DISPERSION ESTIMATES FOR SPHERICAL SCHRÖDINGER EQUATIONS: THE EFFECT OF BOUNDARY CONDITIONS

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Dedicated with great pleasure to Petru A. Cojuhari on the occasion of his 65th birthday

ABSTRACT. We investigate the dependence of the $L^1 \to L^\infty$ dispersive estimates for one-dimensional radial Schrödinger operators on boundary conditions at 0. In contrast to the case of additive perturbations, we show that the change of a boundary condition at zero results in the change of the dispersive decay estimates if the angular momentum is positive, $l \in (0, 1/2)$. However, for nonpositive angular momenta, $l \in (-1/2, 0]$, the standard $O(|t|^{-1/2})$ decay remains true for all self-adjoint realizations.

1. Introduction

We are concerned with the one-dimensional Schrödinger equation

$$\mathrm{i}\dot{\psi}(t,x) = H_\alpha \psi(t,x), \quad H_\alpha := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

with the angular momentum $|l| < \frac{1}{2}$ and self-adjoint boundary conditions at $x = 0$ parameterized by a parameter $\alpha \in [0, \pi)$ (the definition is given in Section 2, see (2.1)–(2.2) — for recent discussion of this family of operators see [1, 4]). More precisely, we are interested in the dependence of the $L^1 \to L^\infty$ dispersive estimates associated to the evolution group $\mathrm{e}^{-\mathrm{i}tH_\alpha}$ on the parameters $\alpha \in [0, \pi)$ and $l \in (-1/2, 1/2)$.

On the whole line such results have a long tradition and we refer to Weder [22], Goldberg and Schlag [9], Egorova, Kopylova, Marchenko and Teschl [5], as well as the reviews [10, 18]. On the half line, the case $l = 0$ with a Dirichlet boundary condition was treated by Weder [23]. The case of general l and the Friedrichs boundary condition at 0 ($\alpha = 0$ in our notation)

$$\lim_{x\to 0} x^l((l+1)f(x) - xf'(x)) = 0, \qquad l \in \Big(-\frac{1}{2}, \frac{1}{2}\Big), \tag{1.2}$$

was recently considered in Kovařík and Truc [14] and they proved (see Theorem 2.4 in [14]) that

$$\|\mathrm{e}^{-\mathrm{i}tH_0}\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t \to \infty. \tag{1.3}$$

It was proved in [13] that this estimate remains true under additive perturbations. More precisely (see [13, Theorem 1.1]), let $H = H_0 + q$, where the potential q is a

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real integrable on $\mathbb{R}_+$ function. If in addition

$$\int_0^1 |q(x)|dx < \infty \quad \text{and} \quad \int_1^\infty x^{\max(2,l+1)}|q(x)|dx < \infty, \tag{1.4}$$

and there is neither a resonance nor an eigenvalue at 0, then

$$\big\|\mathrm{e}^{-\mathrm{i}tH}P_c(H)\big\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty. \tag{1.5}$$

Here $P_c(H)$ is the orthogonal projection in $L^2(\mathbb{R}_+)$ onto the continuous spectrum of H.

The main result of the present paper shows that the decay estimates (1.3) and (1.5) are no longer true for $\alpha\in(0,\pi)$ if $l\in(0,1/2)$. In other words, this means that singular rank one perturbations destroy these decay estimates if $l\in(0,1/2)$ (since the change of a boundary condition can be considered as a rank one perturbation in the resolvent sense). Namely, consider first the operator $H_{\pi/2}$, which is associated with the following boundary condition at $x=0$:

$$\lim_{x\to0} x^{-l-1}(lf(x)+xf'(x)) = 0, \qquad l\in\Big(-\frac12,\frac12\Big). \tag{1.6}$$

Theorem 1.1. *Let $|l|<1/2$. Then*

$$\|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty, \tag{1.7}$$

for all $l\in(-1/2,0]$, and

$$\|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}\|_{L^1(\mathbb{R}_+,\max(x^{-l},1))\to L^\infty(\mathbb{R}_+,\min(x^{l},1))} = \mathcal{O}(|t|^{-1/2+l}), \quad t\to\infty, \tag{1.8}$$

whenever $l\in(0,1/2)$. The last estimate is sharp.

In the remaining case $\alpha\in(0,\pi/2)\cup(\pi/2,\pi)$, the decay estimate is given by the the next theorem.

Theorem 1.2. *Let $|l|<1/2$ and $\alpha\in(0,\pi/2)\cup(\pi/2,\pi)$. Then*

$$\|\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)\|_{L^1(\mathbb{R}_+)\to L^\infty(\mathbb{R}_+)} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty, \tag{1.9}$$

for all $l\in(-1/2,0]$, and

$$\|\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)\|_{L^1(\mathbb{R}_+,\max(x^{-l},1))\to L^\infty(\mathbb{R}_+,\min(x^{l},1))} = \mathcal{O}(|t|^{-1/2}), \quad t\to\infty, \tag{1.10}$$

whenever $l\in(0,1/2)$.

Notice that in the case $l\in(0,1/2)$ we need to consider weighted L^1 and L^∞ spaces since functions contained in the domain of H_α might be unbounded near 0.

Finally, let us briefly outline the content of the paper. In the next section we define the operator H_α and collect its basic spectral properties. Section 3 contains the proof of Theorem 1.1. In particular, we compute explicitly the kernel of the evolution group $\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}$ and this enables us to prove (1.7) and (1.8) by using the estimates for Bessel functions J_ν (all necessary facts on Bessel functions are contained in Appendix A). Theorem 1.2 is proved in Section 4. Its proof is based on the use of a version of the van der Corput lemma, which is given in Appendix B. Also Appendix B contains necessary facts about the Wiener algebras $\mathcal{W}_0(\mathbb{R})$ and $\mathcal{W}(\mathbb{R})$. In the final section we formulate some sufficient conditions for a function $f(H)$ of a 1-D Schrödinger operator H to be an integral operator.

