

ALGEBRO-GEOMETRIC FINITE-BAND SOLUTIONS OF THE ABLOWITZ–LADIK HIERARCHY

FRITZ GESZTESY, HELGE HOLDEN, JOHANNA MICHOR, AND GERALD TESCHL

*To Walter Thirring, on the occasion of his 80th birthday,
theoretical and mathematical physicist extraordinaire.*

ABSTRACT. We provide a detailed derivation of all complex-valued algebro-geometric finite-band solutions of the Ablowitz–Ladik hierarchy. In addition, we survey a recursive construction of the Ablowitz–Ladik hierarchy and its zero-curvature and Lax formalism.

1. INTRODUCTION

In the mid-seventies, Ablowitz and Ladik, in a series of papers [3]–[6] (see also [1], [2, Sect. 3.2.2], [7, Ch. 3], [22]), used inverse scattering methods to analyze certain integrable differential-difference systems. One of their integrable variants of such systems included a discretization of the celebrated AKNS-ZS system, the pair of coupled nonlinear differential difference equations,

$$\begin{aligned} -i\alpha_t - (1 - \alpha\beta)(\alpha^- + \alpha^+) + 2\alpha &= 0, \\ -i\beta_t + (1 - \alpha\beta)(\beta^- + \beta^+) - 2\beta &= 0 \end{aligned} \tag{1.1}$$

with $\alpha = \alpha(n, t)$, $\beta = \beta(n, t)$, $(n, t) \in \mathbb{Z} \times \mathbb{R}$. Here we used the notation $f^\pm(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, for complex-valued sequences $f = \{f(n)\}_{n \in \mathbb{Z}}$. In particular, Ablowitz and Ladik [4] (see also [7, Ch. 3]) showed that in the focusing case, where $\beta = -\bar{\alpha}$, and in the defocusing case, where $\beta = \bar{\alpha}$, (1.1) yields the discrete analog of the nonlinear Schrödinger equation

$$-i\alpha_t - (1 \pm |\alpha|^2)(\alpha^- + \alpha^+) + 2\alpha = 0. \tag{1.2}$$

We will refer to (1.1) as the Ablowitz–Ladik system. The principal theme of this paper will be to derive the algebro-geometric finite-band solutions of the Ablowitz–Ladik (AL) hierarchy, a completely integrable sequence of systems of nonlinear evolution equations on the lattice \mathbb{Z} whose first nonlinear member is the Ablowitz–Ladik system (1.1).

Since the mid-seventies there has been an enormous amount of activity in the area of integrable differential-difference equations. Two principal directions of research are responsible for this development: Originally, the development was driven by the theory of completely integrable systems and its applications to fields such as

2000 *Mathematics Subject Classification.* Primary 37K15, 37K10; Secondary 39A12, 35Q55.

Key words and phrases. Ablowitz–Ladik hierarchy, discrete NLS, algebro-geometric solutions.

Research supported in part by the Research Council of Norway, the US National Science Foundation under Grant No. DMS-0405526, and the Austrian Science Fund (FWF) under Grant No. Y330.

Int. Math. Res. Notices **2007**, Art. ID rnm082, 55 pp (2007).

nonlinear optics, and more recently, it gained additional momentum due to its intimate connections with the theory of orthogonal polynomials. In the following we first briefly recall the development in connection with integrable systems and subsequently turn to the one influenced by research on orthogonal polynomials.

The first systematic discussion of the Ablowitz–Ladik hierarchy appears to be due to Schilling [51] (cf. also [58], [62], [65]); infinitely many conservation laws are derived, for instance, by Ding, Sun, and Xu [26]; the bi-Hamiltonian structure of the AL hierarchy is considered by Ercolani and Lozano [28]; connections between the AL hierarchy and the motion of a piecewise linear curve have been established by Doliwa and Santini [27]; Bäcklund and Darboux transformations were studied by Geng [31] and Vekslerchik [63]; the Hirota bilinear formalism, AL τ -functions, etc., were considered by Vekslerchik [62]. The initial value problem for half-infinite AL systems was discussed by Common [24], for an application of the inverse scattering method to (1.2) we refer to Vekslerchik and Konotop [64]. This just scratches the surface of these developments and the interested reader will find much more material in the references cited in these papers and the ones discussed below.

Algebro-geometric (and periodic) solutions of the AL system (1.1) have briefly been studied by Ahmad and Chowdhury [8], [9], Bogolyubov, Prikarpatkii, and Samoilenko [17], Bogolyubov and Prikarpatkii [18], Chow, Conte, and Xu [23], Geng, Dai, and Cao [32], and Vaninsky [60]. In an effort to analyze models describing oscillations in nonlinear dispersive wave systems, Miller, Ercolani, Krichever, and Levermore [47] (see also [46]) gave a detailed analysis of algebro-geometric solutions of the AL system (1.1). Introducing

$$U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}, \quad V(z) = i \begin{pmatrix} z - 1 - \alpha\beta^- & \alpha - \alpha^- z^{-1} \\ \beta^- z - \beta & 1 + \alpha^- \beta - z^{-1} \end{pmatrix} \quad (1.3)$$

with $z \in \mathbb{C} \setminus \{0\}$ a spectral parameter, the authors in [47] relied on the fact that the Ablowitz–Ladik system (1.1) is equivalent to the zero-curvature equations

$$U_t + UV - V^+U = 0, \quad (1.4)$$

the latter being the compatibility relation for the spatial and temporal linear problems

$$\Phi = U\Phi^-, \quad \Phi_t^- = V\Phi^-. \quad (1.5)$$

Here we extended the notation $f^\pm(n) = f(n \pm 1)$, $n \in \mathbb{Z}$, to \mathbb{C}^2 -valued and 2×2 -matrix-valued sequences with complex-valued entries.

Miller, Ercolani, Krichever, and Levermore [47] then performed a thorough analysis of the solutions $\Phi = \Phi(z, n, t)$ associated with the pair (U, V) and derived the theta function representations of α, β satisfying the AL system (1.1). In the particular focusing and defocusing cases they also discussed periodic and quasi-periodic solutions α with respect to $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Vekslerchik [61] also studied finite-genus solutions for the AL hierarchy by establishing connections with Fay’s identity for theta functions.

The connection between the Ablowitz–Ladik system (1.1) and orthogonal polynomials comes about as follows: Let $\{\alpha(n)\}_{n \in \mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers subject to the condition $|\alpha(n)| < 1$, $n \in \mathbb{N}$, and define the transfer matrix

$$T(z) = \begin{pmatrix} z & \alpha \\ \bar{\alpha}z & 1 \end{pmatrix}, \quad z \in \mathbb{T}, \quad (1.6)$$

with spectral parameter z on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider the system of difference equations

$$\Phi(z, n) = T(z, n)\Phi(z, n-1), \quad (z, n) \in \mathbb{T} \times \mathbb{N}, \quad (1.7)$$

with initial condition $\Phi(z, 0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where

$$\Phi(z, n) = \begin{pmatrix} \varphi(z, n) \\ z^n \overline{\varphi(1/z, n)} \end{pmatrix}, \quad (z, n) \in \mathbb{T} \times \mathbb{N}_0. \quad (1.8)$$

(Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.) Then $\varphi(\cdot, n)$ are monic polynomials of degree n first introduced by Szegő in the 1920's in his seminal work on the asymptotic distribution of eigenvalues of sections of Toeplitz forms [55], [56] (see also [57, Ch. XI]). Szegő's point of departure was the trigonometric moment problem and hence the theory of orthogonal polynomials on the unit circle. Indeed, given a probability measure $d\sigma$ supported on an infinite set on the unit circle, one is interested in finding monic polynomials $\chi(\cdot, n)$ of degree $n \in \mathbb{N}_0$ in $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, such that

$$\int_0^{2\pi} d\sigma(e^{i\theta}) \overline{\chi(e^{i\theta}, m)} \chi(e^{i\theta}, n) = w(n)^{-2} \delta_{m,n}, \quad m, n \in \mathbb{N}_0, \quad (1.9)$$

where $w(0)^2 = 1$, $w(n)^2 = \prod_{j=1}^n (1 - |\alpha(j)|^2)^{-1}$, $n \in \mathbb{N}$. Szegő showed that the corresponding polynomials (1.8) with φ replaced by χ satisfy the recurrence formula (1.7). Early work in this area includes important contributions by Akhiezer, Geronimus, Krein, Tomčuk, Verblunsky, Widom, and others, and is summarized in the books by Akhiezer [10], Geronimus [37], Szegő [57], and especially in the recent two-volume treatise by Simon [52].

Unaware of the paper [47], Geronimo and Johnson [35] studied (1.7) in the case where the coefficients α are random variables. Under appropriate ergodicity assumptions on α and the hypothesis of a vanishing Lyapunov exponent on prescribed spectral arcs on the unit circle \mathbb{T} , Geronimo and Johnson [35] (cf. also [34], [36]) developed the corresponding spectral theory associated with (1.7) and the unitary operator it generates in $\ell^2(\mathbb{Z})$. In this sense the discussion in [35] is a purely stationary one and connections with a zero-curvature formalism, theta function representations, and integrable hierarchies are not made in [35] (but in this context we refer to the discussion concerning [33] in the next paragraph). More recently, the defocusing case with periodic and quasi-periodic coefficients was studied in great detail by Deift [25], Golinskii and Nevai [43], Killip and Nenciu [44], Li [45], Nenciu [49], [50], and Simon [52, Ch. 11], [53], [54].

An important extension of (1.7) was developed by Baxter in a series of papers on Toeplitz forms [11]–[14] in 1960–63. In these papers the transfer matrix T in (1.6) is replaced by the more general (complexified) transfer matrix $U(z) = \begin{pmatrix} z & \alpha \\ \beta z & 1 \end{pmatrix}$ in (1.5), that is, precisely the matrix U responsible for the spatial part in the Ablowitz–Ladik system in its zero-curvature formulation (1.3)–(1.5). Here $\alpha = \{\alpha(n)\}_{n \in \mathbb{N}}$, $\beta = \{\beta(n)\}_{n \in \mathbb{N}}$ are subject to the condition $\alpha(n)\beta(n) \neq 1$, $n \in \mathbb{N}$. Studying the following extension of (1.7),

$$\Psi(z, n) = U(z, n)\Psi(z, n-1), \quad (z, n) \in \mathbb{T} \times \mathbb{N}, \quad (1.10)$$

Baxter was led to biorthogonal polynomials on the unit circle with respect to a complex-valued measure on \mathbb{T} . In this context of biorthogonal Laurent polynomials we refer to [16] and [20]. Reference [16], in particular, deals with isomonodromic

tau functions and is applicable to generalized integrable lattices of the Toda-type. Baxter’s U matrix in (1.5) led to a new hierarchy of nonlinear difference equations, called the Szegő–Baxter (SB) hierarchy in [33], in honor of these two pioneers of orthogonal polynomials on the unit circle. The latter reference also contains an in depth study of algebro-geometric solutions of (1.7).

In addition to these recent developments on the AL system and the AL hierarchy, we offer a variety of results in this paper apparently not covered before. These include:

- An elementary, yet effective recursive construction of the AL hierarchy using Laurent polynomials.
- Explicit formulas for Lax pairs for the AL hierarchy.
- The detailed connection between the AL hierarchy and a “complexified” version of transfer matrices first introduced by Baxter.
- A unified treatment of stationary algebro-geometric finite-band solutions and their theta function representations of the entire AL hierarchy.
- A unified treatment of algebro-geometric solutions and their theta function representations of the time-dependent AL hierarchy by solving the \underline{r} th AL flow with initial data given by stationary algebro-geometric finite-band solutions.

The structure of this paper is as follows: In Section 2 we describe our zero-curvature formalism for the Ablowitz–Ladik (AL) hierarchy. Extending a recursive polynomial approach discussed in great detail in [38] in the continuous case and in [19], [39, Ch. 4], [59, Chs. 6, 12] in the discrete context to the case of Laurent polynomials with respect to the spectral parameter, we derive the AL hierarchy of systems of nonlinear evolution equations whose first nonlinear member is the Ablowitz–Ladik system (1.1). Section 3 is devoted to a detailed study of the stationary AL hierarchy. We employ the recursive Laurent polynomial formalism of Section 2 to describe nonnegative divisors of degree p on a hyperelliptic curve \mathcal{K}_p of genus p associated with the \underline{p} th system in the stationary AL hierarchy. By means of a fundamental meromorphic function ϕ on \mathcal{K}_p (an analog of the Weyl–Titchmarsh function for the system (1.7)) we then proceed to derive the theta function representations of the associated Baker–Akhiezer vector and all stationary algebro-geometric finite-band solutions of the AL hierarchy. The corresponding time-dependent results for the AL hierarchy are presented in detail in Section 4. Appendix A collects relevant material on hyperelliptic curves and their theta functions and introduces the terminology freely used in Sections 3 and 4. Appendix B is of a technical nature and summarizes expansions of various key quantities related to the Laurent polynomial recursion formalism as the spectral parameter tends to zero and to infinity.

2. THE ABLOWITZ–LADIK HIERARCHY, RECURSION RELATIONS, ZERO-CURVATURE PAIRS, AND HYPERELLIPTIC CURVES

In this section we summarize the construction of the Ablowitz–Ladik hierarchy employing a Laurent polynomial recursion formalism and derive the associated sequence of Ablowitz–Ladik zero-curvature pairs (we also hint at Lax pairs). Moreover, we discuss the Burchnell–Chaundy Laurent polynomial in connection with the stationary Ablowitz–Ladik hierarchy and the underlying hyperelliptic curve. For a detailed treatment of this material we refer to [39], [40].

We denote by $\mathbb{C}^{\mathbb{Z}}$ the set of complex-valued sequences indexed by \mathbb{Z} .

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1. *In the stationary case we assume that α, β satisfy*

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0, 1\}, \quad n \in \mathbb{Z}. \quad (2.1)$$

In the time-dependent case we assume that α, β satisfy

$$\begin{aligned} \alpha(\cdot, t), \beta(\cdot, t) \in \mathbb{C}^{\mathbb{Z}}, \quad t \in \mathbb{R}, \quad \alpha(n, \cdot), \beta(n, \cdot) \in C^1(\mathbb{R}), \quad n \in \mathbb{Z}, \\ \alpha(n, t)\beta(n, t) \notin \{0, 1\}, \quad (n, t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \quad (2.2)$$

We denote by S^{\pm} the shift operators acting on complex-valued sequences $f = \{f(n)\}_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ according to

$$(S^{\pm}f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}. \quad (2.3)$$

Moreover, we will frequently use the notation

$$f^{\pm} = S^{\pm}f, \quad f \in \mathbb{C}^{\mathbb{Z}}. \quad (2.4)$$

To construct the Ablowitz–Ladik hierarchy one typically introduces appropriate zero-curvature pairs of 2×2 matrices, denoted by $U(z)$ and $V_{\underline{p}}(z)$, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2$ (with z a certain spectral parameter to be discussed later), defined recursively in the following. We take the shortest route to the construction of $V_{\underline{p}}$ and hence to that of the Ablowitz–Ladik hierarchy by starting from the recursion relation (2.5)–(2.12) below.

Define sequences $\{f_{\ell, \pm}\}_{\ell \in \mathbb{N}_0}$, $\{g_{\ell, \pm}\}_{\ell \in \mathbb{N}_0}$, and $\{h_{\ell, \pm}\}_{\ell \in \mathbb{N}_0}$ recursively by

$$g_{0,+} = \frac{1}{2}c_{0,+}, \quad f_{0,+} = -c_{0,+}\alpha^+, \quad h_{0,+} = c_{0,+}\beta, \quad (2.5)$$

$$g_{\ell+1,+} - g_{\ell+1,+}^- = \alpha h_{\ell,+}^- + \beta f_{\ell,+}, \quad \ell \in \mathbb{N}_0, \quad (2.6)$$

$$f_{\ell+1,+}^- = f_{\ell,+} - \alpha(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \quad (2.7)$$

$$h_{\ell+1,+} = h_{\ell,+}^- + \beta(g_{\ell+1,+} + g_{\ell+1,+}^-), \quad \ell \in \mathbb{N}_0, \quad (2.8)$$

and

$$g_{0,-} = \frac{1}{2}c_{0,-}, \quad f_{0,-} = c_{0,-}\alpha, \quad h_{0,-} = -c_{0,-}\beta^+, \quad (2.9)$$

$$g_{\ell+1,-} - g_{\ell+1,-}^- = \alpha h_{\ell,-} + \beta f_{\ell,-}^-, \quad \ell \in \mathbb{N}_0, \quad (2.10)$$

$$f_{\ell+1,-} = f_{\ell,-}^- + \alpha(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0, \quad (2.11)$$

$$h_{\ell+1,-}^- = h_{\ell,-} - \beta(g_{\ell+1,-} + g_{\ell+1,-}^-), \quad \ell \in \mathbb{N}_0. \quad (2.12)$$

Here $c_{0, \pm} \in \mathbb{C}$ are given constants. For later use we also introduce

$$f_{-1, \pm} = h_{-1, \pm} = 0. \quad (2.13)$$

Remark 2.2. The sequences $\{f_{\ell,+}\}_{\ell \in \mathbb{N}_0}$, $\{g_{\ell,+}\}_{\ell \in \mathbb{N}_0}$, and $\{h_{\ell,+}\}_{\ell \in \mathbb{N}_0}$ can be computed recursively as follows: Assume that $f_{\ell,+}$, $g_{\ell,+}$, and $h_{\ell,+}$ are known. Equation (2.6) is a first-order difference equation in $g_{\ell+1,+}$ that can be solved directly and yields a local lattice function that is determined up to a new constant denoted by $c_{\ell+1,+} \in \mathbb{C}$. Relations (2.7) and (2.8) then determine $f_{\ell+1,+}$ and $h_{\ell+1,+}$, etc. The sequences $\{f_{\ell,-}\}_{\ell \in \mathbb{N}_0}$, $\{g_{\ell,-}\}_{\ell \in \mathbb{N}_0}$, and $\{h_{\ell,-}\}_{\ell \in \mathbb{N}_0}$ are determined similarly.

Upon setting

$$\gamma = 1 - \alpha\beta, \quad (2.14)$$

one explicitly obtains

$$\begin{aligned}
f_{0,+} &= c_{0,+}(-\alpha^+), & f_{1,+} &= c_{0,+}(-\gamma^+\alpha^{++} + (\alpha^+)^2\beta) + c_{1,+}(-\alpha^+), \\
g_{0,+} &= \frac{1}{2}c_{0,+}, & g_{1,+} &= c_{0,+}(-\alpha^+\beta) + \frac{1}{2}c_{1,+}, \\
h_{0,+} &= c_{0,+}\beta, & h_{1,+} &= c_{0,+}(\gamma\beta^- - \alpha^+\beta^2) + c_{1,+}\beta, \\
f_{0,-} &= c_{0,-}\alpha, & f_{1,-} &= c_{0,-}(\gamma\alpha^- - \alpha^2\beta^+) + c_{1,-}\alpha, \\
g_{0,-} &= \frac{1}{2}c_{0,-}, & g_{1,-} &= c_{0,-}(-\alpha\beta^+) + \frac{1}{2}c_{1,-}, \\
h_{0,-} &= c_{0,-}(-\beta^+), & h_{1,-} &= c_{0,-}(-\gamma^+\beta^{++} + \alpha(\beta^+)^2) + c_{1,-}(-\beta^+), \text{ etc.}
\end{aligned} \tag{2.15}$$

Here $\{c_{\ell,\pm}\}_{\ell \in \mathbb{N}}$ denote summation constants which naturally arise when solving the difference equations for $g_{\ell,\pm}$ in (2.6), (2.10). Subsequently, it will also be useful to work with the corresponding homogeneous coefficients $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, defined by the vanishing of all summation constants $c_{k,\pm}$ for $k = 1, \dots, \ell$, and choosing $c_{0,\pm} = 1$,

$$\hat{f}_{0,+} = -\alpha^+, \quad \hat{f}_{0,-} = \alpha, \quad \hat{f}_{\ell,\pm} = f_{\ell,\pm}|_{c_{0,\pm}=1, c_{j,\pm}=0, j=1, \dots, \ell}, \quad \ell \in \mathbb{N}, \tag{2.16}$$

$$\hat{g}_{0,\pm} = \frac{1}{2}, \quad \hat{g}_{\ell,\pm} = g_{\ell,\pm}|_{c_{0,\pm}=1, c_{j,\pm}=0, j=1, \dots, \ell}, \quad \ell \in \mathbb{N}, \tag{2.17}$$

$$\hat{h}_{0,+} = \beta, \quad \hat{h}_{0,-} = -\beta^+, \quad \hat{h}_{\ell,\pm} = h_{\ell,\pm}|_{c_{0,\pm}=1, c_{j,\pm}=0, j=1, \dots, \ell}, \quad \ell \in \mathbb{N}. \tag{2.18}$$

By induction one infers that

$$f_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat{f}_{k,\pm}, \quad g_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat{g}_{k,\pm}, \quad h_{\ell,\pm} = \sum_{k=0}^{\ell} c_{\ell-k,\pm} \hat{h}_{k,\pm}. \tag{2.19}$$

In a slight abuse of notation we will occasionally stress the dependence of $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ on α, β by writing $f_{\ell,\pm}(\alpha, \beta)$, $g_{\ell,\pm}(\alpha, \beta)$, and $h_{\ell,\pm}(\alpha, \beta)$.

One can show (cf. [40]) that all homogeneous elements $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$, $\ell \in \mathbb{N}_0$, are polynomials in α, β , and some of their shifts.

