

# COMPUTATIONAL COMPETITION OF SYMMETRIC MIXED FEM IN LINEAR ELASTICITY\*

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ABSTRACT. The Navier-Lamé equation for linear elasticity has evoked the design of various non-standard finite element methods (FEM) in order to overcome the locking phenomenon. Recent developments of Arnold and Winther in 2002 involve a stable mixed method which strongly fulfils the symmetry constraint. Subsequently, two  $H(\text{div})$  non-conforming symmetric mixed methods arose. This paper comments on the implementation of all those mixed FEM and provides a numerical comparison of the different symmetric mixed schemes for linear elasticity. The computational survey also includes the low-order elements of weak symmetry (PEERS), the non-conforming Kouhia and Stenberg (KS) elements plus the conforming displacement  $P_k$ -FEM for  $k = 1, 2, 3, 4$ . Numerical experiments confirm the theoretical convergence rates for sufficiently smooth solutions and illustrate the superiority of the symmetric MFEM amongst the methods of second or third order.

## 1. INTRODUCTION

The numerical solution of the Navier-Lamé equation with mixed weak formulations allows a robust approximation even if the crucial Lamé parameter passes to the incompressible limit when the Poisson ratio approaches  $1/2$ , see [BF91, Chapter IV §3]. In low-order displacement formulations, the well-known locking effect causes a priori error estimates to deteriorate. While there are many known stable mixed finite element methods (MFEM), the additional symmetry constraint implied for the stress tensor proved to be difficult to impose in numerical schemes. This has resulted in the introduction of discretisations with no or reduced symmetry incorporated in the discrete stress space [ABD84, Ste88, CDFH00]. The first MFEM which were designed

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name	finite element schemes	function spaces	continuity constraints
AW21 (15+6 dofs)		$\Sigma_T := \{ \sigma \in P_2(T; \mathbb{S}) \mid \nu_E \cdot (\sigma \nu_E) \in P_1(E) \forall E \in \partial T \}$ $\mathcal{V}_T := P_1(T; \mathbb{R}^2)$	$\forall E \in \partial T, v \in P_1(E; \mathbb{R}^2)$ $\int_E (\sigma \nu_E) \cdot v \, ds$
AW15 (12+3 dofs)		$\Sigma_T := \{ \sigma \in P_2(T; \mathbb{S}) \mid \operatorname{div} \sigma \in \operatorname{RM}(T), \nu_E \cdot (\sigma \nu_E) \in P_1(E) \forall E \in \partial T \}$ $\mathcal{V}_T := \operatorname{RM}(T)$	$\forall E \in \partial T, v \in P_1(E; \mathbb{R}^2)$ $\int_E (\sigma \nu_E) \cdot v \, ds$
AW30 (24+6 dofs)		$\Sigma_T := \{ \sigma \in P_3(T; \mathbb{S}) \mid \operatorname{div} \sigma \in P_1(T; \mathbb{R}^2) \}$ $\mathcal{V}_T := P_1(T; \mathbb{R}^2)$	$\sigma \nu$
AW24 (21+3 dofs)		$\Sigma_T := \{ \sigma \in P_3(T; \mathbb{S}) \mid \operatorname{div} \sigma \in \operatorname{RM}(T) \}$ $\mathcal{V}_T := \operatorname{RM}(T)$	$\sigma \nu$
S15 (9+6 dofs)		$\Sigma_T := P_1(T; \mathbb{S})$ $\mathcal{V}_T := P_1(T; \mathbb{R}^2)$	$\forall E \in \partial T$ $\nu_E \cdot (\sigma \nu_E)$ $v \cdot \tau_E$
S27 (15+12 dofs)		$\Sigma_T := P_2(T; \mathbb{S})$ $\mathcal{V}_T := P_2(T; \mathbb{R}^2)$	$\forall E \in \partial T$ $\nu_E \cdot (\sigma \nu_E) \in P_1(E)$ $v \cdot \tau_E$

TABLE 1. Symmetric mixed finite elements schemes

especially to fulfil the stress symmetry without the need of a sub-grid are due to Arnold and Winther [AAW08, AFW09, AW02, AW03a, AW03b]. As discussed in [AW03b], the continuity property imposed on the stress field in the conforming MFEM substantially increases complexity of the finite elements. Since (complete) continuity is not required in the mixed formulation of linear elasticity, non-conforming MFEM can be an efficient and easier to implement alternative to conforming elements. Table 1 displays the MFEM compared in this paper. While the conforming lowest-order MFEM of Arnold and Winther AW30 and AW24 have 30 and 24 degrees of freedom and are based on a polynomial basis of degree 3, the non-conforming AW21 and AW15 MFEM are based on a quadratic polynomial basis with 21 and 15 degrees of freedom, as depicted in the first row of Table 1. The recently introduced S15 and S27 MFEM due to Schöberl and Sinwel [SS07] are based on linear and quadratic polynomials and have 15 and 27 degrees of freedom.

This paper is devoted to the computational competition of several FEM displayed in Table 2 for the Navier-Lamé equation of linear elasticity. Our interest hereby lies on the numerical examination of the rather novel MFEM of Arnold and Winther [AW02, AW03b, CGRT08] and the elements introduced by Schöberl and Sinwel [SS07]. For some of these elements we discuss the implementation topics that have not been presented yet. Of particular interest is the numerical competition of the many newly available mixed finite elements of order one up to three with traditional displacement-oriented FEM of the same orders. Since  $P_4$  is locking free [BS92] in contrast to  $P_k$ ,  $k = 1, 2, 3$ , it is included in the survey as well as the low-order MFEM of weak symmetry [ABD84] (PEERS) and the non-conforming KS-FEM [KS95]. We note that there are some recent elements which we could not include in this survey such as [GG11a, GG11b]. Moreover, while there are several MFEM with weak symmetry constraints that might also have been worth to consider in our comparison, see e.g. [AFW07, Guz10, CGG10], our focus lies on symmetric methods. The reason for including PEERS and KS-FEM is the great popularity of these methods in the engineering community. We also mention that, although out of the scope of this paper, higher-order PEERS are available which probably would show a more favourable performance than the lowest-order version used in the examples.

While robustness, locking and computational complexity usually play a pivotal role, singularities in the solution or high regularity may dominate the choice for the method and the mesh-design. The first academic numerical example on the unit square is designed such that the linear  $P_1$  finite element shows locking. The experiments with several Poisson ratios  $\nu$  close to  $1/2$  confirm the theoretical locking-free property of the symmetric MFEM. The theoretical findings of [BS92], that  $P_k$ ,

name	convergence result
AW21	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch\ u\ _{H^2(\Omega)}$
AW15	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch\ u\ _{H^2(\Omega)}$
AW30	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch^m\ \sigma\ _{H^m(\Omega)}$ for $1 \leq m \leq 3$
AW24	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch^m\ \sigma\ _{H^m(\Omega)}$ for $1 \leq m \leq 2$
S15	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch^{m-1}\ u\ _{H^m(\Omega)}$ for $1 \leq m \leq 2$
S27	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch^{m-1}\ u\ _{H^m(\Omega)}$ for $1 \leq m \leq 3$
KS	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch\ \sigma\ _{H^1(\Omega)}$
PEERS	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch\ \sigma\ _{H^1(\Omega)}$
$P_k$	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq C(\lambda)h^k\ u\ _{H^{k+1}(\Omega)}$ for $k = 1, 2, 3$
$P_4$	$\ \sigma - \sigma_h\ _{L^2(\Omega)} \leq Ch^4\ u\ _{H^5(\Omega)}$

TABLE 2. Theoretical convergence rates of different mixed FEM, the non-conforming KS-FEM and the conforming  $P_k$ -FEM. The constant  $C$  is independent of material parameters (except for  $P_k$ ,  $k = 1, 2, 3$ ) and independent of the (sufficiently small) mesh size  $h$ .

$k = 1, 2, 3$ , show locking while  $P_4$  is locking free, are empirically verified. For this smooth example the higher-order schemes show faster convergence rates and higher accuracy. However, this example is not representative from a practical point of view. Experiments for the Cook's membrane and the example with rigid circular inclusion show that singular solutions or curved boundaries can reduce the convergence rates of the methods. For the Cook's membrane problem even the low-order schemes lead to suboptimal convergence rates. This motivates the use of local mesh refinement which is investigated more closely for some L-shaped domain. The experiments show that graded meshes in contrast to uniform meshes lead to optimal convergence rates. However, the right choice of the grading parameter is not known in practise because it depends on the material parameters. The experiments show that a too small grading parameter results in suboptimal convergence rates while a too large value can lead to errors of different order of magnitude in accuracy. Therefore adaptive mesh refinement strategies for the symmetric MFEM have to be investigated which is postponed to forthcoming work.

