DECOUPLING OF DEFICIENCY INDICES AND APPLICATIONS TO SCHRÖDINGER-TYPE OPERATORS WITH POSSIBLY STRONGLY SINGULAR POTENTIALS

FRITZ GESZTESY, MARIUS MITREA, IRINA NENCIU, AND GERALD TESCHL

Abstract. We investigate closed, symmetric $L^2(\mathbb{R}^n)$-realizations H of Schrödinger-type operators $(-\Delta+V)\upharpoonright_{C_0^\infty(\mathbb{R}^n\setminus\Sigma)}$ whose potential coefficient V has a countable number of well-separated singularities on compact sets Σ_j, $j\in J$, of n-dimensional Lebesgue measure zero, with $J\subseteq\mathbb{N}$ an index set and $\Sigma=\bigcup_{j\in J}\Sigma_j$. We show that the defect, $\operatorname{def}(H)$, of H can be computed in terms of the individual defects, $\operatorname{def}(H_j)$, of closed, symmetric $L^2(\mathbb{R}^n)$-realizations of $(-\Delta+V_j)\upharpoonright_{C_0^\infty(\mathbb{R}^n\setminus\Sigma_j)}$ with potential coefficient V_j localized around the singularity Σ_j, $j\in J$, where $V=\sum_{j\in J}V_j$. In particular, we prove

$$\operatorname{def}(H)=\sum_{j\in J}\operatorname{def}(H_j),$$

including the possibility that one, and hence both sides equal ∞. We first develop an abstract approach to the question of decoupling of deficiency indices and then apply it to the concrete case of Schrödinger-type operators in $L^2(\mathbb{R}^n)$.

Moreover, we also show how operator (and form) bounds for V relative to $H_0=-\Delta\upharpoonright_{H^2(\mathbb{R}^n)}$ can be estimated in terms of the operator (and form) bounds of V_j, $j\in J$, relative to H_0. Again, we first prove an abstract result and then show its applicability to Schrödinger-type operators in $L^2(\mathbb{R}^n)$.

Extensions to second-order (locally uniformly) elliptic differential operators on $\mathbb{R}^n$ with a possibly strongly singular potential coefficient are treated as well.

Contents

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1. Introduction

The main theme of this paper is centered around the question of (essential) self-adjointness of Schrödinger-type operators H in $L^2(\mathbb{R}^n)$. Specifically, we are interested in the case where the potential V has a (potentially infinite) number of well-separated singularities, and we want to know how each of these singularities contributes to the deficiency index of the full operator H. We were inspired to work on this question, in particular, by the paper of Felli, Marchini, and Terracini [20], who consider the question of essential self-adjointness of a specific Schrödinger operator with so-called multipolar inverse-square potentials. Unlike in their work, we consider a more general set-up, and proceed by using more abstract localization techniques. More precisely, we consider general compact subsets $\Sigma_j \subset \mathbb{R}^n$ of n-dimensional Lebesgue measure zero as the singularities of potentials V_j, $j \in J$ ($J \subseteq \mathbb{N}$ a suitable index set), respectively, and prove the localization of deficiency indices solely under the additional requirement that the singular sets Σ_j be uniformly separated by some distance $\varepsilon > 0$. In particular, one can then directly conclude that if every Schrödinger operator H_j in $L^2(\mathbb{R}^n)$ with localized potential V_j around the singularity Σ_j is essentially self-adjoint, then so is the full Schrödinger operator H associated with the singularity set $\bigcup_{j\in J} \Sigma_j$.

While we will discuss in Section 5 several applications of our results, one easily presentable consequence is the following fact:

Theorem 1.1. *Let $J \subseteq \mathbb{N}$ be an index set, and $\{x_j\}_{j\in J} \subset \mathbb{R}^n$ be a set of points such that*
$$\inf_{\substack{j,j'\in J\\ j\neq j'}} |x_j - x_{j'}| > 0. \tag{1.1}$$
Fix $\delta > 0$, and consider the multipolar inverse-square potential function:
$$V(x) = V_0(x) + \sum_{j\in J} V_j(|x-x_j|)\chi_{B_n(x_j;\delta)}(x) \ \text{ for all } \ x \in \mathbb{R}^n\backslash\{x_j\}_{j\in J}, \tag{1.2}$$
where $B_n(x_j;\delta)$ denotes the (open) ball in $\mathbb{R}^n$ centered at x_j of radius δ, $V_0 \in L^\infty(\mathbb{R}^n)$, and V_j satisfies
$$V_j(r) = \frac{c_j}{r^2} + \widetilde{V}_j(r), \quad c_j \in \mathbb{R},\ r\widetilde{V}_j(r) \in L^1((0,\delta)) \cap L^\infty_{\text{loc}}((0,\delta]), \quad j \in J. \tag{1.3}$$
Then the Schrödinger operator
$$H = -\Delta + V, \quad \operatorname{dom}(H) = C_0^\infty(\mathbb{R}^n\backslash\{x_j\}_{j\in J}) \tag{1.4}$$
is essentially self-adjoint if and only if
$$c_j \geqslant 1 - \left(\frac{n-2}{2}\right)^2 = -\frac{n(n-4)}{4} \ \text{ for every } \ j \in J. \tag{1.5}$$

In fact, in the above theorem one could replace each $V_j(|x-x_j|)$ by arbitrary potentials (not necessarily radial) which are locally bounded away from x_j such that
$$H_j = -\Delta + V_j\chi_{B_n(x_j;\delta)}, \quad \operatorname{dom}(H_j) = C_0^\infty(\mathbb{R}^n\backslash\{x_j\}), \quad j \in J, \tag{1.6}$$
is essentially self-adjoint in $L^2(\mathbb{R}^n)$. For example, one still gets essential self-adjointness of H_j if we require $V_j(x) \geqslant -n(n-4)/(4|x-x_j|^2)$, $x \in \mathbb{R}^n\backslash\{x_j\}$, and hence essential self-adjointness of H if the latter inequality holds for all $j \in J$.

The only reason for our specific choice (1.3) is that in this case the essential self-adjointness issue is well-understood (cf. [67, Theorem X.30]). Indeed, another natural choice is a dipole potential of the type,

$$V_j(x) = \frac{(x-x_j)\cdot d_j}{|x-x_j|^3}, \quad x \in \mathbb{R}^n \backslash \{x_j\}, \tag{1.7}$$

for fixed $d_j \in \mathbb{R}^n$ (cf. [21], [22], [26]), or a van der Waals-type potential of the form,

$$V_j(x) = \frac{A_j}{|x-x_j|^m} - \frac{B_j}{|x-x_j|^6}, \quad x \in \mathbb{R}^n \backslash \{x_j\}, \tag{1.8}$$

for fixed $A_j \in (0,\infty)$, $B_j \in \mathbb{R}$ (although, physics requires $B_j \in (0,\infty)$), and $m \geqslant 10$, with $m = 12$ giving rise to a Lennard–Jones-type potential (cf., e.g., [47, pp. 53–59], [54, Sect. 3.2]). Moreover, the exceptional one-point sets $\{x_j\}$ in such examples can be generalized to compact subsets, $\Sigma_j \subset \mathbb{R}^n$, of n-dimensional Lebesgue measure zero, uniformly separated by a positive distance. We also note that our hypotheses on the discrete set J are sufficiently general to describe periodic structures (crystals), half-crystals, an infinite graphene sheet, etc.

More generally, and far beyond Theorem 1.1, we shall derive results to the effect that the defect of operators H of the form (1.4) can actually be expressed as the sum over the individual defects of the operators H_j of the form (1.6),

$$\operatorname{def}(H) = \sum_{j\in J} \operatorname{def}(H_j), \tag{1.9}$$

including the possibility that one, and hence both sides of (1.9) equal ∞. Here

$$\operatorname{def}(A) = \big[n_+(A) + n_-(A)\big)\big]/2, \tag{1.10}$$

with $n_\pm(A)$ the standard deficiency indices of the symmetric operator A in a Hilbert space $\mathcal{H}$ (cf. Definition 2.4 for more details). In particular, for A bounded from below, or A commuting with a conjugation (cf. Definition 2.7), $\operatorname{def}(A) = n_\pm(A)$. In fact, this extension of the case of deficiency indices zero to general deficiency indices is one of our principal results. For further details we refer to Sections 4 and 5. At this point, however, we decided to focus on how Theorem 1.1 generalizes and compares to Theorem 8.4 of [20], which is a special case of our result:

- In contrast to our result, [20, Theorem 8.4] considers the special case $\widetilde{V}_j = 0$ for every $j \in J$.
- Most significantly, we are able to completely eliminate their conditions (see (19) in [20, Lemma 3.5]) that:

$$\sum_{j\in J} |x_j|^{-(n-2)} < \infty \tag{1.11}$$

and that there exists a constant $C \in (0,\infty)$ such that

$$\sum_{j\in J\backslash\{j'\}} |x_j - x_{j'}|^{-(n-2)} \leqslant C, \quad j' \in J. \tag{1.12}$$

(Note that we have slightly reformulated the assumptions from [20] in order to fit our notation, as well as the fact that our index set J is not necessarily equal to $\mathbb{N}$.) In particular (as observed in [20, Remark 3.7]), if we wish to place the singularities $\{x_j\}_{j\in J}$ on a lattice $\mathbb{Z}^d \times 0 \subseteq \mathbb{R}^n$, $d \leqslant n$, the two conditions (1.11) and (1.12) above only allow for $d < n-2$. By contrast, our Theorem 1.1 poses no restrictions on the dimension d of the lattice.

• In [20], conditions (1.11) and (1.12) are used in a technical result, yielding the existence of a certain radius $\delta > 0$, which will be their radius of localization around each singularity (what the authors call shattering of reticular singularities). This specific radius ensures that the potential V, defined as in (1.2), but with $c_j = -\lambda > -\big(\frac{n-2}{2}\big)^2$ for all $j \in J$, satisfies a certain minimization condition (see [20, Lemma 3.5]). This then allows the authors to prove their result (see [20, Theorem 8.4]). Since we do not need to impose the two restrictions (1.11), (1.12) on the locations of our singularities, we are free to choose the radius $\delta > 0$ arbitrarily.

• Finally, we note that we prove our result with distinct coupling constants $c_j \in \mathbb{R}$ in each singular potential V_j, and show that in order to guarantee essential self-adjointness of H, all of these coupling constants must satisfy the well-known condition (1.5).

The role played by the radius $\delta > 0$ in our Theorem 1.1 is different from that played in [20, Theorem 8.4]. In our case, the localization by the characteristic function $\chi_{B_n(x_j;\delta)}$ takes place only in order to ensure that the sum defining the potential V in (1.2) is finite at every $x \in \mathbb{R}^n \backslash \{x_j\}_{j\in J}$. In [20, Theorem 8.4], this is achieved by requiring $\delta < 1/2$, and without loss of generality we could have also required that $\delta < \varepsilon_0/2$, with $\varepsilon_0 = \inf_{j\neq j'\in J} \big|x_j - x_{j'}\big| > 0$; in both cases, these conditions would imply that the local potentials $c_j \chi_{B_n(x_j;\delta)}(x)/|x-x_j|^2$ have relatively disjoint supports. However, we know from [4, Cor. 2] that the deficiency indices (and hence the property of being essentially self-adjoint) of a Schrödinger operator remain invariant under the addition of relatively bounded potentials with relative bound less than one. This in fact allows one to consider any potentials V_j with singularity at x_j, as long as their sum $V = \sum_{j\in J} V_j$ is finite at every $x \in \mathbb{R}^n \backslash \{x_j\}_{j\in J}$. The statement of Theorem 1.1 is only an example of such a phenomenon, and its proof (carried out at the end of Section 5) illustrates the general method.

While reference [20] by Felli, Marchini, and Terracini is closest in spirit to our work, and motivated ours, there are many related papers on this subject which we briefly turn to next. This subject has a long history, especially in connection with essential self-adjointness of Schrödinger and Dirac-type operators, that is, in the special case of vanishing deficiency indices. Indeed, the literature devoted to essential self-adjointness of Schrödinger and Dirac-type operators with possibly strongly singular potential coefficients is enormous and no exhaustive list of references can possibly be included here. Instead, we refer to a few representative and classical items and the references cited therein, such as, [3], [5]–[8], [11], [13], [15], [16], [18], [23], [24], [27], [29]–[42], [44], [45], [48]–[51], [58], [61]–[63], [67, Ch. X], [69], [71], [73, Ch. 9], [74], [76]–[81], [83]–[85], [87]. The case of multi-center singularities and the associated decoupling of deficiency indices in the sense of relation (1.9) also have a longer history and go back to sources such as [3], [4], [5], [9], [26], [39], [40], [48], [60], [62]. For more recent activities in connection with boundedness from below and essential self-adjointness of Schrödinger-type operators with multi-center singularities we refer to [10], [12], [20], [21], [22].

Next, we turn to the organization of this paper. In Section 2 we briefly present the functional analytic background of our work; this is very well-known material, but we include it for clarity of notation, and for ease of reference. Section 3 presents an abstract form of several results of Morgan [60], which deal with the question of localization of relative form and operator boundedness for infinite sums of operators. We note that the form results in Section 3 apply to the strong $|x-x_j|^{-2}$-type singularities discussed in Theorem 1.1 and yield relative (form) boundedness (cf. the comments surrounding (3.14)–(3.19)). To prove the equivalence in Theorem 1.1, we first tackle the full problem of decoupling of deficiency indices for infinite sums of operators in Section 4 from an abstract point of view. The main technical and abstract result of our paper, Theorem 4.2, is presented and proved in this section. We emphasize that the hypotheses of Theorem 4.2 can be checked in practice in a fairly straightforward manner as shown in Section 5, where we apply our general decoupling result to both Schrödinger and second-order elliptic differential operators with potential coefficients that can exhibit countably infinitely many (uniformly separated) singularities. The main result of this section is Theorem 5.8, which shows that deficiency indices localize around singularities under extremely general conditions, see Hypothesis 5.2. More concrete examples, as well as the proof of Theorem 1.1, follow then as a simple consequence. We extend this localization result to more general second-order elliptic operators in Theorem 5.11. We close our paper with an appendix containing some background on the notion of support for arbitrary functions on arbitrary subsets in $\mathbb{R}^n$. This is needed in our case in order to deal with the fact that our localization results in Section 5 permit "arbitrarily bad" compact singularity sets Σ_j, $j \in J$.

Finally, we briefly summarize some of the notation used in this paper: Throughout, we denote by $\mathcal{H}$ a separable complex Hilbert space, by $\|\cdot\|_{\mathcal{H}}$ the norm in $\mathcal{H}$, by $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and by $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

Next, if T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The closure of a closable operator S is denoted by $\overline{S}$.

The Banach space of bounded linear operators on the separable complex Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$.

The symbol $\dotplus$ denotes the direct sum in the sense of Banach spaces (to be distinguished from the orthogonal direct sum $\oplus$ in $\mathcal{H}$).

If $J \subseteq \mathbb{N}$ denotes an index set, we denote by $\#(J)$ the cardinality of J. We also employ the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We employ the usual multi-index notation for partial derivatives of functions on $\mathbb{R}^n$, that is, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, etc. Similarly, we will employ the notation $\partial_k = \partial/\partial_{x_k}$, $1 \leqslant k \leqslant n$. The symbol $B_n(x;r) \subset \mathbb{R}^n$ represents the open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$.