Remark 2.3. As an efficient tool to later distinguish between nonhomogeneous and homogeneous quantities $f_{\ell,\pm}$, $g_{\ell,\pm}$, $h_{\ell,\pm}$, and $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, $\hat{h}_{\ell,\pm}$, respectively, we now introduce the notion of degree as follows. Denote

$$f^{(r)} = S^{(r)}f, \quad f = \{f(n)\}_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad S^{(r)} = \begin{cases} (S^+)^r, & r \geq 0, \\ (S^-)^{-r}, & r < 0, \end{cases} \quad r \in \mathbb{Z}, \tag{2.20}$$

and define

$$\deg(\alpha^{(r)}) = r, \quad \deg(\beta^{(r)}) = -r, \quad r \in \mathbb{Z}. \tag{2.21}$$

This then results in

$$\begin{aligned}
\deg(\hat{f}_{\ell,+}^{(r)}) &= \ell + 1 + r, & \deg(\hat{f}_{\ell,-}^{(r)}) &= -\ell + r, & \deg(\hat{g}_{\ell,\pm}^{(r)}) &= \pm\ell, \\
\deg(\hat{h}_{\ell,+}^{(r)}) &= \ell - r, & \deg(\hat{h}_{\ell,-}^{(r)}) &= -\ell - 1 - r, & \ell \in \mathbb{N}_0, & r \in \mathbb{Z}.
\end{aligned} \tag{2.22}$$

We also note the following useful result (cf. [40]): Assume (2.1), then,

$$\begin{aligned}
g_{\ell,+} - g_{\ell,+}^- &= \alpha h_{\ell,+} + \beta f_{\ell,+}^-, & \ell \in \mathbb{N}_0, \\
g_{\ell,-} - g_{\ell,-}^- &= \alpha h_{\ell,-} + \beta f_{\ell,-}, & \ell \in \mathbb{N}_0.
\end{aligned} \tag{2.23}$$

Moreover, we note the following symmetries,

$$\hat{f}_{\ell,\pm}(\alpha, \beta) = \hat{h}_{\ell,\mp}(\beta, \alpha), \quad \hat{g}_{\ell,\pm}(\alpha, \beta) = \hat{g}_{\ell,\mp}(\beta, \alpha), \quad \ell \in \mathbb{N}_0. \tag{2.24}$$

Lemma 2.4. *Let $n \in \mathbb{Z}$. Then the homogeneous coefficients $\{\hat{f}_{\ell,\pm}\}_{\ell \in \mathbb{N}_0}$, $\{\hat{g}_{\ell,\pm}\}_{\ell \in \mathbb{N}_0}$, and $\{\hat{h}_{\ell,\pm}\}_{\ell \in \mathbb{N}_0}$ satisfy the following relations:*

$$\begin{aligned} \hat{f}_{\ell,+}(n) &= \alpha(n)(\delta_n, L^{\ell+1}\delta_n) + \rho(n) \begin{cases} (\delta_{n-1}, L^{\ell+1}\delta_n), & n \text{ even}, \\ (\delta_n, L^{\ell+1}\delta_{n-1}), & n \text{ odd}, \end{cases} & \ell \in \mathbb{N}_0, \\ \hat{f}_{\ell,-}(n) &= \alpha(n)(\delta_n, L^{-\ell}\delta_n) + \rho(n) \begin{cases} (\delta_{n-1}, L^{-\ell}\delta_n), & n \text{ even}, \\ (\delta_n, L^{-\ell}\delta_{n-1}), & n \text{ odd}, \end{cases} & \ell \in \mathbb{N}_0, \\ \hat{g}_{0,\pm} &= 1/2, \quad \hat{g}_{\ell,\pm}(n) = (\delta_n, L^{\pm\ell}\delta_n), & \ell \in \mathbb{N}, \\ \hat{h}_{\ell,+}(n) &= \beta(n)(\delta_n, L^\ell\delta_n) + \rho(n) \begin{cases} (\delta_n, L^\ell\delta_{n-1}), & n \text{ even}, \\ (\delta_{n-1}, L^\ell\delta_n), & n \text{ odd}, \end{cases} & \ell \in \mathbb{N}_0, \\ \hat{h}_{\ell,-}(n) &= \beta(n)(\delta_n, L^{-\ell-1}\delta_n) + \rho(n) \begin{cases} (\delta_n, L^{-\ell-1}\delta_{n-1}), & n \text{ even}, \\ (\delta_{n-1}, L^{-\ell-1}\delta_n), & n \text{ odd}, \end{cases} & \ell \in \mathbb{N}_0. \end{aligned} \tag{2.33}$$

For the proof of Lemma 2.4 and some of its applications in connection with conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy we refer to [42].

Next we define the 2×2 zero-curvature matrices

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix} \tag{2.34}$$

and

$$V_{\underline{p}}(z) = i \begin{pmatrix} G_{\underline{p}}^-(z) & -F_{\underline{p}}^-(z) \\ H_{\underline{p}}^-(z) & -K_{\underline{p}}^-(z) \end{pmatrix}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2, \tag{2.35}$$

for appropriate Laurent polynomials $F_{\underline{p}}(z)$, $G_{\underline{p}}(z)$, $H_{\underline{p}}(z)$, and $K_{\underline{p}}(z)$ in the spectral parameter $z \in \mathbb{C} \setminus \{0\}$ to be determined shortly. By postulating the stationary zero-curvature relation,

$$0 = UV_{\underline{p}} - V_{\underline{p}}^+U, \tag{2.36}$$

one concludes that (2.36) is equivalent with the following relations

$$z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^- = 0, \tag{2.37}$$

$$z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} - K_{\underline{p}} + K_{\underline{p}}^- = 0, \tag{2.38}$$

$$-F_{\underline{p}} + zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + K_{\underline{p}}^-) = 0, \tag{2.39}$$

$$z\beta(G_{\underline{p}}^- + K_{\underline{p}}) - zH_{\underline{p}} + H_{\underline{p}}^- = 0. \tag{2.40}$$

In order to make the connection between the zero-curvature formalism and the recursion relations (2.5)–(2.12), we now define Laurent polynomials $F_{\underline{p}}$, $G_{\underline{p}}$, $H_{\underline{p}}$, and $K_{\underline{p}}$ by¹

$$F_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} f_{p_- - \ell, -} z^{-\ell} + \sum_{\ell=0}^{p_+ - 1} f_{p_+ - 1 - \ell, +} z^\ell, \tag{2.41}$$

$$G_{\underline{p}}(z) = \sum_{\ell=1}^{p_-} g_{p_- - \ell, -} z^{-\ell} + \sum_{\ell=0}^{p_+} g_{p_+ - \ell, +} z^\ell, \tag{2.42}$$

¹In this paper, a sum is interpreted as zero whenever the upper limit in the sum is strictly less than its lower limit.

$$H_{\underline{p}}(z) = \sum_{\ell=0}^{p_- - 1} h_{p_- - 1 - \ell, -} z^{-\ell} + \sum_{\ell=1}^{p_+} h_{p_+ - \ell, +} z^{\ell}, \quad (2.43)$$

$$K_{\underline{p}}(z) = \sum_{\ell=0}^{p_-} g_{p_- - \ell, -} z^{-\ell} + \sum_{\ell=1}^{p_+} g_{p_+ - \ell, +} z^{\ell} = G_{\underline{p}}(z) + g_{p_-, -} - g_{p_+, +}. \quad (2.44)$$

The corresponding homogeneous quantities are defined by ($\ell \in \mathbb{N}_0$)

$$\begin{aligned} \widehat{F}_{0, \mp}(z) &= 0, \quad \widehat{F}_{\ell, -}(z) = \sum_{k=1}^{\ell} \widehat{f}_{\ell-k, -} z^{-k}, \quad \widehat{F}_{\ell, +}(z) = \sum_{k=0}^{\ell-1} \widehat{f}_{\ell-1-k, +} z^k, \\ \widehat{G}_{0, -}(z) &= 0, \quad \widehat{G}_{\ell, -}(z) = \sum_{k=1}^{\ell} \widehat{g}_{\ell-k, -} z^{-k}, \\ \widehat{G}_{0, +}(z) &= \frac{1}{2}, \quad \widehat{G}_{\ell, +}(z) = \sum_{k=0}^{\ell} \widehat{g}_{\ell-k, +} z^k, \\ \widehat{H}_{0, \mp}(z) &= 0, \quad \widehat{H}_{\ell, -}(z) = \sum_{k=0}^{\ell-1} \widehat{h}_{\ell-1-k, -} z^{-k}, \quad \widehat{H}_{\ell, +}(z) = \sum_{k=1}^{\ell} \widehat{h}_{\ell-k, +} z^k, \\ \widehat{K}_{0, -}(z) &= \frac{1}{2}, \quad \widehat{K}_{\ell, -}(z) = \sum_{k=0}^{\ell} \widehat{g}_{\ell-k, -} z^{-k} = \widehat{G}_{\ell, -}(z) + \widehat{g}_{\ell, -}, \\ \widehat{K}_{0, +}(z) &= 0, \quad \widehat{K}_{\ell, +}(z) = \sum_{k=1}^{\ell} \widehat{g}_{\ell-k, +} z^k = \widehat{G}_{\ell, +}(z) - \widehat{g}_{\ell, +}. \end{aligned} \quad (2.45)$$

The stationary zero-curvature relation (2.36), $0 = UV_{\underline{p}} - V_{\underline{p}}^+ U$, is then equivalent to

$$-\alpha(g_{p_+, +} + g_{p_-, -}) + f_{p_+ - 1, +} - f_{p_- - 1, -} = 0, \quad (2.46)$$

$$\beta(g_{p_+, +}^- + g_{p_-, -}) + h_{p_+ - 1, +}^- - h_{p_- - 1, -} = 0. \quad (2.47)$$

Thus, varying $p_{\pm} \in \mathbb{N}_0$, equations (2.46) and (2.47) give rise to the stationary Ablowitz-Ladik (AL) hierarchy which we introduce as follows

$$\begin{aligned} \text{s-AL}_{\underline{p}}(\alpha, \beta) &= \begin{pmatrix} -\alpha(g_{p_+, +} + g_{p_-, -}) + f_{p_+ - 1, +} - f_{p_- - 1, -} \\ \beta(g_{p_+, +}^- + g_{p_-, -}) + h_{p_+ - 1, +}^- - h_{p_- - 1, -} \end{pmatrix} = 0, \\ \underline{p} &= (p_-, p_+) \in \mathbb{N}_0^2. \end{aligned} \quad (2.48)$$

Explicitly (recalling $\gamma = 1 - \alpha\beta$ and taking $p_- = p_+$ for simplicity),

$$\begin{aligned} \text{s-AL}_{(0,0)}(\alpha, \beta) &= \begin{pmatrix} -c_{(0,0)}\alpha \\ c_{(0,0)}\beta \end{pmatrix} = 0, \\ \text{s-AL}_{(1,1)}(\alpha, \beta) &= \begin{pmatrix} -\gamma(c_{0, -}\alpha^- + c_{0, +}\alpha^+) - c_{(1,1)}\alpha \\ \gamma(c_{0, +}\beta^- + c_{0, -}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0, \\ \text{s-AL}_{(2,2)}(\alpha, \beta) &= \begin{pmatrix} -\gamma(c_{0, +}\alpha^{++}\gamma^+ + c_{0, -}\alpha^{--}\gamma^- - \alpha(c_{0, +}\alpha^+\beta^- + c_{0, -}\alpha^-\beta^+)) \\ -\beta(c_{0, -}(\alpha^-)^2 + c_{0, +}(\alpha^+)^2) \\ \gamma(c_{0, -}\beta^{++}\gamma^+ + c_{0, +}\beta^{--}\gamma^- - \beta(c_{0, +}\alpha^+\beta^- + c_{0, -}\alpha^-\beta^+)) \\ -\alpha(c_{0, +}(\beta^-)^2 + c_{0, -}(\beta^+)^2) \end{pmatrix} \end{aligned}$$

$$+ \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \quad \text{etc.}, \quad (2.49)$$

represent the first few equations of the stationary Ablowitz–Ladik hierarchy. Here we introduced

$$c_{\underline{p}} = (c_{p,-,-} + c_{p+,+})/2, \quad p_{\pm} \in \mathbb{N}_0. \quad (2.50)$$

By definition, the set of solutions of (2.48), with p_{\pm} ranging in \mathbb{N}_0 and $c_{\ell,\pm} \in \mathbb{C}$, $\ell \in \mathbb{N}_0$, represents the class of algebro-geometric Ablowitz–Ladik solutions.

In the following we will frequently assume that α, β satisfy the \underline{p} th stationary AL system $\text{s-AL}_{\underline{p}}(\alpha, \beta) = 0$, supposing a particular choice of summation constants $c_{\ell,\pm} \in \mathbb{C}$, $\ell = 0, \dots, p_{\pm}$, $p_{\pm} \in \mathbb{N}_0$, has been made.

In accordance with our notation introduced in (2.16)–(2.18) and (2.45), the corresponding homogeneous stationary Ablowitz–Ladik equations are defined by

$$\text{s-}\widehat{\text{AL}}_{\underline{p}}(\alpha, \beta) = \text{s-AL}_{\underline{p}}(\alpha, \beta)|_{c_{0,\pm}=1, c_{\ell,\pm}=0, \ell=1,\dots,p_{\pm}}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2. \quad (2.51)$$

In addition, one can show (cf. [40, Lemma 2.2]) that $g_{p+,+} = g_{p,-,-}$ up to a lattice constant which can be set equal to zero without loss of generality. Thus, we will henceforth assume that

$$g_{p+,+} = g_{p,-,-}, \quad (2.52)$$

which in turn implies that

$$K_{\underline{p}} = G_{\underline{p}} \quad (2.53)$$

and hence renders $V_{\underline{p}}$ in (2.35) traceless in the stationary context. (We note that equations (2.52) and (2.53) cease to be valid in the time-dependent context, though.)

Remark 2.5. (i) The particular choice $c_{0,+} = c_{0,-} = 1$ yields the stationary Ablowitz–Ladik equation. Scaling $c_{0,\pm}$ with the same constant then amounts to scaling $V_{\underline{p}}$ with this constant which drops out in the stationary zero-curvature equation (2.36).

(ii) Different ratios between $c_{0,+}$ and $c_{0,-}$ will lead to different stationary hierarchies. In particular, the choice $c_{0,+} = 2$, $c_{0,-} = \dots = c_{p-1,-} = 0$, $c_{p,-,-} \neq 0$, yields the stationary Baxter–Szegő hierarchy considered in detail in [33]. However, in this case some parts from the recursion relation for the negative coefficients still remain. In fact, (2.12) reduces to $g_{p,-,-} - g_{p-,-}^- = \alpha h_{p-1,-}$, $h_{p-1,-} = 0$ and thus requires $g_{p,-,-}$ to be a constant in (2.48) and (2.64). Moreover, $f_{p-1,-} = 0$ in (2.48) in this case.

Next, taking into account (2.53), one infers that the expression $R_{\underline{p}}$, defined as

$$R_{\underline{p}} = G_{\underline{p}}^2 - F_{\underline{p}}H_{\underline{p}}, \quad (2.54)$$

is a lattice constant, that is, $R_{\underline{p}} - R_{\underline{p}}^- = 0$, by taking determinants in the stationary zero-curvature equation (2.36). Hence, $R_{\underline{p}}(z)$ only depends on z , and assuming in addition to (2.1) that

$$c_{0,\pm} \in \mathbb{C} \setminus \{0\}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0, 0)\}, \quad (2.55)$$

one may write $R_{\underline{p}}$ as²

$$R_{\underline{p}}(z) = (c_{0,+}^2/4)z^{-2p_-} \prod_{m=0}^{2p_+} (z - E_m), \quad \{E_m\}_{m=0}^{2p_+} \subset \mathbb{C} \setminus \{0\}, \quad (2.56)$$

$$p = p_- + p_+ - 1 \in \mathbb{N}_0.$$

Moreover, multiplying (2.54) by z^{2p_-} and taking $z \rightarrow 0$ yields

$$\prod_{m=0}^{2p_+} E_m = \frac{c_{0,-}^2}{c_{0,+}^2}. \quad (2.57)$$

Relation (2.54) allows one to introduce a hyperelliptic curve $\mathcal{K}_{\underline{p}}$ of (arithmetic) genus $p = p_- + p_+ - 1$ (possibly with a singular affine part), where

$$\mathcal{K}_{\underline{p}}: \mathcal{F}_{\underline{p}}(z, y) = y^2 - 4c_{0,+}^{-2}z^{2p_-}R_{\underline{p}}(z) = y^2 - \prod_{m=0}^{2p_+} (z - E_m) = 0, \quad p = p_- + p_+ - 1. \quad (2.58)$$

Next we turn to the time-dependent Ablowitz–Ladik hierarchy. For that purpose the coefficients α and β are now considered as functions of both the lattice point and time. For each system in the hierarchy, that is, for each p_{\pm} , we introduce a deformation (time) parameter $t_{\underline{p}} \in \mathbb{R}$ in α, β , replacing $\alpha(n), \beta(n)$ by $\alpha(n, t_{\underline{p}}), \beta(n, t_{\underline{p}})$. Moreover, the definitions (2.34), (2.35), and (2.41)–(2.44) of $U, V_{\underline{p}},$ and $F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}, K_{\underline{p}}$, respectively, still apply. Imposing the zero-curvature relation

$$U_{t_{\underline{p}}} + UV_{\underline{p}} - V_{\underline{p}}^+U = 0, \quad \underline{p} \in \mathbb{N}_0^2, \quad (2.59)$$

then results in the equations

$$\alpha_{t_{\underline{p}}} = i(zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + K_{\underline{p}}^-) - F_{\underline{p}}), \quad (2.60)$$

$$\beta_{t_{\underline{p}}} = -i(\beta(G_{\underline{p}}^- + K_{\underline{p}}) - H_{\underline{p}} + z^{-1}H_{\underline{p}}^-), \quad (2.61)$$

$$0 = z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^-, \quad (2.62)$$

$$0 = z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} + K_{\underline{p}}^- - K_{\underline{p}}. \quad (2.63)$$

Varying $\underline{p} \in \mathbb{N}_0^2$, the collection of evolution equations

$$\text{AL}_{\underline{p}}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{\underline{p}}} - \alpha(g_{p_+,+} + g_{p_-,-}^-) + f_{p_+-1,+} - f_{p_--1,-}^- \\ -i\beta_{t_{\underline{p}}} + \beta(g_{p_+,+}^- + g_{p_-,-}) - h_{p_--1,-} + h_{p_+-1,+}^- \end{pmatrix} = 0, \quad (2.64)$$

$$t_{\underline{p}} \in \mathbb{R}, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2,$$

then defines the time-dependent Ablowitz–Ladik hierarchy. Explicitly, taking $p_- = p_+$ for simplicity,

$$\text{AL}_{(0,0)}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{(0,0)}} - c_{(0,0)}\alpha \\ -i\beta_{t_{(0,0)}} + c_{(0,0)}\beta \end{pmatrix} = 0,$$

$$\text{AL}_{(1,1)}(\alpha, \beta) = \begin{pmatrix} -i\alpha_{t_{(1,1)}} - \gamma(c_{0,-}\alpha^- + c_{0,+}\alpha^+) - c_{(1,1)}\alpha \\ -i\beta_{t_{(1,1)}} + \gamma(c_{0,+}\beta^- + c_{0,-}\beta^+) + c_{(1,1)}\beta \end{pmatrix} = 0,$$

$$\text{AL}_{(2,2)}(\alpha, \beta) \quad (2.65)$$

²We use the convention that a product is to be interpreted equal to 1 whenever the upper limit of the product is strictly less than its lower limit.

$$\begin{aligned}
&= \begin{pmatrix} -i\alpha_{t(2,2)} - \gamma(c_{0,+}\alpha^{++}\gamma^+ + c_{0,-}\alpha^{--}\gamma^- - \alpha(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+)) \\ \quad -\beta(c_{0,-}(\alpha^-)^2 + c_{0,+}(\alpha^+)^2) \\ -i\beta_{t(2,2)} + \gamma(c_{0,-}\beta^{++}\gamma^+ + c_{0,+}\beta^{--}\gamma^- - \beta(c_{0,+}\alpha^+\beta^- + c_{0,-}\alpha^-\beta^+)) \\ \quad -\alpha(c_{0,+}(\beta^-)^2 + c_{0,-}(\beta^+)^2) \end{pmatrix} \\
&\quad + \begin{pmatrix} -\gamma(c_{1,-}\alpha^- + c_{1,+}\alpha^+) - c_{(2,2)}\alpha \\ \gamma(c_{1,+}\beta^- + c_{1,-}\beta^+) + c_{(2,2)}\beta \end{pmatrix} = 0, \quad \text{etc.},
\end{aligned}$$

represent the first few equations of the time-dependent Ablowitz–Ladik hierarchy. Here we recall the definition of $c_{\underline{p}}$ in (2.50).

By (2.64), (2.6), and (2.10), the time derivative of $\gamma = 1 - \alpha\beta$ is given by

$$\gamma_{t_{\underline{p}}} = i\gamma((g_{p_+,+} - g_{p_+,+}^-) - (g_{p_-,-} - g_{p_-,-}^-)). \quad (2.66)$$

Remark 2.6. From (2.37)–(2.40) and the explicit computations of the coefficients $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$, one concludes that the zero-curvature equation (2.59) and hence the Ablowitz–Ladik hierarchy is invariant under the scaling transformation

$$\alpha \rightarrow \alpha_c = \{c\alpha(n)\}_{n \in \mathbb{Z}}, \quad \beta \rightarrow \beta_c = \{\beta(n)/c\}_{n \in \mathbb{Z}}, \quad c \in \mathbb{C} \setminus \{0\}. \quad (2.67)$$

Moreover, $R_{\underline{p}} = G_{\underline{p}}^2 - H_{\underline{p}}F_{\underline{p}}$ and hence $\{E_m\}_{m=0}^{2p+1}$ are invariant under this transformation. Furthermore, choosing $c = e^{ic_{\underline{p}}t}$, one verifies that it is no restriction to assume $c_{\underline{p}} = 0$. This also indicates that stationary solutions α, β can only be constructed up to a multiplicative constant (compare Theorem 3.7, in particular, (3.64), (3.65)).