2. Self-adjoint realizations and their spectral properties

Let $l \in (-1/2, 1/2)$ and denote by $H_{\max}$ the maximal operator associated with

$$\tau = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}$$

in $L^2(\mathbb{R}_+)$. Note that τ is limit point at infinity and limit circle at $x = 0$ since $|l| < 1/2$. Therefore, self-adjoint restrictions of $H_{\max}$ (or in other words, self-adjoint realizations of τ in $L^2(\mathbb{R}_+)$) form a 1-parameter family. More precisely (see, e.g., [7] and also [1]), the following limits

$$\Gamma_0 f := \lim_{x\to 0} W_x(f, x^{l+1}), \quad \Gamma_1 f := \frac{-1}{2l+1} \lim_{x\to 0} W_x(f, x^{-l}) \tag{2.1}$$

exist and are finite for all $f \in \operatorname{dom}(H_{\max})$. Self-adjoint restrictions H_α of $H_{\max}$ are parameterized by the following boundary conditions at $x = 0$:

$$\operatorname{dom}(H_\alpha) = \{f \in \operatorname{dom}(H_{\max})\colon \sin(\alpha)\,\Gamma_1 f = \cos(\alpha)\,\Gamma_0 f\}, \quad \alpha \in [0, \pi). \tag{2.2}$$

Note that the case $\alpha = 0$ corresponds to the Friedrichs extension of $H_{\min} = H_{\max}^*$.

Let $\phi(z, x)$ and $\theta(z, x)$ be the fundamental system of solutions of $\tau u = zu$ given by

$$\begin{aligned} \phi(z,x) &= C_l^{-1}\sqrt{\frac{\pi x}{2}}\, z^{-\frac{2l+1}{4}} J_{l+\frac12}(\sqrt{z}x), \\ \theta(z,x) &= C_l\sqrt{\frac{\pi x}{2}}\, \frac{z^{\frac{2l+1}{4}}}{\sin((l+\frac12)\pi)} J_{-l-\frac12}(\sqrt{z}x), \end{aligned} \tag{2.3}$$

where J_ν is the Bessel function of order ν (see Appendix A) and

$$C_l = \frac{\sqrt{\pi}}{\Gamma(l+\frac32)2^{l+1}}. \tag{2.4}$$

The Weyl solution normalized by $\Gamma_0\psi = 1$ is given by

$$\psi(z,x) = \theta(z,x) + m(z)\phi(z,x) = C_l \mathrm{i} z^{\frac{2l+1}{4}} \sqrt{\frac{\pi x}{2}} H^{(1)}_{l+1/2}(\sqrt{z}x) \in L^2(0,\infty), \tag{2.5}$$

where $H^{(1)}_\nu$ is the Hankel function of the first kind [17, Chapter X.2], and

$$m(z) = -C_l^2 \frac{(-z)^{l+1/2}}{\sin((l+\frac12)\pi)}, \quad z \in \mathbb{C}\setminus\mathbb{R}_+, \tag{2.6}$$

is the Weyl function associated with H_0. Here the branch cut of the root is taken along the negative real axis. Notice that

$$d\rho(\lambda) = \frac{C_l^2}{\pi}\mathbb{1}_{[0,\infty)}(\lambda)\lambda^{l+\frac12}\,d\lambda \tag{2.7}$$

is the corresponding spectral measure. It follows from (A.1) that

$$\phi(z,x) = x^{l+1}(1+o(1)), \quad \theta(z,x) = \frac{x^{-l}}{2l+1}(1+o(1)),$$

as $x \to 0$ and, moreover,

$$\Gamma_0\theta = \Gamma_1\phi = 1, \quad \Gamma_1\theta = \Gamma_0\phi = 0.$$

Set

$$\begin{aligned}\phi_\alpha(z,x) &:= \cos(\alpha)\phi(z,x) + \sin(\alpha)\theta(z,x),\\ \theta_\alpha(z,x) &:= \cos(\alpha)\theta(z,x) - \sin(\alpha)\phi(z,x),\end{aligned} \tag{2.8}$$

for all $z \in \mathbb{C}$. Therefore, $W(\theta_\alpha, \phi_\alpha) = 1$ and

$$\psi_\alpha(z,x) := \theta_\alpha(z,x) + m_\alpha(z)\phi_\alpha(z,x), \quad m_\alpha(z) = \frac{m(z)\cos(\alpha) + \sin(\alpha)}{\cos(\alpha) - m(z)\sin(\alpha)}, \tag{2.9}$$

is a Weyl solution normalized by $W(\psi_\alpha, \phi_\alpha) = 1$. Hence

$$G_\alpha(z;x,y) = \begin{cases} \phi_\alpha(z,x)\psi_\alpha(z,y), & x \le y,\\ \phi_\alpha(z,x)\psi_\alpha(z,y), & x \ge y,\end{cases} \tag{2.10}$$

is the Green's function of H_α. The absolutely continuous spectrum remains unchanged, $\sigma_{\mathrm{ac}}(H_\alpha) = [0,\infty)$, but there is one additional eigenvalue

$$E_\alpha = -\left(\frac{\cot(\alpha)\cos(l\pi)}{C_l^2}\right)^{\frac{2}{2l+1}} \tag{2.11}$$

if $\frac{\pi}{2} < \alpha < \pi$. Finally, since

$$\operatorname{Im} m_\alpha(z) = \frac{\operatorname{Im} m(z)}{|\cos(\alpha) - m(z)\sin(\alpha)|^2}, \tag{2.12}$$

we get the absolutely continuous part of the corresponding spectral measure of the operator H_α:

$$\begin{aligned}\rho_\alpha'(\lambda)d\lambda &= \frac{1}{\pi}\operatorname{Im} m_\alpha(\lambda + \mathrm{i}0)d\lambda\\ &= \frac{1}{\pi}\frac{C_l^2\lambda^{l+1/2}\mathbb{1}_{[0,\infty)}(\lambda)}{(\cos(\alpha) - C_l^2\sin(\alpha)\tan(\pi l)\lambda^{l+1/2})^2 + C_l^4\sin^2(\alpha)\lambda^{2l+1}}d\lambda.\end{aligned} \tag{2.13}$$

3. Proof of Theorem 1.1

Similar to the case $\alpha = 0$ (see [14]), the kernel of the evolution group $\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}$ can be computed explicitly.