For $\Omega \subseteq \mathbb{R}^n$ open, $C_0^\infty(\Omega)$ denotes the set of C^∞-functions with compact support contained in Ω, $\mathcal{D}(\Omega)$ represents the corresponding space of test functions obtained via an inductive limit procedure, and $\mathcal{D}'(\Omega)$ denotes the corresponding space of distributions (i.e., continuous linear functionals on $\mathcal{D}(\Omega)$).

For simplicity, if the underlying Lebesgue measure $d^n x$ is understood, we abbreviate $L^p(\mathbb{R}^n) := L^p(\mathbb{R}^n; d^n x)$ and $L^p_{\rm loc}(\mathbb{R}^n) := L^p_{\rm loc}(\mathbb{R}^n; d^n x)$. Similarly, given a locally integrable weight $0 < w \in L^1_{\rm loc}(\mathbb{R}^n)$, we will employ the notation $L^p_w(\mathbb{R}^n) := L^p(\mathbb{R}^n; w d^n x)$, $p \geqslant 1$. For $\Omega \subseteq \mathbb{R}^n$ open, standard L^2-based Sobolev spaces are denoted as usually by $H^k(\Omega)$, and analogously for their local versions, $H^k_{\rm loc}(\Omega)$, $k \in \mathbb{N}$.

The closure of a set $M \subset \mathbb{R}^n$ will be denoted by $\overline{M}$, $\mathring{M}$ denotes the interior of M, and the symbol χ_M is used for the characteristic function of the set $M \subset \mathbb{R}^n$.

2. Functional Analytic Background

We start with a bit of background (cf., e.g., [1, Ch. 8], [19, Part III], [43, Sect. V.3], [67, Sect. X.1], [75, Part VI], [82, Ch. 2], [86, Sect. 5.4, Ch. 8] for details).

Let A be a densely defined operator in $\mathcal{H}$, then A is called *symmetric* if $A \subseteq A^*$. Thus, A^* is also densely defined, rendering A closable with $\overline{A} = (A^*)^*$. In addition, A is *self-adjoint* if $A = A^*$, and *essentially self-adjoint* if $\overline{A}$ is self-adjoint.

Theorem 2.1 (Basic criterion for self-adjointness)**.** *Suppose A is symmetric in $\mathcal{H}$. Then the following statements (i)–(iii) are equivalent:*
(i) *A is self-adjoint.*
(ii) *A is closed and* $\ker(A^* \pm iI) = \{0\}$.
(iii) $\operatorname{ran}(A \pm iI) = \mathcal{H}$.

Theorem 2.2. *Suppose A is symmetric and closed in $\mathcal{H}$. Then the following assertions hold:*
(i) $\dim(\ker(A^* - zI))$ *is independent of* $z \in \mathbb{C}_+ = \{z \in \mathbb{C} \mid \Im(z) > 0\}$.
(ii) $\dim(\ker(A^* - zI))$ *is independent of* $z \in \mathbb{C}_- = \{z \in \mathbb{C} \mid \Im(z) < 0\}$.
(iii) *$\sigma(A)$ is one of the following:*
(iii_1) $\overline{\mathbb{C}_+} = \{z \in \mathbb{C} \mid \Im(z) \geqslant 0\}$. (*if* $\dim(\ker(A^*+iI)) > 0$, $\dim(\ker(A^*-iI)) = 0$).
(iii_2) $\overline{\mathbb{C}_-} = \{z \in \mathbb{C} \mid \Im(z) \leqslant 0\}$. (*if* $\dim(\ker(A^*-iI)) > 0$, $\dim(\ker(A^*+iI)) = 0$).
(iii_3) $\mathbb{C}$. (*if* $\dim(\ker(A^* \pm iI)) > 0$).
(iii_4) *a subset of* $\mathbb{R}$ (*if* $\dim\ker(A^* \pm iI)) = 0$).
(iv) *$A = A^*$ if and only if (iii_4) holds (i.e., if and only if $\sigma(A) \subseteq \mathbb{R}$.).*
(v) *$A = A^*$ if and only if* $\dim(\ker(A^* - zI)) = 0$ *for all* $z \in \mathbb{C}_+ \cup \mathbb{C}_-$.

Corollary 2.3. *Suppose A is symmetric and closed in $\mathcal{H}$ and bounded from below, that is, there exists $c \in \mathbb{R}$ such that*

$$(f, Af)_{\mathcal{H}} \geqslant c\|f\|^2_{\mathcal{H}}, \quad f \in \operatorname{dom}(A). \tag{2.1}$$

Then $\dim(\ker(A^* - zI))$ *is independent of* $z \in \mathbb{C}\backslash[c, \infty)$.

Definition 2.4 (Deficiency indices)**.** *Suppose A is symmetric in $\mathcal{H}$. Define the deficiency subspaces of A (resp., $\overline{A}$) by*

$$\begin{aligned} \mathcal{H}_\pm(A) &= \ker(A^* \mp iI) = (\operatorname{ran}(A \pm iI))^\perp \\ &= \mathcal{H}_\pm(\overline{A}) \quad (\text{since } A \subseteq \overline{A} \subseteq A^* = (\overline{A})^*). \end{aligned} \tag{2.2}$$

Then,

$$\begin{aligned} n_\pm(A) &= \dim(\mathcal{H}_\pm(A)) = \dim(\ker(A^* \mp iI)) \\ &= \dim\big((\operatorname{ran}(A \pm iI)^\perp\big) = n_\pm\big(\overline{A}\big) \end{aligned} \tag{2.3}$$

are called the deficiency indices of A (resp., $\overline{A}$) and one introduces

$$\operatorname{def}(A) = \big[n_+(A) + n_-(A))\big]/2. \tag{2.4}$$

One recalls that for a symmetric and closed operator A one has von Neumann's first formula

$$\operatorname{dom}(A^*) = \operatorname{dom}(A) \dotplus \mathcal{H}_+(A) \dotplus \mathcal{H}_-(A). \tag{2.5}$$

In particular,

$$\begin{aligned} &\operatorname{dom}(A^*) = \big\{ f + f_+ + f_- \mid f \in \operatorname{dom}(A),\ f_\pm \in \mathcal{H}_\pm(A) \big\}, \\ &A^*(f + f_+ + f_-) = Af + if_+ - if_-, \quad f \in \operatorname{dom}(A),\ f_\pm \in \mathcal{H}_\pm(A). \end{aligned} \tag{2.6}$$

If A is symmetric, $A \subseteq A^*$, and $B \supseteq A$ is a symmetric extension of A, then

$$A \subseteq B \subseteq B^* \subseteq A^*. \tag{2.7}$$

Theorem 2.5 (von Neumann)**.** *Suppose A is symmetric and closed in $\mathcal{H}$. Then the closed symmetric extensions of A are in one-to-one correspondence with the set of partial isometries* (*in the usual inner product of $\mathcal{H}$*) *of $\mathcal{H}_+(A)$ into $\mathcal{H}_-(A)$. Let $U : I(U) \to \mathcal{H}_-(A)$ with $I(U) \subseteq \mathcal{H}_+(A)$ be such an isometry with initial space $I(U)$, with $I(U)$ closed in $\mathcal{H}$. Then the corresponding closed and symmetric extension A_U of A is given by*

$$\begin{aligned} &\operatorname{dom}(A_U) = \big\{ f + f_+ + Uf_+ \mid f \in \operatorname{dom}(A),\ f_+ \in I(U) \big\}, \\ &A_U(f + f_+ + Uf_+) = Af + if_+ - iUf_+, \quad f \in \operatorname{dom}(A),\ f_+ \in \mathcal{H}_+(A). \end{aligned} \tag{2.8}$$

If $\dim(I(U)) < \infty$, then

$$n_\pm(A_U) = n_\pm(A) - \dim(I(U)). \tag{2.9}$$

Corollary 2.6. *Let A be closed and symmetric in $\mathcal{H}$. Then the following assertions hold:*
(*i*) *A is self-adjoint if and only if $n_+(A) = n_-(A) = 0$.*
(*ii*) *A has self-adjoint extensions if and only if $n_+(A) = n_-(A)$. In such a scenario, there is a one-to-one correspondence between self-adjoint extensions of A and unitary maps from $\mathcal{H}_+(A)$ onto $\mathcal{H}_-(A)$. Therefore, if $n_\pm(A) = n < \infty$, the family of self-adjoint extensions of A is a real n^2-parameter family.*
(*iii*) *If either $n_+(A) = 0$, or $n_-(A) = 0$, then A has no proper symmetric extensions.* (*Such operators are called maximally symmetric.*)

By Corollary 2.3, any symmetric operator A in $\mathcal{H}$ bounded from below has equal deficiency indices and hence self-adjoint extensions.

Definition 2.7. *An anti linear map $\mathcal{C} : \mathcal{H} \to \mathcal{H}$* (*i.e., $\mathcal{C}(c_1 f_1 + c_2 f_2) = \overline{c}_1 \mathcal{C} f_1 + \overline{c}_2 \mathcal{C} f_2$, $c_j \in \mathbb{C}$, $f_j \in \mathcal{H}$, $j = 1, 2$*) *is called a conjugation if it is norm preserving* (*i.e., $\|\mathcal{C} f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$, $f \in \mathcal{H}$*) *and $\mathcal{C}^2 = I_{\mathcal{H}}$.*

The prime example of a conjugation map is of course complex conjugation in $L^2(\Omega)$, that is, $\mathcal{C} : \begin{cases} L^2(\Omega) \to L^2(\Omega), \\ f \mapsto \mathcal{C} f = \overline{f}. \end{cases}$

Theorem 2.8. *Suppose A is symmetric in $\mathcal{H}$ and $\mathcal{C}$ is a conjugation in $\mathcal{H}$ with $\mathcal{C} \operatorname{dom}(A) \subseteq \operatorname{dom}(A)$ and $A\mathcal{C} = \mathcal{C}A$. Then $n_+(A) = n_-(A)$ and hence A has self-adjoint extensions.*

3. On Relative Form and Operator Boundedness

To set the stage, we briefly recall the notion of relatively bounded and relatively form bounded perturbations of a closed operator A in some complex separable Hilbert space $\mathcal{H}$:

Definition 3.1. (i) *Suppose that A is a closed operator in $\mathcal{H}$. A closable operator B in $\mathcal{H}$ is called relatively bounded with respect to A (in short, B is called A-bounded), if*

$$\begin{aligned} &\operatorname{dom}(A) \subseteq \operatorname{dom}(B), \\ &\textit{and for some constants } a, b \in [0,\infty), \\ &\|Bf\|_{\mathcal{H}} \leqslant a\|Af\|_{\mathcal{H}} + b\|f\|_{\mathcal{H}}, \quad f \in \operatorname{dom}(A). \end{aligned} \tag{3.1}$$

(ii) *Assume, in addition, that A is self-adjoint in $\mathcal{H}$ and bounded from below, that is, $A \geqslant cI_{\mathcal{H}}$ for some $c \in \mathbb{R}$. Then a densely defined and closed operator B in $\mathcal{H}$ is called relatively form bounded with respect to A (in short, B is called A-form bounded), if*

$$\operatorname{dom}\big(|A|^{1/2}\big) \subseteq \operatorname{dom}\big(|B|^{1/2}\big). \tag{3.2}$$

In particular, B is A-form bounded if and only if $|B|$ is. Using the polar decomposition of B (i.e., $B = U|B|$), one observes that B is A-bounded if and only if $|B|$ is A-bounded.

We also recall that in connection with relative boundedness, the first condition in (3.1), $\operatorname{dom}(A) \subseteq \operatorname{dom}(B)$, already implies the second condition, viz., there exist numbers $a, b \in [0,\infty)$ such that $\|Bf\|_{\mathcal{H}} \leqslant a\|Af\|_{\mathcal{H}} + b\|f\|_{\mathcal{H}}$, $f \in \operatorname{dom}(A)$, or equivalently,

$$\begin{aligned} &\text{there exist numbers } \widetilde{a}, \widetilde{b} \in [0,\infty) \text{ such that} \\ &\|Bf\|_{\mathcal{H}}^2 \leqslant \widetilde{a}^2\|Af\|_{\mathcal{H}}^2 + \widetilde{b}^2\|f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(A). \end{aligned} \tag{3.3}$$

We also note that if A is self-adjoint and bounded from below, the number α defined by

$$\alpha = \lim_{\mu\uparrow\infty} \big\|B(A+\mu I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} = \lim_{\mu\uparrow\infty} \big\||B|(A+\mu I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \tag{3.4}$$

equals the greatest lower bound (i.e., the infimum) of the possible values for a in (3.1) (resp., for $\widetilde{a}$ in (3.3)). This number α is called the A-bound of B. Similarly, we call

$$\beta = \lim_{\mu\uparrow\infty} \big\||B|^{1/2}\big(|A|^{1/2}+\mu I_{\mathcal{H}}\big)^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \tag{3.5}$$

the A-form bound of B (resp., $|B|$). If $\alpha = 0$ in (3.4) (resp., $\beta = 0$ in (3.5)) then B is called *infinitesimally bounded* (resp., *infinitesimally form bounded*) with respect to A.

Our first result is an abstract version of Morgan [60, Theorem 2.1] (see also [14, Proposition 3.3], [28], [46, Sect. 4]). Throughout this section, infinite sums are understood in the weak operator topology and $J \subseteq \mathbb{N}$ denotes an index set.