The problem we are concerned with in this paper is assumed on some plane elastic body  $\Omega \subset \mathbb{R}^2$  with boundary  $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$  which consists of some closed part  $\Gamma_D$  of positive length for displacement boundary conditions  $u_D$  and its complement  $\Gamma_N = \Gamma \setminus \Gamma_D$  subject to

surface loads  $g$  with exterior unit normal  $\nu$ . Given a volume force  $f \in L^2(\Omega; \mathbb{R}^2)$  and a traction  $g \in L^2(\Gamma_N; \mathbb{R}^2)$ , we seek the displacement field  $u \in \mathcal{V} := L^2(\Omega; \mathbb{R}^2)$  and the stress tensor  $\sigma \in \Sigma := H(\text{div}, \Omega; \mathbb{S})$  which satisfy

$$(1.1) \quad \begin{aligned} -\text{div } \sigma &= f & \text{and} & \quad \sigma = \mathbb{C}\varepsilon(u) & \text{in } \Omega, \\ u &= u_D & \text{on } \Gamma_D & \quad \text{and} & \quad \sigma\nu = g & \text{on } \Gamma_N. \end{aligned}$$

Here and throughout this paper,  $\varepsilon(v) := \text{sym } Dv$  is the linearised Green strain tensor and  $\mathbb{C}$  is the symmetric fourth-order bounded and positive definite isotropic elasticity tensor. In computations, we seek an approximation  $(\sigma_h, u_h) \in \Sigma_h \times \mathcal{V}_h$  in possible discrete spaces  $\Sigma_h$  and  $\mathcal{V}_h$  discussed in detail in the next section. The standard notation for Lebesgue spaces  $L^2(\Omega)$  and Sobolev spaces  $H^1(\Omega)$  is used throughout the paper. Additionally, in the context of the examined Lamé problem described above,  $\mathbb{S} := \mathbb{R}_{\text{sym}}^{2 \times 2}$  denotes the space of symmetric matrices.

The paper is organised as follows: Section 2 outlines the design and implementation of the two non-conforming Arnold-Winther MFEM [AW03b] as a continuation of the examinations in [CGRT08] for the conforming MFEM of [AW02]. Since the formulation of the MFEM by Schöberl and Sinwel [SS07] is similar to the former MFEM, we suggest a new design of the base functions. The computational competition in Section 3 compares the four Arnold-Winther and two Schöberl-Sinwel symmetric MFEM to the weakly symmetric PEERS MFEM and the KS and  $P_k$ ,  $k = 1, 2, 3, 4$ , displacement FEM. Section 4 outlines some conclusions.

## 2. MIXED NON-CONFORMING FINITE ELEMENT FORMULATION

This section summarises the design of the non-conforming mixed finite element AW21 and the simplified AW15 finite element of [AW03b]. We detail the derivation of the coefficient matrices which determine the discrete basis functions and comment on the discretisation of the (local) operator matrices.

We assume some regular triangulation  $\mathcal{T} \subset \Omega \subset \mathbb{R}^2$  with the set of edges  $\mathcal{E}$  and the set of nodes  $\mathcal{N}$ , i.e., it holds for any triangles  $T_1, T_2 \in \mathcal{T}$

$$T_1 \cap T_2 \in \mathcal{T} \cup \mathcal{E} \cup \mathcal{N} \cup \{\emptyset\}.$$

Unit outer normal and tangent vectors with respect to an edge  $E \in \mathcal{E}$  are then defined to be  $\nu_E$  and  $\tau_E$ , respectively.

In addition to the spaces defined above, the spaces

$$\begin{aligned} \Sigma_0 &:= \left\{ \sigma \in H(\operatorname{div}, \Omega; \mathbb{S}) \mid \right. \\ &\quad \left. \int_{\Gamma_N} \psi \cdot (\sigma \nu) \, ds = 0 \text{ for all } \psi \in \mathcal{D}(\Gamma_N; \mathbb{R}^2) \right\}, \\ \Sigma_g &:= \left\{ \sigma \in H(\operatorname{div}, \Omega; \mathbb{S}) \mid \right. \\ &\quad \left. \int_{\Gamma_N} \psi \cdot (\sigma \nu) \, ds = \int_{\Gamma_N} \psi \cdot g \, ds \text{ for all } \psi \in \mathcal{D}(\Gamma_N; \mathbb{R}^2) \right\} \end{aligned}$$

involve the traction boundary conditions where  $\mathcal{D}$  is the space of test functions.

The weak formulation of (1.1) reads: Given data  $u_D \in H^1(\Omega; \mathbb{R}^2)$ ,  $f \in L^2(\Omega; \mathbb{R}^2)$ ,  $g \in L^2(\Gamma_N; \mathbb{R}^2)$ , seek the solution  $(\sigma, u) \in \Sigma_g \times \mathcal{V}$  with

$$\begin{aligned} (2.1) \quad & \int_{\Omega} \sigma : \mathbb{C}^{-1} \tau \, dx + \int_{\Omega} u \cdot \operatorname{div} \tau \, dx \\ &= \int_{\Gamma_D} u_D \cdot (\tau \nu_{\Omega}) \, ds \quad \text{for all } \tau \in \Sigma_0, \\ & \int_{\Omega} v \cdot \operatorname{div} \sigma \, dx = - \int_{\Omega} f \cdot v \, dx \quad \text{for all } v \in \mathcal{V}. \end{aligned}$$

**2.1. MFEM design (AW21).** According to [AW03b], the discrete spaces for stress and displacement for the non-conforming mixed finite element denoted AW21 in Table 1 are defined on a triangle  $T \in \mathcal{T} \subset \Omega$  by

$$(2.2) \quad \begin{aligned} \Sigma_T &:= \{ \sigma \in P_2(T; \mathbb{S}) \mid \nu_E \cdot (\sigma \nu_E) \in P_1(E) \text{ for each } E \in \partial T \}, \\ \mathcal{V}_T &:= P_1(T; \mathbb{R}^2). \end{aligned}$$

For each triangle  $T \in \mathcal{T}$  and the barycentric coordinates  $\lambda_1, \lambda_2, \lambda_3$  on  $T$  a basis  $(\varphi_1, \dots, \varphi_6)$  of  $P_2(T; \mathbb{R})$  reads

$$\begin{aligned} \varphi_1 &= \lambda_1, & \varphi_2 &= \lambda_2, & \varphi_3 &= \lambda_3, \\ \varphi_4 &= \lambda_1 \lambda_2, & \varphi_5 &= \lambda_2 \lambda_3, & \varphi_6 &= \lambda_1 \lambda_3. \end{aligned}$$

The 18 coefficients  $a_1, \dots, a_6, b_1, \dots, b_6, c_1, \dots, c_6 \in \mathbb{R}$  of an arbitrary stress field  $\sigma_T \in P_2(T; \mathbb{S})$  read

$$(2.3) \quad \sigma_T := \sum_{k=1}^6 \varphi_k \begin{pmatrix} a_k & c_k \\ c_k & b_k \end{pmatrix}.$$

The 15 degrees of freedom  $\xi_1, \dots, \xi_{15}$  of  $\sigma_T \in \Sigma_T$  are specified according to (2.2) by the following.

- (a) The values of the moments of degree 0 and 1 of the two normal components of  $\sigma$  on each edge  $E_1, E_2, E_3$  of  $T$  (12 degrees of freedom), namely, for  $j = 1, 2, 3$ ,

$$|E_j|^{-1} \int_{E_j} \sigma \nu_{E_j} ds = \begin{pmatrix} \xi_{4(j-1)+1} \\ \xi_{4(j-1)+2} \end{pmatrix},$$

$$|E_j|^{-2} \int_{E_j} ((x - \text{mid}(E_j)) \tau_{E_j}) \sigma \nu_{E_j} ds = \begin{pmatrix} \xi_{4(j-1)+3} \\ \xi_{4(j-1)+4} \end{pmatrix}.$$

- (b) The values of the three components of the integral mean of  $\sigma$  on  $T$  (3 degrees of freedom), i.e.,

$$|T|^{-1} \int_T \sigma dx = \begin{pmatrix} \xi_{13} & \xi_{15} \\ \xi_{15} & \xi_{14} \end{pmatrix}.$$

The coefficients of the basis functions in (2.3) for some triangle  $T \in \mathcal{T}$  and  $\sigma_T \in \Sigma_T$  are specified by the linear relation

$$(2.4) \quad C (a_1, b_1, c_1, \dots, a_6, b_6, c_6)^T = (\xi_1, \dots, \xi_{18})^T$$

with the  $18 \times 18$  matrix  $C$ . By the definition (a)–(b) of the degrees of freedom and the constraints with regard to the considered stress space (2.2)

$$C := \frac{1}{60} \begin{pmatrix} R & S \\ 20\tilde{I} & 5\tilde{I} \\ 0 & 60K \end{pmatrix}.$$

The evaluation of the integrals in (a) for the  $P_2$  basis yields the entries in the first row and involves the degrees of freedom  $\xi_1, \dots, \xi_{12}$  given by

$$R := \begin{pmatrix} 30N^{(1)} & 30N^{(1)} & 0 \\ -5M^{(1)} & 5M^{(1)} & 0 \\ 0 & 30N^{(2)} & 30N^{(2)} \\ 0 & -5M^{(2)} & 5M^{(2)} \\ 30N^{(3)} & 0 & 30N^{(3)} \\ 5M^{(3)} & 0 & -5M^{(3)} \end{pmatrix}, \quad S := \begin{pmatrix} 10N^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 10N^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10N^{(3)} \\ 0 & 0 & 0 \end{pmatrix}$$

with the normals matrix

$$N^{(j)} := \begin{pmatrix} \nu_{E_j}(1) & 0 & \nu_{E_j}(2) \\ 0 & \nu_{E_j}(2) & \nu_{E_j}(1) \end{pmatrix} \in \mathbb{R}^{2 \times 3}$$

for the  $k$ th component  $\nu_{E_j}(k)$  of the global unit normal  $\nu_{E_j}$  along the  $j$ th edge  $E_j$  of a triangle  $T$ . We define the element-dependent normals matrix  $M$  which is identical to  $N$  with the difference that, instead of unique global normal vectors, it contains the outer unit normals along the edges of the elements. The conditions (b) for degrees of freedom  $\xi_{13}, \dots, \xi_{15}$  are encoded in the second row of  $C$  with  $3 \times 3$  identity matrices  $I_{3 \times 3}$  and

$$\tilde{I} := (I_{3 \times 3} \quad I_{3 \times 3} \quad I_{3 \times 3}) \in \mathbb{R}^{3 \times 9}.$$