Finally, we briefly hint at explicit expressions of the Lax pair for the Ablowitz–Ladik hierarchy. More details will be presented in [42]. First we need some notation. Let T be a bounded operator in $\ell^2(\mathbb{Z})$. Given the standard basis (2.25) in $\ell^2(\mathbb{Z})$, we represent T by

$$T = (T(m, n))_{(m, n) \in \mathbb{Z}^2}, \quad T(m, n) = (\delta_m, T\delta_n), \quad (m, n) \in \mathbb{Z}^2. \quad (2.68)$$

Moreover, we introduce the upper and lower triangular parts T_{\pm} of T by

$$T_{\pm} = (T_{\pm}(m, n))_{(m, n) \in \mathbb{Z}^2}, \quad T_{\pm}(m, n) = \begin{cases} T(m, n), & \pm(n - m) > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (2.69)$$

Next, consider the finite difference expression $P_{\underline{p}}$ defined by

$$\begin{aligned}
P_{\underline{p}} &= \frac{i}{2} \sum_{\ell=1}^{p_+} c_{p_+-\ell,+} ((L^{\ell})_+ - (L^{\ell})_-) - \frac{i}{2} \sum_{\ell=1}^{p_-} c_{p_--\ell,-} ((L^{-\ell})_+ - (L^{-\ell})_-) \\
&\quad - \frac{i}{2} c_{\underline{p}} Q_d, \quad \underline{p} = (p_-, p_+) \in \mathbb{N}_0^2,
\end{aligned} \quad (2.70)$$

with L given by (2.27), $c_{\underline{p}} = (c_{p_-,-} + c_{p_+,+})/2$, and Q_d denoting the doubly infinite diagonal matrix

$$Q_d = ((-1)^k \delta_{k,\ell})_{k,\ell \in \mathbb{Z}}. \quad (2.71)$$

Then one can show that $(L, P_{\underline{p}})$ represents the Lax pair for the Ablowitz–Ladik equations (2.64) for $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$. In particular, the hierarchy of nonlinear Ablowitz–Ladik evolution equations (2.64) then can alternatively be derived by imposing the Lax commutator equations

$$L_{t_{\underline{p}}}(t_{\underline{p}}) - [P_{\underline{p}}(t_{\underline{p}}), L(t_{\underline{p}})] = 0, \quad t_{\underline{p}} \in \mathbb{R}, \quad \underline{p} \in \mathbb{N}_0^2. \quad (2.72)$$

For additional representations of $P_{\underline{p}}$ in terms of L and a particular factorization of L we refer to [42]. The Ablowitz–Ladik Lax pair in the special defocusing case, where $\beta = \bar{\alpha}$, in the finite-dimensional context, was recently derived by Nenciu [50].

In the special stationary case, where $P_{\underline{p}}$ and L commute, $[P_{\underline{p}}, L] = 0$, they satisfy an algebraic relationship of the type

$$\begin{aligned} P_{\underline{p}}^2 + R_{\underline{p}}(L) &= P_{\underline{p}}^2 + (c_{0,+}^2/4)L^{-2p_-} \prod_{m=0}^{2p+1} (L - E_m) = 0, \\ R_{\underline{p}}(z) &= (c_{0,+}^2/4)z^{-2p_-} \prod_{m=0}^{2p+1} (z - E_m), \quad p = p_- + p_+ - 1. \end{aligned} \tag{2.73}$$

Thus, the expression $P_{\underline{p}}^2 + R_{\underline{p}}(L)$ in (2.73) represents the Burchnell–Chaundy Laurent polynomial of the Lax pair $(L, P_{\underline{p}})$.

3. THE STATIONARY ABLOWITZ–LADIK FORMALISM

This section is devoted to a detailed study of the stationary Ablowitz–Ladik hierarchy and its algebro-geometric solutions. Our principal tools are derived from combining the Laurent polynomial recursion formalism introduced in Section 2 and a fundamental meromorphic function ϕ on a hyperelliptic curve \mathcal{K}_p . With the help of ϕ we study the Baker–Akhiezer vector Ψ , trace formulas, and theta function representations of ϕ , Ψ , α , and β . For proofs of the elementary results of the stationary formalism we refer to [39], [40].

Unless explicitly stated otherwise, we suppose throughout this section that

$$\alpha, \beta \in \mathbb{C}^{\mathbb{Z}}, \quad \alpha(n)\beta(n) \notin \{0, 1\}, \quad n \in \mathbb{Z}, \tag{3.1}$$

and assume (2.5)–(2.13), (2.34)–(2.36), (2.41)–(2.44), (2.48), (2.53), (2.54), (2.56), keeping $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$ fixed.

We recall the hyperelliptic curve

$$\begin{aligned} \mathcal{K}_p: \mathcal{F}_p(z, y) &= y^2 - 4c_{0,+}^{-2}z^{2p_-}R_{\underline{p}}(z) = y^2 - \prod_{m=0}^{2p+1} (z - E_m) = 0, \\ R_{\underline{p}}(z) &= \left(\frac{c_{0,+}}{2z^{p_-}}\right)^2 \prod_{m=0}^{2p+1} (z - E_m), \quad \{E_m\}_{m=0}^{2p+1} \subset \mathbb{C} \setminus \{0\}, \quad p = p_- + p_+ - 1, \end{aligned} \tag{3.2}$$

as introduced in (2.58). Throughout this section we assume the affine part of \mathcal{K}_p to be nonsingular, that is, we suppose that

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 2p+1. \tag{3.3}$$

\mathcal{K}_p is compactified by joining two points $P_{\infty_{\pm}}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by \mathcal{K}_p . Points P on $\mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\cdot)$ is the meromorphic function on \mathcal{K}_p satisfying $\mathcal{F}_p(z, y) = 0$. The complex structure on \mathcal{K}_p is then defined in the usual way, see Appendix A. Hence, \mathcal{K}_p becomes a two-sheeted hyperelliptic Riemann surface of genus p in a standard manner.

We also emphasize that by fixing the curve \mathcal{K}_p (i.e., by fixing E_0, \dots, E_{2p+1}), the summation constants $c_{1,\pm}, \dots, c_{p_{\pm},\pm}$ in $f_{p_{\pm},\pm}, g_{p_{\pm},\pm}$, and $h_{p_{\pm},\pm}$ (and hence in the corresponding stationary s-AL $_{\underline{p}}$ equations) are uniquely determined as is clear

from (B.8) which establishes the summation constants $c_{\ell, \pm}$ as symmetric functions of $E_0^{\pm 1}, \dots, E_{2p+1}^{\pm 1}$.

For notational simplicity we will usually tacitly assume that $p \in \mathbb{N}$ and hence $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0, 0), (0, 1), (1, 0)\}$.

We denote by $\{\mu_j(n)\}_{j=1, \dots, p}$ and $\{\nu_j(n)\}_{j=1, \dots, p}$ the zeros of $(\cdot)^{p-} F_{\underline{p}}(\cdot, n)$ and $(\cdot)^{p-1} H_{\underline{p}}(\cdot, n)$, respectively. Thus, we may write

$$F_{\underline{p}}(z) = -c_{0,+} \alpha^+ z^{-p-} \prod_{j=1}^p (z - \mu_j), \quad (3.4)$$

$$H_{\underline{p}}(z) = c_{0,+} \beta z^{-p-1} \prod_{j=1}^p (z - \nu_j), \quad (3.5)$$

and we recall that (cf. (2.54))

$$R_{\underline{p}} - G_{\underline{p}}^2 = -F_{\underline{p}} H_{\underline{p}}. \quad (3.6)$$

The next step is crucial; it permits us to “lift” the zeros μ_j and ν_j of $(\cdot)^{p-} F_{\underline{p}}$ and $(\cdot)^{p-1} H_{\underline{p}}$ from the complex plane \mathbb{C} to the curve \mathcal{K}_p . From (3.6) one infers that

$$R_{\underline{p}}(z) - G_{\underline{p}}(z)^2 = 0, \quad z \in \{\mu_j, \nu_k\}_{j,k=1, \dots, p}. \quad (3.7)$$

We now introduce $\{\hat{\mu}_j\}_{j=1, \dots, p} \subset \mathcal{K}_p$ and $\{\hat{\nu}_j\}_{j=1, \dots, p} \subset \mathcal{K}_p$ by

$$\hat{\mu}_j(n) = (\mu_j(n), (2/c_{0,+}) \mu_j(n)^{p-} G_{\underline{p}}(\mu_j(n), n)), \quad j = 1, \dots, p, \quad n \in \mathbb{Z}, \quad (3.8)$$

and

$$\hat{\nu}_j(n) = (\nu_j(n), -(2/c_{0,+}) \nu_j(n)^{p-} G_{\underline{p}}(\nu_j(n), n)), \quad j = 1, \dots, p, \quad n \in \mathbb{Z}. \quad (3.9)$$

We also introduce the points $P_{0, \pm}$ by

$$P_{0, \pm} = (0, \pm(c_{0,-}/c_{0,+})) \in \mathcal{K}_p, \quad \frac{c_{0,-}^2}{c_{0,+}^2} = \prod_{m=0}^{2p+1} E_m. \quad (3.10)$$

We emphasize that $P_{0, \pm}$ and $P_{\infty \pm}$ are not necessarily on the same sheet of \mathcal{K}_p .

Next we introduce the fundamental meromorphic function on \mathcal{K}_p by

$$\phi(P, n) = \frac{(c_{0,+}/2) z^{-p-} y + G_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n)} \quad (3.11)$$

$$= \frac{-H_{\underline{p}}(z, n)}{(c_{0,+}/2) z^{-p-} y - G_{\underline{p}}(z, n)}, \quad (3.12)$$

$$P = (z, y) \in \mathcal{K}_p, \quad n \in \mathbb{Z},$$

with divisor $(\phi(\cdot, n))$ of $\phi(\cdot, n)$ given by

$$(\phi(\cdot, n)) = \mathcal{D}_{P_{0,-}\hat{\underline{\nu}}(n)} - \mathcal{D}_{P_{\infty-}\hat{\underline{\mu}}(n)}, \quad (3.13)$$

using (3.4) and (3.5). Here we abbreviated

$$\hat{\underline{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}, \quad \hat{\underline{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_p\} \in \text{Sym}^p(\mathcal{K}_p). \quad (3.14)$$

Given $\phi(\cdot, n)$, the meromorphic stationary Baker–Akhiezer vector $\Psi(\cdot, n, n_0)$ on \mathcal{K}_p is then defined by

$$\Psi(P, n, n_0) = \begin{pmatrix} \psi_1(P, n, n_0) \\ \psi_2(P, n, n_0) \end{pmatrix},$$

$$\psi_1(P, n, n_0) = \begin{cases} \prod_{n'=n_0+1}^n (z + \alpha(n')\phi^-(P, n')), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} (z + \alpha(n')\phi^-(P, n'))^{-1}, & n \leq n_0 - 1, \end{cases} \quad (3.15)$$

$$\psi_2(P, n, n_0) = \phi(P, n_0) \begin{cases} \prod_{n'=n_0+1}^n (z\beta(n')\phi^-(P, n')^{-1} + 1), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} (z\beta(n')\phi^-(P, n')^{-1} + 1)^{-1}, & n \leq n_0 - 1. \end{cases} \quad (3.16)$$

Basic properties of ϕ and Ψ are summarized in the following result.

Lemma 3.1 ([40]). *Suppose that α, β satisfy (3.1) and the \underline{p} th stationary Ablowitz-Ladik system (2.48). Moreover, assume (3.2) and (3.3) and let $P = (z, y) \in \mathcal{K}_{\underline{p}} \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}$, $(n, n_0) \in \mathbb{Z}^2$. Then ϕ satisfies the Riccati-type equation*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \quad (3.17)$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \quad (3.18)$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \quad (3.19)$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p-} \frac{y(P)}{F_{\underline{p}}(z)}. \quad (3.20)$$

The vector Ψ satisfies

$$U(z)\Psi^-(P) = \Psi(P), \quad (3.21)$$

$$V_{\underline{p}}(z)\Psi^-(P) = -(i/2)c_{0,+}z^{-p-}y\Psi^-(P), \quad (3.22)$$

$$\psi_2(P, n, n_0) = \phi(P, n)\psi_1(P, n, n_0), \quad (3.23)$$

$$\psi_1(P, n, n_0)\psi_1(P^*, n, n_0) = z^{n-n_0} \frac{F_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n_0)} \Gamma(n, n_0), \quad (3.24)$$

$$\psi_2(P, n, n_0)\psi_2(P^*, n, n_0) = z^{n-n_0} \frac{H_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n_0)} \Gamma(n, n_0), \quad (3.25)$$

$$\begin{aligned} \psi_1(P, n, n_0)\psi_2(P^*, n, n_0) + \psi_1(P^*, n, n_0)\psi_2(P, n, n_0) \\ = 2z^{n-n_0} \frac{G_{\underline{p}}(z, n)}{F_{\underline{p}}(z, n_0)} \Gamma(n, n_0), \end{aligned} \quad (3.26)$$

$$\begin{aligned} \psi_1(P, n, n_0)\psi_2(P^*, n, n_0) - \psi_1(P^*, n, n_0)\psi_2(P, n, n_0) \\ = -c_{0,+}z^{n-n_0-p-} \frac{y}{F_{\underline{p}}(z, n_0)} \Gamma(n, n_0), \end{aligned} \quad (3.27)$$

where we used the abbreviation

$$\Gamma(n, n_0) = \begin{cases} \prod_{n'=n_0+1}^n \gamma(n'), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} \gamma(n')^{-1}, & n \leq n_0 - 1. \end{cases} \quad (3.28)$$

Combining the Laurent polynomial recursion approach of Section 2 with (3.4) and (3.5) readily yields trace formulas for $f_{\ell,\pm}$ and $h_{\ell,\pm}$ in terms of symmetric functions of the zeros μ_j and ν_k of $(\cdot)^{p-}F_{\underline{p}}$ and $(\cdot)^{p-1}H_{\underline{p}}$, respectively. For simplicity we just record the simplest cases.

Lemma 3.2 ([40]). *Suppose that α, β satisfy (3.1) and the \underline{p} th stationary Ablowitz–Ladik system (2.48). Then,*

$$\frac{\alpha}{\alpha^+} = (-1)^{p+1} \frac{c_{0,+}}{c_{0,-}} \prod_{j=1}^p \mu_j, \quad (3.29)$$

$$\frac{\beta^+}{\beta} = (-1)^{p+1} \frac{c_{0,+}}{c_{0,-}} \prod_{j=1}^p \nu_j, \quad (3.30)$$

$$\sum_{j=1}^p \mu_j = \alpha^+ \beta - \gamma^+ \frac{\alpha^{++}}{\alpha^+} - \frac{c_{1,+}}{c_{0,+}}, \quad (3.31)$$

$$\sum_{j=1}^p \nu_j = \alpha^+ \beta - \gamma \frac{\beta^-}{\beta} - \frac{c_{1,+}}{c_{0,+}}. \quad (3.32)$$

Next we turn to asymptotic properties of ϕ and Ψ in a neighborhood of $P_{\infty\pm}$ and $P_{0,\pm}$.

Lemma 3.3 ([40]). *Suppose that α, β satisfy (3.1) and the \underline{p} th stationary Ablowitz–Ladik system (2.48). Moreover, let $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}$, $(n, n_0) \in \mathbb{Z}^2$. Then ϕ has the asymptotic behavior*

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \beta + \beta^- \gamma \zeta + O(\zeta^2), & P \rightarrow P_{\infty+}, \\ -(\alpha^+)^{-1} \zeta^{-1} + (\alpha^+)^{-2} \alpha^{++} \gamma^+ + O(\zeta), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z, \quad (3.33)$$

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \alpha^{-1} - \alpha^{-2} \alpha^- \gamma \zeta + O(\zeta^2), & P \rightarrow P_{0,+}, \\ -\beta^+ \zeta - \beta^{++} \gamma^+ \zeta^2 + O(\zeta^3), & P \rightarrow P_{0,-}, \end{cases} \quad \zeta = z. \quad (3.34)$$

The components of the Baker–Akhiezer vector Ψ have the asymptotic behavior

$$\psi_1(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{n_0-n} (1 + O(\zeta)), & P \rightarrow P_{\infty+}, \\ \frac{\alpha^+(n)}{\alpha^+(n_0)} \Gamma(n, n_0) + O(\zeta), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z, \quad (3.35)$$

$$\psi_1(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \begin{cases} \frac{\alpha(n)}{\alpha(n_0)} + O(\zeta), & P \rightarrow P_{0,+}, \\ \zeta^{n-n_0} \Gamma(n, n_0) (1 + O(\zeta)), & P \rightarrow P_{0,-}, \end{cases} \quad \zeta = z, \quad (3.36)$$

$$\psi_2(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \begin{cases} \beta(n) \zeta^{n_0-n} (1 + O(\zeta)), & P \rightarrow P_{\infty+}, \\ -\frac{1}{\alpha^+(n_0)} \Gamma(n, n_0) \zeta^{-1} (1 + O(\zeta)), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z, \quad (3.37)$$

$$\psi_2(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \begin{cases} \frac{1}{\alpha(n_0)} + O(\zeta), & P \rightarrow P_{0,+}, \\ -\beta^+(n) \Gamma(n, n_0) \zeta^{n+1-n_0} (1 + O(\zeta)), & P \rightarrow P_{0,-}, \end{cases} \quad \zeta = z. \quad (3.38)$$

The divisors (ψ_j) of ψ_j , $j = 1, 2$, are given by

$$(\psi_1(\cdot, n, n_0)) = \mathcal{D}_{\underline{\mu}(n)} - \mathcal{D}_{\underline{\mu}(n_0)} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty+}}), \quad (3.39)$$

$$(\psi_2(\cdot, n, n_0)) = \mathcal{D}_{\hat{\underline{\nu}}(n)} - \mathcal{D}_{\hat{\underline{\mu}}(n_0)} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty+}}) + \mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty-}}. \quad (3.40)$$

Since nonspecial divisors play a fundamental role in this section and the next, we now take a closer look at them.

Lemma 3.4 ([40]). *Suppose that α, β satisfy (3.1) and the p th stationary Ablowitz–Ladik system (2.48). Moreover, assume (3.2) and (3.3) and let $n \in \mathbb{Z}$. Let $\mathcal{D}_{\hat{\underline{\mu}}}$, $\hat{\underline{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}$, and $\mathcal{D}_{\hat{\underline{\nu}}}$, $\hat{\underline{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_p\}$, be the pole and zero divisors of degree p , respectively, associated with α, β , and ϕ defined according to (3.8) and (3.9), that is,*

$$\begin{aligned} \hat{\mu}_j(n) &= (\mu_j(n), (2/c_{0,+})\mu_j(n)^{p-1}G_{\underline{p}}(\mu_j(n), n)), \quad j = 1, \dots, p, \\ \hat{\nu}_j(n) &= (\nu_j(n), -(2/c_{0,+})\nu_j(n)^{p-1}G_{\underline{p}}(\nu_j(n), n)), \quad j = 1, \dots, p. \end{aligned} \quad (3.41)$$

Then $\mathcal{D}_{\hat{\underline{\mu}}(n)}$ and $\mathcal{D}_{\hat{\underline{\nu}}(n)}$ are nonspecial for all $n \in \mathbb{Z}$.

Next, we shall provide an explicit representation of ϕ, Ψ, α , and β in terms of the Riemann theta function associated with \mathcal{K}_p . We freely employ the notation established in Appendix A. (We recall our tacit assumption $p \in \mathbb{N}$ to avoid the trivial case $p = 0$.)