Theorem 3.2. *Suppose that T, W are self-adjoint operators in $\mathcal{H}$ such that*

$$\operatorname{dom}\big(|T|^{1/2}\big) \subseteq \operatorname{dom}\big(|W|^{1/2}\big), \tag{3.6}$$

and let $c, d \in (0,\infty)$, $e \in [0,\infty)$. *Moreover, suppose* $\Phi_j \in \mathcal{B}(\mathcal{H})$, $j \in J$, *leave* $\operatorname{dom}\big(|T|^{1/2}\big)$ *invariant, that is,*

$$\Phi_j \operatorname{dom}\big(|T|^{1/2}\big) \subseteq \operatorname{dom}\big(|T|^{1/2}\big), \quad j \in J, \tag{3.7}$$

and satisfy the following conditions (i)–(iii)*:*
(i) $\sum_{j\in J} \Phi_j^* \Phi_j \leqslant I_{\mathcal{H}}$.
(ii) $\sum_{j\in J} \Phi_j^* |W| \Phi_j \geqslant c^{-1}|W|$ *on* $\operatorname{dom}\big(|T|^{1/2}\big)$.
(iii) $\sum_{j\in J} \||T|^{1/2}\Phi_j f\|_{\mathcal{H}}^2 \leqslant d\||T|^{1/2} f\|_{\mathcal{H}}^2 + e\|f\|_{\mathcal{H}}^2$, $f \in \operatorname{dom}\big(|T|^{1/2}\big)$.
Then,

$$\big\||W|^{1/2}\Phi_j f\big\|_{\mathcal{H}}^2 \leqslant a\big\||T|^{1/2}\Phi_j f\big\|_{\mathcal{H}}^2 + b\|\Phi_j f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(|T|^{1/2}),\ j \in J, \tag{3.8}$$

implies

$$\big\||W|^{1/2} f\big\|_{\mathcal{H}}^2 \leqslant a\,c\,d\big\||T|^{1/2} f\big\|_{\mathcal{H}}^2 + [a\,c\,e + b\,c]\|f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(|T|^{1/2}). \tag{3.9}$$

Proof. For $f \in \operatorname{dom}\big(|T|^{1/2}\big)$ one computes

$$\begin{aligned}
\big\||W|^{1/2} f\big\|_{\mathcal{H}}^2 &\leqslant c\sum_{j\in J} \big\||W|^{1/2}\Phi_j f\big\|_{\mathcal{H}}^2 \quad (\text{by } (ii)) \\
&\leqslant c\sum_{j\in J}\Big[a\big\||T|^{1/2}\Phi_j f\big\|_{\mathcal{H}}^2 + b\|\Phi_j f\|_{\mathcal{H}}^2\Big] \quad (\text{by } (3.8)) \\
&\leqslant a\,c\,d\big\||T|^{1/2} f\big\|_{\mathcal{H}}^2 + (a\,c\,e + b\,c)\|f\|_{\mathcal{H}}^2 \quad (\text{by } (i) \text{ and } (iii)),
\end{aligned} \tag{3.10}$$

finishing the proof. $\square$

Remark 3.3. The following condition (iii'), viz., for some $\tilde{e} \in (0,\infty)$,
(iii') $\sum_{j\in J} \big\|[|T|^{1/2}, \Phi_j] f\big\|_{\mathcal{H}}^2 \leqslant \tilde{e}\|f\|_{\mathcal{H}}^2$, $f \in \operatorname{dom}\big(|T|^{1/2}\big)$,
together with condition (i), implies condition (iii) with $d = 1+\varepsilon$ ($\varepsilon > 0$ arbitrarily small) and $e = \frac{(1+\varepsilon)\tilde{e}}{\varepsilon}$. (Here $[\,\cdot\,,\,\cdot\,]$ denotes the commutator symbol.) To see this, one uses the triangle inequality,

$$\big\|[|T|^{1/2}, \Phi_j] f\big\|_{\mathcal{H}} \geqslant \big\||T|^{1/2}\Phi_j f\big\|_{\mathcal{H}} - \big\|\Phi_j |T|^{1/2} f\big\|_{\mathcal{H}} \tag{3.11}$$

as well as the observation that, for any real numbers a and b,

$$\frac{1+\varepsilon}{\varepsilon}(a-b)^2 \geqslant a^2 - (1+\varepsilon)b^2 \text{ if and only if } \frac{1}{1+\varepsilon}a^2 - 2ab + (1+\varepsilon)b^2 \geqslant 0. \tag{3.12}$$

While condition (iii') might look slightly more natural in our context, the formulation in condition (iii) is advantageous in cases where T has an explicit factorization as $T = A^*A$, but a straightforward formula for $|T|^{1/2}$ is not available, since in such cases one can use the fact that

$$\big\||T|^{1/2} f\big\|_{\mathcal{H}}^2 = \big\|Af\big\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}\big(|T|^{1/2}\big) = \operatorname{dom}(A). \tag{3.13}$$

Finally, we note that condition (iii') itself is implied by the fully localized condition:
(iii'') $\big\|[|T|^{1/2}, \Phi_j] f\big\|_{\mathcal{H}}^2 \leqslant \tilde{e}\|\Phi_j f\|_{\mathcal{H}}^2$, $f \in \operatorname{dom}\big(|T|^{1/2}\big)$, $j \in J$. $\diamond$

In particular, consider the concrete case of

$$T = -\Delta, \quad \operatorname{dom}(T) = H^2(\mathbb{R}^n), \tag{3.14}$$

in $L^2(\mathbb{R}^n)$, and assume that W, the operator of multiplication with a real-valued function W (with a slight abuse of notation), is relatively form bounded with respect to $T = -\Delta$ (for sufficient conditions on W, see, e.g., [86, Theorems 10.17 (b), 10.18] with $r = 1$). Let $\{\phi_j\}_{j\in J}$, $J \subseteq \mathbb{N}$, be a family of smooth, real-valued functions defined on $\mathbb{R}^n$ in such a manner that for each $x \in \mathbb{R}^n$, there exists an open neighborhood $U_x \subset \mathbb{R}^n$ of x such that there exist only finitely many indices $k \in J$ with $\operatorname{supp}(\phi_k) \cap U_x \neq \emptyset$ and $\phi_k|_{U_x} \neq 0$, as well as

$$\sum_{j\in J} \phi_j(x)^2 = 1, \quad x \in \mathbb{R}^n \tag{3.15}$$

(the sum over $j \in J$ in (3.15) being finite). Finally, let Φ_j be the operator of multiplication by the function ϕ_j, $j \in J$. Then one notes that for these choices, hypothesis (i) holds with equality, and hypothesis (ii) with $c = 1$ follows from (i). Moreover, item (iii) holds with $d = 1$ as long as

$$e = \bigg\| \sum_{j\in J} |\nabla\phi_j(\cdot)|^2 \bigg\|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{3.16}$$

To verify this, one observes that $\||T|^{1/2}\phi f\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} d^n x \, |\nabla(\phi(x)f(x))|^2$ and that the cross terms vanish since $\sum_{j\in J} \phi_j(x)(\nabla\phi_j)(x) = 0$, $x \in \mathbb{R}^n$, by condition (i). (We note again that the latter sum over $j \in J$ contains only finitely many terms in every bounded neighborhood of $x \in \mathbb{R}^n$.) This is precisely [60, Theorem 2.1]. Strongly singular potentials that are covered by Theorem 3.2 are, for instance, of the following form: Let $J \subseteq \mathbb{N}$ be an index set, and $\{x_j\}_{j\in J} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 3$, be a set of points such that

$$\inf_{\substack{j,j'\in J\\ j\neq j'}} |x_j - x_{j'}| > 0. \tag{3.17}$$

In addition, let $\gamma_j \in \mathbb{R}$, $j \in J$, $\gamma, \delta \in (0,\infty)$ with

$$|\gamma_j| \leqslant \gamma < (n-2)^2/4, \; j \in J, \tag{3.18}$$

and

$$W(x) = \sum_{j\in J} \gamma_j \frac{e^{-\delta|x-x_j|}}{|x-x_j|^2}, \quad x \in \mathbb{R}^n \backslash \{x_j\}_{j\in J}. \tag{3.19}$$

Then an application of Hardy's inequality in $\mathbb{R}^n$, $n \geqslant 3$, shows that W is form bounded with respect to T in (3.14) with form bound strictly less than one (cf. [14, p. 28–29]).

Similarly, one obtains the following operator perturbation analog of the form perturbation result in Theorem 3.2.

Theorem 3.4. *Suppose that T, W are symmetric in $\mathcal{H}$ such that*

$$\operatorname{dom}(T) \subseteq \operatorname{dom}(W), \tag{3.20}$$

and let $c > 0$, $d > 0$, $e \geqslant 0$. Moreover, suppose $\Phi_j \in \mathcal{B}(\mathcal{H})$, $j \in J$, leave $\operatorname{dom}(T)$ *invariant, that is,*

$$\Phi_j \operatorname{dom}(T) \subseteq \operatorname{dom}(T), \quad j \in J, \tag{3.21}$$

and satisfy the following conditions (i)–(iii):

(i) $\sum_{j\in J} \Phi_j^*\Phi_j \leqslant I_{\mathcal{H}}$.

(ii) $\sum_{j\in J} \Phi_j^* W^2 \Phi_j \geqslant c^{-1} W^2$ *on* $\operatorname{dom}(T)$.

(iii) $\sum_{j\in J} \|T\Phi_j f\|^2_{\mathcal{H}} \leqslant d\|Tf\|^2_{\mathcal{H}} + e\|f\|^2_{\mathcal{H}}$, $\quad f \in \operatorname{dom}(T)$.

(*Here infinite sums are understood in the weak operator topology.*) *Then,*

$$\|W\Phi_j f\|_{\mathcal{H}}^2 \leqslant a\|T\Phi_j f\|_{\mathcal{H}}^2 + b\|\Phi_j f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(T),\ j \in J, \tag{3.22}$$

implies

$$\|Wf\|_{\mathcal{H}}^2 \leqslant a\,c\,d\|Tf\|_{\mathcal{H}}^2 + [a\,c\,e + b\,c]\|f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(T). \tag{3.23}$$

Proof. For $f \in \operatorname{dom}(T)$ one computes

$$\begin{aligned}
\|Wf\|_{\mathcal{H}}^2 &\leqslant c\sum_{j\in J}\|W\Phi_j f\|_{\mathcal{H}}^2 \quad (\text{by } (ii)) &(3.24)\\
&\leqslant c\sum_{j\in J}\big[a\|T\Phi_j f\|_{\mathcal{H}}^2 + b\|\Phi_j f\|_{\mathcal{H}}^2\big] \quad (\text{by } (3.22)) &(3.25)\\
&\leqslant a\,c\,d\|Tf\|_{\mathcal{H}}^2 + (a\,c\,e + b\,c)\|f\|_{\mathcal{H}}^2 \quad (\text{by } (i) \text{ and } (iii)), &(3.26)
\end{aligned}$$

concluding the proof. □

Again one notes that item (iii) holds with $d = 1+\varepsilon$, $\varepsilon > 0$, and $e > 0$ if

$$\sum_{j\in J}\|[T,\Phi_j]f\|_{\mathcal{H}}^2 \leqslant \frac{\varepsilon^2}{4+2\varepsilon}\|Tf\|_{\mathcal{H}}^2 + \frac{\varepsilon e}{2+\varepsilon}\|f\|_{\mathcal{H}}^2, \quad f \in \operatorname{dom}(T). \tag{3.27}$$

As an immediate consequence of [4, Corollary 2] one obtains the following result.

Corollary 3.5. *In addition to the assumptions of Theorem* 3.4*, suppose that T is closed, and that $a\,c\,d < 1$. Then*

$$n_\pm(T+W) = n_\pm(T). \tag{3.28}$$

In particular, in the case of $T = -\Delta$ in $L^2(\mathbb{R}^n)$ as in (3.14) and W the operator of multiplication with a real-valued function W (again, abusing notation) bounded with respect to $T = -\Delta$ (for sufficient conditions on W, see, e.g., [86, Theorems 10.17 (b), 10.18] with $r = 2$), and Φ_j the operator of multiplication by a smooth function ϕ_j, item (ii) with $c = 1$ follows from (i) in case one has equality on the support of W in item (i). Moreover,

$$[T,\Phi_j]f = (-\Delta\phi_j)f - 2(\nabla\phi_j)\cdot(\nabla f), \quad f \in \operatorname{dom}(T), \tag{3.29}$$

implying

$$\begin{gathered}
\sum_{j\in J}\|[T,\Phi_j]f\|_{L^2(\mathbb{R}^n)}^2 \leqslant \alpha\|f\|_{L^2(\mathbb{R}^n)}^2 + \beta\|\nabla f\|_{L^2(\mathbb{R}^n)^n}^2,\\
\alpha = 2\bigg\|\sum_{j\in J}(\Delta\phi_j)^2\bigg\|_{L^\infty(\mathbb{R}^n)}, \quad \beta = 4\bigg\|\sum_{j\in J}(\nabla\phi_j)^2\bigg\|_{L^\infty(\mathbb{R}^n)}.
\end{gathered} \tag{3.30}$$

Finally, the elementary inequality,

$$\|\nabla f\|_{L^2(\mathbb{R}^n)^n} \leqslant \varepsilon\|\Delta f\|_{L^2(\mathbb{R}^n)} + (2\varepsilon)^{-1}\|f\|_{L^2(\mathbb{R}^n)}, \quad f \in H^2(\mathbb{R}^n),\ \varepsilon > 0, \tag{3.31}$$

shows that (3.27) holds if $\alpha, \beta \in [0,\infty)$.

Next, introducing the space of uniformly locally L^p-functions by

$$L^p_{\mathrm{loc\,unif}}(\mathbb{R}^n) := \Big\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)\,\Big|\, \sup_{x\in\mathbb{R}^n}\|f\chi_{B_n(x;1)}\|_{L^p(\mathbb{R}^n)} < \infty\Big\}, \quad p \in [1,\infty), \tag{3.32}$$

one can derive a quick proof of [68, Theorem XIII.96]:

Corollary 3.6. *Let $V \in L^p_{\mathrm{loc\,unif}}(\mathbb{R}^n)$ be real-valued with $p = 2$ for $n = 1, 2, 3$ and $p > n/2$ for $n \geqslant 4$. Then V is infinitesimally bounded (and hence infinitesimally form bounded) with respect to $H_0 = -\Delta$, $\mathrm{dom}(H_0) = H^2(\mathbb{R}^n)$, in $L^2(\mathbb{R}^n)$.*

Proof. Let ϕ be a nonnegative smooth function which equals 1 in $B_n(0; 1/2)$ and vanishes outside $B_n(0; 1)$. Let x_j, $j \in J$, be the points of a periodic lattice such that $\sum_{j\in J} \phi(x - x_j)^2 \geqslant 1/2$, $x \in \mathbb{R}^n$. Set $\phi_j(x) = \phi(x - x_j)\big[\sum_{j'\in J} \phi(x - x_{j'})^2\big]^{-1/2}$, $x \in \mathbb{R}^n$, $j \in J$, such that $\sum_{j\in J} \phi_j(x)^2 = 1$, $x \in \mathbb{R}^n$. Then items (i)–(iii) hold with $c = 1$, $d = 1 + \varepsilon$ as pointed out above. Moreover, by [67, Theorem X.15 and X.20], for any $\varepsilon > 0$, one can find a corresponding $b(\varepsilon) > 0$, such that

$$\|V\phi_j f\|^2_{L^2(\mathbb{R}^n)} \leqslant \varepsilon\|H_0\phi_j f\|^2_{L^2(\mathbb{R}^n)} + b(\varepsilon)\|\phi_j f\|^2_{L^2(\mathbb{R}^n)}, \quad f \in H^2(\mathbb{R}^n),\ j \in J, \tag{3.33}$$

proving Corollary 3.6. □

For a form version of Corollary 3.6 we refer to the comments surrounding equation (2.10) in Morgan [60, p. 112].

It is well-known that for $n = 1$, $V \in L^2_{\mathrm{loc\,unif}}(\mathbb{R}^n)$ is equivalent to V being relatively bounded with respect to H_0, which in turn is, in fact, equivalent to V being infinitesimally bounded with respect to H_0, see, [72, Theorem 2.7.1]. (See also [25] for more results and literature on the one-dimensional case.) For necessary and sufficient conditions on form boundedness (resp., infinitesimal form boundedness) of V relative to H_0 in the multi-dimensional case we refer to [56, Theorem 4.2] (resp., [57, Theorem III]), see also [55, Sects. 2.5, 11.4].

4. Decoupling of Deficiency Indices. An Abstract Approach

Next we turn to a general scheme of determining deficiency indices particularly suited for dealing with Schrödinger-type operators with potentials that exhibit strong singularities at (possibly, countably many) uniformly separated points (or compact sets of n-dimensional Lebesgue measure zero) to be explored in Section 5.