Lemma 3.1. *Let $|l| < 1/2$. Then the evolution group $\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}$ is an integral operator for all $t \ne 0$ and its kernel is given by*

$$[\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}](x,y) = \frac{\mathrm{i}^{l-1/2}}{2t}\mathrm{e}^{\mathrm{i}\frac{x^2+y^2}{4t}}\sqrt{xy}\, J_{-l-1/2}\left(\frac{xy}{2t}\right), \tag{3.1}$$

for all x, $y > 0$ and $t \ne 0$.

Proof. First, notice that

$$\phi_{\pi/2}(z,x) = \theta(z,x), \quad m_{\pi/2}(z) = -1/m(z),$$

and then define the spectral transformation $U\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+;\rho_{\pi/2})$ by

$$U\colon f \mapsto \hat f, \quad \hat f(\lambda) := \int_{\mathbb{R}_+}\theta(\lambda,x)f(x)dx,$$

for every $f \in L^2_c(\mathbb{R}_+)$. Notice that U extends to an isometry on $L^2(\mathbb{R}_+)$ and its inverse $U^{-1}\colon L^2(\mathbb{R}_+;\rho_{\pi/2}) \to L^2(\mathbb{R}_+)$ is given by

$$U^{-1}\colon g \mapsto \check g, \quad \check g(x) := \int_{\mathbb{R}_+}\theta(\lambda,x)g(\lambda)d\rho_{\pi/2}(\lambda),$$

for all $g \in L^2_c(\mathbb{R}_+;\rho_{\pi/2})$. Therefore, we get by using (2.3) and (2.13)

$$\begin{aligned}(\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}f)(x) =&(U^{-1}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}Uf)(x) = (U^{-1}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\check{f})(x)\\ &= \int_{\mathbb{R}_+}\theta(\lambda,x)\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\int_{\mathbb{R}_+}\theta(\lambda,y)f(y)\,dy\,d\rho_{\pi/2}(\lambda)\\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\frac{\sqrt{xy}}{2}J_{-l-\frac{1}{2}}(\sqrt{\lambda}x)J_{-l-\frac{1}{2}}(\sqrt{\lambda}y)f(y)\,dy\,d\lambda.\end{aligned}$$

Since $|l|<1/2$, (A.1) implies that

$$|J_{-l-1/2}(k)| \le \frac{2^{l+1/2}}{\Gamma(1/2-l)k^{l+1/2}}(1+\mathcal{O}(k)) \tag{3.2}$$

as $k\to 0$. Noting that $f\in L^2_c(\mathbb{R}_+)$ and using (3.2), Fubini's theorem implies

$$(\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}f)(x) = \int_{\mathbb{R}_+}f(y)\int_{\mathbb{R}_+}\mathrm{e}^{-(\mathrm{i}t+\varepsilon)\lambda}\frac{\sqrt{xy}}{2}J_{-l-\frac{1}{2}}(\sqrt{\lambda}x)J_{-l-\frac{1}{2}}(\sqrt{\lambda}y)d\lambda\,dy. \tag{3.3}$$

The integral

$$[\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}](x,y) := \frac{\sqrt{xy}}{2}\int_0^\infty \mathrm{e}^{-\mathrm{i}t\lambda}J_{-l-\frac{1}{2}}(\sqrt{\lambda}x)J_{-l-\frac{1}{2}}(\sqrt{\lambda}y)d\lambda \tag{3.4}$$

is known as Weber's second exponential integral [21, §13.31] (cf. also [6, (4.14.39)]) and hence

$$(\mathrm{e}^{-(\mathrm{i}t+\varepsilon)H_{\pi/2}}f)(x) = \frac{1}{\varepsilon+\mathrm{i}t}\int_0^\infty \mathrm{e}^{-\frac{x^2+y^2}{4(\varepsilon+\mathrm{i}t)}}\frac{\sqrt{xy}}{2}I_{-l-\frac{1}{2}}\Big(\frac{xy}{2(\varepsilon+\mathrm{i}t)}\Big)f(y)dy,$$

where I_ν is the modified Bessel function (see [17, Chapter X] and in particular formula (10.27.6) there)

$$I_\nu(z) = \sum_{n=0}^\infty \frac{(z/2)^{\nu+2n}}{n!\Gamma(\nu+m+1)} = \mathrm{e}^{\mp\mathrm{i}\nu\pi/2}J_\nu(\pm\mathrm{i}z), \quad -\pi\le\arg(z)\le\pi/2. \tag{3.5}$$

The estimate (A.2) implies

$$|J_{-l-1/2}(k)| \le k^{-1/2}(1+\mathcal{O}(k^{-1})) \tag{3.6}$$

as $k\to\infty$. Therefore, there is $C>0$ which depends only on l and such that

$$|\sqrt{k}J_{-l-1/2}(k)| \le C\left(\frac{1+k}{k}\right)^l, \quad k>0. \tag{3.7}$$

By (3.7) we deduce

$$\frac{\sqrt{xy}}{2|\varepsilon+\mathrm{i}t|}\left|\mathrm{e}^{-\frac{x^2+y^2}{4(\varepsilon+\mathrm{i}t)}}I_{-l-\frac{1}{2}}\Big(\frac{xy}{2(\varepsilon+\mathrm{i}t)}\Big)\right| \le C\sqrt{\frac{1}{|\varepsilon+\mathrm{i}t|}}\left|1+\frac{2(\varepsilon+\mathrm{i}t)}{xy}\right|^l,$$

which is uniformly (wrt. ε) bounded on compact sets $K\subset\subset\mathbb{R}_+\times\mathbb{R}_+$. Thus we can apply dominated convergence and hence the claim follows. □

In particular, we immediately arrive at the following estimate.

