ON THE SPATIAL ASYMPTOTICS OF SOLUTIONS OF THE TODA LATTICE

GERALD TESCHL

Abstract. We investigate the spatial asymptotics of decaying solutions of the Toda lattice and show that the asymptotic behavior is preserved by the time evolution. In particular, we show that the leading asymptotic term is time independent. Moreover, we establish infinite propagation speed for the Toda lattice. All results are extended to the entire Toda as well as the Kac–van Moerbeke hierarchy.

1. Introduction

Since the seminal work of Gardner et al. [9] in 1967 it is known that completely integrable wave equations can be solved by virtue of the inverse scattering transform. In particular, this implies that short-range perturbations of the free solution remain short-range during the time evolution. This raises the question to what extend spatial asymptotical properties are preserved during time evolution. In [1], [2] (see also [13]) Bondareva and Shubin considered the initial value problem for the Korteweg–de Vries (KdV) equation in the class of initial conditions which have a prescribed asymptotic expansion in terms of powers of the spatial variable. As part of their analysis they obtained that the leading term of this asymptotic expansion is time independent. Inspired by this intriguing fact, the aim of the present paper is to prove a general result for the Toda equation which contains the analog of this result plus the known results for short-range perturbation alluded to before as a special case.

More specifically, recall the Toda lattice [23] (in Flaschka's variables [8])

$$\begin{aligned} \frac{d}{dt} a(n,t) &= a(n,t)\Big(b(n+1,t) - b(n,t)\Big),\\ \frac{d}{dt} b(n,t) &= 2\Big(a(n,t)^2 - a(n-1,t)^2\Big), \qquad n \in \mathbb{Z}. \end{aligned} \tag{1.1}$$

It is a well studied physical model and the prototypical discrete integrable wave equation. We refer to the monographs [7], [10], [21], [23] or the review articles [14], [22] for further information.

Then our main result, Theorem 2.5 below, implies for example that

$$a(n,t) = \frac{1}{2} + \frac{\alpha}{n^\delta} + O(\frac{1}{n^{\delta+\varepsilon}}), \quad b(n,t) = \frac{\beta}{n^\delta} + O(\frac{1}{n^{\delta+\varepsilon}}), \quad n \to \infty, \tag{1.2}$$

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for all $t \in \mathbb{R}$ provided this holds for the initial condition $t = 0$. Here $\alpha, \beta \in \mathbb{R}$ and $\delta \geq 0$, $0 < \varepsilon \leq 1$.

A few remarks are in order: First of all, it is important to point out that the error terms will in general grow with t (see the discussion after Theorem 2.5 for a rough time dependent bound on the error). An analogous result holds for $n \to -\infty$. Moreover, there is nothing special about the powers $n^{-\delta}$, which can be replaced by any bounded sequence which, roughly speaking, does decay at most exponentially and whose difference is asymptotically of lower order. Finally, similar results hold for the Ablowitz–Ladik equation. However, since the Ablowitz–Ladik system does not have the same difference structure some modifications are neccessary and will be given in Michor [17].

2. The Cauchy problem for the Toda lattice

To set the stage let us recall some basic facts for the Toda lattice. We will only consider bounded solutions and hence require

Hypothesis H.2.1. *Suppose $a(t)$, $b(t)$ satisfy*

$$a(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad b(t) \in \ell^\infty(\mathbb{Z}, \mathbb{R}), \qquad a(n,t) \neq 0, \qquad (n,t) \in \mathbb{Z} \times \mathbb{R},$$

and let $t \mapsto (a(t), b(t))$ be differentiable in $\ell^\infty(\mathbb{Z}) \oplus \ell^\infty(\mathbb{Z})$.

First of all, to see complete integrability it suffices to find a so-called Lax pair [16], that is, two operators $H(t)$, $P(t)$ in $\ell^2(\mathbb{Z})$ such that the Lax equation

$$\frac{d}{dt} H(t) = P(t)H(t) - H(t)P(t) \tag{2.1}$$

is equivalent to (1.1). Here $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable (complex-valued) sequences over $\mathbb{Z}$. One can easily convince oneself that the right choice is

$$\begin{aligned} H(t) &= a(t)S^+ + a^-(t)S^- + b(t), \\ P(t) &= a(t)S^+ - a^-(t)S^-, \end{aligned} \tag{2.2}$$

where $(S^\pm f)(n) = f^\pm(n) = f(n \pm 1)$ are the usual shift operators.