Hypothesis 4.1. *Let $J \subseteq \mathbb{N}$ be an index set and let T, T_j, $j \in J$ be closed symmetric operators in $\mathcal{H}$. Suppose there exist $\Phi_j \in \mathcal{B}(\mathcal{H})$ and $\widetilde{\Phi}_j \in \mathcal{B}(\mathcal{H})$, $j \in J$, such that*

$$\Phi = \sum_{j\in J} \Phi_j \in \mathcal{B}(\mathcal{H}), \tag{4.1}$$

with convergence in the strong operator topology, and

$$\widetilde{\Phi}_j\Phi_j = \Phi_j,\ j \in J, \quad \widetilde{\Phi}_j\Phi_k = 0,\ j, k \in J,\ j \neq k, \tag{4.2}$$

$$g \in \mathrm{dom}(T_j^*) \ \textit{implies}\ \Phi_j g \in \mathrm{dom}(T^*), \quad j \in J, \tag{4.3}$$

$$g \in \mathrm{dom}(T^*) \ \textit{implies}\ \Phi_j g \in \mathrm{dom}(T_j^*), \quad j \in J, \tag{4.4}$$

$$g \in \mathrm{dom}(T_j) \ \textit{implies}\ \widetilde{\Phi}_j g \in \mathrm{dom}(T), \quad j \in J, \tag{4.5}$$

$$g \in \mathrm{dom}(T) \ \textit{implies}\ \widetilde{\Phi}_j g \in \mathrm{dom}(T_j), \quad j \in J, \tag{4.6}$$

$$g \in \mathrm{dom}(T_j^*) \ \textit{implies}\ (I_{\mathcal{H}} - \Phi_j)g \in \mathrm{dom}(T_j), \quad j \in J, \tag{4.7}$$

$$g \in \mathrm{dom}(T^*) \ \textit{implies}\ \Big(I_{\mathcal{H}} - \sum_{j\in J} \Phi_j\Big)g \in \mathrm{dom}(T), \tag{4.8}$$

$$g \in \operatorname{dom}(T^*) \ \textit{ implies } \ \operatorname*{s-lim}_{N\to\infty} T^*\bigg(\sum_{\substack{j\in J\\ |j|\leqslant N}} \Phi_j g\bigg) = T^*\bigg(\sum_{j\in J}\Phi_j g\bigg). \tag{4.9}$$

(*Condition* (4.9) *is redundant if* $\#(J) < \infty$.)

Next, we recall the notion of linear independence with respect to a linear subspace of $\mathcal{H}$: Let $\mathcal{D} \subseteq \mathcal{H}$ be a linear subspace of $\mathcal{H}$. Then the vectors $f_k \in \mathcal{H}$, $1 \leqslant k \leqslant N$, $N \in \mathbb{N}$, are called *linearly independent* $(\operatorname{mod} \mathcal{D})$, if

$$\begin{aligned}&\sum_{k=1}^{N} c_k f_k \in \mathcal{D} \ \text{ for some coefficients } \ c_k \in \mathbb{C},\ 1 \leqslant k \leqslant N,\\ &\quad \text{implies } \ c_k = 0,\ 1 \leqslant k \leqslant N.\end{aligned} \tag{4.10}$$

In addition, with $\mathcal{D}$ and $\mathcal{E}$ linear subspaces of $\mathcal{H}$ with $\mathcal{D} \subseteq \mathcal{E}$, the quotient subspace $\mathcal{E}/\mathcal{D}$ consists of equivalence classes $[f]$ such that $g \in [f]$ if and only if $(f-g) \in \mathcal{D}$, in particular, $f = g \ (\operatorname{mod} \mathcal{D})$ is equivalent to $(f-g) \in \mathcal{D}$. Moreover, the dimension of $\mathcal{E}$ $(\operatorname{mod} \mathcal{D})$, denoted by $\dim_{\mathcal{D}}(\mathcal{E})$, equals $n \in \mathbb{N}$, if there are n, but not more than n, linearly independent vectors in $\mathcal{E}$, such that no linear combination (except, the trivial one) belongs to $\mathcal{D}$. If no such finite $n \in \mathbb{N}$ exists, one defines $\dim_{\mathcal{D}}(\mathcal{E}) = \infty$. Consequently, if A is symmetric and closed, then (2.5) implies

$$\dim_{\operatorname{dom}(A)}(\operatorname{dom}(A^*)) = n_+(A) + n_-(A) = 2\operatorname{def}(A). \tag{4.11}$$

The following result computes the defect of T in terms of those of T_j, $j \in J$.

Theorem 4.2. *Assume Hypothesis* 4.1. *Then*

$$\operatorname{def}(T) = \sum_{j\in J} \operatorname{def}(T_j), \tag{4.12}$$

including the possibility that one, and hence both sides of (4.12) *equal* ∞.

Proof. We start with the special case where

$$\sum_{j\in J} \operatorname{def}(T_j) < \infty, \tag{4.13}$$

that is, for at most finitely many $k \in J$, $0 < \operatorname{def}(T_k) = N_k$ for some $N_k \in \mathbb{N}$. In this context we abbreviate

$$K = \{k \in J \mid \operatorname{def}(T_k) > 0\} \subseteq J, \quad K \ \text{ finite.} \tag{4.14}$$

For each $k \in K$, let

$$f_{k,\ell} \in \operatorname{dom}(T_k^*), \quad 1 \leqslant \ell \leqslant 2\operatorname{def}(T_k), \tag{4.15}$$

be a maximal set of vectors in $\operatorname{dom}(T_k^*)$, linearly independent $(\operatorname{mod} \operatorname{dom}(T_k))$. Then (4.3) yields

$$\Phi_k f_{k,\ell} \in \operatorname{dom}(T^*),\ 1 \leqslant \ell \leqslant 2\operatorname{def}(T_k),\ k \in K. \tag{4.16}$$

Next, let $\beta_{k,\ell} \in \mathbb{C}$, $1 \leqslant \ell \leqslant 2\operatorname{def}(T_k)$, $k \in K$, be constants such that

$$\sum_{k\in K} \sum_{\ell=1}^{2\operatorname{def}(T_k)} \beta_{k,\ell}\Phi_k f_{k,\ell} \in \operatorname{dom}(T). \tag{4.17}$$

(Here and in what follows, note that (4.13) and the fact that K is finite imply that every sum as above has finitely many terms.) Applying (4.6) and subsequently (4.2) then yields that for every $k \in K$,

$$\operatorname{dom}(T_k) \ni \widetilde{\Phi}_k\Bigg(\sum_{k'\in K}\sum_{\ell=1}^{2\operatorname{def}(T'_k)} \beta_{k',\ell}\Phi_{k'}f_{k',\ell}\Bigg) = \sum_{\ell=1}^{2\operatorname{def}(T_k)} \beta_{k,\ell}\Phi_k f_{k,\ell}. \tag{4.18}$$

On the other hand, from (4.7) and the fact that $f_{k,\ell} \in \operatorname{dom}(T_k^*)$, one finds that for every $k \in K$,

$$(I_{\mathcal{H}} - \Phi_k)\sum_{\ell=1}^{2\operatorname{def}(T_k)} \beta_{k,\ell}f_{k,\ell} \in \operatorname{dom}(T_k). \tag{4.19}$$

Combining (4.18) and (4.19) one concludes

$$\sum_{\ell=1}^{2\operatorname{def}(T_k)} \beta_{k,\ell}f_{k,\ell} \in \operatorname{dom}(T_k),\ k \in K, \tag{4.20}$$

and hence

$$\beta_{k,\ell} = 0,\ 1 \leqslant \ell \leqslant 2\operatorname{def}(T_k),\ k \in K, \tag{4.21}$$

since for every $k \in K$, $f_{k,\ell} \in \operatorname{dom}(T_k^*)$, $1 \leqslant \ell \leqslant 2\operatorname{def}(T_k)$, were chosen linearly independent (mod $\operatorname{dom}(T_k)$). Consequently,

$$\Phi_k f_{k,\ell} \in \operatorname{dom}(T^*), \quad 1 \leqslant \ell \leqslant 2\operatorname{def}(T_k),\ k \in K, \tag{4.22}$$

are linearly independent (mod $\operatorname{dom}(T)$), implying

$$2\operatorname{def}(T) = \dim_{\operatorname{dom}(T)}(\operatorname{dom}(T^*)) \geqslant 2\sum_{k\in K}\operatorname{def}(T_k) = 2\sum_{j\in J}\operatorname{def}(T_j). \tag{4.23}$$

Now suppose (by contradiction) that equality does not hold in (4.22), that is,

$$\operatorname{def}(T) > \sum_{k\in K}\operatorname{def}(T_k) = \sum_{j\in J}\operatorname{def}(T_j). \tag{4.24}$$

Since all the $\Phi_k f_{k,l}$, $1 \leqslant \ell \leqslant 2\operatorname{def}(T_k)$, $k \in K$, are linearly independent in $\operatorname{dom}(T^*)$ (mod $\operatorname{dom}(T)$), assumption (4.24) implies that there exists $f \in \operatorname{dom}(T^*)$ such that

$$f, \Phi_k f_{k,\ell}, \quad 1 \leqslant \ell \leqslant 2\operatorname{def}(T_k),\ k \in K, \tag{4.25}$$

are still linearly independent (mod $\operatorname{dom}(T)$). One notes once again that by (4.7), $(I_{\mathcal{H}} - \Phi_k)f_{k,\ell} \in \operatorname{dom}(T_k)$, $1 \leqslant \ell \leqslant 2\operatorname{def}(T_k)$, $k \in K$, and hence one can write,

$$f_{k,\ell} = \Phi_k f_{k,\ell} + g_{k,\ell}, \quad g_{k,\ell} \in \operatorname{dom}(T_k),\ 1 \leqslant \ell \leqslant 2\operatorname{def}(T_k),\ k \in K. \tag{4.26}$$

Applying (4.4), one concludes that $\Phi_j f \in \operatorname{dom}(T_j^*)$, $j \in J$.

If $j \in K$, then there exist coefficients $c_{k,\ell} \in \mathbb{C}$, $1 \leqslant \ell \leqslant 2\operatorname{def}(T_k)$, $k \in K$, and an element $\widetilde{g}_j \in \operatorname{dom}(T_j)$, such that

$$\Phi_j f = \sum_{\ell=1}^{2\operatorname{def}(T_j)} c_{j,\ell}f_{j,\ell} + \widetilde{g}_j = \sum_{\ell=1}^{2\operatorname{def}(T_j)} c_{j,\ell}\Phi_j f_{j,\ell} + g_j, \tag{4.27}$$

where in the second identity we have used (4.26), and we have set

$$g_j = \Bigg(\sum_{\ell=1}^{2\operatorname{def}(T_j)} c_{j,\ell}g_{j,\ell} + \widetilde{g}_j\Bigg) \in \operatorname{dom}(T_j). \tag{4.28}$$

Hypothesis (4.2) then implies that $\widetilde{\Phi}_j \Phi_j f = \Phi_j f$, which translates into

$$\sum_{\ell=1}^{2\,\mathrm{def}(T_j)} c_{j,\ell} \widetilde{\Phi}_j \Phi_j f_{j,\ell} + \widetilde{\Phi}_j g_j = \sum_{\ell=1}^{2\,\mathrm{def}(T_j)} c_{j,\ell} \Phi_j f_{j,\ell} + g_j, \tag{4.29}$$

and hence, using (4.2) again on the terms under the sum,

$$\widetilde{\Phi}_j g_j = g_j, \quad j \in K. \tag{4.30}$$

If $j \in J\backslash K$, we simply set

$$g_j = \Phi_j f \in \mathrm{dom}(T_j^*) = \mathrm{dom}(T_j), \tag{4.31}$$

since in this case $\mathrm{def}(T_j) = 0$. Another straightforward application of (4.2) then yields the rest of the cases (with $j \in J\backslash K$) since

$$\widetilde{\Phi}_j g_j = g_j, \quad j \in J. \tag{4.32}$$

By (4.5) one obtains that

$$g_j \in \mathrm{dom}(T), \quad j \in J. \tag{4.33}$$

Next, by (4.8), one concludes that $\big(I_{\mathcal{H}} - \sum_{j\in J} \Phi_j\big) f \in \mathrm{dom}(T)$ and hence $f = \sum_{j\in J} \Phi_j f \pmod{\mathrm{dom}(T)}$ implies that there exists $g_0 \in \mathrm{dom}(T)$ such that

$$f = \sum_{j\in J} \Phi_j f + g_0 = \sum_{k\in K} \left(\sum_{\ell=1}^{2\,\mathrm{def}(T_k)} c_{k,\ell} \Phi_k f_{k,\ell} \right) + \sum_{j\in J} g_j + g_0. \tag{4.34}$$

If $\#(J) < \infty$, then (4.33) implies that $\sum_{j\in J} g_j \in \mathrm{dom}(T)$, which in turn implies that f and $\Phi_k f_{k,\ell}$, $1 \leqslant \ell \leqslant 2\,\mathrm{def}(T_k)$, $k \in K$, are linearly dependent (mod $\mathrm{dom}(T)$), contradicting (4.25).

If $\#(J) = \infty$, (4.9) implies that $T^*\big(\sum_{j\in J} \Phi_j f\big)$ is well-defined, and hence so is $T^*\big(\sum_{j\in J} \Psi_j\big)$. Since the partial sums are in $\mathrm{dom}(T)$, so is their limit, as T is closed. Consequently, (4.12) also holds in this case.

Finally, we consider the case where

$$\sum_{j\in J} \mathrm{def}(T_j) = \infty. \tag{4.35}$$

In this case, for any $N \in \mathbb{N}$ there exists then a finite subset $K_N \subset J$, such that for each $k \in K_N$ there exists an integer $N_k \in \mathbb{N}$ with $N_k \leqslant \mathrm{def}(T_k)$ and

$$\sum_{k\in K_N} N_k \geqslant N. \tag{4.36}$$

(The integer N_k is only needed in the situation where the corresponding $\mathrm{def}(T_k) = \infty$.) For each $k \in K_N$, let

$$f_{k,\ell} \in \mathrm{dom}(T_k^*), \quad 1 \leqslant \ell \leqslant N_k, \tag{4.37}$$

be linearly independent (mod $\mathrm{dom}(T_k)$). Then, following verbatim the first part of our proof above, one concludes again that $\Phi_k f_{k,\ell} \in \mathrm{dom}(T^*)$, $1 \leqslant \ell \leqslant 2N_k$, $k \in K_N$, are linearly independent (mod $\mathrm{dom}(T)$). Consequently, and by the choice of $K_N \subset J$,

$$2\,\mathrm{def}(T) \geqslant \sum_{k\in K_N} 2N_k \geqslant 2N. \tag{4.38}$$

Since $N \in \mathbb{N}$ was arbitrary, $\mathrm{def}(T) = \infty$, completing the proof of (4.12). $\square$

Remark 4.3. (*i*) It might be surprising at first sight that our conditions (4.2)–(4.9) (i.e., the full hypotheses of Theorem 4.2) only involve operator domains, and do not require any further information on the operators themselves. Note however that this is consistent with our point of view on the deficiency index (see (4.11)), and that, as will be shown explicitly in Section 5, the conditions in Hypothesis 4.1 can all be realized very naturally due to locality properties of Schrödinger-type operators and second-order elliptic operators (with, possibly, strongly singular potential coefficients). Furthermore, while we focus exclusively on the case of second-order elliptic partial differential operators in Section 5, the case of first-order Dirac-type operators can be discussed along analogous lines (cf. [3], [39], [40]). In fact, the first-order case is technically quite a bit simpler than the second-order situation discussed in this paper as the analog of the term $-2(\nabla\phi_j)\cdot(\nabla f)$, see, for instance, (5.27), and hence all the difficulties surrounding it, does not occur in the context of first-order partial differential operators.
(*ii*) As will be illustrated in Section 5 in a fairly straightforward manner, Hypothesis 4.1 is sufficiently flexible to permit a total decoupling of singularities, with respect to their contribution to the total defect $\operatorname{def}(T)$ of T, as long as the singularities in partial differential operators are separated by a strictly positive distance.
(*iii*) The abstract approach developed in this section was inspired by the concrete case of Dirac-type operators treated in [3], [30], [39], [40] (see also [9]). $\diamond$

5. Applications to Schrödinger-Type and Second-Order Elliptic Partial Differential Operators and Decoupling of Singularities

In this section we apply the abstract approach developed in Theorem 4.2 to the concrete case of Schrödinger operators and second-order elliptic partial differential operators, each possibly containing a strongly singular potential term. Our results illustrate the concept of decoupling of singularities with respect to deficiency index computations whenever the singularities are separated by a fixed minimal positive distance.