Since any stress field  $\sigma_T \in \Sigma_T$  has to satisfy  $\nu_{E_j} \cdot (\sigma \nu_{E_j}) \in P_1(E_j)$  on the edges  $E_1, E_2, E_3$  of  $T$ , the remaining degrees of freedom  $\xi_{16}, \dots, \xi_{18}$  vanish and the respective conditions are included in  $C$  by

$$K^{(j)} := ((\nu_{E_j}(1))^2, (\nu_{E_j}(2))^2, 2\nu_{E_j}(1)\nu_{E_j}(2)) \in \mathbb{R}^3$$

with

$$K := \begin{pmatrix} K^{(1)} & 0 & 0 \\ 0 & K^{(2)} & 0 \\ 0 & 0 & K^{(3)} \end{pmatrix} \in \mathbb{R}^{3 \times 9}.$$

**2.2. Local stiffness matrix (AW21).** The fourth-order material tensor  $\mathbb{C}$  and its inverse  $\mathbb{C}^{-1}$  read

$$\mathbb{C} := \begin{pmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \mathbb{C}^{-1} := \begin{pmatrix} \frac{\lambda+2\mu}{4\mu(\lambda+\mu)} & \frac{-\lambda}{4\mu(\lambda+\mu)} & 0 \\ \frac{-\lambda}{4\mu(\lambda+\mu)} & \frac{\lambda+2\mu}{4\mu(\lambda+\mu)} & 0 \\ 0 & 0 & \frac{1}{\mu} \end{pmatrix}.$$

On each triangle  $T \in \mathcal{T}$  the local stiffness matrix, resulting from (2.1),

$$\text{STIMA}(T) := \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \in \mathbb{R}_{\text{sym}}^{21 \times 21}$$

requires the computation of

$$(2.5a) \quad A_{jk} := \int_T \sigma_j : \mathbb{C}^{-1} \sigma_k \, dx \quad \text{for } j, k = 1, \dots, 15;$$

$$(2.5b) \quad B_{jk} := \int_T \text{div } \sigma_j \cdot v_k \, dx \quad \text{for } j = 1, \dots, 15; \, k = 1, \dots, 6.$$

Any basis function  $\sigma_1, \dots, \sigma_{15}$  of  $\Sigma_T$  resulting from the coefficient evaluation in (2.4) can be represented as

$$\sigma_j = \begin{pmatrix} \alpha_j & \gamma_j \\ \gamma_j & \beta_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha_j \\ \beta_j \\ \gamma_j \end{pmatrix} := \begin{pmatrix} \sum_{k=1}^6 a_j^{(k)} \varphi_k \\ \sum_{k=1}^6 b_j^{(k)} \varphi_k \\ \sum_{k=1}^6 c_j^{(k)} \varphi_k \end{pmatrix}.$$

The entry  $A_{jk}$  of the local stiffness matrix is given for the base functions  $\sigma_j$  and  $\sigma_k$  by

$$\begin{aligned} A_{jk} &= \int_T \sigma_j : \mathbb{C}^{-1} \sigma_k \, dx = \int_T (\alpha_j, \beta_j, \gamma_j) \mathbb{C}^{-1} (\alpha_k, \beta_k, \gamma_k)^T \, dx \\ &= \mathbb{C}_{1,1}^{-1} \int_T (\alpha_j \alpha_k + \beta_j \beta_k) \, dx + \mathbb{C}_{1,2}^{-1} \int_T (\alpha_j \beta_k + \beta_j \alpha_k) \, dx \\ &\quad + \mathbb{C}_{3,3}^{-1} \int_T \gamma_j \gamma_k \, dx. \end{aligned}$$

The integrals occurring in the  $A_{jk}$  can be integrated exactly on any  $T \in \mathcal{T}$

$$\int_T \alpha_j \beta_k \, dx = \sum_{m=1}^6 \sum_{n=1}^6 a_m^{(j)} b_n^{(k)} \int_T \varphi_m \varphi_n \, dx = |T| a^{(j)T} M b^{(k)}$$



with the mass matrix

$$M := \frac{1}{180} \begin{pmatrix} 30 & 15 & 15 & 6 & 3 & 6 \\ 15 & 30 & 15 & 6 & 6 & 3 \\ 15 & 15 & 30 & 3 & 6 & 6 \\ 6 & 6 & 3 & 2 & 1 & 1 \\ 3 & 6 & 6 & 1 & 2 & 1 \\ 6 & 3 & 6 & 1 & 1 & 2 \end{pmatrix}.$$

For the evaluation of the second part of the stiffness matrix (2.5b) we recall the conditions imposed on elements of  $\Sigma_T$  and note that the divergence of any base function  $\sigma_j$  yields the form

$$\operatorname{div} \sigma_j = \begin{pmatrix} d_1^{(j)} \varphi_1 + d_3^{(j)} \varphi_2 + d_5^{(j)} \varphi_3 \\ d_2^{(j)} \varphi_1 + d_4^{(j)} \varphi_2 + d_6^{(j)} \varphi_3 \end{pmatrix}$$

for some coefficients  $d_k^{(j)}$ . From this we derive the coefficient matrix  $D := (d^{(1)}, \dots, d^{(18)})$  given by

$$D := \begin{pmatrix} L_1 & L_2 & L_3 & L_2 & 0 & L_3 \\ L_1 & L_2 & L_3 & L_1 & L_3 & 0 \\ L_1 & L_2 & L_3 & 0 & L_2 & L_1 \end{pmatrix} X$$

with the derivative matrices  $L_1, L_2, L_3$ ,

$$L_k := \begin{pmatrix} D_x \lambda_k & 0 & D_y \lambda_k \\ 0 & D_y \lambda_k & D_x \lambda_k \end{pmatrix}$$

and

$$X = (x^{(1)}, \dots, x^{(15)}) \in \mathbb{R}^{18 \times 15},$$

$$x^{(j)} = (a_1^{(j)}, b_1^{(j)}, c_1^{(j)}, \dots, a_6^{(j)}, b_6^{(j)}, c_6^{(j)})^T \in \mathbb{R}^{18}.$$

The second part of the stiffness matrix (2.5b) reads

$$B := \frac{|T|}{12} \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} D.$$

*Remark 2.1.* Neumann boundary conditions are essential conditions in the mixed formulation (2.1) and have to be imposed on the discrete stresses. Opposite to the treatment with Lagrange multipliers in [CGRT08], the current implementation eliminates respective boundary degrees of freedom by condensation.

**2.3. MFEM design (AW15).** In addition to the element described above, a simplified non-conforming element was derived in [AW03b] which only includes the reduced velocity space  $\mathcal{V}_T := \text{RM}(T)$  of discontinuous piecewise rigid body motions, see AW15 in Table 1. More specifically, a function in  $\text{RM}(T)$  has a representation  $(a, b)^T + c(-y, x)^T$  for some coefficients  $a, b, c \in \mathbb{R}$ . The reduced local stress space  $\widehat{\Sigma}_T$  is given by

$$\widehat{\Sigma}_T := \{\tau \in \Sigma_T \mid \text{div } \tau \in \text{RM}(T)\}.$$

When compared to the non-conforming element of Section 2.1, the degrees of freedom corresponding to the first two moments of the normal derivatives on the edges are still used while the integral mean values on the element were dropped. Moreover, the constraint on the divergence of the stress fields now is more restricted by means of the space  $\text{RM}(T)$ .

In order to obtain the unknown coefficients  $a_1, b_1, c_1, \dots, a_6, b_6, c_6$  of the basis functions for a discrete  $\sigma_T \in \widehat{\Sigma}_T$  on a triangle  $T$ , we solve a linear system

$$(2.6) \quad C (a_1, b_1, c_1, \dots, a_6, b_6, c_6)^T = (\xi_1, \dots, \xi_{18})^T.$$

The constraints of  $\widehat{\Sigma}_T$  lead to the coefficient matrix

$$C := \frac{1}{60} \begin{pmatrix} R & S \\ 0 & 60\tilde{Q} \\ 0 & 60K \end{pmatrix} \in \mathbb{R}^{18 \times 18}$$

which encodes 12 base functions in terms of the aforementioned  $P_2$  basis. The first row represents the degrees of freedom resulting from the first two moments of the normal derivatives on the edges of a triangle with matrices  $R, S \in \mathbb{R}^{12 \times 9}$  from Section 2.1. The constraints of the space  $\text{RM}$  are encoded in the second row of  $C$  by means of the matrix

$$Q^{(j)} := \text{diam}(T) \begin{pmatrix} D_x v^{(j)} & 0 & D_y v^{(j)} \\ 0 & D_y w^{(j)} & D_x w^{(j)} \\ D_x w^{(j)} & D_y v^{(j)} & D_y w^{(j)} + D_x v^{(j)} \end{pmatrix}$$

with

$$\begin{aligned} v^{(1)\dots(3)} &:= ((\lambda_1 - \lambda_2), \lambda_3, -\lambda_3), \\ w^{(1)\dots(3)} &:= (-\lambda_2, \lambda_2, (\lambda_1 - \lambda_3)), \\ \tilde{Q} &:= (Q^{(1)}, Q^{(2)}, Q^{(3)}) \in \mathbb{R}^{3 \times 9}. \end{aligned}$$

Note that the matrix  $\tilde{Q}$  is scaled in relation to the size of the triangle in order to avoid ill-conditioning of the coefficient matrix. The remaining vanishing quadratic degrees of freedom are encoded in the matrix  $K \in \mathbb{R}^{3 \times 9}$  from Section 2.1.