Corollary 3.2. *Let $|l| < 1/2$. Then there is a constant $C > 0$ which depends only on l and such that the inequality*

$$\left|[\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}](x,y)\right| \le \frac{C}{\sqrt{2t}} \left(\frac{2t+xy}{xy}\right)^l \tag{3.8}$$

holds for all x, $y > 0$ and $t > 0$.

Proof. Applying (3.7) to (3.1), we arrive at (3.8). □

Remark 3.3. *For any fixed x and $y \in \mathbb{R}_+$, we get from* (A.1)

$$\left|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}(x,y)\right| \sim \frac{\sqrt{xy}}{2t}\left(\frac{xy}{4t}\right)^{-l-1/2} = \frac{1}{t^{1/2-l}}\left(\frac{xy}{2}\right)^{-l} \tag{3.9}$$

Moreover, in view of (A.1) *one can see that*

$$\left|\mathrm{e}^{-\mathrm{i}tH_{\pi/2}}(x,y)\right| \ge c_l\, t^{l-1/2}\left(\frac{xy}{2}\right)^{-l}, \tag{3.10}$$

whenever $xy < t$ with some constant $c_l > 0$, which depends only on l.

Now we are ready to prove our first main result.

Proof of Theorem 1.1. If $l \in (-1/2, 0]$, then

$$\left(\frac{2t+xy}{xy}\right)^l \le 1$$

for all x,$y > 0$ and $t \ge 0$. This immediately implies (1.7).

Assume now that $l \in (0, 1/2)$. Clearly,

$$\frac{2t+xy}{xy} = 1 + 2\frac{t}{xy} \le 3t\max(x^{-1},1)\max(y^{-1},1)$$

for all $t \ge 1$ and x, $y > 0$. Indeed, the latter follows from the weaker estimate

$$\frac{t}{xy} \le t\max(x^{-1},1)\max(y^{-1},1), \quad t \ge 1,\ x,y > 0,$$

which is equivalent to $1 \le \max(x,1)\max(y,1)$ for all x, $y > 0$. Therefore,

$$\left(\frac{2t+xy}{xy}\right)^l \le 3t^l\max(x^{-l},1)\max(y^{-l},1), \quad t \ge 1,\ x,y > 0,$$

which proves (1.8). Remark 3.3 shows that (1.8) is sharp. □

4. Proof of Theorem 1.2

Let us consider the following improper integrals:

$$I_1(t;x,y) := \sqrt{xy}\int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J_{l+\frac12}(kx)J_{l+\frac12}(ky)\,\mathrm{Im}\, m_\alpha(k^2)\, k^{-2l}dk, \tag{4.1}$$

$$I_2(t;x,y) := \sqrt{xy}\int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J_{l+\frac12}(kx)J_{-l-\frac12}(ky)\,\mathrm{Im}\, m_\alpha(k^2)\, kdk, \tag{4.2}$$

$$I_3(t;x,y) := \sqrt{xy}\int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J_{-l-\frac12}(kx)J_{-l-\frac12}(ky)\,\mathrm{Im}\, m_\alpha(k^2)\, k^{2l+2}dk, \tag{4.3}$$

where x, $y > 0$ and $t \ne 0$. Moreover, here and below we shall use the convention $\mathrm{Im}\, m_\alpha(k^2) := \mathrm{Im}\, m_\alpha(k^2 + \mathrm{i}0) = \lim_{\varepsilon\downarrow 0}\mathrm{Im}\, m_\alpha(k^2 + \mathrm{i}\varepsilon)$ for all $k \in \mathbb{R}$. Denote the corresponding integrand by A_j, that is, $I_j(t) = \int_{\mathbb{R}_+}\mathrm{e}^{-\mathrm{i}tk^2}A_j(k;x,y)dk$. Our aim is to use Lemma B.2 (plus the remarks after this lemma) and hence we need to show that each A_j belongs to the Wiener algebra $\mathcal{W}(\mathbb{R})$, that is, coincide with a function which is the Fourier transform of a finite measure.

We also need the following estimates, which follow from (2.13)

$$\operatorname{Im} m_\alpha(k^2) = \begin{cases} C_l^2|k|^{2l+1}, & \alpha = 0, \\ \frac{\cos^2(\pi l)}{C_l^2\sin^2(\alpha)}|k|^{-2l-1} + \mathcal{O}(|k|^{-4l-2}), & \alpha \neq 0, \end{cases} \quad k \to \infty, \tag{4.4}$$

and

$$\operatorname{Im} m_\alpha(k^2) = \begin{cases} \frac{C_l^2}{\cos(\alpha)^2}|k|^{2l+1} + \mathcal{O}(|k|^{4l+2}), & \alpha \neq \pi/2, \\ C_l^{-2}\cos^2(\pi l)|k|^{-2l-1}, & \alpha = \pi/2, \end{cases} \quad k \to 0. \tag{4.5}$$

4.1. **The integral I_1.** Consider the function

$$J(r) := \sqrt{r}\, J_{l+\frac{1}{2}}(r) = \frac{r^{l+1}}{2^{l+1/2}} \sum_{n=0}^{\infty} \frac{(-r^2/4)^n}{n!\Gamma(\nu+n+1)}, \quad r \geq 0.$$

Note that $J(r) \sim r^{l+1}$ as $r \to 0$ and $J(r) = \sqrt{\frac{2}{\pi}}\sin(r - \frac{l\pi}{2}) + O(r^{-1})$ as $r \to +\infty$ (see (A.2)). Moreover, $J'(r) \sim r^l$ as $r \to 0$ and $J'(r) = \sqrt{\frac{2}{\pi}}\cos(r - \frac{l\pi}{2}) + O(r^{-1})$ as $r \to +\infty$ (see (A.4)). In particular, $\tilde{J}(r) := J(r) - \sqrt{\frac{2}{\pi}}\sin(r - \frac{l\pi}{2})$ is in $H^1(\mathbb{R}_+)$. Moreover, we can define $J(r)$ for $r < 0$ such that it is locally in H^1 and $J(r) = \sqrt{\frac{2}{\pi}}\sin(r - \frac{l\pi}{2})$ for $r < -1$. By construction we then have $\tilde{J} \in H^1(\mathbb{R})$ and thus $\tilde{J}$ is the Fourier transform of an integrable function (see Lemma B.3). Moreover, $\sin(r - \frac{l\pi}{2})$ is the Fourier transform of the sum of two Dirac delta measures and so J is the Fourier transform of a finite measure. By scaling, the total variation of the measures corresponding to $J(kx)$ is independent of x.