Let θ denote the Riemann theta function associated with \mathcal{K}_p and introduce a fixed homology basis $\{a_j, b_j\}_{j=1, \dots, p}$ on \mathcal{K}_p . Choosing as a convenient fixed base point one of the branch points, $Q_0 = (E_{m_0}, 0)$, the Abel maps \underline{A}_{Q_0} and $\underline{\alpha}_{Q_0}$ are defined by (A.20) and (A.21) and the Riemann vector $\underline{\Xi}_{Q_0}$ is given by (A.31). Let $\omega_{P_+, P_-}^{(3)}$ be the normal differential of the third kind holomorphic on $\mathcal{K}_p \setminus \{P_+, P_-\}$ with simple poles at P_{\pm} and residues ± 1 , respectively. In particular, one obtains for $\omega_{P_{0,-}, P_{\infty\pm}}^{(3)}$,

$$\omega_{P_{0,-}, P_{\infty\pm}}^{(3)} = \left(\frac{y + y_{0,-}}{z} \mp \prod_{j=1}^p (z - \lambda_{\pm, j}) \right) \frac{dz}{2y}, \quad P_{0,-} = (0, y_{0,-}), \quad (3.42)$$

where the constants $\{\lambda_{\pm, j}\}_{j=1}^p \subset \mathbb{C}$ are uniquely determined by employing the normalization

$$\int_{a_j} \omega_{P_{0,-}, P_{\infty\pm}}^{(3)} = 0, \quad j = 1, \dots, p. \quad (3.43)$$

The explicit formula (3.42) then implies the following asymptotic expansions (using the local coordinate $\zeta = z$ near $P_{0,\pm}$ and $\zeta = 1/z$ near $P_{\infty\pm}$),

$$\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)} \underset{\zeta \rightarrow 0}{=} \left\{ \begin{array}{c} 0 \\ \ln(\zeta) \end{array} \right\} + \omega_0^{0,\pm}(P_{0,-}, P_{\infty-}) + O(\zeta) \text{ as } P \rightarrow P_{0,\pm}, \quad (3.44)$$

$$\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)} \underset{\zeta \rightarrow 0}{=} \left\{ \begin{array}{c} 0 \\ -\ln(\zeta) \end{array} \right\} + \omega_0^{\infty,\pm}(P_{0,-}, P_{\infty-}) + O(\zeta) \text{ as } P \rightarrow P_{\infty\pm}, \quad (3.45)$$

$$\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty+}}^{(3)} \underset{\zeta \rightarrow 0}{=} \left\{ \begin{array}{c} 0 \\ \ln(\zeta) \end{array} \right\} + \omega_0^{0,\pm}(P_{0,-}, P_{\infty+}) + O(\zeta) \text{ as } P \rightarrow P_{0,\pm}, \quad (3.46)$$

$$\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty+}}^{(3)} \underset{\zeta \rightarrow 0}{=} \left\{ \begin{array}{c} -\ln(\zeta) \\ 0 \end{array} \right\} + \omega_0^{\infty,\pm}(P_{0,-}, P_{\infty+}) + O(\zeta) \text{ as } P \rightarrow P_{\infty\pm}. \quad (3.47)$$

Lemma 3.5. *With $\omega_0^{\infty\sigma}(P_{0,-}, P_{\infty\pm})$ and $\omega_0^{0,\sigma'}(P_{0,-}, P_{\infty\pm})$, $\sigma, \sigma' \in \{+, -\}$, defined as in (3.44)–(3.47) one has*

$$\begin{aligned} & \exp(\omega_0^{0,-}(P_{0,-}, P_{\infty\pm}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty\pm}) \\ & \quad - \omega_0^{\infty-}(P_{0,-}, P_{\infty\pm}) + \omega_0^{0,+}(P_{0,-}, P_{\infty\pm})) = 1. \end{aligned} \quad (3.48)$$

Proof. Pick $Q_{1,\pm} = (z_1, \pm y_1) \in \mathcal{K}_p \setminus \{P_{\infty\pm}\}$ in a neighborhood of $P_{\infty\pm}$ and $Q_{2,\pm} = (z_2, \pm y_2) \in \mathcal{K}_p \setminus \{P_{0,\pm}\}$ in a neighborhood of $P_{0,\pm}$. Without loss of generality one may assume that $P_{\infty+}$ and $P_{0,+}$ lie on the same sheet. Then by (3.42),

$$\begin{aligned} & \int_{Q_0}^{Q_{2,-}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} - \int_{Q_0}^{Q_{1,+}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} - \int_{Q_0}^{Q_{1,-}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} + \int_{Q_0}^{Q_{2,+}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} \\ & = \int_{Q_0}^{Q_{2,+}} \frac{dz}{z} - \int_{Q_0}^{Q_{1,+}} \frac{dz}{z} = \ln(z_2) - \ln(z_1) + 2\pi ik, \end{aligned} \quad (3.49)$$

for some $k \in \mathbb{Z}$. On the other hand, by (3.44)–(3.47) one obtains

$$\begin{aligned} & \int_{Q_0}^{Q_{2,-}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} - \int_{Q_0}^{Q_{1,+}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} - \int_{Q_0}^{Q_{1,-}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} + \int_{Q_0}^{Q_{2,+}} \omega_{P_{0,-}, P_{\infty-}}^{(3)} \\ & = \ln(z_2) + \ln(1/z_1) + \omega_0^{0,-}(P_{0,-}, P_{\infty-}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty-}) \\ & \quad - \omega_0^{\infty-}(P_{0,-}, P_{\infty-}) + \omega_0^{0,+}(P_{0,-}, P_{\infty-}) + O(z_2) + O(1/z_1), \end{aligned} \quad (3.50)$$

and hence the part of (3.48) concerning $\omega_{P_{0,-}, P_{\infty-}}^{(3)}$ follows. The corresponding result for $\omega_{P_{0,-}, P_{\infty+}}^{(3)}$ is proved analogously. \square

In the following it will be convenient to use the abbreviation

$$\begin{aligned} \underline{z}(P, \underline{Q}) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ P &\in \mathcal{K}_p, \quad \underline{Q} = \{Q_1, \dots, Q_p\} \in \text{Sym}^p(\mathcal{K}_p). \end{aligned} \quad (3.51)$$

We note that $\underline{z}(\cdot, \underline{Q})$ is independent of the choice of base point Q_0 .

For later use we state the following result.

Lemma 3.6. *The following relations hold:*

$$\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+) = \underline{z}(P_{\infty-}, \hat{\underline{\nu}}) = \underline{z}(P_{0,-}, \hat{\underline{\mu}}) = \underline{z}(P_{0,+}, \hat{\underline{\nu}}^+), \quad (3.52)$$

$$\underline{z}(P_{\infty+}, \hat{\underline{\nu}}^+) = \underline{z}(P_{0,-}, \hat{\underline{\nu}}), \quad \underline{z}(P_{0,+}, \hat{\underline{\mu}}^+) = \underline{z}(P_{\infty-}, \hat{\underline{\mu}}). \quad (3.53)$$

Proof. We indicate the proof of some of the relations to be used in (3.69) and (3.70). Let $\hat{\underline{\lambda}}$ denote either $\hat{\underline{\mu}}$ or $\hat{\underline{\nu}}$. Then,

$$\begin{aligned} \underline{z}(P_{0,+}, \hat{\underline{\lambda}}^+) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\lambda}}^+}) \\ &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty+}) \\ &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\lambda}}}) \\ &= \underline{z}(P_{\infty-}, \hat{\underline{\lambda}}), \end{aligned} \quad (3.54)$$

$$\begin{aligned} \underline{z}(P_{\infty+}, \hat{\underline{\lambda}}^+) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\lambda}}^+}) \\ &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\lambda}}}) + \underline{A}_{P_{0,-}}(P_{\infty+}) \\ &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\lambda}}}) \end{aligned}$$

$$= \underline{z}(P_{0,-}, \hat{\lambda}), \text{ etc.} \quad (3.55)$$

Here we used $\underline{A}_{Q_0}(P^*) = -\underline{A}_{Q_0}(P)$, $P \in \mathcal{K}_p$, since Q_0 is a branch point of \mathcal{K}_p , and $\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\lambda}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\lambda}}) + \underline{A}_{P_{0,-}}(P_{\infty+})$. The latter equality immediately follows from (3.39) in the case $\hat{\lambda} = \hat{\mu}$ and from combining (3.13) and (3.40) in the case $\hat{\lambda} = \hat{\nu}$. \square

Given these preparations, the theta function representations of ϕ , ψ_1 , ψ_2 , α , and β then read as follows.

Theorem 3.7. *Suppose that α, β satisfy (3.1) and the p th stationary Ablowitz–Ladik system (2.48). Moreover, assume hypothesis (3.2) and (3.3), and let $P \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}$ and $(n, n_0) \in \mathbb{Z}^2$. Then for each $n \in \mathbb{Z}$, $\mathcal{D}_{\hat{\mu}(n)}$ and $\mathcal{D}_{\hat{\nu}(n)}$ are nonspecial. Moreover,*

$$\phi(P, n) = C(n) \frac{\theta(\underline{z}(P, \hat{\nu}(n)))}{\theta(\underline{z}(P, \hat{\mu}(n)))} \exp \left(\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)} \right), \quad (3.56)$$

$$\psi_1(P, n, n_0) = C(n, n_0) \frac{\theta(\underline{z}(P, \hat{\mu}(n)))}{\theta(\underline{z}(P, \hat{\mu}(n_0)))} \exp \left((n - n_0) \int_{Q_0}^P \omega_{P_{0,-}, P_{\infty+}}^{(3)} \right), \quad (3.57)$$

$$\begin{aligned} \psi_2(P, n, n_0) &= C(n) C(n, n_0) \frac{\theta(\underline{z}(P, \hat{\nu}(n)))}{\theta(\underline{z}(P, \hat{\mu}(n_0)))} \\ &\quad \times \exp \left(\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)} + (n - n_0) \int_{Q_0}^P \omega_{P_{0,-}, P_{\infty+}}^{(3)} \right), \end{aligned} \quad (3.58)$$

where

$$\begin{aligned} C(n) &= (-1)^{n-n_0} \exp \left((n - n_0) (\omega_0^{0,-}(P_{0,-}, P_{\infty-}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty-})) \right) \\ &\quad \times \frac{1}{\alpha(n_0)} \exp \left(-\omega_0^{0,+}(P_{0,-}, P_{\infty-}) \frac{\theta(\underline{z}(P_{0,+}, \hat{\mu}(n_0)))}{\theta(\underline{z}(P_{0,+}, \hat{\nu}(n_0)))} \right), \end{aligned} \quad (3.59)$$

$$C(n, n_0) = \exp \left(-(n - n_0) \omega_0^{\infty+}(P_{0,-}, P_{\infty+}) \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(n_0)))}{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(n)))} \right). \quad (3.60)$$

The Abel map linearizes the auxiliary divisors $\mathcal{D}_{\hat{\mu}(n)}$ and $\mathcal{D}_{\hat{\nu}(n)}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n_0)}) + (n - n_0) \underline{A}_{P_{0,-}}(P_{\infty+}), \quad (3.61)$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\nu}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\nu}(n_0)}) + (n - n_0) \underline{A}_{P_{0,-}}(P_{\infty+}), \quad (3.62)$$

in addition,

$$\begin{aligned} \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\nu}(n)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n)}) - \underline{A}_{Q_0}(P_{0,-}) + \underline{A}_{Q_0}(P_{\infty-}) \\ &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n)}) + \underline{A}_{P_{0,-}}(P_{\infty-}). \end{aligned} \quad (3.63)$$

Finally, α, β are of the form

$$\begin{aligned} \alpha(n) &= \alpha(n_0) (-1)^{n-n_0} \exp \left(-(n - n_0) (\omega_0^{0,-}(P_{0,-}, P_{\infty-}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty-})) \right) \\ &\quad \times \frac{\theta(\underline{z}(P_{0,+}, \hat{\nu}(n_0))) \theta(\underline{z}(P_{0,+}, \hat{\mu}(n)))}{\theta(\underline{z}(P_{0,+}, \hat{\mu}(n_0))) \theta(\underline{z}(P_{0,+}, \hat{\nu}(n)))}, \end{aligned} \quad (3.64)$$

$$\begin{aligned} \beta(n) &= \beta(n_0) (-1)^{n-n_0} \exp \left((n - n_0) (\omega_0^{0,-}(P_{0,-}, P_{\infty-}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty-})) \right) \\ &\quad \times \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(n_0))) \theta(\underline{z}(P_{\infty+}, \hat{\nu}(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\nu}(n_0))) \theta(\underline{z}(P_{\infty+}, \hat{\mu}(n)))}, \end{aligned} \quad (3.65)$$

$$\begin{aligned} \alpha(n)\beta(n) &= \exp(\omega_0^{\infty+}(P_{0,-}, P_{\infty-}) - \omega_0^{0,+}(P_{0,-}, P_{\infty-})) \\ &\quad \times \frac{\theta(\underline{z}(P_{0,+}, \hat{\underline{\mu}}(n)))\theta(\underline{z}(P_{\infty+}, \hat{\underline{\nu}}(n)))}{\theta(\underline{z}(P_{0,+}, \hat{\underline{\nu}}(n)))\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))}, \end{aligned} \quad (3.66)$$

$$\begin{aligned} \Gamma(n, n_0) &= \exp((n - n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty+}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty+}))) \\ &\quad \times \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n)))\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n_0)))\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))}. \end{aligned} \quad (3.67)$$

Proof. Applying Abel's theorem (cf. Theorem A.1, (A.29)) to (3.13) proves (3.63), and applying it to (3.39), (3.40) results in (3.61) and (3.62). By Lemma 3.4, $\mathcal{D}_{\hat{\underline{\mu}}}$ and $\mathcal{D}_{\hat{\underline{\nu}}}$ are nonspecial. By equation (3.13) and Theorem A.3, $\phi(P, n) \exp(-\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)})$ must be of the type

$$\phi(P, n) \exp\left(-\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)}\right) = C(n) \frac{\theta(\underline{z}(P, \hat{\underline{\nu}}(n)))}{\theta(\underline{z}(P, \hat{\underline{\mu}}(n)))} \quad (3.68)$$

for some constant $C(n)$. A comparison of (3.68) and the asymptotic relations (3.33) then yields, with the help of (3.44), (3.45) and (3.52), (3.53), the following expressions for α and β :

$$\begin{aligned} (\alpha^+)^{-1} &= C^+ e^{\omega_0^{0,+}(P_{0,-}, P_{\infty-})} \frac{\theta(\underline{z}(P_{0,+}, \hat{\underline{\nu}}^+))}{\theta(\underline{z}(P_{0,+}, \hat{\underline{\mu}}^+))} \\ &= C^+ e^{\omega_0^{0,+}(P_{0,-}, P_{\infty-})} \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\nu}}))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}))} \\ &= -C e^{\omega_0^{\infty-}(P_{0,-}, P_{\infty-})} \frac{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\nu}}))}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}))}. \end{aligned} \quad (3.69)$$

Similarly one obtains

$$\begin{aligned} \beta^+ &= C^+ e^{\omega_0^{\infty+}(P_{0,-}, P_{\infty-})} \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\nu}}^+))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}^+))} \\ &= C^+ e^{\omega_0^{\infty+}(P_{0,-}, P_{\infty-})} \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\nu}}))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}))} \\ &= -C e^{\omega_0^{0,-}(P_{0,-}, P_{\infty-})} \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\nu}}))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}))}. \end{aligned} \quad (3.70)$$

Here we used (3.61) and (3.62), more precisely,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}}) + \underline{A}_{P_{0,-}}(P_{\infty+}), \quad \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}^+}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}}) + \underline{A}_{P_{0,-}}(P_{\infty+}). \quad (3.71)$$

Thus, one concludes

$$C(n+1) = -\exp[\omega_0^{0,-}(P_{0,-}, P_{\infty-}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty-})]C(n), \quad n \in \mathbb{Z}, \quad (3.72)$$

and

$$C(n+1) = -\exp[\omega_0^{\infty-}(P_{0,-}, P_{\infty-}) - \omega_0^{0,+}(P_{0,-}, P_{\infty-})]C(n), \quad n \in \mathbb{Z}, \quad (3.73)$$

which is consistent with (3.48). The first-order difference equation (3.72) then implies

$$C(n) = (-1)^{(n-n_0)} \exp[(n - n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty-}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty-}))]C(n_0), \quad n, n_0 \in \mathbb{Z}. \quad (3.74)$$

Thus one infers (3.64) and (3.65). Moreover, (3.74) and taking $n = n_0$ in the first line in (3.69) yield (3.59). Dividing the first line in (3.70) by the first line in (3.69) then proves (3.66).

By (3.39) and Theorem A.3, $\psi_1(P, n, n_0)$ must be of the type (3.57). A comparison of (3.15), (3.33), and (3.57) as $P \rightarrow P_{\infty+}$ (with local coordinate $\zeta = 1/z$) then yields

$$\psi_1(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \zeta^{n_0-n}(1 + O(\zeta)) \quad (3.75)$$

and

$$\begin{aligned} \psi_1(P, n, n_0) \underset{\zeta \rightarrow 0}{=} & C(n, n_0) \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n_0)))} \\ & \times \exp[(n - n_0)\omega_0^{\infty+}(P_{0,-}, P_{\infty+})] \zeta^{n_0-n}(1 + O(\zeta)) \end{aligned} \quad (3.76)$$

proving (3.60). Equation (3.58) is clear from (3.23), (3.56), and (3.57).

Finally, a comparison of (3.36) and (3.57) as $P \rightarrow P_{0,-}$ (with local coordinate $\zeta = z$) yields

$$\psi_1(P, n, n_0) \underset{\zeta \rightarrow 0}{=} \Gamma(n, n_0) \zeta^{n-n_0}(1 + O(\zeta)) \quad (3.77)$$

$$\begin{aligned} \underset{\zeta \rightarrow 0}{=} & C(n, n_0) \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n)))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n_0)))} \exp((n - n_0)\omega_0^{0,-}(P_{0,-}, P_{\infty+})) \\ & \times \zeta^{n-n_0}(1 + O(\zeta)) \end{aligned} \quad (3.78)$$

and hence

$$\begin{aligned} \Gamma(n, n_0) &= C(n, n_0) \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n)))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n_0)))} \exp((n - n_0)\omega_0^{0,-}(P_{0,-}, P_{\infty+})) \\ &= \exp((n - n_0)(\omega_0^{0,-}(P_{0,-}, P_{\infty+}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty+}))) \\ &\quad \times \frac{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n)))\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n_0)))}{\theta(\underline{z}(P_{0,-}, \hat{\underline{\mu}}(n_0)))\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n)))} \end{aligned} \quad (3.79)$$

using (3.60). \square

We note that the apparent n_0 -dependence of $C(n)$ in the right-hand side of (3.59) actually drops out to ensure the n_0 -independence of ϕ in (3.56).

The theta function representations (3.64), (3.65) for α, β and that for Γ in (3.67) also show that $\gamma(n) \notin \{0, 1\}$ for all $n \in \mathbb{Z}$, and hence condition (3.1) is satisfied for the stationary algebro-geometric AL solutions discussed in this section, provided the associated divisors $\mathcal{D}_{\hat{\underline{\mu}}}(n)$ and $\mathcal{D}_{\underline{z}}(n)$ stay away from $P_{\infty\pm}, P_{0,\pm}$ for all $n \in \mathbb{Z}$.

The stationary algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data, that is, the construction of α and β by starting from a set of initial data (nonspecial divisors) of full measure, will be presented in [41].

4. THE TIME-DEPENDENT ABLOWITZ–LADIK FORMALISM

In this section we extend the algebro-geometric analysis of Section 3 to the time-dependent Ablowitz–Ladik hierarchy. For proofs of the elementary results of the time-dependent formalism we refer to [39], [40].

For most of this section we assume the following hypothesis.

Hypothesis 4.1. (i) Suppose that α, β satisfy

$$\begin{aligned} \alpha(\cdot, t), \beta(\cdot, t) &\in \mathbb{C}^{\mathbb{Z}}, \quad t \in \mathbb{R}, \quad \alpha(n, \cdot), \beta(n, \cdot) \in C^1(\mathbb{R}), \quad n \in \mathbb{Z}, \\ \alpha(n, t)\beta(n, t) &\notin \{0, 1\}, \quad (n, t) \in \mathbb{Z} \times \mathbb{R}. \end{aligned} \quad (4.1)$$

(ii) Assume that the hyperelliptic curve \mathcal{K}_p satisfies (3.2) and (3.3).

The basic problem in the analysis of algebro-geometric solutions of the Ablowitz–Ladik hierarchy consists of solving the time-dependent \underline{r} th Ablowitz–Ladik flow with initial data a stationary solution of the \underline{p} th system in the hierarchy. More precisely, given $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$ we consider a solution $\alpha^{(0)}, \beta^{(0)}$ of the \underline{p} th stationary Ablowitz–Ladik system $\text{s-AL}_{\underline{p}}(\alpha^{(0)}, \beta^{(0)}) = 0$, $\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0, 0)\}$, associated with the hyperelliptic curve \mathcal{K}_p and a corresponding set of summation constants $\{c_{\ell, \pm}\}_{\ell=1, \dots, p_{\pm}} \subset \mathbb{C}$. Next, let $\underline{r} = (r_-, r_+) \in \mathbb{N}_0^2$; we intend to construct a solution α, β of the Ablowitz–Ladik flow $\text{AL}_{\underline{r}}(\alpha, \beta) = 0$ with $\alpha(t_{0, \underline{r}}) = \alpha^{(0)}, \beta(t_{0, \underline{r}}) = \beta^{(0)}$ for some $t_{0, \underline{r}} \in \mathbb{R}$. To emphasize that the summation constants in the definitions of the stationary and the time-dependent Ablowitz–Ladik equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\tilde{V}_{\underline{r}}, \tilde{F}_{\underline{r}}, \tilde{G}_{\underline{r}}, \tilde{H}_{\underline{r}}, \tilde{K}_{\underline{r}}, \tilde{f}_{s, \pm}, \tilde{g}_{s, \pm}, \tilde{h}_{s, \pm}, \tilde{c}_{s, \pm}$, in order to distinguish them from $V_{\underline{p}}, F_{\underline{p}}, G_{\underline{p}}, H_{\underline{p}}, K_{\underline{p}}, f_{\ell, \pm}, g_{\ell, \pm}, h_{\ell, \pm}, c_{\ell, \pm}$, in the following. In addition, we will follow a more elaborate notation inspired by Hirota’s τ -function approach and indicate the individual \underline{r} th Ablowitz–Ladik flow by a separate time variable $t_{\underline{r}} \in \mathbb{R}$.