**2.4. Local stiffness matrix (AW15).** The evaluation of the local stiffness matrix follows the derivation in Section 2.2. While  $A$  is evaluated as before, by definition of the base functions and the constraints on the stress space, the divergence of some  $\sigma_j$  has the form

$$\operatorname{div} \sigma_j = \begin{pmatrix} d_1^j - d_3^j \varphi_3 \\ d_2^j + d_3^j \varphi_2 \end{pmatrix}$$

for some coefficients  $d_k^{(j)} \in \mathbb{R}$ . The coefficient matrix  $D := (d^{(1)}, \dots, d^{(15)})$  reads

$$D := \begin{pmatrix} F_1 & F_2 & F_3 & F_2 & 0 & F_3 \\ G_1 & G_2 & G_3 & G_2 & 0 & G_3 \\ 0 & 0 & 0 & G_1 - G_2 & G_3 & -G_3 \end{pmatrix} X$$

with

$$\begin{aligned} F_k &:= (D_x \lambda_k \quad 0 \quad D_y \lambda_k), \\ G_k &:= (0 \quad D_y \lambda_k \quad D_x \lambda_k) \end{aligned}$$

and

$$\begin{aligned} X &= (x^{(1)}, \dots, x^{(12)}) \in \mathbb{R}^{18 \times 12}, \\ x^{(j)} &= (a_1^{(j)}, b_1^{(j)}, c_1^{(j)}, \dots, a_6^{(j)}, b_6^{(j)}, c_6^{(j)})^T \in \mathbb{R}^{18}. \end{aligned}$$

The second part of the stiffness matrix simplifies to

$$B := \frac{|T|}{3} \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix} D.$$

**2.5. Other symmetric MFEM.** In addition to providing details about the implementation of the non-conforming Arnold-Winther elements in the previous section, we also aim to compare it to other symmetric MFEM developed recently. A detailed account on the conforming Arnold-Winther element [AW02] was given in [CGRT08] and we only summarise the main properties for the sake of completeness. Another set of symmetric element is due to [SS07]. Instead of using the explicit base functions listed in the reference, our implementation follows the lines of the Arnold-Winther element implementation. It thus is instructive to provide some details about the required modifications for the design of the shape functions.

**2.5.1. Conforming Arnold-Winther elements.** The conforming elements of Arnold and Winther for the mixed formulation are based on polynomials of degree 3 for the symmetric stresses and of degree 1 for the displacement, see AW30 and AW24 in Table 1. In order to obtain continuity of the stress field in normal direction along the boundary of elements, it was shown in [AW02] that vertex degrees of freedom cannot be circumvented. The non-conforming elements AW21 and AW15 described above on the one hand lack normal-stress continuity due to

missing vertex degrees of freedom which, on the other hand, enables to employ a lower polynomial degree for the stress space. For the conforming AW30 element, a reduced AW24 element was described as well. Since the design of the AW24 element is similar to the one of the AW30 element [CGRT08] and to the ones presented above, we omit the details. Experimental results for the two lowest order conforming Arnold-Winther elements as depicted in Table 1 are presented in Section 3.

*2.5.2. Schöberl-Sinwel TD-NNS elements.* In [SS07], a class of mixed finite elements were presented which employ subspaces of  $H(\text{curl}, \Omega; \mathbb{R}^2)$  for the displacement and of  $H(\text{div div}, \Omega; \mathbb{S})$  for the stress. According to the continuity requirements, the method is coined Tangential-Displacement-Normal-Normal-Stress (TD-NNS) formulation of elasticity. In this paper, for the sake of simpler notation, we call these elements S15 and S27 of polynomial degree 1 and 2 and overall number of degrees of freedom 15 and 27 as depicted in Table 1 for comparison with Arnold-Winther elements. The homogeneous spaces of order  $k$  are defined by

$$\begin{aligned} \Sigma_{h,k} &:= \{ \sigma \in L^2_{\text{sym}}(\Omega) \mid \sigma|_T \in P_k(T; \mathbb{S}), \nu_E \cdot (\sigma \nu_E) = 0 \text{ on } \Gamma_N \\ &\quad \forall E \in \partial T : \nu_E \cdot (\sigma \nu_E) \in P_{\max(1,k-1)}(E) \cap C^0(\Omega) \}, \\ \mathcal{V}_{h,k} &:= \{ v \in L^2(\Omega)^2 \mid v|_T \in P_k(T; \mathbb{R}^2), v \cdot \tau_E \in C^0(\Omega), v \cdot \tau_E = 0 \text{ on } \Gamma_D \}. \end{aligned}$$

Note that  $\mathcal{V}_{h,k}$  is the second family of Nédélec edge elements. The homogeneous problem is given by: Find  $(\sigma, u) \in \Sigma_{h,k} \times \mathcal{V}_{h,k}$  such that

$$\begin{aligned} a(\sigma, \rho) + b(\rho, u) &= 0 & \forall \rho \in \Sigma_{h,k} \\ b(\sigma, v) &= -(f, v) & \forall v \in \mathcal{V}_{h,k}. \end{aligned}$$

with the bilinear forms

$$\begin{aligned} a(\sigma, \rho) &:= \int_{\Omega} \sigma : \mathbb{C}^{-1} \rho \, dx, \\ b(\rho, v) &:= \langle \text{div} \rho, v \rangle = \sum_{T \in \mathcal{T}} \int_T \text{div} \rho \cdot v \, dx - \int_{\partial T} (\tau \cdot (\rho \nu)) (v \cdot \tau) \, ds. \end{aligned}$$

Opposite to the basis functions derived in [SS07], we prefer an approach similar to the design of the Arnold-Winther mixed FEM presented above and in [CGRT08]. In what follows, we discuss the design of the S27 element of order 2. The design of the order 1 element S15 is a mere reduction of the shown approach.

As in Section 2.3, some stress field  $\sigma_T \in P_k(T; \mathbb{S})$  and some displacement field  $u_T \in P_k(T; \mathbb{R}^2)$  on some triangle  $T$  have the form

$$\sigma_T := \sum_{j=1}^{N_k} \varphi_j \begin{pmatrix} a_k & c_k \\ c_k & b_k \end{pmatrix} \quad \text{and} \quad u_T := \sum_{j=1}^{N_k} \varphi_j \begin{pmatrix} d_k \\ e_k \end{pmatrix}$$

with  $(3 + 2)N_k$  coefficients overall, according to degree  $k$  of the polynomial space. The 15+12 degrees of freedom for the S27 element are defined by (a)–(d).

- (a) The values of the moments of degree 0 and 1 of the two normal-normal components of  $\sigma$  on each edge  $E_1, E_2, E_3$  of  $T$  (6 degrees of freedom), namely, for  $j = 1, 2, 3$ ,

$$|E_j|^{-1} \int_{E_j} (\sigma \nu_{E_j}) \cdot \nu_{E_j} ds = \xi_{2(j-1)+1},$$

$$|E_j|^{-2} \int_{E_j} ((x - \text{mid}(E_j)) \tau_{E_j}) (\sigma \nu_{E_j}) \cdot \nu_{E_j} ds = \xi_{2(j-1)+2}.$$

- (b) The values of the three components of the integral mean of  $\sigma$  on  $T$  tested with a basis  $\{\varphi_j\}$  of  $P_1(T; \mathbb{S})$  (9 degrees of freedom)

$$|T|^{-1} \int_T \sigma : \varphi_j dx = \xi_{j+6} \quad \text{for } j = 1, \dots, 9.$$

For instance a basis of  $P_1(T; \mathbb{S})$  is given by  $\left\{ \begin{pmatrix} \lambda_k & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \lambda_k \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 0 & \lambda_k \\ \lambda_k & 0 \end{pmatrix} \right\}$ ,  $k = 1, 2, 3$ .

- (c) The values of the moments of degrees 0, 1 and 2 of the tangential components of  $u$  on each edge  $E_1, E_2, E_3$  of  $T$  (9 degrees of freedom), namely, for  $j = 1, 2, 3$ ,

$$|E_j|^{-1} \int_{E_j} u \cdot \tau_{E_j} ds = \xi_{3(j-1)+16},$$

$$|E_j|^{-2} \int_{E_j} ((x - \text{mid}(E_j)) \tau_{E_j}) u \cdot \tau_{E_j} ds = \xi_{3(j-1)+17},$$

$$|E_j|^{-3} \int_{E_j} ((x - \text{mid}(E_j)) \tau_{E_j})^2 u \cdot \tau_{E_j} ds = \xi_{3(j-1)+18}.$$

- (d) The values of the integral mean on the element tested with a basis  $\{q_j\}$  of the space  $P_0(T; \mathbb{R}^2) \oplus xP_0(T)$  such as  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \lambda_2 \\ \lambda_3 \end{pmatrix} \right\}$  (3 degrees of freedom)

$$|T|^{-1} \int_T u \cdot q_j dx = \xi_{j+24} \quad \text{for } j = 1, 2, 3.$$

With the notation from Section 2.1 the coefficient matrix for the stress space  $\Sigma_T := \Sigma_{h,2}|_T$  reads

$$C_{\Sigma_T} := \frac{1}{60} \begin{pmatrix} R & S \\ I_R & I_S \\ 0 & 60K \end{pmatrix} \in \mathbb{R}^{18 \times 18}.$$

