations ρ_i; this has formula: $\rho_1^* \otimes \rho_2(g) = \rho_1^*(g) \otimes \rho_2(g)$ in this case).

Finally, some questions that are more group theory than linear algebra:

Question 5. Let G be a finite group.

a) Show that for any irreducible representation ρ of G with character χ, the set

$$\ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$$

is a normal subgroup of G;

b) Show that each normal subgroup $N \lhd G$ is the intersection of subgroups of the form $\ker(\chi)$, where χ is the character of an irreducible representation.

Question 6. (Tricky)Let p be a prime number. Show that any p-group G has a faithful irreducible representation if and only if the centre $Z(G)$ is cyclic.