Summing up, we are looking for solutions α, β of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} \tilde{\text{AL}}_{\underline{r}}(\alpha, \beta) &= \begin{pmatrix} -i\alpha_{t_{\underline{r}}} - \alpha(\tilde{g}_{r_+, +} + \tilde{g}_{r_-, -}) + \tilde{f}_{r_+ - 1, +} - \tilde{f}_{r_+ - 1, -} \\ -i\beta_{t_{\underline{r}}} + \beta(\tilde{g}_{r_+, +} + \tilde{g}_{r_-, -}) - \tilde{h}_{r_+ - 1, -} + \tilde{h}_{r_+ - 1, +} \end{pmatrix} = 0, \\ (\alpha, \beta)|_{t=t_{0, \underline{r}}} &= (\alpha^{(0)}, \beta^{(0)}), \end{aligned} \quad (4.2)$$

$$\text{s-AL}_{\underline{p}}(\alpha^{(0)}, \beta^{(0)}) = \begin{pmatrix} -\alpha^{(0)}(g_{p_+, +} + g_{p_-, -}) + f_{p_+ - 1, +} - f_{p_+ - 1, -} \\ \beta^{(0)}(g_{p_+, +} + g_{p_-, -}) - h_{p_+ - 1, -} + h_{p_+ - 1, +} \end{pmatrix} = 0 \quad (4.3)$$

for some $t_{0, \underline{r}} \in \mathbb{R}$, where $\alpha = \alpha(n, t_{\underline{r}}), \beta = \beta(n, t_{\underline{r}})$ satisfy (4.1) and a fixed curve \mathcal{K}_p is associated with the stationary solutions $\alpha^{(0)}, \beta^{(0)}$ in (4.3). Here,

$$\underline{p} = (p_-, p_+) \in \mathbb{N}_0^2 \setminus \{(0, 0)\}, \quad \underline{r} = (r_-, r_+) \in \mathbb{N}_0^2, \quad p = p_- + p_+ - 1. \quad (4.4)$$

In terms of the zero-curvature formulation this amounts to solving

$$U_{t_{\underline{r}}}(z, t_{\underline{r}}) + U(z, t_{\underline{r}})\tilde{V}_{\underline{r}}(z, t_{\underline{r}}) - \tilde{V}_{\underline{r}}^+(z, t_{\underline{r}})U(z, t_{\underline{r}}) = 0, \quad (4.5)$$

$$U(z, t_{0, \underline{r}})V_{\underline{p}}(z, t_{0, \underline{r}}) - V_{\underline{p}}^+(z, t_{0, \underline{r}})U(z, t_{0, \underline{r}}) = 0. \quad (4.6)$$

One can show (cf. [41]) that the stationary Ablowitz–Ladik system (4.6) is actually satisfied for all times $t_{\underline{r}} \in \mathbb{R}$. Thus, we impose

$$U_{t_{\underline{r}}}(z, t_{\underline{r}}) + U(z, t_{\underline{r}})\tilde{V}_{\underline{r}}(z, t_{\underline{r}}) - \tilde{V}_{\underline{r}}^+(z, t_{\underline{r}})U(z, t_{\underline{r}}) = 0, \quad (4.7)$$

$$U(z, t_{\underline{r}})V_{\underline{p}}(z, t_{\underline{r}}) - V_{\underline{p}}^+(z, t_{\underline{r}})U(z, t_{\underline{r}}) = 0, \quad (4.8)$$

instead of (4.5) and (4.6). For further reference, we recall the relevant quantities here (cf. (2.34), (2.35), (2.41)–(2.43), (2.53)):

$$U(z) = \begin{pmatrix} z & \alpha \\ z\beta & 1 \end{pmatrix}, \quad (4.9)$$

$$V_{\underline{p}}(z) = i \begin{pmatrix} G_{\underline{p}}^-(z) & -F_{\underline{p}}^-(z) \\ H_{\underline{p}}^-(z) & -G_{\underline{p}}^-(z) \end{pmatrix}, \quad \tilde{V}_{\underline{r}}(z) = i \begin{pmatrix} \tilde{G}_{\underline{r}}^-(z) & -\tilde{F}_{\underline{r}}^-(z) \\ \tilde{H}_{\underline{r}}^-(z) & -\tilde{K}_{\underline{r}}^-(z) \end{pmatrix},$$

and

$$F_{\underline{p}}(z) = \sum_{\ell=1}^{p-} f_{p-,\ell,-} z^{-\ell} + \sum_{\ell=0}^{p+} f_{p+,-1-\ell,+} z^{\ell} = -c_{0,+} \alpha^+ z^{-p-} \prod_{j=1}^p (z - \mu_j),$$

$$G_{\underline{p}}(z) = \sum_{\ell=1}^{p-} g_{p-,\ell,-} z^{-\ell} + \sum_{\ell=0}^{p+} g_{p+,-\ell,+} z^{\ell},$$

$$H_{\underline{p}}(z) = \sum_{\ell=0}^{p-} h_{p-,-1-\ell,-} z^{-\ell} + \sum_{\ell=1}^{p+} h_{p+,-\ell,+} z^{\ell} = c_{0,+} \beta z^{-p-+1} \prod_{j=1}^p (z - \nu_j),$$

$$\tilde{F}_{\underline{r}}(z) = \sum_{s=1}^{r-} \tilde{f}_{r-,-s,-} z^{-s} + \sum_{s=0}^{r+} \tilde{f}_{r+,-1-s,+} z^s, \quad (4.10)$$

$$\tilde{G}_{\underline{r}}(z) = \sum_{s=1}^{r-} \tilde{g}_{r-,-s,-} z^{-s} + \sum_{s=0}^{r+} \tilde{g}_{r+,-s,+} z^s,$$

$$\tilde{H}_{\underline{r}}(z) = \sum_{s=0}^{r-} \tilde{h}_{r-,-1-s,-} z^{-s} + \sum_{s=1}^{r+} \tilde{h}_{r+,-s,+} z^s,$$

$$\tilde{K}_{\underline{r}}(z) = \sum_{s=0}^{r-} \tilde{g}_{r-,-s,-} z^{-s} + \sum_{s=1}^{r+} \tilde{g}_{r+,-s,+} z^s = \tilde{G}_{\underline{r}}(z) + \tilde{g}_{r-,-} - \tilde{g}_{r+,+}$$

for fixed $\underline{p} \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, $\underline{r} \in \mathbb{N}_0^2$. Here $f_{\ell,\pm}$, $\tilde{f}_{s,\pm}$, $g_{\ell,\pm}$, $\tilde{g}_{s,\pm}$, $h_{\ell,\pm}$, and $\tilde{h}_{s,\pm}$ are defined as in (2.5)–(2.12) with appropriate sets of summation constants $c_{\ell,\pm}$, $\ell \in \mathbb{N}_0$, and $\tilde{c}_{k,\pm}$, $k \in \mathbb{N}_0$. Explicitly, (4.7) and (4.8) are equivalent to (cf. (2.37)–(2.40), (2.60)–(2.63)),

$$\alpha_{t_{\underline{r}}} = i(z\tilde{F}_{\underline{r}}^- + \alpha(\tilde{G}_{\underline{r}} + \tilde{K}_{\underline{r}}^-) - \tilde{F}_{\underline{r}}), \quad (4.11)$$

$$\beta_{t_{\underline{r}}} = -i(\beta(\tilde{G}_{\underline{r}} + \tilde{K}_{\underline{r}}^-) - \tilde{H}_{\underline{r}} + z^{-1}\tilde{H}_{\underline{r}}^-), \quad (4.12)$$

$$0 = z(\tilde{G}_{\underline{r}}^- - \tilde{G}_{\underline{r}}) + z\beta\tilde{F}_{\underline{r}} + \alpha\tilde{H}_{\underline{r}}^-, \quad (4.13)$$

$$0 = z\beta\tilde{F}_{\underline{r}}^- + \alpha\tilde{H}_{\underline{r}} + \tilde{K}_{\underline{r}}^- - \tilde{K}_{\underline{r}}, \quad (4.14)$$

$$0 = z(G_{\underline{p}}^- - G_{\underline{p}}) + z\beta F_{\underline{p}} + \alpha H_{\underline{p}}^-, \quad (4.15)$$

$$0 = z\beta F_{\underline{p}}^- + \alpha H_{\underline{p}} - G_{\underline{p}} + G_{\underline{p}}^-, \quad (4.16)$$

$$0 = -F_{\underline{p}} + zF_{\underline{p}}^- + \alpha(G_{\underline{p}} + G_{\underline{p}}^-), \quad (4.17)$$

$$0 = z\beta(G_{\underline{p}} + G_{\underline{p}}^-) - zH_{\underline{p}} + H_{\underline{p}}^-, \quad (4.18)$$

respectively. In particular, (2.54) holds in the present $t_{\underline{r}}$ -dependent setting, that is,

$$G_{\underline{p}}^2 - F_{\underline{p}} H_{\underline{p}} = R_{\underline{p}}. \quad (4.19)$$

As in the stationary context (3.8), (3.9) we introduce

$$\hat{\mu}_j(n, t_{\underline{r}}) = (\mu_j(n, t_{\underline{r}}), (2/c_{0,+})\mu_j(n, t_{\underline{r}})^{p-} G_{\underline{p}}(\mu_j(n, t_{\underline{r}}), n, t_{\underline{r}})) \in \mathcal{K}_p, \quad (4.20)$$

$$j = 1, \dots, p, (n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R},$$

and

$$\hat{\nu}_j(n, t_{\underline{r}}) = (\nu_j(n, t_{\underline{r}}), -(2/c_{0,+})\nu_j(n, t_{\underline{r}})^{p-} G_{\underline{p}}(\nu_j(n, t_{\underline{r}}), n, t_{\underline{r}})) \in \mathcal{K}_p, \quad (4.21)$$

$$j = 1, \dots, p, (n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R},$$

and note that the regularity assumptions (4.1) on α, β imply continuity of μ_j and ν_k with respect to $t_{\underline{r}} \in \mathbb{R}$ (away from collisions of these zeros, μ_j and ν_k are of course C^∞).

In analogy to (3.11), (3.12), one defines the following meromorphic function $\phi(\cdot, n, t_{\underline{r}})$ on \mathcal{K}_p ,

$$\phi(P, n, t_{\underline{r}}) = \frac{(c_{0,+}/2)z^{-p-}y + G_{\underline{p}}(z, n, t_{\underline{r}})}{F_{\underline{p}}(z, n, t_{\underline{r}})} \quad (4.22)$$

$$= \frac{-H_{\underline{p}}(z, n, t_{\underline{r}})}{(c_{0,+}/2)z^{-p-}y - G_{\underline{p}}(z, n, t_{\underline{r}})}, \quad (4.23)$$

$$P = (z, y) \in \mathcal{K}_p, (n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R},$$

with divisor $(\phi(\cdot, n, t_{\underline{r}}))$ of $\phi(\cdot, n, t_{\underline{r}})$ given by

$$(\phi(\cdot, n, t_{\underline{r}})) = \mathcal{D}_{P_{0,-}\hat{\nu}(n, t_{\underline{r}})} - \mathcal{D}_{P_{\infty-}\hat{\mu}(n, t_{\underline{r}})}. \quad (4.24)$$

The time-dependent Baker–Akhiezer vector is then defined in terms of ϕ by

$$\Psi(P, n, n_0, t_{\underline{r}}, t_{0,\underline{x}}) = \begin{pmatrix} \psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{x}}) \\ \psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{x}}) \end{pmatrix}, \quad (4.25)$$

$$\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{x}}) = \exp\left(i \int_{t_{0,\underline{x}}}^{t_{\underline{r}}} ds (\tilde{G}_{\underline{r}}(z, n_0, s) - \tilde{F}_{\underline{r}}(z, n_0, s)\phi(P, n_0, s))\right)$$

$$\times \begin{cases} \prod_{n'=n_0+1}^n (z + \alpha(n', t_{\underline{r}})\phi^-(P, n', t_{\underline{r}})), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} (z + \alpha(n', t_{\underline{r}})\phi^-(P, n', t_{\underline{r}}))^{-1}, & n \leq n_0 - 1, \end{cases} \quad (4.26)$$

$$\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{x}}) = \exp\left(i \int_{t_{0,\underline{x}}}^{t_{\underline{r}}} ds (\tilde{G}_{\underline{r}}(z, n_0, s) - \tilde{F}_{\underline{r}}(z, n_0, s)\phi(P, n_0, s))\right)$$

$$\times \phi(P, n_0, t_{\underline{r}}) \begin{cases} \prod_{n'=n_0+1}^n (z\beta(n', t_{\underline{r}})\phi^-(P, n', t_{\underline{r}})^{-1} + 1), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} (z\beta(n', t_{\underline{r}})\phi^-(P, n', t_{\underline{r}})^{-1} + 1)^{-1}, & n \leq n_0 - 1, \end{cases} \quad (4.27)$$

$$P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}, (n, t_{\underline{r}}) \in \mathbb{Z} \times \mathbb{R}.$$

One observes that

$$\psi_1(P, n, n_0, t_{\underline{r}}, \tilde{t}_{\underline{r}}) = \psi_1(P, n_0, n_0, t_{\underline{r}}, \tilde{t}_{\underline{r}})\psi_1(P, n, n_0, t_{\underline{r}}, t_{\underline{r}}), \quad (4.28)$$

$$P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}, (n, n_0, t_{\underline{r}}, \tilde{t}_{\underline{r}}) \in \mathbb{Z}^2 \times \mathbb{R}^2.$$

The following lemma records basic properties of ϕ and Ψ in analogy to the stationary case discussed in Lemma 3.1.

Lemma 4.2 ([40]). *Assume Hypothesis 4.1 and suppose that (4.7), (4.8) hold. In addition, let $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}\}$, $(n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) \in \mathbb{Z}^2 \times \mathbb{R}^2$. Then ϕ satisfies*

$$\alpha\phi(P)\phi^-(P) - \phi^-(P) + z\phi(P) = z\beta, \quad (4.29)$$

$$\phi_{t_{\underline{r}}}(P) = i\tilde{F}_{\underline{r}}\phi^2(P) - i(\tilde{G}_{\underline{r}}(z) + \tilde{K}_{\underline{r}}(z))\phi(P) + i\tilde{H}_{\underline{r}}(z), \quad (4.30)$$

$$\phi(P)\phi(P^*) = \frac{H_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \quad (4.31)$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{\underline{p}}(z)}{F_{\underline{p}}(z)}, \quad (4.32)$$

$$\phi(P) - \phi(P^*) = c_{0,+}z^{-p-}\frac{y(P)}{F_{\underline{p}}(z)}. \quad (4.33)$$

Moreover, assuming $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}$, then Ψ satisfies

$$\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) = \phi(P, n, t_{\underline{r}})\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}), \quad (4.34)$$

$$U(z)\Psi^-(P) = \Psi(P), \quad (4.35)$$

$$V_{\underline{p}}(z)\Psi^-(P) = -(i/2)c_{0,+}z^{-p-}y\Psi^-(P), \quad (4.36)$$

$$\Psi_{t_{\underline{r}}}(P) = \tilde{V}_{\underline{r}}^+(z)\Psi(P), \quad (4.37)$$

$$\psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_1(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) = z^{n-n_0}\frac{F_{\underline{p}}(z, n, t_{\underline{r}})}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}), \quad (4.38)$$

$$\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) = z^{n-n_0}\frac{H_{\underline{p}}(z, n, t_{\underline{r}})}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}), \quad (4.39)$$

$$\begin{aligned} & \psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) + \psi_1(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) \\ &= 2z^{n-n_0}\frac{G_{\underline{p}}(z, n, t_{\underline{r}})}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}), \end{aligned} \quad (4.40)$$

$$\begin{aligned} & \psi_1(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) - \psi_1(P^*, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})\psi_2(P, n, n_0, t_{\underline{r}}, t_{0,\underline{r}}) \\ &= -c_{0,+}z^{n-n_0-p-}\frac{y}{F_{\underline{p}}(z, n_0, t_{0,\underline{r}})}\Gamma(n, n_0, t_{\underline{r}}), \end{aligned} \quad (4.41)$$

where

$$\Gamma(n, n_0, t_{\underline{r}}) = \begin{cases} \prod_{n'=n_0+1}^n \gamma(n', t_{\underline{r}}), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n+1}^{n_0} \gamma(n', t_{\underline{r}})^{-1}, & n \leq n_0 - 1. \end{cases} \quad (4.42)$$

In addition, as long as the zeros $\mu_j(n_0, s)$ of $(\cdot)^{p-}F_{\underline{p}}(\cdot, n_0, s)$ are all simple and distinct from zero for $s \in \mathcal{I}_{\mu}$, $\mathcal{I}_{\mu} \subseteq \mathbb{R}$ an open interval, $\Psi(\cdot, n, n_0, t_{\underline{r}}, t_{0,\underline{r}})$ is meromorphic on $\mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}$ for $(n, t_{\underline{r}}, t_{0,\underline{r}}) \in \mathbb{Z} \times \mathcal{I}_{\mu}^2$.

Next we consider the $t_{\underline{r}}$ -dependence of $F_{\underline{p}}$, $G_{\underline{p}}$, and $H_{\underline{p}}$.

Lemma 4.3 ([40]). *Assume Hypothesis 4.1 and suppose that (4.7), (4.8) hold. In addition, let $(z, n, t_{\underline{r}}) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{R}$. Then,*

$$F_{\underline{p}, t_{\underline{r}}} = -2iG_{\underline{p}}\tilde{F}_{\underline{r}} + i(\tilde{G}_{\underline{r}} + \tilde{K}_{\underline{r}})F_{\underline{p}}, \quad (4.43)$$

$$G_{\underline{p}, t_{\underline{r}}} = iF_{\underline{p}}\tilde{H}_{\underline{r}} - iH_{\underline{p}}\tilde{F}_{\underline{r}}, \quad (4.44)$$

$$H_{\underline{p}, t_{\underline{r}}} = 2iG_{\underline{p}}\tilde{H}_{\underline{r}} - i(\tilde{G}_{\underline{r}} + \tilde{K}_{\underline{r}})H_{\underline{p}}. \quad (4.45)$$

In particular, (4.43)–(4.45) are equivalent to

$$V_{\underline{p}, t_{\underline{r}}} = [\tilde{V}_{\underline{r}}, V_{\underline{p}}]. \quad (4.46)$$

Next we turn to the Dubrovin equations for the time variation of the zeros μ_j of $(\cdot)^{p-1}F_{\underline{p}}$ and ν_j of $(\cdot)^{p-1}H_{\underline{p}}$ governed by the $\tilde{\text{AL}}_{\underline{r}}$ flow.

Lemma 4.4 ([40]). *Assume Hypothesis 4.1 and suppose that (4.7), (4.8) hold on $\mathbb{Z} \times \mathcal{I}_{\mu}$ with $\mathcal{I}_{\mu} \subseteq \mathbb{R}$ an open interval. In addition, assume that the zeros μ_j , $j = 1, \dots, p$, of $(\cdot)^{p-1}F_{\underline{p}}(\cdot)$ remain distinct and nonzero on $\mathbb{Z} \times \mathcal{I}_{\mu}$. Then $\{\hat{\mu}_j\}_{j=1, \dots, p}$, defined in (4.20), satisfies the following first-order system of differential equations on $\mathbb{Z} \times \mathcal{I}_{\mu}$,*

$$\mu_{j, t_{\underline{r}}} = -i\tilde{F}_{\underline{r}}(\mu_j)y(\hat{\mu}_j)(\alpha^+)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^p (\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, p, \quad (4.47)$$

with

$$\hat{\mu}_j(n, \cdot) \in C^\infty(\mathcal{I}_{\mu}, \mathcal{K}_p), \quad j = 1, \dots, p, \quad n \in \mathbb{Z}. \quad (4.48)$$

For the zeros ν_j , $j = 1, \dots, p$, of $(\cdot)^{p-1}H_{\underline{p}}(\cdot)$, identical statements hold with μ_j and \mathcal{I}_{μ} replaced by ν_j and \mathcal{I}_{ν} , etc. (with $\mathcal{I}_{\nu} \subseteq \mathbb{R}$ an open interval). In particular, $\{\hat{\nu}_j\}_{j=1, \dots, p}$, defined in (4.21), satisfies the first-order system on $\mathbb{Z} \times \mathcal{I}_{\nu}$,

$$\nu_{j, t_{\underline{r}}} = i\tilde{H}_{\underline{r}}(\nu_j)y(\hat{\nu}_j)(\beta\nu_j)^{-1} \prod_{\substack{k=1 \\ k \neq j}}^p (\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, p, \quad (4.49)$$

with

$$\hat{\nu}_j(n, \cdot) \in C^\infty(\mathcal{I}_{\nu}, \mathcal{K}_p), \quad j = 1, \dots, p, \quad n \in \mathbb{Z}. \quad (4.50)$$

When attempting to solve the Dubrovin systems (4.47) and (4.49), they must be augmented with appropriate divisors $\mathcal{D}_{\hat{\mu}(n_0, t_{0, \underline{r}})} \in \text{Sym}^p \mathcal{K}_p$, $t_{0, \underline{r}} \in \mathcal{I}_{\mu}$, and $\mathcal{D}_{\hat{\nu}(n_0, t_{0, \underline{r}})} \in \text{Sym}^p \mathcal{K}_p$, $t_{0, \underline{r}} \in \mathcal{I}_{\nu}$, as initial conditions. The algebro-geometric initial value problem for the AL hierarchy with appropriate initial divisors will be discussed in detail in [41].

Next, we turn to the asymptotic expansions of ϕ and Ψ in a neighborhood of $P_{\infty_{\pm}}$ and $P_{0, \pm}$. Since this is a bit more involved we provide some details.