It is formed according to the moments and constraints described in (a) and (b). The evaluation of the integrals in (a) for the  $P_2$  basis yields

the entries in the first row and involves the degrees of freedom  $\xi_1, \dots, \xi_6$  given by

$$R := \begin{pmatrix} 30N^{(1)} & 30N^{(1)} & 0 \\ -5M^{(1)} & 5M^{(1)} & 0 \\ 0 & 30N^{(2)} & 30N^{(2)} \\ 0 & -5M^{(2)} & 5M^{(2)} \\ 30N^{(3)} & 0 & 30N^{(3)} \\ 5M^{(3)} & 0 & -5M^{(3)} \end{pmatrix}, \quad S := \begin{pmatrix} 10N^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 10N^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10N^{(3)} \\ 0 & 0 & 0 \end{pmatrix}$$

with the normal-normal vectors

$$N^{(j)} := \left( (\nu_{E_j}(1))^2, (\nu_{E_j}(2))^2, 2\nu_{E_j}(1)\nu_{E_j}(2) \right) \in \mathbb{R}^3$$

for the  $k$ th component  $\nu_{E_j}(k)$  of the globally unique normal  $\nu_{E_j}$  along the  $j$ th edge  $E_j$  of a triangle  $T$ . As mentioned before in Section 2.1, the matrix  $M$  contains the normal-normal entries with element-dependent outer unit normals. The conditions (b) for degrees of freedom  $\xi_7, \dots, \xi_{15}$  are encoded in the second row of  $C_{\Sigma_T}$  with  $3 \times 3$  identity matrices  $I$  and

$$I_R := \begin{pmatrix} 10I & 5I & 5I \\ 5I & 10I & 5I \\ 5I & 5I & 10I \end{pmatrix} \quad \text{and} \quad I_S := \begin{pmatrix} 2I & 1I & 2I \\ 2I & 2I & 1I \\ 1I & 2I & 2I \end{pmatrix} \in \mathbb{R}^{9 \times 9}.$$

The constraints  $\nu_{E_j} \cdot (\sigma \nu_{E_j}) \in P_1(E_j)$  are encoded in the matrix  $K \in \mathbb{R}^{3 \times 9}$  from Section 2.1. Note that the coefficient matrix  $C_{\Sigma_T}$  is somehow similar to the one for the non-conforming Arnold-Winther elements.

The coefficient matrix

$$C_{\nu_T} := \frac{1}{120} \begin{pmatrix} \tilde{R} & \tilde{S} \\ \tilde{I}_R & \tilde{I}_S \end{pmatrix} \in \mathbb{R}^{12 \times 12}$$

corresponds to the displacement degrees of freedom described in (c) and (d). The evaluation of the integrals in (c) for the  $P_2$  basis yields the entries in the first row and involves the degrees of freedom  $\xi_{16}, \dots, \xi_{24}$  given by

$$\tilde{R} := \begin{pmatrix} 60N^{(1)} & 60N^{(1)} & 0 \\ -10M^{(1)} & 10M^{(1)} & 0 \\ 5N^{(1)} & 5N^{(1)} & 0 \\ 0 & 60N^{(2)} & 60N^{(2)} \\ 0 & -10M^{(2)} & 10M^{(2)} \\ 0 & 5N^{(2)} & 5N^{(2)} \\ 60N^{(3)} & 0 & 60N^{(3)} \\ 10M^{(3)} & 0 & -10M^{(3)} \\ 5N^{(3)} & 0 & 5N^{(3)} \end{pmatrix}, \quad \tilde{S} := \begin{pmatrix} 20N^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ N^{(1)} & 0 & 0 \\ 0 & 20N^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & N^{(2)} & 0 \\ 0 & 0 & 20N^{(3)} \\ 0 & 0 & 0 \\ 0 & 0 & N^{(3)} \end{pmatrix}$$

with the global transpose tangents

$$N^{(j)} := (-\nu_{E_j}(2), \nu_{E_j}(1)) \in \mathbb{R}^2$$

and the element-dependent equivalent  $M$ . The conditions (d) for degrees of freedom  $\xi_{25}, \dots, \xi_{27}$  are encoded in the second row of  $C_{\mathcal{V}_T}$  and

$$\begin{aligned} \tilde{I}_R &:= \begin{pmatrix} 40 & 0 & 40 & 0 & 40 & 0 \\ 0 & 40 & 0 & 40 & 0 & 40 \\ 10 & 10 & 20 & 10 & 10 & 20 \end{pmatrix}, \\ \tilde{I}_S &:= \begin{pmatrix} 10 & 0 & 10 & 0 & 10 & 0 \\ 0 & 10 & 0 & 10 & 0 & 10 \\ 4 & 2 & 4 & 4 & 2 & 4 \end{pmatrix} \in \mathbb{R}^{3 \times 6}. \end{aligned}$$

### 3. NUMERICAL EXPERIMENTS

This section presents the computational competition of the standard continuous  $P_k$ ,  $k = 1, \dots, 4$ , displacement finite elements, the weakly symmetric mixed finite element PEERS [ABD84], the non-conforming displacement finite element KS [KS95] and the symmetric mixed methods AW30, AW24, AW21, AW15, S15 and S27 described above. We begin with an academic example which represents an ideal benchmark for the locking phenomenon. Next more realistic benchmark problems such as the Cook's membrane problem and the example of a rigid circular inclusion in an infinite plate are considered. Finally, the rotated L-shaped domain with singular solution motivates the need of local mesh refinement.

All methods will be compared with respect to the stress tensor error in the energy norm  $\|\sigma\| := \|\mathbb{C}^{-1/2}(\sigma)\|_{L^2(\Omega)}$ . The implementation of the AW30 MFEM is presented in [CGRT08] and due to the close relationship, a few modifications lead to an implementation of the AW24 MFEM. The implementation of PEERS is described in [CDFH00], where the continuity constraints are enforced using Lagrange multipliers. In order to compare the results of PEERS to all other methods, the error is plotted with respect to the theoretically necessary degrees of freedom and not to the size of the discrete system involving a high number of degrees of freedom for the Lagrange multipliers. For the KS non-conforming finite element, the first component is chosen to be non-conforming and the second component is chosen to be continuous. Note that in [BS92] it is proven that  $P_k$  shows locking for  $k = 1, 2, 3$  and is locking free for  $k = 4$ , which is also confirmed by the following numerical examples.

Throughout this section, let  $N$  denote the number of degrees of freedom, i.e., the number of unknowns of the algebraic systems to be solved.

**3.1. Academic example.** As a first academic example consider the model problem (1.1) on the unit square  $\Omega = (0, 1) \times (0, 1)$  with homogeneous Dirichlet boundary conditions. The elasticity modulus is set to  $E = 10^5$  and the Poisson ratio is chosen out of  $\nu \in \{0.3, 0.49, 0.4999\}$ .

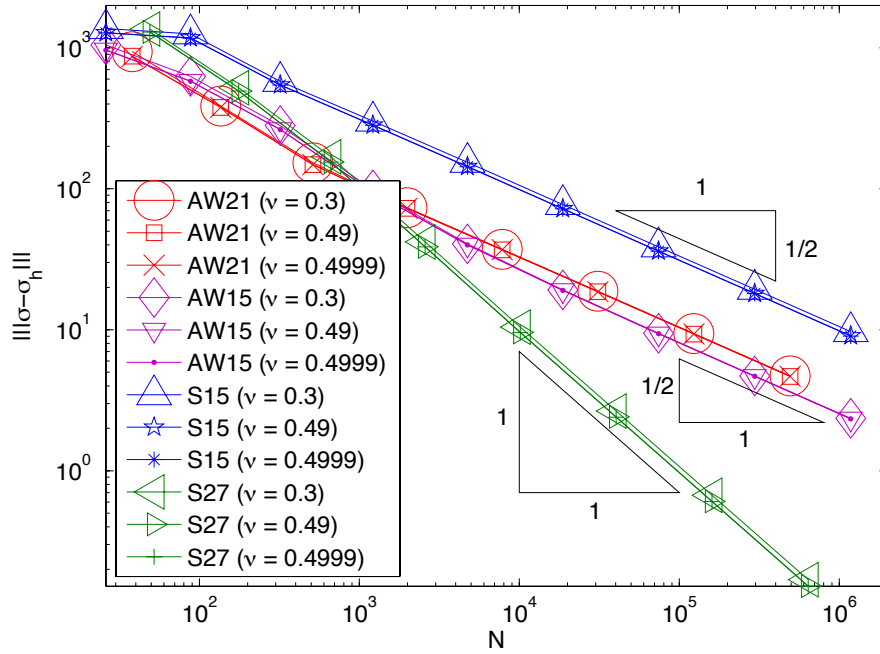


FIGURE 3.1. Energy error  $\|\sigma - \sigma_h\|$  for the academic example with Poisson ratio  $\nu = 0.3, 0.49, 0.4999$ .