We start with the following auxiliary result. (It is surely well-known – the case of Lipschitz functions is covered by [59, Theorem 4.12] – but we provide its proof for the convenience of the reader.)

Lemma 5.1. *Assume $F_0, F_1 \subset \mathbb{R}^n$ are such that* $\operatorname{dist}(F_0,F_1) \geqslant \varepsilon$ *for some* $\varepsilon > 0$. *Then there exists a function $\phi \in C^\infty(\mathbb{R}^n)$ satisfying*

$$\begin{aligned} &0 \leqslant \phi \leqslant 1 \text{ on } \mathbb{R}^n, \quad \phi\big|_{F_0} = 0, \quad \phi\big|_{F_1} = 1, \text{ and} \\ &\|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \leqslant c_{n,\alpha}\, \varepsilon^{-|\alpha|} \text{ for each multi-index } \alpha \in \mathbb{N}_0^n, \end{aligned} \tag{5.1}$$

where the constant $c_{n,\alpha} \in (0,\infty)$ depends only on n and α.

Proof. We start by introducing

$$\widetilde{F}_1 = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x,F_1) \leqslant \varepsilon/4\}. \tag{5.2}$$

Consider a function

$$0 \leqslant \theta \in C_0^\infty(\mathbb{R}^n), \quad \operatorname{supp}(\theta) \subseteq B_n(0;1), \quad \int_{\mathbb{R}^n} d^n x\, \theta(x) = 1, \tag{5.3}$$

then define ϕ via

$$\phi(x) = \Big(\frac{\varepsilon}{4}\Big)^{-n} \int_{\widetilde{F}_1} d^n y\, \theta(4(x-y)/\varepsilon), \quad x \in \mathbb{R}^n. \tag{5.4}$$

Obviously, $\phi \in C^\infty(\mathbb{R}^n)$ and for each $x \in \mathbb{R}^n$ one has

$$0 \leqslant \phi(x) \leqslant \Big(\frac{\varepsilon}{4}\Big)^{-n} \int_{\mathbb{R}^n} d^n y\, \theta(4(x-y)/\varepsilon) = \int_{\mathbb{R}^n} d^n y\, \theta(y) = 1. \tag{5.5}$$

One observes that if $x \in F_0$, then for each $y \in \widetilde{F}_1$ one necessarily has $|x-y| \geqslant \varepsilon/4$. Since by construction $\operatorname{supp}(\theta) \subseteq B_n(0;1)$, this forces $\theta(4(x-y)/\varepsilon) = 0$. One therefore obtains $\phi(x) = 0$ for each $x \in F_0$, and hence, $\phi\big|_{F_0} = 0$. Similarly, if $x \in F_1$, then necessarily

$$\operatorname{supp}\big(\theta(4(x-\cdot)/\varepsilon)\big) \subset B_n(x;\varepsilon/4) \subset \widetilde{F}_1. \tag{5.6}$$

Consequently, for each $x \in F_1$ one has

$$\phi(x) = \Big(\frac{\varepsilon}{4}\Big)^{-n} \int_{\mathbb{R}^n} d^n y\, \theta(4(x-y)/\varepsilon) = \int_{\mathbb{R}^n} d^n y\, \theta(y) = 1, \tag{5.7}$$

and hence, $\phi\big|_{F_1} = 1$.

Finally, for every multi-index α one estimates

$$\begin{aligned} |(\partial^\alpha \phi)(x)| &= \Big(\frac{\varepsilon}{4}\Big)^{-n-|\alpha|} \bigg| \int_{\widetilde{F}_1} d^n y\, (\partial^\alpha \theta)(4(x-y)/\varepsilon) \bigg| \\ &\leqslant \Big(\frac{\varepsilon}{4}\Big)^{-n-|\alpha|} \int_{|x-y|\leqslant \varepsilon/4} d^n y\, |(\partial^\alpha \theta)(4(x-y)/\varepsilon)| \\ &\leqslant \|\partial^\alpha \theta\|_{L^\infty(\mathbb{R}^n)} \Big(\frac{\varepsilon}{4}\Big)^{-n-|\alpha|} \int_{|y|\leqslant \varepsilon/4} d^n y \\ &= c_{n,\alpha}\, \varepsilon^{-|\alpha|}, \quad x \in \mathbb{R}^n, \end{aligned} \tag{5.8}$$

for some finite constant $c_{n,\alpha} > 0$ depending only on n and α. $\square$

To set up the type of Schrödinger operators we are interested in, we next collect the following set of assumptions.

Hypothesis 5.2. *Let $J \subseteq \mathbb{N}$ be an index set and $n \in \mathbb{N}$, $n \geqslant 2$.*
(i) Consider compact sets $\Sigma_j \subset \mathbb{R}^n$ of n-dimensional Lebesgue measure zero, $j \in J$.
(ii) Consider $V_j \in L^2_{\text{loc}}(\mathbb{R}^n\backslash\Sigma_j)$ real-valued and with bounded support, $j \in J$.
(iii) Suppose there exists $\varepsilon > 0$ such that

$$\operatorname{dist}(\{\operatorname{supp}(V_j) \cup \Sigma_j\}, \{\operatorname{supp}(V_{j'}) \cup \Sigma_{j'}\}) \geqslant \varepsilon, \quad j,j' \in J, j \neq j'. \tag{5.9}$$

Granted Hypothesis 5.2, we introduce

$$\Sigma = \bigcup_{j\in J} \Sigma_j, \tag{5.10}$$

$$A_j = \operatorname{supp}(V_j) \cup \Sigma_j,\ j \in J, \quad A = \bigcup_{j\in J} A_j = \bigcup_{j\in J} \operatorname{supp}(V_j) \cup \Sigma, \tag{5.11}$$

$$V(x) = \sum_{j\in J} V_j(x) \ \text{ for a.e. } x \in \mathbb{R}^n\backslash\Sigma. \tag{5.12}$$

One notes that A_j are compact sets and, as a consequence of the uniform separation of sets, properly stated in (5.9), that Σ and A are closed subsets of $\mathbb{R}^n$. In addition, Σ is of n-dimensional Lebesgue measure zero, V is real-valued, and, due to the uniform separation of the A_j's, $V \in L^2_{\text{loc}}(\mathbb{R}^n\backslash\Sigma)$.

Next, we introduce the symmetric Schrödinger operators in $L^2(\mathbb{R}^n)$,

$$\dot{H}_j f = (-\Delta f) + V_j f, \quad f \in \operatorname{dom}(\dot{H}_j) = C_0^\infty(\mathbb{R}^n\backslash\Sigma_j), \quad j \in J, \tag{5.13}$$

$$\dot{H} f = (-\Delta f) + V f, \quad f \in \operatorname{dom}(\dot{H}) = C_0^\infty(\mathbb{R}^n\backslash\Sigma), \tag{5.14}$$

whose closures in $L^2(\mathbb{R}^n)$ are denoted by H_j, $j \in J$, and H, respectively, and whose adjoints are then given by (cf., e.g., [43], [70, Sect. 2.1])

$$\begin{aligned} &H_j^* f = (-\Delta f) + V_j f \text{ in } \mathcal{D}(\mathbb{R}^n\backslash\Sigma_j)', \\ &f \in \operatorname{dom}(H_j^*) = \big\{g \in L^2(\mathbb{R}^n) \,\big|\, [-(\Delta g) + V_j g] \in L^2(\mathbb{R}^n)\big\}, \quad j \in J, \end{aligned} \tag{5.15}$$

$$\begin{aligned} &H^* f = (-\Delta f) + V f \text{ in } \mathcal{D}(\mathbb{R}^n\backslash\Sigma)', \\ &f \in \operatorname{dom}(H^*) = \big\{g \in L^2(\mathbb{R}^n) \,\big|\, [-(\Delta g) + V g] \in L^2(\mathbb{R}^n)\big\}. \end{aligned} \tag{5.16}$$

Next we will show that the abstract Theorem 4.2 applies to H and H_j, $j \in J$, by proving a series of results that verify each item in Hypothesis 4.1; in fact, we will typically prove slightly stronger results. Moreover, note that by Theorem 2.8 (with $\mathcal{C}$ the standard complex conjugation of complex-valued functions) one has $\operatorname{def}(H) = n_+(H) = n_-(H)$ as well as $\operatorname{def}(H_j) = n_+(H_j) = n_-(H_j)$, $j \in J$.

We start with the following auxiliary result.

Lemma 5.3. *Assume Hypothesis* 5.2. *There exist real-valued* $\phi_j, \widetilde{\phi}_j \in C_0^\infty(\mathbb{R}^n)$, $j \in J$, *such that the following conditions* (*i*)–(*v*) *hold:*
(*i*) $\partial^\alpha \phi_j \in L^\infty(\mathbb{R}^n)$, $0 \leqslant |\alpha| \leqslant 2$, $\phi_j\big|_{A_j} = 1$, $j \in J$.
(*ii*) $\operatorname{supp}(\phi_j) \cap \operatorname{supp}(\phi_{j'}) = \emptyset$, $j, j' \in J$, $j \neq j'$.
(*iii*) *For some* $0 < \delta < \varepsilon/2$, $\operatorname{dist}(\operatorname{supp}(1-\phi_j), A_j) \geqslant \delta$, $j \in J$.
(*iv*) $\partial^\alpha \widetilde{\phi}_j \in L^\infty(\mathbb{R}^n)$, $0 \leqslant |\alpha| \leqslant 2$, $\widetilde{\phi}_j\big|_{\operatorname{supp}(\phi_j)} = 1$, $j \in J$.
(*v*) $\operatorname{supp}\big(\widetilde{\phi}_j\big) \cap \operatorname{supp}\big(\widetilde{\phi}_{j'}\big) = \emptyset$, $j, j' \in J$, $j \neq j'$.

Proof. Fix $j \in J$ and define $U_{j,\eta} = \bigcup_{a \in A_j} B_n(a;\eta)$, $\eta > 0$. Then $U_{j,\varepsilon/4}$ is an open neighborhood of A_j and

$$E_{j,\varepsilon/4} = \overline{U_{j,\varepsilon/4}}, \quad F_{j,\varepsilon/4} = \mathbb{R}^n\backslash U_{j,\varepsilon/2}, \quad j \in J, \tag{5.17}$$

are closed and disjoint. By Lemma 5.1 one can find $\phi_j \in C^\infty(\mathbb{R}^n)$ such that

$$\phi_j\big|_{E_{j,\varepsilon/4}} = 1, \quad \phi_j\big|_{F_{j,\varepsilon/2}} = 0, \quad \partial^\alpha \phi_j \in L^\infty(\mathbb{R}^n), \ 0 \leqslant |\alpha| \leqslant 2. \tag{5.18}$$

It is now clear that one can choose $\delta = \varepsilon/4$. This shows the existence of $\phi_j \in C_0^\infty(\mathbb{R}^n)$, $j \in J$, satisfying properties (*i*)–(*iii*); the existence of $\widetilde{\phi}_j \in C_0^\infty(\mathbb{R}^n)$, $j \in J$, satisfying items (*iv*)–(*v*) follows analogously. $\square$

In the following we identify Φ_j and $\widetilde{\Phi}_j$ with the operator of multiplication by the bounded, real-valued functions ϕ_j and $\widetilde{\phi}_j$, $j \in J$, defined on all of $L^2(\mathbb{R}^n)$, respectively.

For simplicity, we focus on the case $n \in \mathbb{N}$, $n \geqslant 2$, throughout this section. The case $n = 1$ is obviously analogous (and by far simpler).

The next result verifies the analogs of conditions (4.3) and (4.4) (in fact, it proves additional facts).

Lemma 5.4. *Assume Hypothesis* 5.2. *Then for all $j \in J$, the following conditions (i)–(ii) hold:*
(i) $f \in \operatorname{dom}(H_j^*)$ implies $\phi_j f \in \operatorname{dom}(H_j^*) \cap \operatorname{dom}(H^*)$.
(ii) $f \in \operatorname{dom}(H^*)$ implies $\phi_j f \in \operatorname{dom}(H^*) \cap \operatorname{dom}(H_j^*)$.
In both cases,

$$H_j^*(\phi_j f) = H^*(\phi_j f), \quad j \in J. \tag{5.19}$$

All statements also hold with ϕ_j replaced by $\widetilde{\phi}_j$.