Lemma 4.5. *Assume Hypothesis 4.1 and suppose that (4.7), (4.8) hold. Moreover, let $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$, $(n, n_0, t_{\underline{r}}, t_{0, \underline{r}}) \in \mathbb{Z}^2 \times \mathbb{R}^2$. Then ϕ has the asymptotic behavior*

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \beta + \beta^- \gamma \zeta + O(\zeta^2), & P \rightarrow P_{\infty_+}, \\ -(\alpha^+)^{-1} \zeta^{-1} + (\alpha^+)^{-2} \alpha^{++} \gamma^+ + O(\zeta), & P \rightarrow P_{\infty_-}, \end{cases} \quad \zeta = 1/z, \quad (4.51)$$

$$\phi(P) \underset{\zeta \rightarrow 0}{=} \begin{cases} \alpha^{-1} - \alpha^{-2} \alpha^- \gamma \zeta + O(\zeta^2), & P \rightarrow P_{0,+}, \\ -\beta^+ \zeta - \beta^{++} \gamma^+ \zeta^2 + O(\zeta^3), & P \rightarrow P_{0,-}, \end{cases} \quad \zeta = z. \quad (4.52)$$

The component ψ_1 of the Baker–Akhiezer vector Ψ has the asymptotic behavior

$$\psi_1(P, n, n_0, t_{\underline{r}}, t_{0, \underline{r}}) \underset{\zeta \rightarrow 0}{=} \exp\left(\pm \frac{i}{2}(t_{\underline{r}} - t_{0, \underline{r}}) \sum_{s=0}^{r_+} \tilde{c}_{r_+ - s, +} \zeta^{-s}\right) (1 + O(\zeta))$$

$$\times \begin{cases} \zeta^{n_0-n}, & P \rightarrow P_{\infty+}, \\ \Gamma(n, n_0, t_{\underline{x}}) \frac{\alpha^+(n, t_{\underline{x}})}{\alpha^+(n_0, t_{0, \underline{x}})} \\ \times \exp\left(i \int_{t_{0, \underline{x}}}^{t_{\underline{x}}} ds (\tilde{g}_{r_+,+}(n_0, s) - \tilde{g}_{r_-, -}(n_0, s))\right), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z, \quad (4.53)$$

$$\psi_1(P, n, n_0, t_{\underline{x}}, t_{0, \underline{x}}) \underset{\zeta \rightarrow 0}{=} \exp\left(\pm \frac{i}{2}(t_{\underline{x}} - t_{0, \underline{x}}) \sum_{s=0}^{r_-} \tilde{c}_{r_-, -s, -} \zeta^{-s}\right) (1 + O(\zeta)) \\ \times \begin{cases} \frac{\alpha(n, t_{\underline{x}})}{\alpha(n_0, t_{0, \underline{x}})}, & P \rightarrow P_{0,+}, \\ \Gamma(n, n_0, t_{\underline{x}}) \zeta^{n-n_0} \\ \times \exp\left(i \int_{t_{0, \underline{x}}}^{t_{\underline{x}}} ds (\tilde{g}_{r_+,+}(n_0, s) - \tilde{g}_{r_-, -}(n_0, s))\right), & P \rightarrow P_{0,-}, \end{cases} \quad \zeta = z. \quad (4.54)$$

Proof. Since by the definition of ϕ in (4.22) the time parameter $t_{\underline{x}}$ can be viewed as an additional but fixed parameter, the asymptotic behavior of ϕ remains the same as in Lemma 3.3. Similarly, also the asymptotic behavior of $\psi_1(P, n, n_0, t_{\underline{x}}, t_{\underline{x}})$ is derived in an identical fashion to that in Lemma 3.3. This proves (4.53) and (4.54) for $t_{0, \underline{x}} = t_{\underline{x}}$, that is,

$$\psi_1(P, n, n_0, t_{\underline{x}}, t_{\underline{x}}) \underset{\zeta \rightarrow 0}{=} \begin{cases} \zeta^{n_0-n} (1 + O(\zeta)), & P \rightarrow P_{\infty+}, \\ \Gamma(n, n_0, t_{\underline{x}}) \frac{\alpha^+(n, t_{\underline{x}})}{\alpha^+(n_0, t_{\underline{x}})} + O(\zeta), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z, \quad (4.55)$$

$$\psi_1(P, n, n_0, t_{\underline{x}}, t_{\underline{x}}) \underset{\zeta \rightarrow 0}{=} \begin{cases} \frac{\alpha(n, t_{\underline{x}})}{\alpha(n_0, t_{\underline{x}})} + O(\zeta), & P \rightarrow P_{0,+}, \\ \Gamma(n, n_0, t_{\underline{x}}) \zeta^{n-n_0} (1 + O(\zeta)), & P \rightarrow P_{0,-}, \end{cases} \quad \zeta = z. \quad (4.56)$$

Remaining to be investigated is

$$\psi_1(P, n_0, n_0, t_{\underline{x}}, t_{0, \underline{x}}) = \exp\left(i \int_{t_{0, \underline{x}}}^{t_{\underline{x}}} dt (\tilde{G}_{\underline{x}}(z, n_0, t) - \tilde{F}_{\underline{x}}(z, n_0, t) \phi(P, n_0, t))\right). \quad (4.57)$$

The asymptotic expansion of the integrand is derived using Theorem B.1. Focusing on the homogeneous coefficients first, one computes as $P \rightarrow P_{\infty\pm}$,

$$\begin{aligned} \hat{G}_{s,+} - \hat{F}_{s,+} \phi &= \hat{G}_{s,+} - \hat{F}_{s,+} \frac{G_{\underline{p}} + (c_{0,+}/2)z^{-p-y}}{F_{\underline{p}}} \\ &= \hat{G}_{s,+} - \hat{F}_{s,+} \left(\frac{2z^{p-} G_{\underline{p}}}{c_{0,+} y} + 1\right) \left(\frac{2z^{p-} F_{\underline{p}}}{c_{0,+} y}\right)^{-1} \\ &\underset{\zeta \rightarrow 0}{=} \pm \frac{1}{2} \zeta^{-s} + \frac{\hat{g}_{0,+} \mp \frac{1}{2}}{\hat{f}_{0,+}} \hat{f}_{s,+} + O(\zeta), \quad P \rightarrow P_{\infty\pm}, \quad \zeta = 1/z. \end{aligned} \quad (4.58)$$

Since

$$\tilde{F}_{\underline{x}} \underset{\zeta \rightarrow 0}{=} \sum_{s=0}^{r_+} \tilde{c}_{r_+, -s, +} \hat{F}_{s,+} + O(\zeta), \quad \tilde{G}_{\underline{x}} \underset{\zeta \rightarrow 0}{=} \sum_{s=0}^{r_+} \tilde{c}_{r_+, -s, +} \hat{G}_{s,+} + O(\zeta), \quad (4.59)$$

one infers from (4.51)

$$\tilde{G}_{\underline{L}} - \tilde{F}_{\underline{L}}\phi \underset{\zeta \rightarrow 0}{=} \frac{1}{2} \sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+} \zeta^{-s} + O(\zeta), \quad P \rightarrow P_{\infty_+}, \quad \zeta = 1/z. \quad (4.60)$$

Insertion of (4.60) into (4.57) then proves (4.53) as $P \rightarrow P_{\infty_+}$.

As $P \rightarrow P_{\infty_-}$, we need one additional term in the asymptotic expansion of $\tilde{F}_{\underline{L}}$, that is, we will use

$$\tilde{F}_{\underline{L}} \underset{\zeta \rightarrow 0}{=} \sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+} \hat{F}_{s,+} + \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-} \hat{f}_{s-1,-} \zeta + O(\zeta^2). \quad (4.61)$$

This then yields

$$\tilde{G}_{\underline{L}} - \tilde{F}_{\underline{L}}\phi \underset{\zeta \rightarrow 0}{=} -\frac{1}{2} \sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+} \zeta^{-s} - (\alpha^+)^{-1} (\tilde{f}_{r_+,+} - \tilde{f}_{r_-1,-}) + O(\zeta). \quad (4.62)$$

Invoking (2.7) and (4.2) one concludes that

$$\tilde{f}_{r_-1,-} - \tilde{f}_{r_+,+} = -i\alpha_{t_{\underline{L}}}^+ + \alpha^+ (\tilde{g}_{r_+,+} - \tilde{g}_{r_-,-}) \quad (4.63)$$

and hence

$$\tilde{G}_{\underline{L}} - \tilde{F}_{\underline{L}}\phi \underset{\zeta \rightarrow 0}{=} -\frac{1}{2} \sum_{s=0}^{r_+} \tilde{c}_{r_+-s,+} \zeta^{-s} - \frac{i\alpha_{t_{\underline{L}}}^+}{\alpha^+} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-} + O(\zeta). \quad (4.64)$$

Insertion of (4.64) into (4.57) then proves (4.53) as $P \rightarrow P_{\infty_-}$.

Using Theorem B.1 again, one obtains in the same manner as $P \rightarrow P_{0,\pm}$,

$$\hat{G}_{s,-} - \hat{F}_{s,-}\phi \underset{\zeta \rightarrow 0}{=} \pm \frac{1}{2} \zeta^{-s} - \hat{g}_{s,-} + \frac{\hat{g}_{0,-} \pm \frac{1}{2} \hat{f}_{s,-}}{\hat{f}_{0,-}} + O(\zeta). \quad (4.65)$$

Since

$$\tilde{F}_{\underline{L}} \underset{\zeta \rightarrow 0}{=} \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-} \hat{F}_{s,-} + \tilde{f}_{r_+-1,+} + O(\zeta), \quad P \rightarrow P_{0,\pm}, \quad \zeta = z, \quad (4.66)$$

$$\tilde{G}_{\underline{L}} \underset{\zeta \rightarrow 0}{=} \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-} \hat{G}_{s,-} + \tilde{g}_{r_+,+} + O(\zeta), \quad P \rightarrow P_{0,\pm}, \quad \zeta = z, \quad (4.67)$$

(4.65)–(4.67) yield

$$\tilde{G}_{\underline{L}} - \tilde{F}_{\underline{L}}\phi \underset{\zeta \rightarrow 0}{=} \pm \frac{1}{2} \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-} \zeta^{-s} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-} - \frac{\hat{g}_{0,-} \pm \frac{1}{2}}{\hat{f}_{0,-}} (\tilde{f}_{r_+-1,+} - \tilde{f}_{r_-,-}) + O(\zeta), \quad (4.68)$$

where we again used (4.52), (2.19), and (4.2). As $P \rightarrow P_{0,-}$, one thus obtains

$$\tilde{G}_{\underline{L}} - \tilde{F}_{\underline{L}}\phi \underset{\zeta \rightarrow 0}{=} -\frac{1}{2} \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-} \zeta^{-s} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-}, \quad P \rightarrow P_{0,-}, \quad \zeta = z. \quad (4.69)$$

Insertion of (4.69) into (4.57) then proves (4.54) as $P \rightarrow P_{0,-}$.

As $P \rightarrow P_{0,+}$, one obtains

$$\tilde{G}_{\underline{L}} - \tilde{F}_{\underline{L}}\phi \underset{\zeta \rightarrow 0}{=} \frac{1}{2} \sum_{s=0}^{r_-} \tilde{c}_{r_--s,-} \zeta^{-s} + \tilde{g}_{r_+,+} - \tilde{g}_{r_-,-} - \frac{1}{\alpha} (\tilde{f}_{r_+-1,+} - \tilde{f}_{r_-,-}) + O(\zeta),$$

$$\stackrel{\equiv}{=} \frac{1}{2} \sum_{s=0}^{r_-} \tilde{c}_{r_-,s,-} \zeta^{-s} - \frac{i\alpha_{t_{\mathbb{Z}}}}{\alpha} + O(\zeta), \quad P \rightarrow P_{0,+}, \quad \zeta = z, \quad (4.70)$$

using $\tilde{f}_{r_-,-} = \tilde{f}_{r_-,-1,-} + \alpha(\tilde{g}_{r_-,-} - \tilde{g}_{r_-,-}^-)$ (cf. (2.11)) and (4.2). Insertion of (4.70) into (4.57) then proves (4.54) as $P \rightarrow P_{0,+}$. \square

Next, we note that Lemmas 3.2 and 3.4 on trace formulas and nonspecial divisors in the stationary context immediately extend to the present time-dependent situation since $t_{\mathbb{Z}} \in \mathbb{R}$ just plays the role of a parameter. We thus omit the details.

Finally, we turn to the principal result of this section, the representation of ϕ , Ψ , α , and β in terms of the Riemann theta function associated with \mathcal{K}_p , assuming $p \in \mathbb{N}$ for the remainder of this section.

In addition to (3.42)–(3.48), let $\omega_{P_{\infty\pm},q}^{(2)}$ and $\omega_{P_{0,\pm},q}^{(2)}$ be the normalized differentials of the second kind with a unique pole at $P_{\infty\pm}$ and $P_{0,\pm}$, respectively, and principal parts

$$\omega_{P_{\infty\pm},q}^{(2)} \stackrel{\equiv}{=}_{\zeta \rightarrow 0} (\zeta^{-2-q} + O(1))d\zeta, \quad P \rightarrow P_{\infty\pm}, \quad \zeta = 1/z, \quad q \in \mathbb{N}_0, \quad (4.71)$$

$$\omega_{P_{0,\pm},q}^{(2)} \stackrel{\equiv}{=}_{\zeta \rightarrow 0} (\zeta^{-2-q} + O(1))d\zeta, \quad P \rightarrow P_{0,\pm}, \quad \zeta = z, \quad q \in \mathbb{N}_0, \quad (4.72)$$

with vanishing a -periods,

$$\int_{a_j} \omega_{P_{\infty\pm},q}^{(2)} = \int_{a_j} \omega_{P_{0,\pm},q}^{(2)} = 0, \quad j = 1, \dots, p. \quad (4.73)$$

Moreover, we define

$$\begin{aligned} \tilde{\Omega}_{\mathbb{Z}}^{(2)} = & \frac{i}{2} \left(\sum_{s=1}^{r_-} s \tilde{c}_{r_-,s,-} (\omega_{P_{0,+},s-1}^{(2)} - \omega_{P_{0,-},s-1}^{(2)}) \right. \\ & \left. + \sum_{s=1}^{r_+} s \tilde{c}_{r_+,s,+} (\omega_{P_{\infty,+},s-1}^{(2)} - \omega_{P_{\infty,-},s-1}^{(2)}) \right), \end{aligned} \quad (4.74)$$

where $\tilde{c}_{\ell,\pm}$ are the summation constants in $\tilde{F}_{\mathbb{Z}}$. The corresponding vector of b -periods of $\tilde{\Omega}_{\mathbb{Z}}^{(2)}/(2\pi i)$ is then denoted by

$$\tilde{U}_{\mathbb{Z}}^{(2)} = (\tilde{U}_{\mathbb{Z},1}^{(2)}, \dots, \tilde{U}_{\mathbb{Z},p}^{(2)}), \quad \tilde{U}_{\mathbb{Z},j}^{(2)} = \frac{1}{2\pi i} \int_{b_j} \tilde{\Omega}_{\mathbb{Z}}^{(2)}, \quad j = 1, \dots, p. \quad (4.75)$$

Finally, we abbreviate

$$\tilde{\Omega}_{\mathbb{Z}}^{\infty,\pm} = \lim_{P \rightarrow P_{\infty\pm}} \left(\int_{Q_0}^P \tilde{\Omega}_{\mathbb{Z}}^{(2)} \pm \frac{i}{2} \sum_{s=0}^{r_+} \tilde{c}_{r_+,s,+} \zeta^{-s} \right), \quad (4.76)$$

$$\tilde{\Omega}_{\mathbb{Z}}^{0,\pm} = \lim_{P \rightarrow P_{0,\pm}} \left(\int_{Q_0}^P \tilde{\Omega}_{\mathbb{Z}}^{(2)} \pm \frac{i}{2} \sum_{s=0}^{r_-} \tilde{c}_{r_-,s,-} \zeta^{-s} \right). \quad (4.77)$$

Theorem 4.6. *Assume Hypothesis 4.1 and suppose that (4.7), (4.8) hold. In addition, let $P \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}$ and $(n, n_0, t_{\mathbb{Z}}, t_{0,\mathbb{Z}}) \in \mathbb{Z}^2 \times \mathbb{R}^2$. Then for each $(n, t_{\mathbb{Z}}) \in \mathbb{Z} \times \mathbb{R}$, $\mathcal{D}_{\hat{\mu}(n, t_{\mathbb{Z}})}$ and $\mathcal{D}_{\hat{\nu}(n, t_{\mathbb{Z}})}$ are nonspecial. Moreover,*

$$\phi(P, n, t_{\mathbb{Z}}) = C(n, t_{\mathbb{Z}}) \frac{\theta(\underline{z}(P, \hat{\nu}(n, t_{\mathbb{Z}})))}{\theta(\underline{z}(P, \hat{\mu}(n, t_{\mathbb{Z}})))} \exp \left(\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)} \right), \quad (4.78)$$

$$\psi_1(P, n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) = C(n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) \frac{\theta(\underline{z}(P, \hat{\underline{\mu}}(n, t_{\underline{x}})))}{\theta(\underline{z}(P, \hat{\underline{\mu}}(n_0, t_{0,\underline{x}})))} \quad (4.79)$$

$$\times \exp \left((n - n_0) \int_{Q_0}^P \omega_{P_{0,-}, P_{\infty+}}^{(3)} - (t_{\underline{x}} - t_{0,\underline{x}}) \int_{Q_0}^P \tilde{\Omega}_{\underline{x}}^{(2)} \right),$$

$$\psi_2(P, n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) = C(n, t_{\underline{x}}) C(n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) \frac{\theta(\underline{z}(P, \hat{\underline{\nu}}(n, t_{\underline{x}})))}{\theta(\underline{z}(P, \hat{\underline{\nu}}(n_0, t_{0,\underline{x}})))} \quad (4.80)$$

$$\times \exp \left(\int_{Q_0}^P \omega_{P_{0,-}, P_{\infty-}}^{(3)} + (n - n_0) \int_{Q_0}^P \omega_{P_{0,-}, P_{\infty+}}^{(3)} - (t_{\underline{x}} - t_{0,\underline{x}}) \int_{Q_0}^P \tilde{\Omega}_{\underline{x}}^{(2)} \right),$$

where

$$C(n, t_{\underline{x}}) = \frac{(-1)^{n-n_0}}{\alpha(n_0, t_{\underline{x}})} \exp \left((n - n_0) (\omega_0^{0,-}(P_{0,-}, P_{\infty-}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty-})) \right) \quad (4.81)$$

$$\times \exp \left(-\omega_0^{0,+}(P_{0,-}, P_{\infty-}) \frac{\theta(\underline{z}(P_{0,+}, \hat{\underline{\mu}}(n_0, t_{\underline{x}})))}{\theta(\underline{z}(P_{0,+}, \hat{\underline{\nu}}(n_0, t_{\underline{x}})))} \right),$$

$$C(n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) = \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n_0, t_{0,\underline{x}})))}{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n, t_{\underline{x}})))} \quad (4.82)$$

$$\times \exp \left((t_{\underline{x}} - t_{0,\underline{x}}) \tilde{\Omega}_{\underline{x}}^{\infty+} - (n - n_0) \omega_0^{\infty+}(P_{0,-}, P_{\infty+}) \right).$$

The Abel map linearizes the auxiliary divisors $\mathcal{D}_{\hat{\underline{\mu}}(n, t_{\underline{x}})}$ and $\mathcal{D}_{\hat{\underline{\nu}}(n, t_{\underline{x}})}$ in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(n, t_{\underline{x}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(n_0, t_{0,\underline{x}})}) + (n - n_0) \underline{A}_{P_{0,-}}(P_{\infty+}) + (t_{\underline{x}} - t_{0,\underline{x}}) \tilde{U}_{\underline{x}}^{(2)}, \quad (4.83)$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}(n, t_{\underline{x}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}(n_0, t_{0,\underline{x}})}) + (n - n_0) \underline{A}_{P_{0,-}}(P_{\infty+}) + (t_{\underline{x}} - t_{0,\underline{x}}) \tilde{U}_{\underline{x}}^{(2)}. \quad (4.84)$$

Finally, α, β are of the form

$$\alpha(n, t_{\underline{x}}) = \alpha(n_0, t_{0,\underline{x}}) \exp \left((n - n_0) (\omega_0^{0,+}(P_{0,-}, P_{\infty+}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty+})) \right) \quad (4.85)$$

$$\times \exp \left((t_{\underline{x}} - t_{0,\underline{x}}) (\tilde{\Omega}_{\underline{x}}^{\infty+} - \tilde{\Omega}_{\underline{x}}^{0,+}) \frac{\theta(\underline{z}(P_{0,+}, \hat{\underline{\mu}}(n, t_{\underline{x}}))) \theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n_0, t_{0,\underline{x}})))}{\theta(\underline{z}(P_{0,+}, \hat{\underline{\mu}}(n_0, t_{0,\underline{x}}))) \theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n, t_{\underline{x}})))} \right),$$

$$\beta(n, t_{\underline{x}}) = \frac{1}{\alpha(n_0, t_{0,\underline{x}})} \exp \left((n - n_0) (\omega_0^{0,+}(P_{0,-}, P_{\infty+}) - \omega_0^{\infty+}(P_{0,-}, P_{\infty+})) \right)$$

$$\times \exp \left(\omega_0^{\infty+}(P_{0,-}, P_{\infty-}) - \omega_0^{0,+}(P_{0,-}, P_{\infty-}) \right) \quad (4.86)$$

$$\times \exp \left(-(t_{\underline{x}} - t_{0,\underline{x}}) (\tilde{\Omega}_{\underline{x}}^{\infty+} - \tilde{\Omega}_{\underline{x}}^{0,+}) \frac{\theta(\underline{z}(P_{0,+}, \hat{\underline{\mu}}(n_0, t_{0,\underline{x}}))) \theta(\underline{z}(P_{\infty+}, \hat{\underline{\nu}}(n, t_{\underline{x}})))}{\theta(\underline{z}(P_{0,+}, \hat{\underline{\nu}}(n_0, t_{0,\underline{x}}))) \theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n, t_{\underline{x}})))} \right),$$

and

$$\begin{aligned} \alpha(n, t_{\underline{x}}) \beta(n, t_{\underline{x}}) &= \exp \left(\omega_0^{\infty+}(P_{0,-}, P_{\infty-}) - \omega_0^{0,+}(P_{0,-}, P_{\infty-}) \right) \\ &\times \frac{\theta(\underline{z}(P_{0,+}, \hat{\underline{\mu}}(n, t_{\underline{x}}))) \theta(\underline{z}(P_{\infty+}, \hat{\underline{\nu}}(n, t_{\underline{x}})))}{\theta(\underline{z}(P_{0,+}, \hat{\underline{\nu}}(n, t_{\underline{x}}))) \theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(n, t_{\underline{x}})))}. \end{aligned} \quad (4.87)$$

Proof. As in Theorem 3.7 one concludes that $\phi(P, n, t_{\underline{x}})$ is of the form (4.78) and that for $t_{0,\underline{x}} = t_{\underline{x}}$, $\psi_1(P, n, n_0, t_{\underline{x}}, t_{\underline{x}})$ is of the form

$$\psi_1(P, n, n_0, t_{\underline{x}}, t_{\underline{x}}) = C(n, n_0, t_{\underline{x}}, t_{\underline{x}}) \frac{\theta(\underline{z}(P, \hat{\underline{\mu}}(n, t_{\underline{x}})))}{\theta(\underline{z}(P, \hat{\underline{\mu}}(n_0, t_{\underline{x}})))} \exp \left((n - n_0) \int_{Q_0}^P \omega_{P_{0,-}, P_{\infty+}}^{(3)} \right). \quad (4.88)$$

To discuss $\psi_1(P, n, n_0, t_{\underline{x}}, t_{0,\underline{x}})$ we recall (4.28), that is,

$$\psi_1(P, n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) = \psi_1(P, n, n_0, t_{\underline{x}}, t_{\underline{x}})\psi_1(P, n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}}), \quad (4.89)$$

and hence remaining to be studied is

$$\psi_1(P, n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}}) = \exp\left(i \int_{t_{0,\underline{x}}}^{t_{\underline{x}}} ds (\tilde{G}_{\underline{x}}(z, n_0, s) - \tilde{F}_{\underline{x}}(z, n_0, s)\phi(P, n_0, s))\right). \quad (4.90)$$