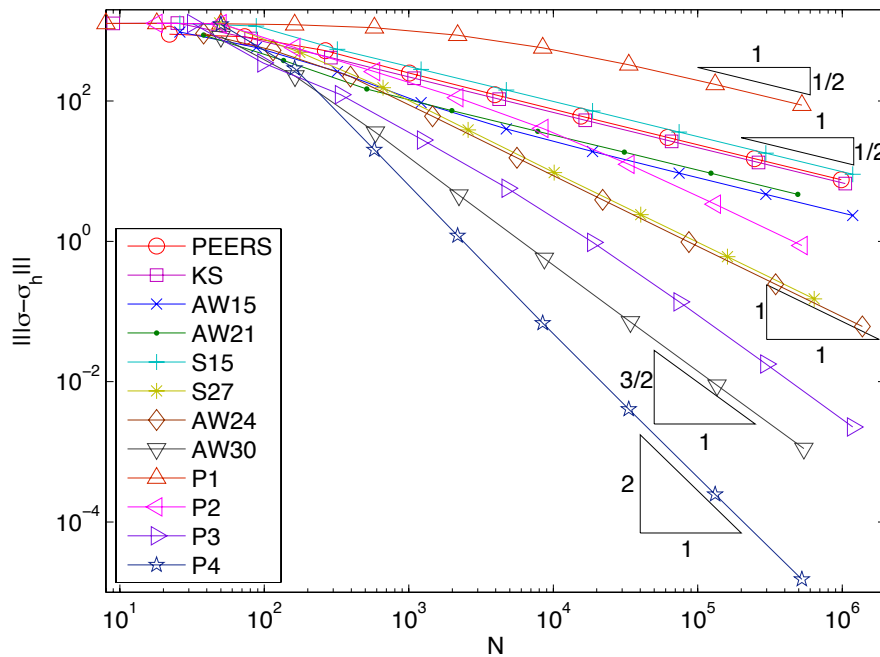


FIGURE 3.2. Survey of the energy error  $\|\sigma - \sigma_h\|$  for the academic example with Poisson ratio  $\nu = 0.499$ .



The right hand side is defined as

$$\begin{aligned} f_x(x, y) &= -2\mu\pi^3 \cos(\pi y) \sin(\pi y)(2 \cos(2\pi x) - 1), \\ f_y(x, y) &= 2\mu\pi^3 \cos(\pi x) \sin(\pi x)(2 \cos(2\pi y) - 1), \end{aligned}$$

and depends only on the Lamé parameter  $\mu$  and not on the critical Lamé parameter  $\lambda$ . The corresponding displacement solution is given as

$$\begin{aligned} u_x(x, y) &= \pi \cos(\pi y) \sin^2(\pi x) \sin(\pi y), \\ u_y(x, y) &= -\pi \cos(\pi x) \sin(\pi x) \sin^2(\pi y). \end{aligned}$$

Notice that for the displacement solution it holds  $\operatorname{div}(u) \equiv 0$ . The experiments of Figure 3.1 verify the theoretical findings of [AW03b] and [SS07]. Since the convergence graphs for  $\nu = 0.3$ ,  $\nu = 0.49$  and  $\nu = 0.4999$  are very close or even cover each other, these results demonstrate empirically the locking-free property of the methods, i.e., the robustness for  $\nu$  tending to  $1/2$ . Moreover, for this convex example with smooth solution, uniform refinement leads to optimal order of convergence. Figure 3.2 shows that  $P_1$ ,  $P_2$  and  $P_3$  lead to suboptimal convergence rates for coarser meshes because of the locking effect. However since locking is a pre-asymptotic phenomenon, the graphs show optimal convergence rates for larger number of degrees of freedom. It can be observed that all other first order methods lead to smaller errors with optimal convergence without pre-asymptotic ranges and thus no locking phenomena. Among those, the AW15 MFEM shows the smallest errors. Note that the KS non-conforming FEM and the S15 and PEERS MFEM lead to significantly larger errors than the AW21 or AW15 MFEM. The locking of the  $P_2$  element leads to the fact that the AW24 and S27 MFEM, which are of the same order of convergence, show similarly better results. The AW30 MFEM leads to smaller errors than the  $P_3$  FEM. Since the fourth-order  $P_4$  FEM is locking free, it shows faster convergence rates and smaller errors than the third-order AW30 MFEM.

**3.2. Cook's membrane problem.** The Cook's membrane benchmark problem considers the model problem (1.1) with  $\Omega$  depicted in Figure 3.3. The domain describes a tapered panel which is clamped on the left side and subject to a surface load in vertical direction on the right side. The interior load is zero,  $f \equiv 0$ . For  $(x, y) \in \Gamma_N$ , the surface load is given by  $g(x, y) = (0, 1)$  if  $x = 48$  and  $g(x, y) = 0$  elsewhere. Since the plate is clamped,  $u_D \equiv 0$  on  $\Gamma_D$ . This benchmark problem is a standard test for bending dominated response. In the experiments, the elasticity modulus is  $E = 10^5$  and the Poisson ratio  $\nu = 0.499$ . Figure 3.4 shows the displacement solution magnified by a factor of  $2 \cdot 10^3$  and the modulus of the deviatoric part  $|\operatorname{dev}(\sigma)|$  pictured as grey scales for the AW15 element. The unknown solution is singular because of

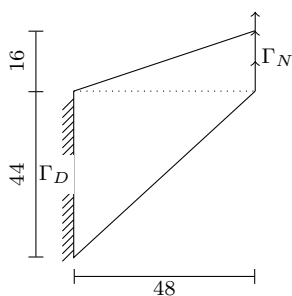


FIGURE 3.3. The Cook's membrane.

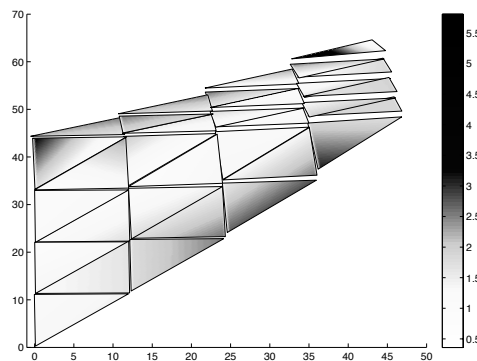


FIGURE 3.4. Solution of the Cook's membrane benchmark.

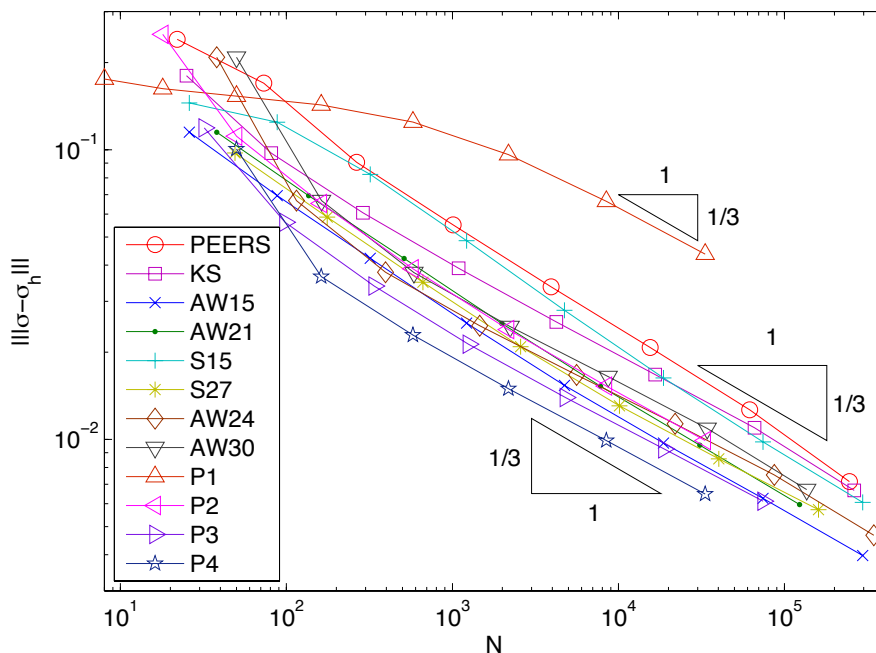


FIGURE 3.5. Energy errors  $\|\sigma - \sigma_h\|$  for the Cook's membrane problem with Poisson ratio  $\nu = 0.499$ .

the chosen mixed boundary conditions. Therefore, the error is approximated by the difference between the discrete solution and a fine grid approximation. The fine grid approximation is computed on a uniform refinement of the grid for the last level. For the  $P_k$ ,  $k = 1, \dots, 4$ , FEM, the fine grid solution is computed additionally with one order higher polynomials. Figure 3.5 shows the convergence history for uniform refined meshes. The first observation is that the convergence rate is suboptimal for all FEM due to the regularity constraints. Remark that it is unclear how to use graded meshes to improve the convergence

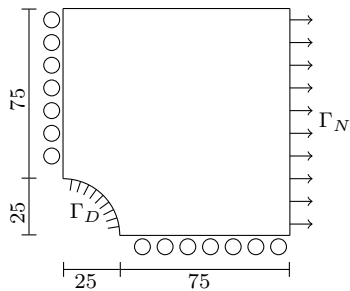


FIGURE 3.6. Domain circular inclusion.

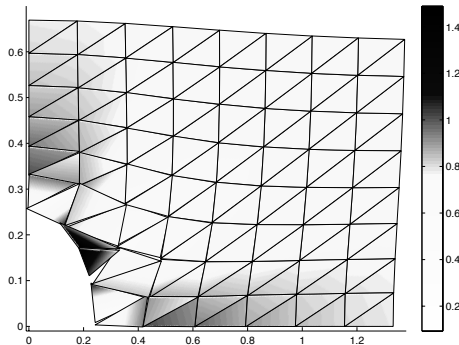


FIGURE 3.7. Solution of the circular inclusion benchmark.

rates. Hence, adaptive refinement strategies have to be investigated. For this benchmark example it turns out to be a crucial point for the KS FEM, that the first component is non-conforming. Choosing the second component to be non-conforming and hence the first continuous, results in singular discrete algebraic systems. This could be avoided by bisection the upper right triangle such that not all nodes of the resulting triangles lay on the Neumann boundary. The locking-effect is visible only for the linear  $P_1$  FEM because of the dominating regularity constraints. The  $P_4$  FEM shows the best results. The  $P_3$  or  $P_2$  FEM lead to similar errors as the MFEM or the KS FEM. Among the MFEM and the KS FEM, the AW15 shows the best results while PEERS and on the finest grid also KS show the largest error.