Proof. (1) Let $f \in \operatorname{dom}(H_j^*)$ and $\psi_j \in C_0^\infty(\mathbb{R}^n \backslash A_j)$, $j \in J$. Then $H_j^* f = [(-\Delta f) + V_j f] \in L^2(\mathbb{R}^n)$ implies

$$\psi_j(H_j^* f) = \psi_j(-\Delta f) \in L^2(\mathbb{R}^n), \tag{5.20}$$

which together with the fact that $\psi_j \in C_0^\infty(\mathbb{R}^n \backslash A_j)$ is arbitrary, implies that

$$\Delta f \in L^2_{\text{loc}}(\mathbb{R}^n \backslash A_j). \tag{5.21}$$

Thus (cf., e.g., [43, Theorem 1]) also

$$\nabla f \in L^2_{\text{loc}}(\mathbb{R}^n \backslash A_j)^n, \tag{5.22}$$

and hence

$$f \in H^2_{\text{loc}}(\mathbb{R}^n \backslash A_j), \tag{5.23}$$

which in turn implies

$$\operatorname{dom}(H_j^*) \subseteq H^2_{\text{loc}}(\mathbb{R}^n \backslash A_j). \tag{5.24}$$

(2) Let $f \in \operatorname{dom}(H^*)$ and $\psi \in C_0^\infty(\mathbb{R}^n \backslash A)$. Then by precisely the same arguments one concludes that $\psi(H^* f) = \psi(-\Delta f) \in L^2(\mathbb{R}^n)$, and

$$\nabla f \in L^2_{\text{loc}}(\mathbb{R}^n \backslash A)^n, \quad \Delta f \in L^2_{\text{loc}}(\mathbb{R}^n \backslash A), \tag{5.25}$$

and hence

$$\operatorname{dom}(H^*) \subseteq H^2_{\text{loc}}(\mathbb{R}^n \backslash A). \tag{5.26}$$

(3) Let $f \in \operatorname{dom}(H_j^*)$, then

$$\begin{aligned} (-\Delta + V_j)(\phi_j f) &= (-\Delta + V)(\phi_j f) \\ &= \phi_j(-\Delta + V_j) f - 2(\nabla \phi_j) \cdot (\nabla f) - (\Delta \phi_j) f. \end{aligned} \tag{5.27}$$

Since

$$\phi_j, |\nabla \phi_j|, (\Delta \phi_j) \in L^\infty(\mathbb{R}^n), \quad \nabla \phi_j \in C_0^\infty(\mathbb{R}^n \backslash A)^n, \tag{5.28}$$

in fact,

$$\nabla \phi_j\big|_{E_{j,\varepsilon/8}} = 0, \quad \nabla \phi_j\big|_{F_{j,\varepsilon/4}} = 0, \tag{5.29}$$

with $E_{j,\varepsilon/8} \supset A_j$, and $\nabla f \in L^2_{\text{loc}}(\mathbb{R}^n \backslash A_j)^n$ by item (1), one concludes that

$$(\nabla \phi_j) \cdot (\nabla f) = 0 \text{ in a neighborhood of } A_j \text{ (in fact, of } A). \tag{5.30}$$

Thus,

$$(\nabla \phi_j) \cdot (\nabla f) \in L^2(\mathbb{R}^n), \tag{5.31}$$

and hence

$$(-\Delta + V_j)(\phi_j f) = (-\Delta + V)(\phi_j f) \in L^2(\mathbb{R}^n) \tag{5.32}$$

proving item (i).
(4) Let $f \in \mathrm{dom}(H^*)$ be arbitrary. Then reasoning precisely along the lines in item (3) one obtains

$$\begin{aligned} (-\Delta + V)(\phi_j f) &= (-\Delta + V_j)(\phi_j f) \\ &= \phi_j(-\Delta + V)f - 2(\nabla\phi_j)\cdot(\nabla f) - (\Delta\phi_j)f, \quad j \in J. \end{aligned} \tag{5.33}$$

Since $\nabla f \in L^2_{\mathrm{loc}}(\mathbb{R}^n\backslash A)^n$, (5.29) with $E_{j,\varepsilon/8} \supset A_j$ once more yields $(\nabla\phi_j)\cdot(\nabla f) = 0$ in a neighborhood of A and hence

$$(\nabla\phi_j)\cdot(\nabla f) \in L^2(\mathbb{R}^n), \quad j \in J. \tag{5.34}$$

Thus,

$$(-\Delta + V)(\phi_j f) = (-\Delta + V_j)(\phi_j f) \in L^2(\mathbb{R}^n) \tag{5.35}$$

proving item (ii). Equations (5.32) and (5.35) also prove (5.19). $\square$

The following result verifies the analogs of conditions (4.5) and (4.6) (again, additional facts are derived).

Lemma 5.5. *Assume Hypothesis* 5.2. *Then for all $j \in J$, the following conditions (i)–(ii) hold:*
(i) $f \in \mathrm{dom}(H_j)$ implies $\phi_j f \in \mathrm{dom}(H_j) \cap \mathrm{dom}(H)$.
(ii) $f \in \mathrm{dom}(H)$ implies $\phi_j f \in \mathrm{dom}(H) \cap \mathrm{dom}(H_j)$.
In both cases,

$$H_j(\phi_j f) = H(\phi_j f), \quad j \in J. \tag{5.36}$$

All statements also hold with ϕ_j replaced by $\widetilde{\phi}_j$.

Proof. (1) Let $f \in \mathrm{dom}(H_j)$. Since $H_j = \overline{\dot H_j}$, there exists a sequence $\{f_m\}_{m\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n\backslash\Sigma_j)$ such that $\text{s-lim}_{m\to\infty} f_m = f$ and $\text{s-lim}_{m\to\infty} \dot H_j f_m = H_j f$. Consequently, $\phi_j f_m \in C_0^\infty(\mathbb{R}^n\backslash\Sigma_j)$, $\text{s-lim}_{m\to\infty} \phi_j f_m = \phi_j f$, and

$$\begin{aligned} \dot H_j(\phi_j f_m) &= (-\Delta + V_j)(\phi_j f_m) = (-\Delta + V)(\phi_j f_m) = \dot H(\phi_j f_m) \\ &= \phi_j(-\Delta + V_j)f_m - 2(\nabla\phi_j)\cdot(\nabla f_m) - (\Delta\phi_j)f_m. \end{aligned} \tag{5.37}$$

Clearly,

$$\operatorname*{s-lim}_{m\to\infty} \phi_j(-\Delta + V_j)f_m = \operatorname*{s-lim}_{m\to\infty} \phi_j \dot H_j f_m = \phi_j H_j f, \quad \operatorname*{s-lim}_{m\to\infty} (\Delta\phi_j) f_m = (\Delta\phi_j) f. \tag{5.38}$$

Next, let $\psi \in C_0^\infty(\mathbb{R}^n\backslash A)$ be real-valued. Then $f \in H^2_{\mathrm{loc}}(\mathbb{R}^n\backslash A_j)$ (cf. (5.24)) implies that

$$\begin{aligned} &\int_{\mathbb{R}^n} d^n x\, \psi(x)^2 |\nabla(f_m(x) - f(x))|^2 \\ &\quad = -2\int_{\mathbb{R}^n} d^n x\, \overline{[f_m(x) - f(x)]}\psi(x)(\nabla\psi)(x)\cdot(\nabla(f_m - f))(x) \\ &\qquad - \int_{\mathbb{R}^n} d^n x\, \overline{[f_m(x) - f(x)]}\psi(x)^2(\Delta(f_m - f))(x), \end{aligned} \tag{5.39}$$

and hence,

$$\begin{aligned} &\|\psi|\nabla(f_m - f)|\|^2_{L^2(\mathbb{R}^n)} \\ &\quad \leqslant 2\|\nabla\psi\|_{L^\infty(\mathbb{R}^n)}\|f_m - f\|_{L^2(\mathbb{R}^n)}\|\psi|\nabla(f_m - f)|\|_{L^2(\mathbb{R}^n)} \\ &\qquad + \|f_m - f\|_{L^2(\mathbb{R}^n)}\big\|\psi^2[\Delta(f_m - f)]\big\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{5.40}$$

Since $\psi \in C_0^\infty(\mathbb{R}^n\backslash A)$, one concludes that $\psi^2[\Delta(f_m - f)] = \psi^2[H_j(f_m - f)]$ and hence $\lim_{m\to\infty} \big\|\psi^2[\Delta(f_m - f)]\big\|_{L^2(\mathbb{R}^n;d^nx)} = 0$. Inequality (5.40) is of the type

$$A_m^2 \leqslant c_m|A_m| + d_m, \text{ with } \lim_{m\to\infty} c_m = \lim_{m\to\infty} d_m = 0. \tag{5.41}$$

Consequently, the real sequence $\{A_m\}_{m\in\mathbb{N}}$ is bounded, that is, for some $C > 0$, $|A_m| \leqslant C$, and thus, actually,

$$\lim_{m\to\infty} A_m = 0. \tag{5.42}$$

Employing (5.41), (5.42) in (5.40) then yields

$$\lim_{m\to\infty} \|\psi|\nabla(f_m - f)|\|_{L^2(\mathbb{R}^n)} = 0, \tag{5.43}$$

and choosing $\psi = (\partial_k\phi_j)$, $1 \leqslant k \leqslant n$, implies

$$\lim_{m\to\infty} \|(\nabla\phi_j)\cdot(\nabla(f_m - f))\|_{L^2(\mathbb{R}^n))} = 0. \tag{5.44}$$

Combining (5.37), (5.38), and (5.44), yields

$$\operatorname*{s-lim}_{m\to\infty} \dot{H}_j(\phi_j f_m) = \operatorname*{s-lim}_{m\to\infty} \dot{H}(\phi_j f_m) = \phi_j H_j f - 2(\nabla\phi_j)\cdot(\nabla f) - (\Delta\phi_j)f, \tag{5.45}$$

and since H_j and H are closed operators, one concludes $\phi_j f \in \operatorname{dom}(H_j)\cap\operatorname{dom}(H)$ and $H_j(\phi_j f) = H(\phi_j f)$, proving item (i).
(2) Interchanging H_j and H, Σ_j and Σ, A_j and A, noticing that $j \in J$ was arbitrary in part (1), yields item (ii) along precisely the same steps. $\square$

The next result verifies the analogs of conditions (4.7) and (4.8).

Lemma 5.6. *Assume Hypothesis* 5.2. *Then the following conditions* (i)*–*(ii) *hold:*
(i) $f \in \operatorname{dom}(H_j^*)$ *implies* $(1-\phi_j)f \in \operatorname{dom}(H_j)$, $j \in J$.
(ii) $f \in \operatorname{dom}(H^*)$ *implies* $\big(1 - \sum_{j\in J}\phi_j\big)f \in \operatorname{dom}(H)$.

Proof. (1) Let $f \in \operatorname{dom}(H_j^*)$ be arbitrary. Then $(1-\phi_j)f \in \operatorname{dom}(H_j^*)$ by Lemma 5.4 (i). Hence, $(1-\phi_j)f \in L^2(\mathbb{R}^n)$ and $H_j^*[(1-\phi_j)f] = -\Delta[(1-\phi_j)f] \in L^2(\mathbb{R}^n)$, implying

$$\nabla[(1-\phi_j)f] \in L^2(\mathbb{R}^n)^n, \tag{5.46}$$

and hence

$$(1-\phi_j)f \in H^2(\mathbb{R}^n). \tag{5.47}$$

(To verify the claims (5.46) and (5.47) it suffices to employ the standard Fourier transform $\mathcal{F}$ in $L^2(\mathbb{R}^n)$, and denoting $\widehat{u} = \mathcal{F}u$, one concludes that $\big(1+|\xi|^2\big)\widehat{u} \in L^2(\mathbb{R}^n;d^n\xi)$ implies $|\xi|\,\widehat{u} \in L^2(\mathbb{R}^n;d^n\xi)$, $\xi_\ell\,\xi_m\,\widehat{u} \in L^2(\mathbb{R}^n;d^n\xi)$, $1 \leqslant \ell, m \leqslant n$.)

Next, we recall that for $\Omega_k \subseteq \mathbb{R}^n$ open, $k = 1,2$, with $\Omega_1 \subset \Omega_2$, $f \in H^{m,2}(\Omega_2)$ implies that $f\big|_{\Omega_1} \in H^{m,2}(\Omega_1)$, $m \in \mathbb{N}\cup\{0\}$ (see, e.g., [68, p. 253–254]). Hence, using the fact that $\operatorname{dist}(\operatorname{supp}(1-\phi_j), A_j) \geqslant \varepsilon/8$, one concludes that

$$(1-\phi_j)f \in H^2(\mathbb{R}^n) \text{ implies } (1-\phi_j)f\big|_{\mathbb{R}^n\backslash A_j} \in H^2(\mathbb{R}^n\backslash A_j), \tag{5.48}$$

and an application of [17, Theorem V.3.4] yields

$$\eta_m(1-\phi_j)f \in H_0^2(\mathbb{R}^n\backslash A_j), \tag{5.49}$$

where

$$\eta_m \in C_0^\infty(\mathbb{R}^n), \quad 0 \leqslant \eta_m \leqslant 1, \quad \eta_m(x) = 1, \ |x| \leqslant m, \ m \in \mathbb{N}, \tag{5.50}$$

is a suitable cutoff function. Having established (5.49), one concludes the existence of a sequence $\{g_m\}_{m\in\mathbb{N}} \in C_0^\infty(\mathbb{R}^n\backslash A_j)$ such that $g_m \underset{m\to\infty}{\longrightarrow} (1-\phi_j)f$ in $H^2(\mathbb{R}^n\backslash A_j)$-norm. Consequently, $\operatorname{s-lim}_{m\to\infty} g_m = (1-\phi_j)f$ and

$$\operatorname*{s-lim}_{m\to\infty} \dot{H}_j g_m = \operatorname*{s-lim}_{m\to\infty} (-\Delta g_m) = -\Delta[(1-\phi_j)f] = H_j^*[(1-\phi_j)f] \in L^2(\mathbb{R}^n), \quad (5.51)$$

implying

$$H_j[(1-\phi_j)f] = H_j^*[(1-\phi_j)f] \in L^2(\mathbb{R}^n), \qquad (5.52)$$

since $H_j = \overline{\dot{H}_j}$ is a closed operator. This proves item (i).
(2) Replacing H_j^* by H^*, $\dot{H}_j$ by $\dot{H}$, A_j by A, $(1-\phi_j)$ by $\big(1-\sum_{j\in J}\phi_j\big)$, and noticing that $j \in J$ was arbitrary in step (1), one can now follow the strategy of proof above line by line to arrive at a proof of item (ii). $\square$

For variants of Lemmas 5.4–5.6 (with somewhat different proofs) we refer to [9], [11], [17, Sects. VII.2, VII.3], [52], [64], and [66]. These results clearly demonstrate the local nature of the operators H_j, H_j^*, $j\in J$, H, and H^* (cf. also [2, Sect. 2.5], [65, Sects. 6.4, 10.2]).

Finally, we now prove that also condition (4.9) holds in the present context of Schrödinger operators.

Lemma 5.7. *Assume Hypothesis* 5.2. *If* $f \in \operatorname{dom}(H^*)$, *then*

$$\operatorname*{s-lim}_{N\to\infty} H^*\bigg(\sum_{\substack{j\in J\\ |j|\leqslant N}} \phi_j f\bigg) = H^*\bigg(\sum_{j\in J}\phi_j f\bigg). \qquad (5.53)$$

Proof. Given $f\in\operatorname{dom}(H^*)$, it suffices to write

$$H^*\bigg(\sum_{\substack{j\in J\\ |j|\leqslant N}} \phi_j f\bigg) = \bigg(\sum_{\substack{j\in J\\ |j|\leqslant N}} \phi_j\bigg) H^* f - 2\sum_{\substack{j\in J\\ |j|\leqslant N}} (\nabla\phi_j)\cdot(\nabla f) - \bigg(\sum_{\substack{j\in J\\ |j|\leqslant N}} (\Delta\phi_j)\bigg) f, \qquad (5.54)$$

noticing that $\nabla\phi_j \in C_0^\infty(\mathbb{R}^n\backslash A)^n$, $\nabla f\in L^2_{\rm loc}(\mathbb{R}^n\backslash A)^n$, and $\operatorname{supp}(\phi_j)\cap\operatorname{supp}(\phi_k) = \emptyset$ for $j,k\in J$, $j\neq k$, and hence,

$$\begin{aligned} &\operatorname*{s-lim}_{N\to\infty} \sum_{\substack{j\in J\\ |j|\leqslant N}} \phi_j = \sum_{j\in J}\phi_j, \quad \operatorname*{s-lim}_{N\to\infty} \sum_{\substack{j\in J\\ |j|\leqslant N}} (\partial_k\phi_j) = \sum_{j\in J}(\partial_k\phi_j), \ 1\leqslant k\leqslant n, \\ &\operatorname*{s-lim}_{N\to\infty} \sum_{\substack{j\in J\\ |j|\leqslant N}} (\Delta\phi_j) = \sum_{j\in J}(\Delta\phi_j), \end{aligned} \qquad (5.55)$$

concluding the proof. $\square$

Combining Lemmas 5.3–5.7 then shows that all items in Hypothesis 4.1 are satisfied and hence Theorem 4.2 yields the following result for Schrödinger operators with a possibly strongly singular potential:

Theorem 5.8. *Assume Hypothesis* 5.2. *Then*

$$\operatorname{def}(H) = \sum_{j\in J}\operatorname{def}(H_j), \qquad (5.56)$$

including the possibility that one, and hence both sides of (5.56) *equal* ∞.