Introducing $\hat{\psi}_1(P)$ on $\mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}\}$ by

$$\begin{aligned} \hat{\psi}_1(P, n_0, t_{\underline{x}}, t_{0,\underline{x}}) &= C(n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}}) \frac{\theta(\underline{z}(P, \hat{\mu}(n_0, t_{\underline{x}})))}{\theta(\underline{z}(P, \hat{\mu}(n_0, t_{0,\underline{x}})))} \\ &\quad \times \exp\left(- (t_{\underline{x}} - t_{0,\underline{x}}) \int_{Q_0}^P \tilde{\Omega}_{\underline{x}}^{(2)}\right), \end{aligned} \quad (4.91)$$

we intend to prove that

$$\begin{aligned} \psi_1(P, n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}}) &= \hat{\psi}_1(P, n_0, t_{\underline{x}}, t_{0,\underline{x}}), \\ P \in \mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}\}, n_0 \in \mathbb{Z}, t_{\underline{x}}, t_{0,\underline{x}} \in \mathbb{R}, \end{aligned} \quad (4.92)$$

for an appropriate choice of the normalization constant $C(n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}})$ in (4.91). We start by noting that a comparison of (4.53), (4.54) and (4.71), (4.72), (4.74), (4.79) shows that ψ_1 and $\hat{\psi}_1$ have the same essential singularities at $P_{\infty\pm}$ and $P_{0,\pm}$. Thus, we turn to the local behavior of ψ_1 and $\hat{\psi}_1$. By (4.91), $\hat{\psi}_1$ has zeros and poles at $\hat{\mu}(n_0, t_{\underline{x}})$ and $\hat{\mu}(n_0, t_{0,\underline{x}})$, respectively. Similarly, by (4.90), ψ_1 can have zeros and poles only at poles of $\phi(P, n_0, s)$, $s \in [t_{0,\underline{x}}, t_{\underline{x}}]$ (resp., $s \in [t_{\underline{x}}, t_{0,\underline{x}}]$). In the following we temporarily restrict $t_{0,\underline{x}}$ and $t_{\underline{x}}$ to a sufficiently small nonempty interval $I \subseteq \mathbb{R}$ and pick $n_0 \in \mathbb{Z}$ such that for all $s \in I$, $\mu_j(n_0, s) \neq \mu_k(n_0, s)$ for all $j \neq k$, $j, k = 1, \dots, p$. One computes

$$\begin{aligned} &i\tilde{G}_{\underline{x}}(z, n_0, s) - i\tilde{F}_{\underline{x}}(z, n_0, s)\phi(P, n_0, s) \\ &= i\tilde{G}_{\underline{x}}(z, n_0, s) - i\tilde{F}_{\underline{x}}(z, n_0, s) \frac{(e_{0,+}/2)z^{-p-y} + G_{\underline{x}}(z, n_0, s)}{F_{\underline{x}}(z, n_0, s)} \\ &= \frac{i\tilde{F}_{\underline{x}}(\mu_j(n_0, s), n_0, s)y(\hat{\mu}_j(n_0, s))}{\alpha^+(n_0, s)(z - \mu_j(n_0, s)) \prod_{\substack{k=1 \\ k \neq j}}^p (\mu_j(n_0, s) - \mu_k(n_0, s))} + O(1) \\ &= \frac{\partial}{\partial s} \ln(\mu_j(n_0, s) - z) + O(1). \end{aligned} \quad (4.93)$$

Restricting P to a sufficiently small neighborhood $\mathcal{U}_j(n_0)$ of $\{\hat{\mu}_j(n_0, s) \in \mathcal{K}_p \mid s \in [t_{0,\underline{x}}, t_{\underline{x}}] \subseteq I\}$ such that $\hat{\mu}_k(n_0, s) \notin \mathcal{U}_j(n_0)$ for all $s \in [t_{0,\underline{x}}, t_{\underline{x}}] \subseteq I$ and all $k \in \{1, \dots, p\} \setminus \{j\}$, (4.91) and (4.93) imply

$$\begin{aligned} &\psi_1(P, n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}}) \\ &= \begin{cases} (\mu_j(n_0, t_{\underline{x}}) - z)O(1) & \text{as } P \rightarrow \hat{\mu}_j(n_0, t_{\underline{x}}) \neq \hat{\mu}_j(n_0, t_{0,\underline{x}}), \\ O(1) & \text{as } P \rightarrow \hat{\mu}_j(n_0, t_{\underline{x}}) = \hat{\mu}_j(n_0, t_{0,\underline{x}}), \\ (\mu_j(n_0, t_{0,\underline{x}}) - z)^{-1}O(1) & \text{as } P \rightarrow \hat{\mu}_j(n_0, t_{0,\underline{x}}) \neq \hat{\mu}_j(n_0, t_{\underline{x}}), \end{cases} \\ &\quad P = (z, y) \in \mathcal{K}_p, \end{aligned} \quad (4.94)$$

with $O(1) \neq 0$. Thus, ψ_1 and $\hat{\psi}_1$ have the same local behavior and identical essential singularities at $P_{\infty_{\pm}}$ and $P_{0,\pm}$. By Lemma A.4, ψ_1 and $\hat{\psi}_1$ coincide up to a multiple constant (which may depend on $n_0, t_{\underline{x}}, t_{0,\underline{x}}$). This proves (4.92) for $t_{0,\underline{x}}, t_{\underline{x}} \in I$ and for n_0 as restricted above. By continuity with respect to divisors this extends to all $n_0 \in \mathbb{Z}$ since by hypothesis $\mathcal{D}_{\hat{\mu}(n,s)}$ remain nonspecial for all $(n,s) \in \mathbb{Z} \times \mathbb{R}$. Moreover, since by (4.90), for fixed P and n_0 , $\psi_1(P, n_0, n_0, \cdot, t_{0,\underline{x}})$ is entire in $t_{\underline{x}}$ (and this argument is symmetric in $t_{\underline{x}}$ and $t_{0,\underline{x}}$), (4.92) holds for all $t_{\underline{x}}, t_{0,\underline{x}} \in \mathbb{R}$ (for an appropriate choice of $C(n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}})$). Together with (4.89), this proves (4.79) for all $(n, t_{\underline{x}}), (n_0, t_{0,\underline{x}}) \in \mathbb{Z} \times \mathbb{R}$. The expression (4.80) for ψ_2 then immediately follows from (4.78) and (4.79).

To determine the constant $C(n, n_0, t_{\underline{x}}, t_{0,\underline{x}})$ one compares the asymptotic expansions of $\psi_1(P, n, n_0, t_{\underline{x}}, t_{0,\underline{x}})$ for $P \rightarrow P_{\infty_+}$ in (4.53) and (4.79)

$$\begin{aligned} C(n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) &= \exp((t_{\underline{x}} - t_{0,\underline{x}})\tilde{\Omega}_{\underline{x}}^{\infty,+} - (n - n_0)\omega_0^{\infty,+}(P_{0,-}, P_{\infty_+})) \\ &\quad \times \frac{\theta(\underline{z}(P_{\infty_+}, \hat{\mu}(n_0, t_{0,\underline{x}})))}{\theta(\underline{z}(P_{\infty_+}, \hat{\mu}(n, t_{\underline{x}})))}. \end{aligned} \quad (4.95)$$

Remaining to be computed are the expressions for α and β . Comparing the asymptotic expansions of $\psi_1(P, n, n_0, t_{\underline{x}}, t_{0,\underline{x}})$ for $P \rightarrow P_{0,+}$ in (4.54) and (4.79) shows

$$\begin{aligned} \frac{\alpha(n, t_{\underline{x}})}{\alpha(n_0, t_{0,\underline{x}})} &= C(n, n_0, t_{\underline{x}}, t_{0,\underline{x}}) \exp((n - n_0)(\omega_0^{0,+}(P_{0,-}, P_{\infty_+}) - (t_{\underline{x}} - t_{0,\underline{x}})\tilde{\Omega}_{\underline{x}}^{0,+})) \\ &\quad \times \frac{\theta(\underline{z}(P_{0,+}, \hat{\mu}(n, t_{\underline{x}})))}{\theta(\underline{z}(P_{0,+}, \hat{\mu}(n_0, t_{0,\underline{x}})))} \end{aligned} \quad (4.96)$$

and inserting $C(n, n_0, t_{\underline{x}}, t_{0,\underline{x}})$ proves (4.85). Equation (4.81) for $C(n, t_{\underline{x}})$ follows as in the stationary case since $t_{\underline{x}}$ can be viewed as an additional but fixed parameter. By the first line of (3.69),

$$\alpha(n, t_{\underline{x}}) = \frac{1}{C(n, t_{\underline{x}})} \exp(-\omega_0^{0,+}(P_{0,-}, P_{\infty_-})) \frac{\theta(\underline{z}(P_{0,+}, \hat{\mu}(n, t_{\underline{x}})))}{\theta(\underline{z}(P_{0,+}, \hat{\mu}(n, t_{\underline{x}})))}. \quad (4.97)$$

Inserting the result (4.85) for $\alpha(n, t_{\underline{x}})$ into (4.97) then yields (using Lemma 3.6)

$$\begin{aligned} C(n, t_{\underline{x}}) &= \frac{1}{\alpha(n_0, t_{0,\underline{x}})} \frac{\theta(\underline{z}(P_{0,+}, \hat{\mu}(n_0, t_{0,\underline{x}})))}{\theta(\underline{z}(P_{\infty_+}, \hat{\mu}(n_0, t_{0,\underline{x}})))} \exp((t_{\underline{x}} - t_{0,\underline{x}})(\tilde{\Omega}_{\underline{x}}^{0,+} - \tilde{\Omega}_{\underline{x}}^{\infty,+})) \\ &\quad \times \exp((n - n_0)(\omega_0^{\infty,+}(P_{0,-}, P_{\infty_+}) - \omega_0^{0,+}(P_{0,-}, P_{\infty_+}) - \omega_0^{0,+}(P_{0,-}, P_{\infty_-})). \end{aligned} \quad (4.98)$$

Also, since the first line of (3.70) holds,

$$\beta(n, t_{\underline{x}}) = C(n, t_{\underline{x}}) \exp(\omega_0^{\infty,+}(P_{0,-}, P_{\infty_-})) \frac{\theta(\underline{z}(P_{\infty_+}, \hat{\mu}(n, t_{\underline{x}})))}{\theta(\underline{z}(P_{\infty_+}, \hat{\mu}(n, t_{\underline{x}})))}, \quad (4.99)$$

an insertion of (4.98) into (4.99), observing Lemma 3.6, yields equation (4.86) for $\beta(n, t_{\underline{x}})$. Finally, multiplying (4.97) and (4.99) proves (4.87).

Single-valuedness of $\psi_1(\cdot, n_0, n_0, t_{\underline{x}}, t_{0,\underline{x}})$ on \mathcal{K}_p implies

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n_0, t_{\underline{x}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n_0, t_{0,\underline{x}})}) + i(t_{\underline{x}} - t_{0,\underline{x}})\tilde{\underline{U}}_{\underline{x}}^{(2)}. \quad (4.100)$$

Inserting (4.100) into (3.61),

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n, t_{\underline{x}})}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(n_0, t_{\underline{x}})}) + \underline{A}_{P_{0,-}}(P_{\infty_+})(n - n_0), \quad (4.101)$$

one obtains the result (4.83). \square

Again we note that the apparent n_0 -dependence of $C(n, t_r)$ in the right-hand side of (4.81) actually drops out to ensure the n_0 -independence of ϕ in (4.78).

The theta function representations (4.85) and (4.86) for α and β , and the one for $\Gamma(\cdot, \cdot, t_r)$ analogous to that in (4.87) also show that $\gamma(n, t_r) \notin \{0, 1\}$ for all $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$. Hence, condition (4.1) is satisfied for the time-dependent algebro-geometric AL solutions discussed in this section, provided the associated divisors $\mathcal{D}_{\hat{\mu}}(n, t_r)$ and $\mathcal{D}_{\hat{\nu}}(n, t_r)$ stay away from $P_{\infty_{\pm}}, P_{0, \pm}$ for all $(n, t_r) \in \mathbb{Z} \times \mathbb{R}$.

The time-dependent algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy with complex-valued initial data, that is, the construction of α and β by starting from a set of initial data (nonspecial divisors) of full measure, will be presented in [41].

APPENDIX A. HYPERELLIPTIC CURVES AND THEIR THETA FUNCTIONS

We provide a very brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [29], [30], and [48], as well as monographs dedicated to integrable systems such as [15, Ch. 2], [38, App. A, B].

Fix $p \in \mathbb{N}$. The hyperelliptic curve \mathcal{K}_p of genus p used in Sections 2 and 3 is defined by

$$\mathcal{K}_p: \mathcal{F}_p(z, y) = y^2 - R_{2p+2}(z) = 0, \quad R_{2p+2}(z) = \prod_{m=0}^{2p+1} (z - E_m), \quad (\text{A.1})$$

$$\{E_m\}_{m=0, \dots, 2p+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, \dots, 2p+1. \quad (\text{A.2})$$

The curve (A.1) is compactified by adding the points P_{∞_+} and P_{∞_-} , $P_{\infty_+} \neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $p+1$ nonintersecting cuts \mathcal{C}_j joining $E_{m(j)}$ and $E_{m'(j)}$ and denotes

$$\mathcal{C} = \bigcup_{j \in \{1, \dots, p+1\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \quad (\text{A.3})$$

Defining the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \quad (\text{A.4})$$

and introducing the holomorphic function

$$R_{2p+2}(\cdot)^{1/2}: \Pi \rightarrow \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2p+1} (z - E_m) \right)^{1/2} \quad (\text{A.5})$$

on Π with an appropriate choice of the square root branch in (A.5), one considers

$$\mathcal{M}_p = \{(z, \sigma R_{2p+2}(z)^{1/2}) \mid z \in \mathbb{C}, \sigma \in \{\pm 1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \quad (\text{A.6})$$

by extending $R_{2p+2}(\cdot)^{1/2}$ to \mathcal{C} . The hyperelliptic curve \mathcal{K}_p is then the set \mathcal{M}_p with its natural complex structure obtained upon gluing the two sheets of \mathcal{M}_p crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_p)$ of \mathcal{K}_p is given by

$$\mathcal{B}(\mathcal{K}_p) = \{(E_m, 0)\}_{m=0, \dots, 2p+1} \quad (\text{A.7})$$

and finite points P on \mathcal{K}_p are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on \mathcal{K}_p satisfying $\mathcal{F}_p(z, y) = y^2 - R_{2p+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_p \setminus (\mathcal{B}(\mathcal{K}_p) \cup \{P_{\infty_+}, P_{\infty_-}\})$ are given by $\zeta_{P_0} = z - z_0$,

near $P_{\infty_{\pm}}$ by $\zeta_{P_{\infty_{\pm}}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_p)$ by $\zeta_{(E_{m_0}, 0)} = (z - E_{m_0})^{1/2}$. The Riemann surface \mathcal{K}_p defined in this manner has topological genus p . Moreover, we introduce the holomorphic sheet exchange map (involution)

$$*: \mathcal{K}_p \rightarrow \mathcal{K}_p, \quad P = (z, y) \mapsto P^* = (z, -y), \quad P_{\infty_{\pm}} \mapsto P_{\infty_{\mp}}^* = P_{\infty_{\mp}}. \quad (\text{A.8})$$

One verifies that dz/y is a holomorphic differential on \mathcal{K}_p with zeros of order $p - 1$ at $P_{\infty_{\pm}}$ and hence

$$\eta_j = \frac{z^{j-1} dz}{y}, \quad j = 1, \dots, p, \quad (\text{A.9})$$

form a basis for the space of holomorphic differentials on \mathcal{K}_p . Introducing the invertible matrix C in \mathbb{C}^p ,

$$C = (C_{j,k})_{j,k=1,\dots,p}, \quad C_{j,k} = \int_{a_k} \eta_j, \quad (\text{A.10})$$

$$\underline{c}(k) = (c_1(k), \dots, c_p(k)), \quad c_j(k) = C_{j,k}^{-1}, \quad j, k = 1, \dots, p,$$

the corresponding basis of normalized holomorphic differentials ω_j , $j = 1, \dots, p$ on \mathcal{K}_p is given by

$$\omega_j = \sum_{\ell=1}^p c_j(\ell) \eta_{\ell}, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j, k = 1, \dots, p. \quad (\text{A.11})$$

Here $\{a_j, b_j\}_{j=1,\dots,p}$ is a homology basis for \mathcal{K}_p with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \dots, p. \quad (\text{A.12})$$

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,p}$ we also recall the canonical dissection of \mathcal{K}_p along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_p$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_p$ given by

$$\partial\widehat{\mathcal{K}}_p = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_p^{-1} b_p^{-1}. \quad (\text{A.13})$$

Let $\mathcal{M}(\mathcal{K}_p)$ and $\mathcal{M}^1(\mathcal{K}_p)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on \mathcal{K}_p . Holomorphic differentials are also called Abelian differentials of the first kind. Abelian differentials of the second kind, $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_p)$, are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their a -periods vanish, that is, $\int_{a_j} \omega^{(2)} = 0$, $j = 1, \dots, p$. Any meromorphic differential $\omega^{(3)}$ on \mathcal{K}_p not of the first or second kind is said to be of the third kind. A differential of the third kind $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_p)$ is usually normalized by the vanishing of its a -periods, that is, $\int_{a_j} \omega^{(3)} = 0$, $j = 1, \dots, p$. A normal differential of the third kind $\omega_{P_1, P_2}^{(3)}$ associated with two points $P_1, P_2 \in \widehat{\mathcal{K}}_p$, $P_1 \neq P_2$, by definition, has simple poles at P_j with residues $(-1)^{j+1}$, $j = 1, 2$ and vanishing a -periods.

Next, define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,p}$ by

$$\tau_{j,\ell} = \int_{b_{\ell}} \omega_j, \quad j, \ell = 1, \dots, p. \quad (\text{A.14})$$

Then

$$\text{Im}(\tau) > 0 \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j, \ell = 1, \dots, p. \quad (\text{A.15})$$

Associated with τ one introduces the period lattice

$$L_p = \{\underline{z} \in \mathbb{C}^p \mid \underline{z} = \underline{m} + \underline{n}\tau, \underline{m}, \underline{n} \in \mathbb{Z}^p\} \quad (\text{A.16})$$

and the Riemann theta function associated with \mathcal{K}_p and the given homology basis $\{a_j, b_j\}_{j=1, \dots, p}$,

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^p} \exp(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \underline{n}\tau)), \quad \underline{z} \in \mathbb{C}^p, \quad (\text{A.17})$$

where $(\underline{u}, \underline{v}) = \overline{\underline{u}} \underline{v}^\top = \sum_{j=1}^p \overline{u_j} v_j$ denotes the scalar product in \mathbb{C}^p . It has the fundamental properties

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_p) = \theta(\underline{z}), \quad (\text{A.18})$$

$$\theta(\underline{z} + \underline{m} + \underline{n}\tau) = \exp(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \underline{n}\tau))\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^p. \quad (\text{A.19})$$

Next, fix a base point $Q_0 \in \mathcal{K}_p \setminus \{P_{0,\pm}, P_{\infty,\pm}\}$, denote by $J(\mathcal{K}_p) = \mathbb{C}^p/L_p$ the Jacobi variety of \mathcal{K}_p , and define the Abel map \underline{A}_{Q_0} by

$$\underline{A}_{Q_0}: \mathcal{K}_p \rightarrow J(\mathcal{K}_p), \quad \underline{A}_{Q_0}(P) = \left(\int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_p \right) \pmod{L_p}, \quad P \in \mathcal{K}_p. \quad (\text{A.20})$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0}: \text{Div}(\mathcal{K}_p) \rightarrow J(\mathcal{K}_p), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_p} \mathcal{D}(P) \underline{A}_{Q_0}(P), \quad (\text{A.21})$$

where $\text{Div}(\mathcal{K}_p)$ denotes the set of divisors on \mathcal{K}_p . Here $\mathcal{D}: \mathcal{K}_p \rightarrow \mathbb{Z}$ is called a divisor on \mathcal{K}_p if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_p$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_p)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on \mathcal{K}_p we shall employ the following (additive) notation,

$$\begin{aligned} \mathcal{D}_{Q_0} \underline{Q} &= \mathcal{D}_{Q_0} + \mathcal{D}_Q, & \mathcal{D}_Q &= \mathcal{D}_{Q_1} + \dots + \mathcal{D}_{Q_m}, \\ \underline{Q} &= \{Q_1, \dots, Q_m\} \in \text{Sym}^m \mathcal{K}_p, & Q_0 &\in \mathcal{K}_p, \quad m \in \mathbb{N}, \end{aligned} \quad (\text{A.22})$$

where for any $Q \in \mathcal{K}_p$,

$$\mathcal{D}_Q: \mathcal{K}_p \rightarrow \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_p \setminus \{Q\}, \end{cases} \quad (\text{A.23})$$

and $\text{Sym}^n \mathcal{K}_p$ denotes the n th symmetric product of \mathcal{K}_p . In particular, $\text{Sym}^m \mathcal{K}_p$ can be identified with the set of nonnegative divisors $0 \leq \mathcal{D} \in \text{Div}(\mathcal{K}_p)$ of degree m .

For $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω) , respectively. Two divisors $\mathcal{D}, \mathcal{E} \in \text{Div}(\mathcal{K}_p)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of \mathcal{D} is then given by $[\mathcal{D}] = \{\mathcal{E} \in \text{Div}(\mathcal{K}_p) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0, \quad \deg((\omega)) = 2(p-1), \quad f \in \mathcal{M}(\mathcal{K}_p) \setminus \{0\}, \quad \omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}, \quad (\text{A.24})$$

where the degree $\deg(\mathcal{D})$ of \mathcal{D} is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_p} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_p) \mid f = 0 \text{ or } (f) \geq \mathcal{D}\}, \quad r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \quad (\text{A.25})$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_p) \mid \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \quad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}), \quad (\text{A.26})$$

with $i(\mathcal{D})$ the index of speciality of \mathcal{D} , one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of \mathcal{D} . Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \text{Div}(\mathcal{K}_p)$, $\omega \in \mathcal{M}^1(\mathcal{K}_p) \setminus \{0\}$. Then,*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad p \in \mathbb{N}_0. \quad (\text{A.27})$$

The Riemann–Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - p + 1, \quad p \in \mathbb{N}_0. \quad (\text{A.28})$$

By Abel’s theorem, $\mathcal{D} \in \text{Div}(\mathcal{K}_p)$, $p \in \mathbb{N}$ is principal if and only if

$$\deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \quad (\text{A.29})$$

Finally, assume $p \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \text{Div}(\mathcal{K}_p) \rightarrow J(\mathcal{K}_p)$ is surjective (Jacobi’s inversion theorem).