**3.3. Circular inclusion.** As a third example, consider a rigid circular inclusion in an infinite plate for the domain  $\Omega$  as shown in Figure 3.6. The exact solution [KS95] to the model problem (1.1) in polar coordinates  $(r, \phi)$  reads

$$u_r = \frac{1}{8\mu r} \left( (\kappa - 1)r^2 + 2\gamma a^2 + \left( 2r^2 - \frac{2(\kappa + 1)a^2}{\kappa} + \frac{2a^4}{\kappa r^2} \right) \cos(2\phi) \right),$$

$$u_\phi = -\frac{1}{8\mu r} \left( 2r^2 - \frac{2(\kappa - 1)a^2}{\kappa} - \frac{2a^4}{\kappa r^2} \right) \sin(2\phi),$$

where  $\kappa = 3 - 4\nu$ ,  $\gamma = 2\nu - 1$ ,  $a = 1/4$  and  $\mu$  is the Lamé parameter determined by  $E = 10^5$  and the Poisson ratio  $\nu = 0.499$ . Figure 3.7 shows the displacement solution magnified by a factor of  $5 \cdot 10^4$  and the modulus of the deviatoric part  $|\text{dev}(\sigma)|$  as grey scales for the AW21 element. The numerical comparison in Figure 3.8 shows optimal convergence rates for the first-order methods while suboptimal convergence rates for the higher-order schemes. This is caused by the linear approximation of the boundary at the circular inclusion which is not sufficiently accurate for the higher-order methods. Therefore, for problems with curved boundary, parametric elements have to be used in order to get

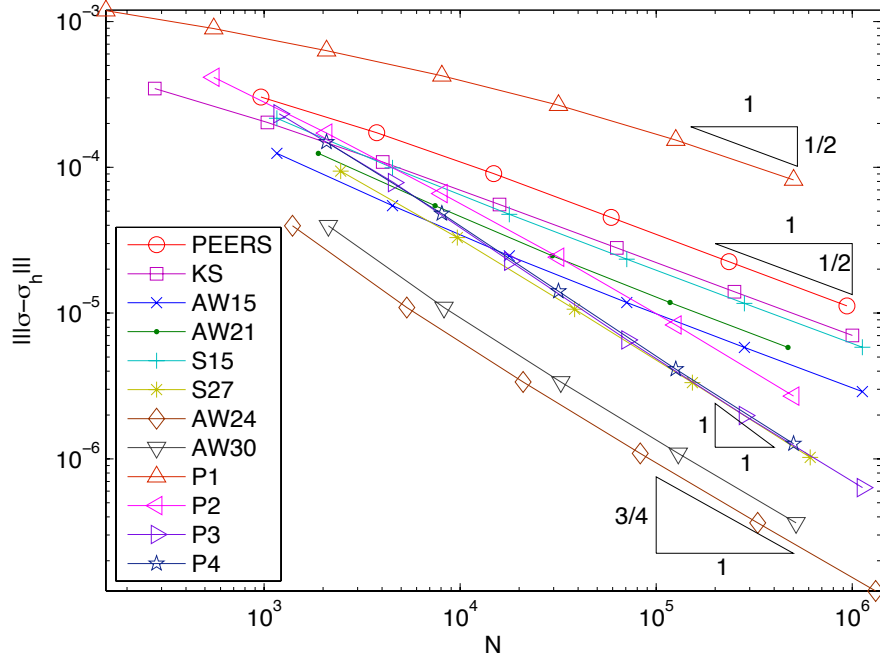


FIGURE 3.8. Energy errors  $\|\sigma - \sigma_h\|$  for the circular inclusion example with Poisson ratio  $\nu = 0.499$ .

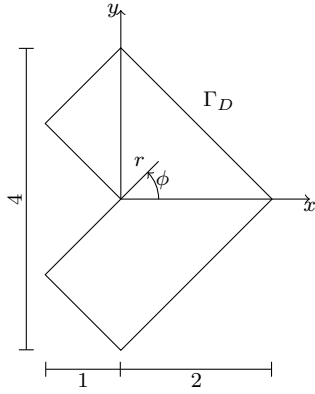


FIGURE 3.9. The L-shaped domain.

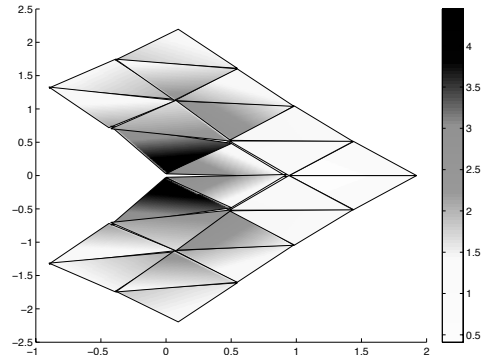


FIGURE 3.10. Solution of the L-shaped benchmark.

optimal convergence rates. Note that the implementation of parametric elements for the MFEM is somewhat more involved than the presented simplified implementation. The experiments illustrate that  $P_1$  and  $P_2$  show locking while for  $P_3$  and  $P_4$  the constraint of the boundary approximation dominates the error. For the first-order stable methods, the AW15 finite element exhibits the best results. The overall smallest error shows the AW24 MFEM.

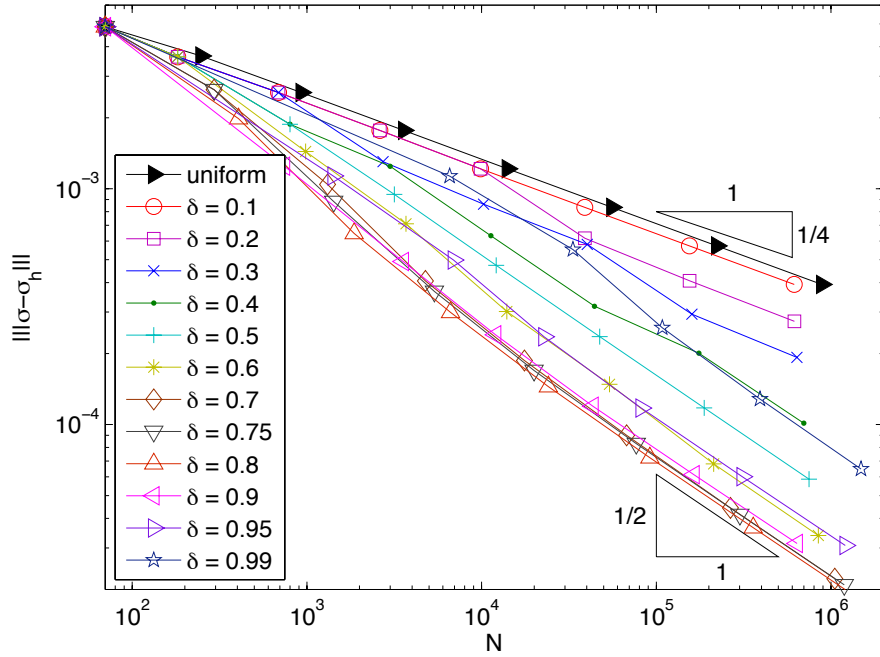


FIGURE 3.11. Energy errors  $\|\sigma - \sigma_h\|$  for different grading parameters  $\delta$  for the low-order AW15 MFEM for the L-shaped example with Poisson ratio  $\nu = 0.499$ .

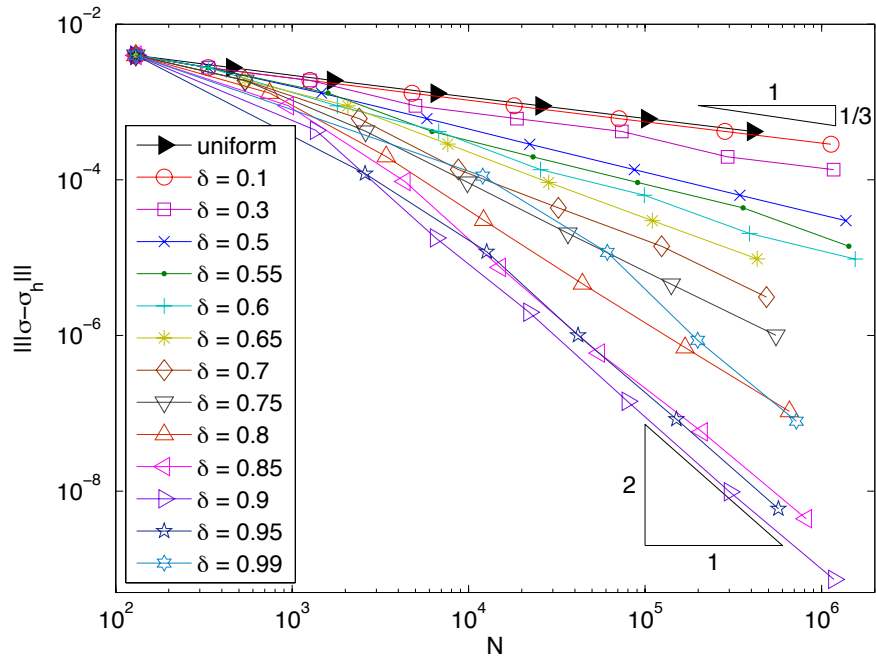


FIGURE 3.12. Energy errors  $\|\sigma - \sigma_h\|$  for different grading parameters  $\delta$  for the high-order AW30 MFEM for the L-shaped example with Poisson ratio  $\nu = 0.499$ .