Remark 5.9. The statement of Theorem 5.8 remains valid if one adds L^∞ potentials to the singular potentials V and/or V_j, $j \in J$. Indeed, this follows directly from the stability of the deficiency indices under perturbations (see, for example, [4]). This will be particularly relevant in the discussion on Example 5.12. $\diamond$

Next, we extend Theorem 5.8 to more general second-order elliptic partial differential operators in $L^2_w(\mathbb{R}^n)$ with possibly strongly singular potential coefficients as follows. For simplicity, we will again focus on the case $n \in \mathbb{N}$, $n \geqslant 2$, only.

Hypothesis 5.10. *Let $n \in \mathbb{N}$, $n \geqslant 2$. In addition to Hypothesis 5.2, assume the following conditions on the coefficients w, $a_{k,\ell}$, b_k, $1 \leqslant k, \ell \leqslant n$, and V_j, $j \in J$.*
(i) $w \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, $w > 0$ a.e. on $\mathbb{R}^n$, and $w^{-1} \in L^\infty_{\text{loc}}(\mathbb{R}^n)$.
(ii) $a_{k,\ell} = a_{\ell,k} \in C^2(\mathbb{R}^n)$ are real-valued, $1 \leqslant k, \ell \leqslant n$ and $a = \{a_{j,k}\}_{1 \leqslant j,k \leqslant n}$ satisfies the local uniform ellipticity condition,

$$\sum_{1 \leqslant k,\ell \leqslant n} a_{k,\ell}(x)\xi_k\xi_\ell \geqslant \lambda(x)\|\xi\|^2_{\mathbb{R}^n} \quad x, \xi \in \mathbb{R}^n, \tag{5.57}$$

where $\lambda > 0$ and continuous on $\mathbb{R}^n$.
(iii) $b_k \in C^1(\mathbb{R}^n)$ are real-valued, $1 \leqslant k \leqslant n$.
(iv) For fixed $\varepsilon_0 > 0$ and $\alpha_j \in [\varepsilon_0, 1]$, $V_j \in Q_{\alpha_j,\text{loc}}(\mathbb{R}^n)$, $j \in J$.
(v) Suppose the maximally defined operator in $L^2_w(\mathbb{R}^n)$ generated by the differential expression

$$L_0(a,b) := \frac{1}{w(x)} \sum_{1 \leqslant k,\ell \leqslant n} D_k a_{k,\ell}(x) D_\ell, \quad D_k = i\partial_k + b_k(x),\ 1 \leqslant k \leqslant n, \quad x \in \mathbb{R}^n, \tag{5.58}$$

is self-adjoint, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, more precisely, assume that

$$\dot{H}_0(a,b)f = L_0(a,b)f, \quad f \in \operatorname{dom}(\dot{H}_0(a,b)) = C_0^\infty(\mathbb{R}^n), \tag{5.59}$$

is essentially self-adjoint in $L^2_w(\mathbb{R}^n)$, and its closure, denoted by $H_0(a,b)$,

$$\begin{aligned} &H_0(a,b)f = \overline{\dot{H}_0(a,b)}f = L_0(a,b)f, \\ &f \in \operatorname{dom}(H_0(a,b)) = \big\{g \in L^2_w(\mathbb{R}^n) \,\big|\, g \in H^2_{\text{loc}}(\mathbb{R}^n);\ L_0(a,b)g \in L^2_w(\mathbb{R}^n)\big\}, \end{aligned} \tag{5.60}$$

is self-adjoint in $L^2_w(\mathbb{R}^n)$.

In this context, the local Stummel space $Q_{\alpha,\text{loc}}(\mathbb{R}^n)$, $\alpha \in (0,1]$ is defined by

$$Q_{\alpha,\text{loc}}(\mathbb{R}^n) := \begin{cases} L^2_{\text{loc}}(\mathbb{R}^n), \quad n = 2,3, \\ \Big\{W \in L^2_{\text{loc}}(\mathbb{R}^n) \,\Big|\, \text{for all } K \in \mathbb{R}^n \text{ compact, there exists} \\ \qquad C_{W,K} > 0 \text{ such that for all } x \in K, \\ \quad \int_{K \cap \overline{B_n(x;1)}} d^n y\, |x-y|^{4-n-\alpha} |W(y)|^2 \leqslant C_{W,f}\Big\}, \quad n \geqslant 4. \end{cases} \tag{5.61}$$

Since the principal target in this paper are strongly singular (electric) potentials V_j, $j \in J$, V, we introduced Hypothesis 5.10 (v) for the second-order part $-\frac{1}{w}\sum_{1 \leqslant k,\ell \leqslant n} D_k a_{j,k} D_\ell$. For a variety of sufficient conditions on $a_{k,\ell}$, b_k, $1 \leqslant k, \ell \leqslant n$, for Hypothesis 5.10 (v) to hold, we refer, for instance, to [80].

Given Hypotheses 5.2 and 5.10, we now introduce in analogy to (5.13), (5.14), the following symmetric operators in $L^2_w(\mathbb{R}^n)$,

$$\dot H_j(a,b)f = L_0(a,b)f + (V_j + V_{0,j})f, \quad f \in \mathrm{dom}(\dot H_j(a,b)) = C_0^\infty(\mathbb{R}^n\backslash\Sigma_j), \quad j \in J, \tag{5.62}$$

$$\dot H(a,b)f = L_0(a,b)f + (V + V_0)f, \quad f \in \mathrm{dom}(\dot H(a,b)) = C_0^\infty(\mathbb{R}^n\backslash\Sigma), \tag{5.63}$$

whose closures in $L^2_w(\mathbb{R}^n)$ are denoted by $H_j(a,b)$, $j \in J$, and $H(a,b)$, respectively. Their adjoints are then given by (cf. [80])

$$\begin{aligned} &H_j(a,b)^*f = L_0(a,b)f + (V_j + V_{0,j})f \text{ in } \mathcal{D}(\mathbb{R}^n\backslash\Sigma_j)', \\ &f \in \mathrm{dom}(H_j(a,b)^*) = \big\{g \in L^2_w(\mathbb{R}^n) \,\big|\, g \in H^2_{\mathrm{loc}}\mathbb{R}^n\backslash\Sigma_j); \\ &\qquad\qquad [L_0(a,b)g + (V_j + V_{0,j})g] \in L^2_w(\mathbb{R}^n)\big\}, \quad j \in J, \end{aligned} \tag{5.64}$$

$$\begin{aligned} &H(a,b)^*f = L_0(a,b)f + (V + V_0)f \text{ in } \mathcal{D}(\mathbb{R}^n\backslash\Sigma)', \\ &f \in \mathrm{dom}(H(a,b)^*) = \big\{g \in L^2_w(\mathbb{R}^n) \,\big|\, g \in H^2_{\mathrm{loc}}\mathbb{R}^n\backslash\Sigma); \\ &\qquad\qquad [L_0(a,b)g + (V + V_0)g] \in L^2_w(\mathbb{R}^n)\big\}. \end{aligned} \tag{5.65}$$

Repeatedly employing product identities of the type

$$\begin{aligned} L_0(a,b)(\psi f) = \psi L_0(a,b)f &+ \frac{2i}{w}\sum_{1\leqslant k,\ell\leqslant n}(\partial_k\psi)a_{k,\ell}(D_\ell f) \\ &- \frac{1}{w}\sum_{1\leqslant k,\ell\leqslant n}(\partial_k a_{k,\ell}\partial_\ell\psi)f, \quad f,\psi \in C^2(\mathbb{R}^n), \end{aligned} \tag{5.66}$$

and closely following the arguments leading to Theorem 5.8, the analog of the latter now reads as follows:

Theorem 5.11. *Assume Hypothesis* 5.10*. Then*

$$\mathrm{def}\,(H(a,b)) = \sum_{j\in J}\mathrm{def}(H_j(a,b)), \tag{5.67}$$

including the possibility that one, and hence both sides of (5.67) *equal* ∞*.*

Proof. It suffices to sketch the necessary modifications in the proofs of Lemmas 5.4–5.7, replacing H_j by $H_j(a,b)$, $j \in J$, and H by $H(a,b)$, respectively.
Lemma 5.4: Due to the fact $g \in H^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\Sigma_j)$, $j \in J$, respectively, $g \in H^2_{\mathrm{loc}}(\mathbb{R}^n\backslash\Sigma)$, items (1) and (2) are clear from the outset. The analog of identity (5.27) now reads

$$\begin{aligned} \frac{1}{w}\bigg(\sum_{1\leqslant k,\ell\leqslant n} D_k a_{j,k} D_\ell + V_j\bigg)(\phi_j f) &= \frac{1}{w}\bigg(\sum_{1\leqslant k,\ell\leqslant n} D_k a_{j,k} D_\ell + V\bigg)(\phi_j f) \\ = \phi_j\frac{1}{w}\bigg(\sum_{1\leqslant k,\ell\leqslant n} D_k a_{j,k} D_\ell + V_j\bigg)f &+ \frac{2i}{w}\sum_{1\leqslant k,\ell\leqslant n}(\partial_k\phi_j)a_{k,\ell}(D_\ell f) \\ - \frac{1}{w}\sum_{1\leqslant k,\ell\leqslant n}(\partial_k a_{k,\ell}\partial_\ell\phi_j)f, &\quad f \in \mathrm{dom}(H_j(a,b)^*),\ j \in J, \end{aligned} \tag{5.68}$$

and the rest of item (3) proceeds along the same lines. Item (4) follows in the same manner.
Lemma 5.5: One can follow the arguments in (5.36)–(5.45) line by line, replacing the term $(\nabla\phi_j)\cdot(\nabla f_m - f)$ by $\frac{-i}{w}\sum_{1\leqslant k,\ell\leqslant n}(\partial_k\phi_j)a_{k,\ell}D_\ell(f_m - f)$, once more choosing

$\psi = (\partial_k \phi_j)$, and observing that $(\partial_k \phi_j) \in C_0^\infty(\mathbb{R}^n \backslash A)$ (which neutralizes the effect of $b_k \in C^1(\mathbb{R}^n)$ in D_k), $1 \leqslant k \leqslant n$.

Lemma 5.6: Once more employing the fact that $g \in H^2_{\text{loc}}(\mathbb{R}^n \backslash \Sigma_j)$, $j \in J$ (resp, $g \in H^2_{\text{loc}}(\mathbb{R}^n \backslash \Sigma)$), one again obtains the crucial inclusion (5.49), and then concludes the argument along the lines of (5.50)–(5.52), substituting $\frac{1}{w} \sum_{1 \leqslant k, \ell \leqslant n} D_k a_{k,\ell}(x) D_\ell$ for $-\Delta$ in (5.51).

Lemma 5.7: Replacing identity (5.54) by

$$\begin{aligned} H(a)^* \bigg(\sum_{\substack{j \in J \\ |j| \leqslant N}} \phi_j f \bigg) &= \bigg(\sum_{\substack{j \in J \\ |j| \leqslant N}} \phi_j \bigg) H(a)^* f + \frac{2i}{w} \sum_{\substack{j \in J \\ |j| \leqslant N}} \sum_{1 \leqslant k, \ell \leqslant n} (\partial_k \phi_j) a_{k,\ell} (D_\ell f) \\ &\quad - \frac{1}{w} \sum_{\substack{j \in J \\ |j| \leqslant N}} \sum_{1 \leqslant k, \ell \leqslant n} (\partial_k a_{k,\ell} \partial_\ell \phi_j) f, \quad f \in \operatorname{dom}(H(a)^*), \end{aligned} \tag{5.69}$$

one proceeds along the lines leading up to (5.55), replacing the latter by

$$\begin{aligned} &\operatorname*{s-lim}_{N \to \infty} \sum_{\substack{j \in J \\ |j| \leqslant N}} \phi_j = \sum_{j \in J} \phi_j, \quad \operatorname*{s-lim}_{N \to \infty} \frac{1}{w} \sum_{\substack{j \in J \\ |j| \leqslant N}} (\partial_k \phi_j) = \frac{1}{w} \sum_{j \in J} (\partial_k \phi_j), \ 1 \leqslant k \leqslant n, \\ &\operatorname*{s-lim}_{N \to \infty} \frac{1}{w} \sum_{\substack{j \in J \\ |j| \leqslant N}} \sum_{1 \leqslant k, \ell \leqslant n} (\partial_k a_{k,\ell} \partial_\ell \phi_j) = \frac{1}{w} \sum_{j \in J} \sum_{1 \leqslant k, \ell \leqslant n} (\partial_k a_{k,\ell} \partial_\ell \phi_j), \end{aligned} \tag{5.70}$$

completing the proof. □

We conclude this section illustrating the scope of Theorems 5.8 and 5.11 in connection with Schrödinger-type operators H as well as second-order elliptic partial differential operators $H(a)$:

Example 5.12. *Let $J \subseteq \mathbb{N}$ be an index set.*
(i) Assume that the set of points $\{x_j\}_{j \in J} \subset \mathbb{R}^n$ satisfies

$$\inf_{\substack{j, j' \in J \\ j \neq j'}} |x_j - x_{j'}| > 0. \tag{5.71}$$

Then concrete examples of potential coefficients V with strong point-like singularities in H and $H(a,b)$ are, for instance, given by

$$\begin{aligned} V(x) := \sum_{j \in J} V_j(x) = \sum_{j \in J} c_j \left(\frac{x - x_j}{|x - x_j|} \right) |x - x_j|^{-\alpha_j} \chi_{B_n(x_j; \varepsilon/4)}(x), \\ x \in \mathbb{R}^n \backslash \{x_j\}_{j \in J}, \end{aligned} \tag{5.72}$$

where

$$\alpha_j \geqslant 0, \quad c_j \in L^\infty(S^{n-1}) \ \text{real-valued}, \quad j \in J. \tag{5.73}$$

In this case,

$$\Sigma_j = \{x_j\}, \quad j \in J, \quad \Sigma = \bigcup_{j \in J} \Sigma_j = \{x_j\}_{j \in J}, \tag{5.74}$$

and the potential V_j comprises the jth term on the right-hand side of (5.72).

(ii) As an example with strong shell-like singularities in H and $H(a,b)$ we mention, for instance,

$$\begin{aligned} V(x) \equiv \sum_{j\in J} V_j(x) = \sum_{j\in J} \beta_j ||x-x_j|-r_j|^{-\gamma_j} \chi_{B_n(x_j;\delta_j)}(x), \\ x \in \mathbb{R}^n \Big\backslash \bigcup_{j\in J} \{x\in\mathbb{R}^n \,|\, |x-x_j| = r_j\}, \end{aligned} \tag{5.75}$$

where

$$\begin{aligned} & 0 < r_j < \delta_j < \varepsilon/2, \quad \beta_j \in \mathbb{R}, \quad \gamma_j \geqslant 0, \quad j \in J, \\ & \{x\in\mathbb{R}^n \,|\, |x-x_j| \leqslant r_j\} \cap \{x\in\mathbb{R}^n \,|\, |x-x_{j'}| \leqslant r_{j'}\} = \emptyset, \quad j \neq j',\ j,j' \in J \end{aligned} \tag{5.76}$$

(e.g., for some $\eta \in (0,1/4)$, $r_j \leqslant \varepsilon\eta$, $j \in J$). In this case,

$$\begin{aligned} & \Sigma_j = \{x\in\mathbb{R}^n \,|\, |x-x_j| = r_j\}, \quad j \in J, \\ & \Sigma = \bigcup_{j\in J} \Sigma_j = \bigcup_{j\in J} \{x\in\mathbb{R}^n \,|\, |x-x_j| = r_j\}, \end{aligned} \tag{5.77}$$

and V_j comprises the jth term on the right-hand side of (5.75)*.*

These examples clearly illustrate the notion of decoupling of singularities when computing deficiency indices as long as all singularities are separated by a minimal positive distance.