Theorem A.2. *Let $\mathcal{D}_{\underline{Q}} \in \text{Sym}^p \mathcal{K}_p$, $\underline{Q} = \{Q_1, \dots, Q_p\}$. Then,*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s \quad (\text{A.30})$$

if and only if $\{Q_1, \dots, Q_p\}$ contains s pairings of the type $\{P, P^*\}$. (This includes, of course, branch points for which $P = P^*$.) One has $s \leq p/2$.

Denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,p}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^p \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^P \omega_j, \quad j = 1, \dots, p. \quad (\text{A.31})$$

Theorem A.3. *Let $\underline{Q} = \{Q_1, \dots, Q_p\} \in \text{Sym}^p \mathcal{K}_p$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_p\}. \quad (\text{A.32})$$

Lemma A.4. [19, Lemmas 5.4 and 6.1] *Let $(n, t_{\underline{x}}), (n_0, t_{0,\underline{x}}) \in \Omega$ for some $\Omega \subseteq \mathbb{Z} \times \mathbb{R}$. Assume $\psi(\cdot, n, t_{\underline{x}})$ to be meromorphic on $\mathcal{K}_p \setminus \{P_{\infty+}, P_{\infty-}, P_{0,+}, P_{0,-}\}$ with possible essential singularities at $P_{\infty\pm}, P_{0,\pm}$ such that $\tilde{\psi}(\cdot, n, t_{\underline{x}})$ defined by*

$$\tilde{\psi}(P, n, t_{\underline{x}}) = \psi(P, n, t_{\underline{x}}) \exp\left((t_{\underline{x}} - t_{0,\underline{x}}) \int_{Q_0}^P \tilde{\Omega}_{\underline{x}}^{(2)}\right) \quad (\text{A.33})$$

is meromorphic on \mathcal{K}_p and its divisor satisfies

$$(\tilde{\psi}(\cdot, n, t_{\underline{x}})) \geq -\mathcal{D}_{\underline{\mu}(n_0, t_{0,\underline{x}})} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty+}}) \quad (\text{A.34})$$

for some positive divisor $\mathcal{D}_{\underline{\mu}(n_0, t_{0,\underline{x}})}$ of degree p . Here $\tilde{\Omega}_{\underline{x}}^{(2)}$ is defined in (4.74) and the path of integration is chosen identical to that in the Abel maps³ (A.20) and (A.21). Define a divisor $\mathcal{D}_0(n, t_{\underline{x}})$ by

$$(\tilde{\psi}(\cdot, n, t_{\underline{x}})) = \mathcal{D}_0(n, t_{\underline{x}}) - \mathcal{D}_{\underline{\mu}(n_0, t_{0,\underline{x}})} + (n - n_0)(\mathcal{D}_{P_{0,-}} - \mathcal{D}_{P_{\infty+}}). \quad (\text{A.35})$$

³This is to avoid multi-valued expressions and hence the use of the multiplicative Riemann–Roch theorem in the proof of Lemma A.4.

Then

$$\mathcal{D}_0(n, t_r) \in \text{Sym}^p \mathcal{K}_p, \quad \mathcal{D}_0(n, t_r) > 0, \quad \deg(\mathcal{D}_0(n, t_r)) = p. \quad (\text{A.36})$$

Moreover, if $\mathcal{D}_0(n, t_r)$ is nonspecial for all $(n, t_r) \in \Omega$, that is, if

$$i(\mathcal{D}_0(n, t_r)) = 0, \quad (n, t_r) \in \Omega, \quad (\text{A.37})$$

then $\psi(\cdot, n, t_r)$ is unique up to a constant multiple (which may depend on the parameters $(n, t_r), (n_0, t_{0,r}) \in \Omega$).

APPENDIX B. ASYMPTOTIC SPECTRAL PARAMETER EXPANSIONS

In this appendix we consider asymptotic spectral parameter expansions of F_p/y , G_p/y , and H_p/y , the resulting recursion relations for the homogeneous coefficients \hat{f}_ℓ , \hat{g}_ℓ , and \hat{h}_ℓ , their connection with the nonhomogeneous coefficients f_ℓ , g_ℓ , and h_ℓ , and the connection between $c_{\ell, \pm}$ and $c_\ell(\underline{E}^{\pm 1})$ (cf. (B.3)). For detailed proofs of the material in this section we refer to [39], [40]. We will employ the notation

$$\underline{E}^{\pm 1} = (E_0^{\pm 1}, \dots, E_{2p+1}^{\pm 1}). \quad (\text{B.1})$$

We start with the following elementary results (consequences of the binomial expansion) assuming $\eta \in \mathbb{C}$ such that $|\eta| < \min\{|E_0|^{-1}, \dots, |E_{2p+1}|^{-1}\}$:

$$\left(\prod_{m=0}^{2p+1} (1 - E_m \eta) \right)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E}) \eta^k, \quad (\text{B.2})$$

where

$$c_0(\underline{E}) = 1, \\ c_k(\underline{E}) = \sum_{\substack{j_0, \dots, j_{2p+1}=0 \\ j_0 + \dots + j_{2p+1} = k}} \frac{(2j_0)! \cdots (2j_{2p+1})! E_0^{j_0} \cdots E_{2p+1}^{j_{2p+1}}}{2^{2k} (j_0!)^2 \cdots (j_{2p+1}!)^2 (2j_0 - 1) \cdots (2j_{2p+1} - 1)}, \quad k \in \mathbb{N}. \quad (\text{B.3})$$

The first few coefficients explicitly are given by

$$c_0(\underline{E}) = 1, \quad c_1(\underline{E}) = -\frac{1}{2} \sum_{m=0}^{2p+1} E_m, \quad c_2(\underline{E}) = \frac{1}{4} \sum_{\substack{m_1, m_2=0 \\ m_1 < m_2}}^{2p+1} E_{m_1} E_{m_2} - \frac{1}{8} \sum_{m=0}^{2p+1} E_m^2, \quad \text{etc.} \quad (\text{B.4})$$

Next we turn to asymptotic expansions. We recall the convention $y(P) = \mp \zeta^{-p-1} + O(\zeta^{-p})$ near $P_{\infty, \pm}$ (where $\zeta = 1/z$) and $y(P) = \pm(c_{0,-}/c_{0,+}) + O(\zeta)$ near $P_{0, \pm}$ (where $\zeta = z$).

Theorem B.1 ([40]). *Assume (3.1), $s\text{-AL}_{\underline{p}}(\alpha, \beta) = 0$, and suppose $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty,+}, P_{\infty,-}\}$. Then $z^{p-} F_p/y$, $z^{p-} G_p/y$, and $z^{p-} H_p/y$ have the following convergent expansions as $P \rightarrow P_{\infty, \pm}$, respectively, $P \rightarrow P_{0, \pm}$,*

$$\frac{z^{p-} F_p(z)}{c_{0,+} y} = \begin{cases} \mp \sum_{\ell=0}^{\infty} \hat{f}_{\ell,+} \zeta^{\ell+1}, & P \rightarrow P_{\infty, \pm}, & \zeta = 1/z, \\ \pm \sum_{\ell=0}^{\infty} \hat{f}_{\ell,-} \zeta^{\ell}, & P \rightarrow P_{0, \pm}, & \zeta = z, \end{cases} \quad (\text{B.5})$$

$$\frac{z^{p-} G_p(z)}{c_{0,+} y} = \begin{cases} \mp \sum_{\ell=0}^{\infty} \hat{g}_{\ell,+} \zeta^{\ell}, & P \rightarrow P_{\infty, \pm}, & \zeta = 1/z, \\ \pm \sum_{\ell=0}^{\infty} \hat{g}_{\ell,-} \zeta^{\ell}, & P \rightarrow P_{0, \pm}, & \zeta = z, \end{cases} \quad (\text{B.6})$$

$$\frac{z^{p_-} H_{\underline{p}}(z)}{c_{0,+} y} = \begin{cases} \mp \sum_{\ell=0}^{\infty} \hat{h}_{\ell,+} \zeta^{\ell}, & P \rightarrow P_{\infty\pm}, & \zeta = 1/z, \\ \pm \sum_{\ell=0}^{\infty} \hat{h}_{\ell,-} \zeta^{\ell+1}, & P \rightarrow P_{0,\pm}, & \zeta = z, \end{cases} \quad (\text{B.7})$$

where $\zeta = 1/z$ (resp., $\zeta = z$) is the local coordinate near $P_{\infty\pm}$ (resp., $P_{0,\pm}$) and $\hat{f}_{\ell,\pm}$, $\hat{g}_{\ell,\pm}$, and $\hat{h}_{\ell,\pm}$ are the homogeneous versions of the coefficients $f_{\ell,\pm}$, $g_{\ell,\pm}$, and $h_{\ell,\pm}$ introduced in (2.16)–(2.18).

Moreover, the E_m -dependent summation constants $c_{\ell,\pm}$, $\ell = 0, \dots, p_{\pm}$, in $F_{\underline{p}}$, $G_{\underline{p}}$, and $H_{\underline{p}}$ are given by

$$c_{\ell,\pm} = c_{0,\pm} c_{\ell}(E^{\pm 1}), \quad \ell = 0, \dots, p_{\pm}. \quad (\text{B.8})$$

Acknowledgments. We thank P. D. Miller for helpful discussions in connection with the Ablowitz–Ladik hierarchy. F.G., J.M., and G.T. gratefully acknowledge the extraordinary hospitality of the Department of Mathematical Sciences of the Norwegian University of Science and Technology, Trondheim, during extended stays in the summers of 2004–2006, where parts of this paper were written.

REFERENCES

- [1] M. J. Ablowitz. Nonlinear evolution equations – continuous and discrete. *SIAM Rev.* 19:663–684, 1977.
- [2] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, Cambridge, 1991.
- [3] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.* 16:598–603, 1975.
- [4] M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.* 17:1011–1018, 1976.
- [5] M. J. Ablowitz and J. F. Ladik. A nonlinear difference scheme and inverse scattering. *Studies Appl. Math.* 55:213–229, 1976.
- [6] M. J. Ablowitz and J. F. Ladik. On the solution of a class of nonlinear partial difference equations. *Studies Appl. Math.* 57:1–12, 1977.
- [7] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. *Discrete and Continuous Nonlinear Schrödinger Systems*. London Mathematical Society Lecture Note Series, Vol. 302, Cambridge University Press, Cambridge, 2004.
- [8] S. Ahmad and A. Roy Chowdhury. On the quasi-periodic solutions to the discrete non-linear Schrödinger equation. *J. Phys. A* 20:293–303, 1987.
- [9] S. Ahmad and A. Roy Chowdhury. The quasi-periodic solutions to the discrete non-linear Schrödinger equation. *J. Math. Phys.* 28:134–1137, 1987.
- [10] N. I. Akhiezer. *The Classical Moment Problem*. Oliver & Boyd., Edinburgh, 1965.
- [11] G. Baxter. Polynomials defined by a difference system. *Bull. Amer. Math. Soc.* 66:187–190, 1960.
- [12] G. Baxter. Polynomials defined by a difference system. *J. Math. Anal. Appl.* 2:223–263, 1961.
- [13] G. Baxter. A convergence equivalence related to polynomials orthogonal on the unit circle. *Trans. Amer. Math. Soc.* 99:471–487, 1961.
- [14] G. Baxter. A norm inequality for a “finite-section” Wiener-Hopf equation. *Illinois J. Math.* 7:97–103, 1963.
- [15] E. D. Belokolos, A. I. Bobenko, V. Z. Enol’skii, A. R. Its, and V. B. Matveev. *Algebro-Geometric Approach to Nonlinear Integrable Equations*. Springer, Berlin, 1994.
- [16] M. Bertola and M. Gekhtman. Biorthogonal Laurent polynomials, Töplitz determinants, minimal Toda orbits and isomonodromic tau functions. *Constr. Approx.*, 26:383–430, 2007.
- [17] N. N. Bogolyubov, A. K. Prikarpatskii, and V. G. Samoilenko. Discrete periodic problem for the modified nonlinear Korteweg–de Vries equation. *Sov. Phys. Dokl.* 26:490–492, 1981.
- [18] N. N. Bogolyubov and A. K. Prikarpatskii. The inverse periodic problem for a discrete approximation of a nonlinear Schrödinger equation. *Sov. Phys. Dokl.* 27:113–116, 1982.
- [19] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl. Algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van Moerbeke hierarchy. *Mem. Amer. Math. Soc.* no 641, 135:1–79, 1998.

- [20] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal Rational Functions*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 1999.
- [21] M. J. Cantero, L. Moral, and L. Velázquez. Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Lin. Algebra Appl.* 362:29–56, 2003.
- [22] S.-C. Chiu and J. F. Ladik. Generating exactly soluble nonlinear discrete evolution equations by a generalized Wronskian technique. *J. Math. Phys.* 18:690–700, 1977.
- [23] K. W. Chow, R. Conte, and N. Xu. Analytic doubly periodic wave patterns for the integrable discrete nonlinear Schrödinger (Ablovitz–Ladik) model. *Physics Letters A* 349:422–429, 2006.
- [24] A. K. Common. A solution of the initial value problem for half-infinite integrable systems. *Inverse Problems* 8:393–408, 1992.
- [25] P. Deift. Riemann–Hilbert methods in the theory of orthogonal polynomials. *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday. Ergodic Schrödinger Operators, Singular Spectrum, Orthogonal Polynomials, and Inverse Spectral Theory*, F. Gesztesy, P. Deift, C. Galvez, P. Perry, and W. Schlag (eds.), Proceedings of Symposia in Pure Mathematics, Vol. 76/2, Amer. Math. Soc., Providence, RI, 2007, pp. 715–740.
- [26] H.-Y. Ding, Y.-P. Sun, and X.-X. Xu. A hierarchy of nonlinear lattice soliton equations, its integrable coupling systems and infinitely many conservation laws. *Chaos, Solitons and Fractals* 30:227–234, 2006.
- [27] A. Doliwa and P. M. Santini. Integrable dynamics of a discrete curve and the Ablowitz–Ladik hierarchy. *J. Math. Phys.* 36:1259–1273, 1995.
- [28] N. M. Ercolani and G. I. Lozano. A bi-Hamiltonian structure for the integrable, discrete non-linear Schrödinger system. *Physica D* 218:105–121, 2006.
- [29] H. M. Farkas and I. Kra. *Riemann Surfaces*. Springer, New York, 2nd ed., 1992.
- [30] J. Fay. *Theta Functions on Riemann Surfaces*. Lecture Notes in Mathematics, Vol. 352, Springer, New York, 1973.
- [31] X. Geng. Darboux transformation of the discrete Ablowitz–Ladik eigenvalue problem. *Acta Math. Sci.* 9:21–26, 1989.
- [32] X. Geng, H. H. Dai, and C. Cao. Algebro-geometric constructions of the discrete Ablowitz–Ladik flows and applications. *J. Math. Phys.* 44:4573–4588, 2003.
- [33] J. S. Geronimo, F. Gesztesy, and H. Holden. Algebro-geometric solutions of the Baxter–Szegő difference equation. *Commun. Math. Phys.* 258:149–177, 2005.
- [34] J. S. Geronimo and R. Johnson. Rotation number associated with difference equations satisfied by polynomials orthogonal on the unit circle. *J. Differential Equations* 132:140–178, 1996.
- [35] J. S. Geronimo and R. Johnson. An inverse problem associated with polynomials orthogonal on the unit circle. *Commun. Math. Phys.* 193:125–150, 1998.
- [36] J. S. Geronimo and A. Teplyaev. A difference equation arising from the trigonometric moment problem having random reflection coefficients—an operator theoretic approach. *J. Funct. Anal.* 123:12–45, 1994.
- [37] Ya. L. Geronimus. *Orthogonal Polynomials*. Consultants Bureau, New York, 1961.
- [38] F. Gesztesy and H. Holden. *Soliton Equations and Their Algebro-Geometric Solutions. Volume I: (1+1)-Dimensional Continuous Models*. Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge University Press, Cambridge, 2003.
- [39] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. *Soliton Equations and Their Algebro-Geometric Solutions. Volume II: (1+1)-Dimensional Discrete Models*. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, in preparation.
- [40] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. The Ablowitz–Ladik hierarchy revisited. To appear in *Operator Theory, Advances and Applications*, Birkhäuser, Basel; arXiv:nlin/0702058.
- [41] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. The algebro-geometric initial value problem for the Ablowitz–Ladik hierarchy, arXiv:nlin/07063370.
- [42] F. Gesztesy, H. Holden, J. Michor, and G. Teschl. Local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy, in preparation.
- [43] L. Golinskii and P. Nevai. Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle. *Commun. Math. Phys.* 223:223–259, 2001.

- [44] R. Killip and I. Nenciu. CMV: The unitary analogue of Jacobi matrices. *Commun. Pure Appl. Math.* 59:1–41, 2006.
- [45] L.-C. Li. Some remarks on CMV matrices and dressing orbits. *Int. Math. Res. Notices* 40:2437–2446, 2005.
- [46] P. D. Miller. Macroscopic behavior in the Ablowitz–Ladik equations. Nonlinear Evolution Equations & Dynamical Systems, V. G. Makhankov, A. R. Bishop, and D. D. Holm (eds.), World Scientific, Singapore, 1995, pp. 158–167.
- [47] P. D. Miller, N. M. Ercolani, I. M. Krichever, and C. D. Levermore. Finite genus solutions to the Ablowitz–Ladik equations. *Comm. Pure Appl. Math.* 48:1369–1440, 1995.
- [48] D. Mumford. *Tata Lectures on Theta II*. Birkhäuser, Boston, 1984.
- [49] I. Nenciu. Lax pairs for the Ablowitz–Ladik system via orthogonal polynomials on the unit circle. *Int. Math. Res. Notices* 11:647–686, 2005.
- [50] I. Nenciu. CMV matrices in random matrix theory and integrable systems: a survey. *J. Phys. A* 39:8811–8822, 2006.
- [51] R. J. Schilling. A systematic approach to the soliton equations of a discrete eigenvalue problem. *J. Math. Phys.* 30:1487–1501, 1989.
- [52] B. Simon. *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Part 2: Spectral Theory*, AMS Colloquium Publication Series, Vol. 54, Amer. Math. Soc., Providence, R.I., 2005.
- [53] B. Simon. OPUC on one foot. *Bull. Amer. Math. Soc.* 42:431–460, 2005.
- [54] B. Simon. CMV matrices: Five years after. *J. Comp. Appl. Math.* 208:120–154, 2007.
- [55] G. Szegő. Beiträge zur Theorie der Toeplitzschen Formen I. *Math. Z.* 6:167–202, 1920.
- [56] G. Szegő. Beiträge zur Theorie der Toeplitzschen Formen II. *Math. Z.* 9:167–190, 1921.
- [57] G. Szegő. *Orthogonal Polynomials*. Amer. Math. Soc. Colloq. Publ. Vol. 23, Amer. Math. Soc., Providence, R.I., 1978.
- [58] K. M. Tamizhmani and Wen-Xiu Ma. Master symmetries from Lax operators for certain lattice soliton hierarchies. *J. Phys. Soc. Japan* 69:351–361, 2000.
- [59] G. Teschl. *Jacobi Operators and Completely Integrable Nonlinear Lattices*. Math. Surveys Monographs, Vol. 72, Amer. Math. Soc., Providence, R.I., 2000.
- [60] K. L. Vaninsky. An additional Gibbs’ state for the cubic Schrödinger equation on the circle. *Comm. Pure Appl. Math.* 54:537–582, 2001.
- [61] V. E. Vekslerchik. Finite genus solutions for the Ablowitz–Ladik hierarchy. *J. Phys. A* 32:4983–4994, 1999.
- [62] V. E. Vekslerchik. Functional representation of the Ablowitz–Ladik hierarchy. II. *J. Nonlin. Math. Phys.* 9:157–180, 2002.
- [63] V. E. Vekslerchik. Implementation of the Bäcklund transformations for the Ablowitz–Ladik hierarchy. *J. Phys. A* 39:6933–6953, 2006.
- [64] V. E. Vekslerchik and V. V. Konotop. Discrete nonlinear Schrödinger equation under non-vanishing boundary conditions. *Inverse Problems* 8:889–909, 1992.
- [65] Y. Zeng and S. Rauch-Wojciechowski. Restricted flows of the Ablowitz–Ladik hierarchy and their continuous limits. *J. Phys. A* 28:113–134, 1995.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA

E-mail address: gesztesyf@missouri.edu

URL: <http://www.math.missouri.edu/personnel/faculty/gesztesyf.html>

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO–7491 TRONDHEIM, NORWAY

E-mail address: holden@math.ntnu.no

URL: <http://www.math.ntnu.no/~holden/>

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA, AND INTERNATIONAL ERWIN SCHRÖDINGER INSTITUTE FOR MATHEMATICAL PHYSICS, BOLTZMANNGASSE 9, 1090 WIEN, AUSTRIA

E-mail address: Johanna.Michor@esi.ac.at

URL: <http://www.mat.univie.ac.at/~jmichor/>

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA, AND INTERNATIONAL ERWIN SCHRÖDINGER INSTITUTE FOR MATHEMATICAL PHYSICS, BOLTZMANNGASSE 9, 1090 WIEN, AUSTRIA

E-mail address: Gerald.Teschl@univie.ac.at

URL: <http://www.mat.univie.ac.at/~gerald/>