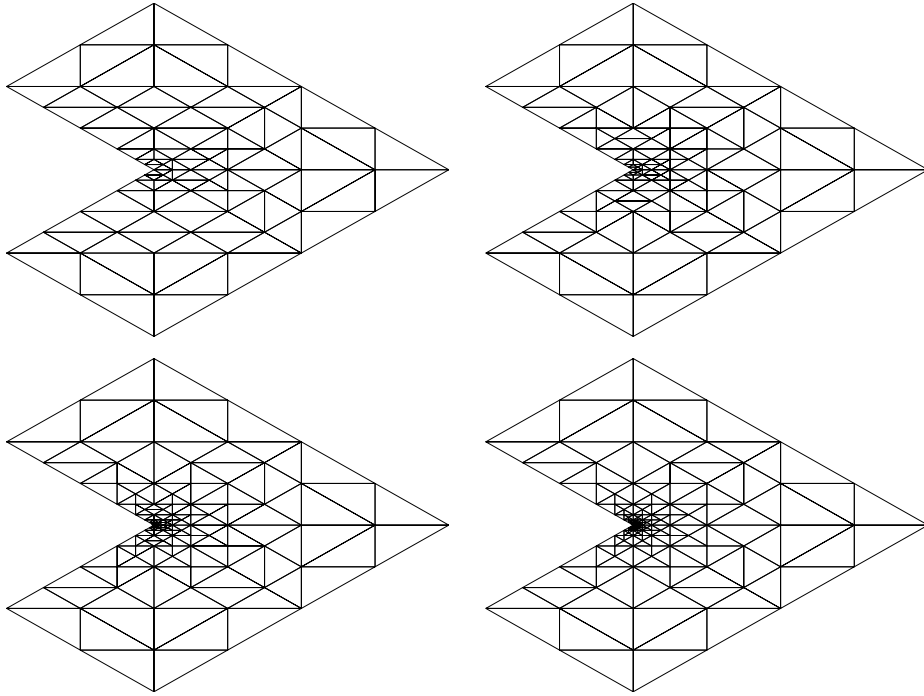


FIGURE 3.13. Graded meshes for  $\delta = 0.6, 0.7, 0.8, 0.9$  (top left to bottom right) for global mesh-size  $1/4$ .

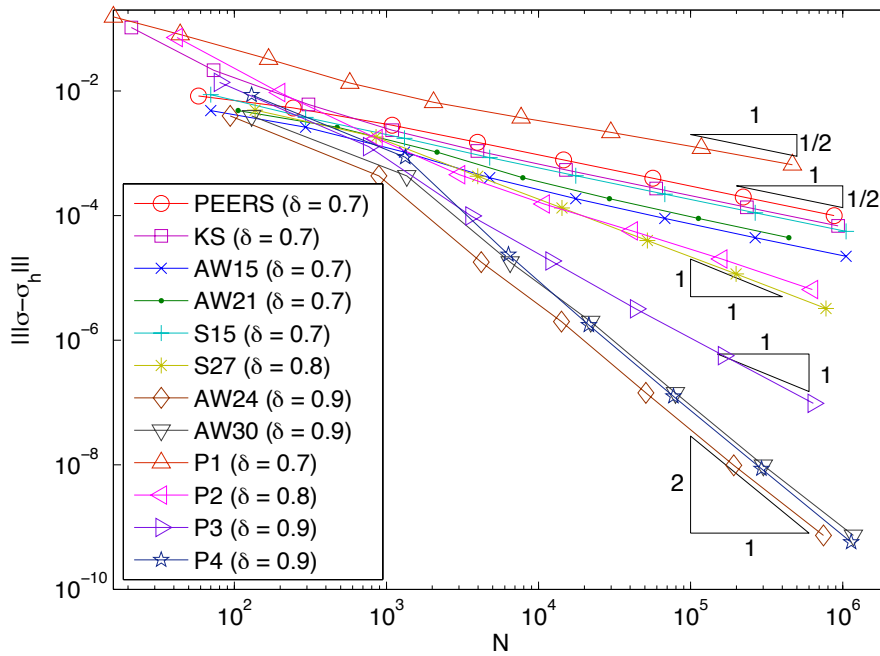


FIGURE 3.14. Survey of the energy errors  $\|\sigma - \sigma_h\|$  for the L-shaped example with Poisson ratio  $\nu = 0.499$  on graded meshes.

**3.4. L-shaped benchmark.** The last example considers the model problem (1.1) where  $\Omega$  is a rotated L-shaped domain as depicted in Figure 3.9. The exact singular solution in radial components is given by

$$u_r(r, \phi) = \frac{r^\alpha}{2\mu} (-(\alpha + 1) \cos((\alpha + 1)\phi) + (C_2 - \alpha - 1)C_1 \cos((\alpha - 1)\phi)),$$

$$u_\phi(r, \phi) = \frac{r^\alpha}{2\mu} ((\alpha + 1) \sin((\alpha + 1)\phi) + (C_2 + \alpha - 1)C_1 \sin((\alpha - 1)\phi)),$$

in the polar coordinate system  $(r, \phi)$  with  $-\pi < \phi \leq \pi$ . The constants read  $C_1 := -\cos((\alpha + 1)\omega)/\cos((\alpha - 1)\omega)$  and  $C_2 := 2(\lambda + 2\mu)/(\lambda + \mu)$ , where  $\alpha \approx 0.544483736782$  is the positive solution of  $\alpha \sin(2\omega) + \sin(2\omega\alpha) = 0$  for  $\omega = 3\pi/4$  and with Lamé parameter  $\lambda, \mu$ . Here, the volume force is zero,  $f \equiv 0$ , and the Dirichlet boundary conditions are computed according to the exact solution. As in the previous examples, the elastic modulus is set to  $E = 10^5$  and the Poisson ratio to  $\nu = 0.499$ . Figure 3.10 shows the displacement solution magnified by a factor of  $5 \cdot 10^3$  and the modulus of the deviatoric part  $|\text{dev}(\sigma)|$  as grey scales for the S15 element. In this example, the re-entrant corner leads to singular solutions which results in slower convergence for uniform meshes as shown in Figures 3.11 and 3.12 for the AW15 and AW30 MFEM. Therefore, local mesh refinement is needed for optimal convergence rates. It is known that  $\beta$ -graded meshes can lead to better convergence rates but suffer from small angles. Therefore, the algorithm GRADMESH of [CGRT08] in combination with the red-green-blue refinement strategy [Car04] is used to create meshes which are more refined towards the re-entrant corner while preserving the mesh quality. Figures 3.11 and 3.12 show the results of some experiments to determine the best grading parameter  $0 < \delta < 1$  for the first-order AW15 and the third-order AW30 MFEM. For the first-order methods, a grading parameter  $\delta = 0.7$  is already sufficient while for the higher-order schemes only a larger value of  $\delta = 0.9$  leads to optimal convergence rates. It is observed that higher grading parameters lead to larger energy errors while lower values lead to suboptimal convergence rates. Since the right choice of the grading parameter is not known a priori, other local mesh refinement strategies such as adaptive finite element methods have to be investigated. Note that the large grading parameter  $\delta = 0.9$  for the high-order methods leads to strong refinement towards the origin, c.f. Figure 3.13. The numerical experiments of Figure 3.14 show that the locking effect reduces the convergence rates for  $P_k$ ,  $k = 1, 2, 3$ , and that  $P_4$  is locking-free. Among the first-order methods, the AW15 finite element shows the best results with significantly smaller errors than for PEERS. It is remarkable that the AW30 and the AW24 MFEM exhibits some superconvergence rates of  $\mathcal{O}(N^{-2})$ . This is in agreement of the conjecture in [CGRT08]. Hence,

the errors of the AW30 and the AW24 MFEM are similar as for the one order higher  $P_4$  FEM.

#### 4. CONCLUSIONS

- i.) The numerical experiments verify the theoretical convergence rates and robustness for  $\nu \rightarrow 1/2$  for the symmetric mixed finite element methods AW30, AW24, AW21, AW15, S15 and S27 as well as for PEERS and the KS-nonconforming and the  $P_4$ -conforming FEM.
- ii.) We presented a short and simple way for implementing the rather complicated mixed schemes AW21, AW15, S15 and S27.
- iii.) The numerical examples confirm that  $P_4$  is robust, while  $P_k$ ,  $k = 1, 2, 3$ , shows locking.
- iv.) Among the first-order methods  $P_1$ , PEERS, KS, AW21, AW15 and S15, the AW15 MFEM shows the best results overall.
- v.) Among the second-order methods  $P_2$ , AW24 and S27, the AW24 MFEM shows the best results overall. Note that for the first two examples the S27 MFEM shows similar results but larger errors for the last two.
- vi.) Among the third-order methods  $P_3$  and AW30, the locking-free AW30 MFEM shows the better results.
- vii.) The comparison of conforming FEM and the MFEM under consideration shows that only the robust version  $P_4$  is competitive and performs best. However, the comparison of the fourth-order scheme with some third-order method clearly shows superiority of the higher-order scheme provided the exact solution is sufficiently smooth.
- viii.) The experiments for the L-shaped domain illustrate that uniform refinement results in suboptimal convergence rates while graded meshes recover optimal rates. The experiments show that some too small grading parameters lead to suboptimal convergence rates while higher values result in errors of several orders of magnitude in accuracy. Since the optimal grading parameter is, in general, not known a priori, adaptive refinement strategies have to be investigated.
- ix.) The experiments for the L-shaped domain leads to the conjecture that for  $f \equiv 0$  the AW30 and AW24 exhibit some superconvergence phenomenon of order  $\mathcal{O}(N^{-2})$ . This is in agreement with the conjecture of [CGRT08]. These results of the AW30 and AW24 MFEM can compete with those of the one order higher  $P_4$  displacements solution. Surprisingly, the S15 and S27 MFEM do not show such properties.



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