Next consider the function

$$F(k) := \frac{\operatorname{Im} m_\alpha(k^2)}{|k|^{2l+1}} = \frac{C_l^2}{(\cos(\alpha) - C_l^2\sin(\alpha)\tan(\pi l)|k|^{2l+1})^2 + C_l^4\sin^2(\alpha)|k|^{4l+2}}.$$

By Corollary B.6, F is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$.

Now it remains to note that

$$I_1(t) = \int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} A_1(k^2; x, y)dk = \int_{\mathbb{R}_+} \mathrm{e}^{-\mathrm{i}tk^2} J(kx)J(ky)F(k)dk, \tag{4.6}$$

and applying Lemma B.2 we end up with the estimate

$$|I_1(t; x, y)| \leq Ct^{-1/2}, \quad t > 0, \tag{4.7}$$

with a positive constant $C > 0$ independent of x, $y > 0$.

4.2. **The integral I_2.** Assume first that $l \in (0, 1/2)$ and write

$$A_2(k^2; x, y) = J(kx)Y(ky)\frac{\chi_l(k)}{\chi_l(ky)}\frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l(k)},$$

where

$$J(r) = \sqrt{r}\, J_{l+\frac{1}{2}}(r), \quad Y(r) = \chi_l(r)\sqrt{r}\, J_{-l-\frac{1}{2}}(r), \quad \chi_l(r) = \frac{|r|^l}{1+|r|^l}.$$

The asymptotic behavior (4.4) and (4.5) of $\operatorname{Im} m_\alpha$ shows that

$$M(k) = \frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l(k)} = \begin{cases} |k|^{1+l}, & k \to 0, \\ |k|^{-2l-1}, & |k| \to \infty, \end{cases}$$

and hence $M \in H^1(\mathbb{R})$, which implies that M is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$.

We continue $J(r)$, $Y(r)$ to the region $r < 0$ such that they are continuously differentiable and satisfy

$$J(r) = \sqrt{\frac{2}{\pi}} \sin\left(r - \frac{\pi l}{2}\right), \quad Y(r) = \sqrt{\frac{2}{\pi}} \cos\left(r + \frac{\pi l}{2}\right),$$

for $r < -1$. Then $\tilde{J}(r) := J(r) - \sqrt{\frac{2}{\pi}} \sin(r - \frac{\pi l}{2})$ and $\tilde{Y}(r) := Y(r) - \sqrt{\frac{2}{\pi}} \cos\left(r + \frac{\pi l}{2}\right)$ are in $H^1(\mathbb{R})$. In fact, they are continuously differentiable and hence it suffices to look at their asymptotic behavior. For $r < -1$ they are zero and for $r > 1$ they are $O(r^{-1})$ and their derivative is $O(r^{-1})$ as can be seen from the asymptotic behavior of Bessel functions (see Appendix A). Hence both J and Y are Fourier transforms of finite measures. By scaling the total variation of the measures corresponding to $J(kx)$ and $Y(ky)$ are independent of x and y, respectively.

It remains to consider the function $\chi_l(k)/\chi_l(ky)$. Observe that

$$h_{y,l}(k) := 1 - \frac{\chi_l(k)}{\chi_l(ky)} = 1 - \frac{1 + |ky|^l}{y^l + |ky|^l} = \frac{1 - y^{-l}}{1 + |k|^l} = (1 - y^{-l})(1 - \chi_l(k)).$$

By Corollary B.6, $1 - \chi_l \in \mathcal{W}_0(\mathbb{R})$. Therefore, applying Lemma B.2, we obtain the following estimate

$$|I_2(t; x, y)| \le C t^{-1/2} \max(1, y^{-l}), \quad t > 0, \tag{4.8}$$

whenever $l \in (0, 1/2)$.

Consider now the remaining case $l \in (-1/2, 0]$. Write

$$A_2(k^2; x, y) = J(kx) Y(ky) \operatorname{Im} m_\alpha(k^2),$$

where

$$J(r) = \sqrt{r}\, J_{l+\frac{1}{2}}(r), \quad Y(r) = \sqrt{r}\, J_{-l-\frac{1}{2}}(r).$$

Noting that $Y(r) \sim r^{-l}$ as $r \to 0$ and using Lemma B.3, we can continue J and Y to the region $r < 0$ such that both J and Y are Fourier transforms of finite measures.

It remains to consider $\operatorname{Im} m_\alpha(k^2)$ given by (2.13). However, by Corollary B.6, this function is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$ and hence applying Lemma B.2, we end up with the estimate

$$|I_2(t; x, y)| \le C t^{-1/2}, \quad t > 0, \tag{4.9}$$

whenever $l \in (-1/2, 0]$.