As a concrete example, we conclude this section with the proof of Theorem 1.1 :

Proof of Theorem 1.1. Let $\delta > 0$ be arbitrary, and note that, for each $j \in J$, one can write,

$$V_j(|x-x_j|)\chi_{B_n(x_j;\delta)}(x) = V_{\mathrm{loc},j}(x) + V_{0,j}(x), \tag{5.78}$$

with

$$\begin{aligned} & V_{\mathrm{loc},j}(x) = V_j(|x-x_j|)\chi_{B_n(x_j;\varepsilon/2)}(x), \\ & V_{0,j}(x) = V_j(|x-x_j|)\chi_{B_n(x_j;\delta)\backslash B_n(x_j;\varepsilon/2)}(x), \quad x \in \mathbb{R}^n\backslash\{x_j\}. \end{aligned} \tag{5.79}$$

Since the supports of the functions $V_{0,j}$ form a locally finite family of sets, and the functions $V_{0,j}$ are bounded uniformly in both $x \in \mathbb{R}^n$ and $j \in J$, one concludes that their sum is well-defined and bounded,

$$\widetilde{V}_0 = V_0 + \sum_{j\in J} V_{0,j} \in L^\infty(\mathbb{R}^n). \tag{5.80}$$

Thus the potential function V from (1.2) can be written as

$$V = \widetilde{V}_0 + \sum_{j\in J} V_{\mathrm{loc},j}, \tag{5.81}$$

and so, by the main result in [4],

$$\mathrm{def}(H) = \mathrm{def}(H_{\mathrm{loc}}), \tag{5.82}$$

with the "localized" operator H_{loc} being

$$H_{\mathrm{loc}} = -\Delta + \sum_{j\in J} V_{\mathrm{loc},j}, \quad \mathrm{dom}(H_{\mathrm{loc}}) = C_0^\infty(\mathbb{R}^n\backslash\{x_j\}_{j\in J}). \tag{5.83}$$

At this point one can apply Theorem 5.8 since Hypothesis 5.2 is satisfied with $\Sigma_j = \{x_j\}$ for each $j \in J$, and the singular potentials $V_{\text{loc},j}$ as defined in (5.79). Thus, one concludes that

$$\operatorname{def}(H_{\text{loc}}) = \sum_{j\in J} \operatorname{def}(H_{\text{loc},j}), \tag{5.84}$$

with

$$H_{\text{loc},j} = -\Delta + V_{\text{loc},j}, \quad \operatorname{dom}(H_{\text{loc},j}) = C_0^\infty(\mathbb{R}^n\backslash\{x_j\}). \tag{5.85}$$

Finally, we note again that for each $j \in J$, one has $\operatorname{def}(H_{\text{loc},j}) = 0$ if and only if (1.5) holds. Indeed, since $H_{\text{loc},j}$ commutes with rotations one can use separation of variables in spherical coordinates and $\operatorname{def}(H_{\text{loc},j}) = 0$ if and only if this holds for each angular momentum operator, which in turn holds if and only if

$$\frac{(n-1)(n-3)}{4} + c_j \geqslant \frac{3}{4} \tag{5.86}$$

by [53, Theorem 2.4]. Thus,

$$\operatorname{def}(H) = \sum_{j\in J} \operatorname{def}(H_j), \tag{5.87}$$

finishing the proof of Theorem 1.1. □

As explained in the Introduction, it was this example and the expectation that uniformly separated singularities of the potential (cf. (5.71)) decouple in the context of deficiency index computations that motivated our interest in this circle of ideas.

Appendix A. The Support of an Arbitrary Function Defined in an Arbitrary Subset of $\mathbb{R}^n$

In this appendix we provide a discussion of the notion of support for arbitrary functions on arbitrary subsets of $\mathbb{R}^n$.

Definition A.1. *Given an arbitrary set $E \subseteq \mathbb{R}^n$ and an arbitrary function $f : E \to \mathbb{C} \cup \{\infty\}$, define the support of f as the set*

$$\operatorname{supp}(f) := \big\{x \in E \,|\, \textit{there is no } r > 0 \textit{ so that } f = 0 \textit{ a.e. in } B_n(x,r) \cap E\big\}, \tag{A.1}$$

where "a.e." is interpreted with respect to the n-dimensional Lebesgue measure in $\mathbb{R}^n$.

In addition, given an arbitrary set $E \subseteq \mathbb{R}^n$ introduce

$$\begin{aligned} E^+ := \{x \in \mathbb{R}^n \,|\, &\text{there is no } r > 0 \text{ so that } B_n(x,r) \cap E \text{ is contained} \\ &\text{in a set of } n\text{-dimensional Lebesgue measure zero}\}, \end{aligned} \tag{A.2}$$

and observe that

$$\mathring{E} \subseteq E^+ \subseteq \overline{E}, \tag{A.3}$$

$$E^+ = \overline{E} \quad \text{if } E \text{ is open.} \tag{A.4}$$

Throughout, if $A \subseteq \mathbb{R}^n$ is measurable, we denote by $|A|$ its n-dimensional Lebesgue measure.

Lemma A.2. *For an arbitrary set $E \subseteq \mathbb{R}^n$ and two arbitrary functions $f, g : E \to \mathbb{C} \cup \{\infty\}$, the following properties hold:*

$$E^+ \text{ is a closed subset of } \mathbb{R}^n \text{ and } |E \backslash E^+| = 0, \tag{A.5}$$

$$\text{for each } F \subseteq E \text{ the function } \chi_F : E \to \mathbb{C} \text{ satisfies } \operatorname{supp}(\chi_F) = F^+ \cap E, \tag{A.6}$$

$$\operatorname{supp}(f) \text{ is a relatively closed subset of } E^+ \cap E \text{ (hence of } E\text{)}, \tag{A.7}$$

$$f = 0 \text{ a.e. in } E \backslash \operatorname{supp}(f), \tag{A.8}$$

$$\operatorname{supp}(f) \subseteq F \text{ if } F \text{ relatively closed subset of } E \text{ and } f = 0 \text{ a.e on } E \backslash F, \tag{A.9}$$

$$\operatorname{supp}\left(f\big|_F\right) \subseteq F^+ \cap F \cap \operatorname{supp}(f) \text{ for each } F \subseteq E, \tag{A.10}$$

$$\operatorname{supp}(f) = \operatorname{supp}(g) \text{ if } f = g \text{ a.e. on } E, \tag{A.11}$$

$$\operatorname{supp}(fg) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g), \tag{A.12}$$

$$\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g). \tag{A.13}$$

In addition, if the set $E \subseteq \mathbb{R}^n$ is open and the function $f : E \to \mathbb{C}$ is continuous, then $\operatorname{supp}(f)$ *may be described as the relative closure in E of the set $\{x \in E \,|\, f(x) \neq 0\}$, which is precisely the standard notion of support in this context.*

Furthermore, if $E \subseteq \mathbb{R}^n$ is open and if $f \in L^1_{\rm loc}(E)$ then one has $\operatorname{supp}(u_f) = \operatorname{supp}(f)$*, where u_f is the distribution canonically associated with the locally integrable function f in the open set E.*

Proof. By (A.2) and the fact that the Lebesgue measure is complete, it follows that for each point $x \in \mathbb{R}^n \backslash E^+$ there exists some number $r_x > 0$ such that $|B_n(x, r_x) \cap E| = 0$. We claim that for the family $\{r_x\}_{x \in \mathbb{R}^n \backslash E^+}$ as above, one has

$$\mathbb{R}^n \backslash E^+ = \bigcup_{x \in \mathbb{R}^n \backslash E^+} B_n(x, r_x). \tag{A.14}$$

The left-to-right inclusion is obvious. To justify the opposite one, one notes that if y belongs to the right-hand side of (A.14), then $y \in \mathbb{R}^n$ and there exists some $x \in \mathbb{R}^n$ such that $|B_n(x, r_x) \cap E| = 0$ and $y \in B_n(x, r_x)$. Consequently, choosing $r := r_x - |x - y| > 0$ forces $B_n(y, r) \subseteq B_n(x, r_x)$, hence $|B_n(y, r) \cap E| = 0$. This shows that $y \notin E^+$, concluding the proof of (A.14). In turn, (A.14) implies that $\mathbb{R}^n \backslash E^+$ is open, thus E^+ is a closed subset of $\mathbb{R}^n$. This takes care of the first claim in (A.5).

Next, one observes that since $\mathbb{R}^n$ is a strongly Lindelöf space (as a second-countable topological space), the union in the right-hand side of (A.14) may be refined to a countable one. Thus, one can find a sequence of points $\{x_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^n \backslash E^+$ along with a sequence of numbers $\{r_j\}_{j \in \mathbb{N}} \subset (0, \infty)$ such that

$$|B_n(x_j, r_j) \cap E| = 0 \text{ for each } j \in \mathbb{N}, \tag{A.15}$$

$$\text{and } E \backslash E^+ = E \cap \Bigg(\bigcup_{j \in \mathbb{N}} B_n(x_j, r_j)\Bigg). \tag{A.16}$$

As such, the last claim in (A.5) readily follows from (A.15)–(A.16).

As far as (A.6) is concerned, pick some $F \subseteq E$. Then $x \in E \backslash \operatorname{supp}(\chi_F)$ if and only if $x \in E$ and there exists $r > 0$ such that $\chi_F = 0$ a.e. on $B_n(x, r) \cap E$. Since $\chi_F = 1$ on $B_n(x, r) \cap F \subseteq B_n(x, r) \cap E$, the latter condition is further equivalent to $|B_n(x, r) \cap F| = 0$, which ultimately is equivalent to $x \notin F^+$. This reasoning shows that $E\setminus \operatorname{supp}(\chi_F) = E\setminus F^+$, so by passing to complements (relative to E, and keeping in mind that $\operatorname{supp}(\chi_F) \subseteq E$) one obtains $\operatorname{supp}(\chi_F) = E\setminus(E\setminus F^+) = E \cap F^+$, concluding the proof of (A.6).

Next, we note that by design,

$$\operatorname{supp}(f) \subseteq E^+ \cap E. \tag{A.17}$$

Also, for each $x \in (E^+ \cap E)\setminus \operatorname{supp}(f)$ there exists a number $r_x > 0$ such that $f = 0$ a.e. in $B_n(x, r_x) \cap E$, and we claim that

$$(E^+ \cap E)\setminus \operatorname{supp}(f) = (E^+ \cap E) \cap \Bigg(\bigcup_{x\in(E^+\cap E)\setminus \operatorname{supp}(f)} B_n(x, r_x) \Bigg). \tag{A.18}$$

Indeed, the left-to-right inclusion is tautological, so we focus on the opposite one. In this regard, if y belongs to the right-hand side of (A.18), then $y \in E^+ \cap E$ and there exist a point $x \in E^+ \cap E$ such that $f = 0$ a.e. in $B_n(x, r_x) \cap E$ and $y \in B_n(x, r_x)$. Then, if $r := r_x - |x - y| > 0$, it follows that $B_n(y, r) \subseteq B_n(x, r_x)$, hence $f = 0$ a.e. in $B_n(y, r) \cap E$. This shows that $y \notin \operatorname{supp}(f)$, finishing the proof of (A.18). In turn, from (A.17) and (A.18) (and (A.5)) one deduces that (A.7) holds.

As before, based on the fact that $\mathbb{R}^n$ is a strongly Lindelöf space, one can find a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset (E^+ \cap E)\setminus \operatorname{supp}(f)$ and a sequence of numbers $\{r_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ such that

$$f = 0 \text{ a.e. in } B_n(x_j, r_j) \cap E \text{ for each } j \in \mathbb{N}, \tag{A.19}$$

and

$$(E^+ \cap E)\setminus \operatorname{supp}(f) = (E^+ \cap E) \cap \Bigg(\bigcup_{j\in\mathbb{N}} B_n(x_j, r_j) \Bigg). \tag{A.20}$$

Thus, (A.8) readily follows from (A.19)–(A.20) and (A.5).

Next, assume that F is a relatively closed subset of E with the property that $f = 0$ a.e. on $E\setminus F$, and pick an arbitrary point $x \in E\setminus F$. Given that $E\setminus F$ is relatively open in E, it follows that there exists a number $r > 0$ such that $B_n(x, r) \cap E \subseteq E\setminus F$. This implies $f = 0$ a.e. on $B_n(x, r) \cap E$ which, in turn, implies $x \notin \operatorname{supp}(f)$. Thus, $\operatorname{supp}(f) \subseteq F$, establishing (A.9). Finally, (A.10) is readily implied by (A.7) and (A.1), while properties (A.11)–(A.13) are seen directly from (A.1).

Suppose now that $E \subseteq \mathbb{R}^n$ is open. Then (A.1) yields

$$\begin{aligned} E\setminus \operatorname{supp}(f) = \{x \in E \,|\, &\text{there exists } r > 0 \text{ so that } B_n(x, r) \subseteq E \\ &\text{and } f = 0 \text{ a.e. in } B_n(x, r)\}. \end{aligned} \tag{A.21}$$

In particular, if $f \in L^1_{\mathrm{loc}}(E)$, from (A.21) one obtains $E\setminus \operatorname{supp}(f) = E\setminus \operatorname{supp}(u_f)$, where u_f is the distribution canonically associated with the function f in the open set E. If in addition f is continuous, then (A.21) further becomes

$$\begin{aligned} E\setminus \operatorname{supp}(f) &= \{x \in E \,|\, \text{there exists } r > 0 \text{ so that } B_n(x, r) \subseteq E \\ &\qquad\qquad \text{and } f = 0 \text{ in } B_n(x, r)\} \\ &= E\setminus\overline{\{x \in E \,|\, f(x) \neq 0\}}, \end{aligned} \tag{A.22}$$

as was to be shown. $\square$

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: gesztesyf@missouri.edu
URL: https://www.math.missouri.edu/people/gesztesy

Present address: Department of Mathematics, Baylor University, One Bear Place #97328, Waco, TX 76798-7328, USA
E-mail address: Fritz_Gesztesy@baylor.edu
URL: http://www.baylor.edu/math/index.php?id=935340

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: mitream@missouri.edu
URL: https://www.math.missouri.edu/people/mitream

Department of Mathematics, University of Illinois at Chicago, Chicago, IL, USA, and Institute of Mathematics "Simon Stoilow" of the Romanian Academy, Bucharest, Romania
E-mail address: nenciu@uic.edu
URL: http://www.math.uic.edu/~nenciu

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria
E-mail address: Gerald.Teschl@univie.ac.at
URL: http://www.mat.univie.ac.at/~gerald/