4.3. **The integral I_3.** Again let us consider two cases. Assume first that $l \in (-1/2, 0]$ and then write

$$A_3(k^2; x, y) = Y(kx) Y(ky) \operatorname{Im} m_\alpha(k^2) k^{2l+1},$$

where

$$Y(r) = \sqrt{r}\, J_{-l-\frac{1}{2}}(r), \quad r > 0.$$

Notice that

$$|k|^{2l+1} \operatorname{Im} m_\alpha(k^2) = \frac{C_l^2 k^{4l+2}}{(\cos(\alpha) - C_l^2 \sin(\alpha) \tan(\pi l) k^{2l+1})^2 + C_l^4 \sin^2(\alpha) k^{4l+2}},$$

which is the sum of a constant and a function of the form (B.5), and hence it belongs to the Wiener algebra $\mathcal{W}(\mathbb{R})$ by Corollary B.6. Arguing as in the previous subsection and applying Lemma B.2, we arrive at the following estimate

$$|I_3(t;x,y)| \le Ct^{-1/2}, \quad t>0, \tag{4.10}$$

whenever $l \in (-1/2, 0]$.

If $l \in (0, 1/2)$, write

$$A_3(k^2;x,y) = Y(kx)Y(ky)\frac{\chi_l(k)}{\chi_l(kx)}\frac{\chi_l(k)}{\chi_l(ky)}\frac{\operatorname{Im} m_\alpha(k^2)}{\chi_l^2(k)},$$

where

$$Y(r) = \chi_l(r)\sqrt{r}\, J_{-l-\frac{1}{2}}(r), \quad \chi_l(r) = \frac{|r|^l}{1+|r|^l}.$$

Notice that

$$\begin{aligned} M(k) :=& \frac{\operatorname{Im} m_\alpha(k^2)|k|^{2l+1}}{\chi_l^2(k)} \\ =& \frac{C_l^2|k|^{2l+2}(1+k^l)^2}{(\cos(\alpha)-C_l^2\sin(\alpha)\tan(\pi l)|k|^{2l+1})^2 + C_l^4\sin^2(\alpha)|k|^{4l+2}} \end{aligned}$$

Clearly, by Corollary B.6, $M \in \mathcal{W}(\mathbb{R})$. Therefore, similar to the previous subsection, we end up with the estimate

$$|I_3(t;x,y)| \le Ct^{-1/2}\max(1,x^{-l})\max(1,y^{-l}), \quad t>0, \tag{4.11}$$

whenever $l \in (0, 1/2)$.

4.4. **Proof of Theorem 1.2.** We begin with the representation of the integral kernel of the evolution group.

Lemma 4.1. *Let $|l| < 1/2$ and $\alpha \in [0,\pi)$. Then the evolution group $\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)$ is an integral operator and its kernel is given by*

$$[\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)](x,y) = \frac{2}{\pi}\int_{\mathbb{R}_+}\mathrm{e}^{-\mathrm{i}tk^2}\,\phi_\alpha(k^2,x)\phi_\alpha(k^2,y)\operatorname{Im} m_\alpha(k^2)k\,dk, \tag{4.12}$$

where the integral is to be understood as an improper integral.

Proof. By (2.3) and (2.8),

$$\begin{aligned} \phi_\alpha(k^2,x) &= \cos(\alpha)\phi(k^2,x) + \sin(\alpha)\theta(k^2,x) \\ &= \sqrt{\frac{\pi x}{2}}\left(C_l^{-1}\cos(\alpha)k^{-l-1/2}J_{l+\frac{1}{2}}(kx) + C_l k^{l+1/2}\frac{\sin(\alpha)}{\cos(\pi l)}J_{-l-\frac{1}{2}}(kx)\right), \end{aligned}$$

and hence

$$\begin{aligned} \phi_\alpha(k^2,x)\phi_\alpha(k^2,y) = \frac{\pi}{2}\sqrt{xy}\Bigg(&\frac{\cos^2(\alpha)}{C_l^2}k^{-2l-1}J_{l+\frac{1}{2}}(kx)J_{l+\frac{1}{2}}(ky) && (4.13) \\ &+ \frac{\sin(2\alpha)}{2\cos(\pi l)}(J_{l+\frac{1}{2}}(kx)J_{-l-\frac{1}{2}}(ky) + J_{-l-\frac{1}{2}}(kx)J_{l+\frac{1}{2}}(ky)) && (4.14) \\ &+ C_l^2 k^{2l+1}\frac{\sin^2(\alpha)}{\cos^2(\pi l)}J_{-l-\frac{1}{2}}(kx)J_{-l-\frac{1}{2}}(ky)\Bigg). && (4.15) \end{aligned}$$

By our considerations in the previous subsections, we have

$$\phi_\alpha(k^2,x)\phi_\alpha(k^2,y)\operatorname{Im} m_\alpha(k^2)k \in \mathcal{W}(\mathbb{R})$$

with norm uniformly bounded for x, y restricted to any compact subset of $(0,\infty)$. Moreover, we have $\mathrm{e}^{-\mathrm{i}(t-\mathrm{i}\varepsilon)H_\alpha}P_c(H_\alpha) \to \mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)$ as $\varepsilon \downarrow 0$ in the strong operator topology. By Lemma C.1, $\mathrm{e}^{-\mathrm{i}(t-\mathrm{i}\varepsilon)H_\alpha}P_c(H_\alpha)$ is an integral operator for all $\varepsilon > 0$ and, moreover, the kernel converges uniformly on compact sets by Lemma C.2. Hence $\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)$ is an integral operator whose kernel is given by the limits of the kernels of the approximating operators, that is, by (4.12). □

Proof of Theorem 1.2. Combining (4.7), (4.8), (4.9), (4.10) and (4.11), we arrive at the following decay estimate for the kernel of the evolution group

$$\Big|[\mathrm{e}^{-\mathrm{i}tH_\alpha}P_c(H_\alpha)](x,y)\Big| \le Ct^{-1/2} \times \begin{cases} 1, & l \in (-1/2,0], \\ \max(1,x^{-l})\max(1,y^{-l}), & l\in(0,1/2). \end{cases} \tag{4.16}$$

This completes the proof of Theorem 1.2. □

Appendix A. Bessel functions

Here we collect basic formulas and information on Bessel functions (see, e.g., [17, 21]). We start with the definition:

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{n=0}^\infty \frac{(-z^2/4)^n}{n!\Gamma(\nu+n+1)}. \tag{A.1}$$

The asymptotic behavior as $|z|\to\infty$ is given by

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\Big(\cos(z-\nu\pi/2-\pi/4) + \mathrm{e}^{|\operatorname{Im} z|}\mathcal{O}(|z|^{-1})\Big), \quad |\arg z| < \pi. \tag{A.2}$$

Noting that

$$J_\nu'(z) = -J_{\nu+1}(z) + \frac{\nu}{z}J_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z), \tag{A.3}$$

one can show that the derivative of the reminder satisfies

$$\Big(\sqrt{\frac{\pi z}{2}}J_\nu(z) - \cos(z-\frac{1}{2}\nu\pi - \frac{1}{4}\pi)\Big)' = \mathrm{e}^{|\operatorname{Im} z|}\mathcal{O}(|z|^{-1}), \quad |z|\to\infty. \tag{A.4}$$

Appendix B. The van der Corput Lemma and the Wiener algebra

We will need the classical van der Corput lemma (see, e.g., [19, page 334]):

Lemma B.1. *Consider the oscillatory integral*

$$I(t) = \int_a^b \mathrm{e}^{\mathrm{i}tk^2+\mathrm{i}ck}A(k)dk.$$

If $A\in \mathrm{AC}(a,b)$, *then*

$$|I(t)| \le C_2\,|t|^{-1/2}\,(\|A\|_\infty + \|A'\|_1), \quad |t|\ge 1,$$

where $C_2 \le 2^{8/3}$ *is a universal constant.*

Note that we can apply the above result with $(a,b) = (-\infty,\infty)$ by considering the limit $(-a,a)\to(-\infty,\infty)$.

Our proof will be based on the following variant of the van der Corput lemma (see, e.g., [13, Lemma A.2]).

Lemma B.2. *Let $(a,b) \subseteq \mathbb{R}$ and consider the oscillatory integral*

$$I(t) = \int_a^b \mathrm{e}^{\mathrm{i}tk^2} A(k) dk.$$

If $A \in \mathcal{W}(\mathbb{R})$, i.e., A is the Fourier transform of a signed measure

$$A(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} d\alpha(p),$$

then the above integral exists as an improper integral and satisfies

$$|I(t)| \le C_2 \, |t|^{-1/2} \, \|A\|_{\mathcal{W}}, \quad |t| > 0.$$

where $\|A\|_{\mathcal{W}} := \|\alpha\| = |\alpha|(\mathbb{R})$ denotes the total variation of α and C_2 is the constant from the van der Corput lemma.

In this respect we note that if A_1 and A_2 are two such functions, then (cf. p. 208 in [2])

$$(A_1 A_2)(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} d(\alpha_1 * \alpha_2)(p)$$

is associated with the convolution

$$\alpha_1 * \alpha_2(\Omega) = \iint \mathbb{1}_\Omega(x+y) d\alpha_1(x) d\alpha_2(y),$$

where $\mathbb{1}_\Omega$ is the indicator function of a set Ω. Note that

$$\|\alpha_1 * \alpha_2\| \le \|\alpha_1\| \|\alpha_2\|.$$

Let $\mathcal{W}_0(\mathbb{R})$ be the Wiener algebra of functions $C(\mathbb{R})$ which are Fourier transforms of L^1 functions,

$$\mathcal{W}_0(\mathbb{R}) = \Big\{ f \in C(\mathbb{R}) : \, f(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx} g(x) dx, \ g \in L^1(\mathbb{R}) \Big\}.$$

Clearly, $\mathcal{W}_0(\mathbb{R}) \subset \mathcal{W}(\mathbb{R})$. Moreover, by the Riemann–Lebesgue lemma, $f \in C_0(\mathbb{R})$, that is, $f(k) \to 0$ as $k \to \infty$ if $f \in \mathcal{W}_0(\mathbb{R})$. A comprehensive survey of necessary and sufficient conditions for $f \in C(\mathbb{R})$ to be in the Wiener algebras $\mathcal{W}_0(\mathbb{R})$ and $\mathcal{W}(\mathbb{R})$ can be found in [15], [16]. We need the following statements.

Lemma B.3. *If $f \in L^2(\mathbb{R})$ is locally absolutely continuous and $f' \in L^p(\mathbb{R})$ with $p \in (1,2]$, then f is in the Wiener algebra $\mathcal{W}_0(\mathbb{R})$ and*

$$\|f\|_{\mathcal{W}} \le C_p \big(\|f\|_{L^2(\mathbb{R})} + \|f'\|_{L^p(\mathbb{R})} \big), \tag{B.1}$$

where $C_p > 0$ is a positive constant, which depends only on p.

Proof. Since the Fourier transform is unitary on $L^2(\mathbb{R})$, it suffices to show that $\hat{f} \in L^1(\mathbb{R})$. First of all, the Cauchy–Schwarz inequality implies $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$ and, in particular,

$$\int_{-1}^1 |\hat{f}(\lambda)| d\lambda \le \sqrt{2} \left(\int_{-1}^1 |\hat{f}(\lambda)|^{1/2} d\lambda \right)^2 \le \sqrt{2} \|f\|_{L^2(\mathbb{R})}. \tag{B.2}$$

On the other hand, $f' \in L^p(\mathbb{R})$ and hence the Hausdorff–Young inequality implies $\lambda \hat{f}(\lambda) \in L^q(\mathbb{R})$ with $1/p + 1/q = 1$. Applying the Hölder inequality and then the

Hausdorff–Young inequality once again, we get

$$\begin{aligned}\int_{|\lambda|>1} |\hat{f}(\lambda)|d\lambda &\le 2\int_{|\lambda|>1} \frac{1}{1+|\lambda|}|\lambda\hat{f}(\lambda)|d\lambda \\ &\le 2\left(\int_{\mathbb{R}} \frac{1}{(1+|\lambda|)^p}d\lambda\right)^{1/p}\left(\int_{\mathbb{R}} |\lambda\hat{f}(\lambda)|^q d\lambda\right)^{1/q} \le C_p'\|f'\|_{L^p(\mathbb{R})},\end{aligned}$$

which completes the proof. □

Remark B.4. *The case $p = 2$ is due to Beurling* [15, Theorem 5.3]*. A similar result was obtained by S. G. Samko. Namely, if $f \in L^1(\mathbb{R}) \cap AC_{\mathrm{loc}}(\mathbb{R})$ is such that $f,\ f' \in L^p(\mathbb{R})$ with some $p \in (1,2]$, then $f \in \mathcal{W}_0(\mathbb{R})$ (see Theorem 6.8 in* [15]*).*

The next result is also due to Beurling (see, e.g., Theorem 5.4 in [15]).

Theorem B.5 (Beurling)**.** *Let $f \in C_0(\mathbb{R})$ be even and $f,\ f' \in AC_{\mathrm{loc}}(\mathbb{R})$. If*

$$C := \int_{\mathbb{R}_+} k|f''(k)|dk < \infty, \tag{B.3}$$

then $f \in \mathcal{W}_0(\mathbb{R})$ and $\|f\|_{\mathcal{W}} \le C$.

Consider the following functions, which appear in Section 4:

$$\chi_l(k) = \frac{|k|^l}{1+|k|^l}, \quad l > 0, \tag{B.4}$$

$$f_{l,p}(k) = \frac{|k|^p}{a + b|k|^l + |k|^{2l}}, \quad 2l > p \ge 0, \tag{B.5}$$

where $a,\ b \in \mathbb{R}$ are such that $a + b|k|^p + |k|^{2p} > 0$ for all $k \in \mathbb{R}$. As an immediate corollary of Beurling's result we get

Corollary B.6. *$\chi_l \in \mathcal{W}(\mathbb{R})$, $1 - \chi_l \in \mathcal{W}_0(\mathbb{R})$, and $f_{l,p} \in \mathcal{W}_0(\mathbb{R})$.*

Appendix C. Integral kernels

There are various criteria for operators in L^p spaces to be integral operators (see, e.g., [3]). Below we present a simple sufficient condition on a function K for $K(H)$ to be an integral operator, where H is a one-dimensional Schrödinger operator. More precisely, let H be a singular Schrödinger operator on $L^2(a,b)$ as in [11] or [12] with corresponding entire system of solutions $\theta(z,x)$ and $\phi(z,x)$. Recall

$$(H-z)^{-1}f(x) = \int_a^b G(z,x,y)f(y)dy, \tag{C.1}$$

where

$$G(z,x,y) = \begin{cases} \phi(z,x)\psi(z,y), & y \ge x, \\ \phi(z,y)\psi(z,x), & y \le x, \end{cases} \tag{C.2}$$

is the Green function of H and $\psi(z,x)$ is the Weyl solution normalized by $W(\theta,\psi) = 1$ (cf. [20, Lem. 9.7]). We start with a simple lemma ensuring that a function $K(H)$ is an integral operator. To this end recall that $K(H)$ is defined as $U^{-1}KU$ with K the multiplication operator in $L^2(\mathbb{R}, d\rho)$, ρ the associates spectral measure, and $U : L^2(a,b) \to L^2(\mathbb{R}, d\rho)$ the spectral transformation

$$(Uf)(\lambda) = \int_a^b \phi(\lambda,x)f(x)dx. \tag{C.3}$$

Lemma C.1. *Suppose H is bounded from below and $|K(\lambda)| \le C(1+|\lambda|)^{-1}$ or otherwise $|K(\lambda)| \le C(1+|\lambda|)^{-2}$. Then $K(H)$ is an integral operator*

$$(K(H)f)(x) = \int_a^b K(x,y) f(y) dy, \tag{C.4}$$

with kernel

$$K(x,y) = \int_{\mathbb{R}} K(\lambda)\phi(\lambda,x)\phi(\lambda,y) d\rho(\lambda). \tag{C.5}$$

In particular, $(1+|.|)^{-1/2}\phi(.,x) \in L^2(\mathbb{R}, d\rho)$ and $K(x,.) \in L^2(a,b)$ for every $x \in (a,b)$.

Proof. Note that (cf. [11, Lemma 3.6])

$$(UG(z;x,.))(\lambda) = \frac{\phi(\lambda,x)}{z-\lambda}.$$

If H is bounded from below then $G(z;x,.)$ is in the form domain of H for fixed x and every $z \in \mathbb{C}\setminus\sigma(H)$ (cf. [8, (A.6)]) and we obtain from [11, Lemma 3.6] that $(1+|\lambda|)^{-1/2}\phi(\lambda,x) \in L^2(\mathbb{R}, d\rho)$. In the general case we at least have $G(z;x,.) \in L^2(a,b)$ and thus $(1+|\lambda|)^{-1}\phi(\lambda,x) \in L^2(\mathbb{R}, d\rho)$. Hence we can use Fubini's theorem to evaluate

$$\begin{aligned} K(H)f(x) = U^{-1}KUf(x) &= \int_{\mathbb{R}} \phi(x,\lambda)K(\lambda)\left(\int_a^b \phi(\lambda,y)f(y)dy\right) d\rho(\lambda) \\ &= \int_a^b K(x,y)f(y)dy. \end{aligned}$$

□

As a consequence we obtain that (4.12) holds at least for $\mathrm{Im}(t) < 0$. To take the limit $\mathrm{Im}(t) \to 0$ we need the following result which follows from [5, Lemma 3.1].

Lemma C.2. *Consider the improper integral*

$$F(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(t+\mathrm{i}\varepsilon)k^2} f(k) dk, \qquad \varepsilon \le 0,$$

where

$$f(k) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kp} d\alpha(p), \qquad |\alpha|(\mathbb{R}) < \infty.$$

Then

$$F(\varepsilon) = \frac{1}{\sqrt{4\pi\mathrm{i}(t+\mathrm{i}\varepsilon)}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{p^2}{4(t+\mathrm{i}\varepsilon)}} d\alpha(p).$$

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