Now the Lax equation (2.1) implies that the operators $H(t)$ for different $t \in \mathbb{R}$ are unitarily equivalent (cf. [21, Thm. 12.4]):

Theorem 2.2. *Let $P(t)$ be a family of bounded skew-adjoint operators, such that $t \mapsto P(t)$ is differentiable. Then there exists a family of unitary propagators $U(t,s)$ for $P(t)$, that is,*

$$\frac{d}{dt} U(t,s) = P(t)U(t,s), \qquad U(s,s) = \mathbb{1}. \tag{2.3}$$

Moreover, the Lax equation (2.1) *implies*

$$H(t) = U(t,s)H(s)U(t,s)^{-1}. \tag{2.4}$$

As pointed out in [19], this result immediately implies global existence of bounded solutions of the Toda lattice as follows: Considering the Banach space of all bounded real-valued coefficients $(a(n), b(n))$ (with the sup norm), local existence is a consequence of standard results for differential equations in Banach spaces. Moreover, Theorem 2.2 implies that the norm $\|H(t)\|$ is constant, which in turn provides a uniform bound on the coefficients of $H(t)$,

$$\|a(t)\|_\infty + \|b(t)\|_\infty \le 2\|H(t)\| = 2\|H(0)\|. \tag{2.5}$$

Hence solutions of the Toda lattice cannot blow up and are global in time (see [21, Sect. 12.2] for details):

Theorem 2.3. *Suppose $(a_0, b_0) \in M = \ell^\infty(\mathbb{Z}, \mathbb{R}) \oplus \ell^\infty(\mathbb{Z}, \mathbb{R})$. Then there exists a unique integral curve $t \mapsto (a(t), b(t))$ in $C^\infty(\mathbb{R}, M)$ of the Toda lattice* (1.1) *such that $(a(0), b(0)) = (a_0, b_0)$.*

However, more can be shown. In fact, when considering the inverse scattering transform for the Toda lattice it is desirable to establish existence of solutions within the Marchenko class, that is, solutions satisfying

$$\sum_{n\in\mathbb{Z}} (1 + |n|)\Big(|a(n,t) - \frac{1}{2}| + |b(n,t)|\Big) < \infty \tag{2.6}$$

for all $t \in \mathbb{R}$. That this is indeed true was first established in [20] and rediscovered in [11] using a different method. Furthermore, the weight $1 + |n|$ can be replaced by an (almost) arbitrary weight function $w(n)$.

Lemma 2.4. *Suppose $a(n,t)$, $b(n,t)$ is some bounded solution of the Toda lattice* (1.1) *satisfying* (2.7) *for one $t_0 \in \mathbb{R}$. Then*

$$\sum_{n\in\mathbb{Z}} w(n)\Big(|a(n,t) - \frac{1}{2}| + |b(n,t)|\Big) < \infty, \tag{2.7}$$

holds for all $t \in \mathbb{R}$, where $w(n) \ge 1$ is some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$.

Moreover, as was demonstrated in [5] (see also [6]), one can even replace $|a(n,t) - \frac{1}{2}| + |b(n,t)|$ by $|a(n,t) - \bar{a}(n,t)| + |b(n,t) - \bar{b}(n,t)|$, where $\bar{a}(n,t)$, $\bar{b}(n,t)$ is some other bounded solution of the Toda lattice. See also [12], where similar results are shown.

This result shows that the asymptotic behavior as $n \to \pm\infty$ is preserved to leading order by the Toda lattice. The purpose of this paper is to show that even the leading term is preserved (i.e., time independent) by the time evolution.

Set

$$\|(a,b)\|_{w,p} = \begin{cases} \Big(\sum\limits_{n\in\mathbb{Z}} w(n)\Big(|a(n)|^p + |b(n)|^p\Big)\Big)^{1/p}, & 1 \le p < \infty \\ \sup\limits_{n\in\mathbb{Z}} w(n)\Big(|a(n)| + |b(n)|\Big), & p = \infty. \end{cases} \tag{2.8}$$

Then one has the following result:

Theorem 2.5. *Let $w(n) \ge 1$ be some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$ and fix some $1 \le p \le \infty$. Suppose a_0, b_0 and $\tilde{a}_0$, $\tilde{b}_0$ are bounded sequences such that*

$$\|(a_0^+ - a_0, b_0^+ - b_0)\|_{w,p} < \infty \quad \textit{and} \quad \|(\tilde{a}_0, \tilde{b}_0)\|_{w,p} < \infty. \tag{2.9}$$

Suppose $a(t), b(t)$ is the unique solution of the Toda lattice (1.1) *corresponding to the initial conditions*

$$a(0) = a_0 + \tilde{a}_0 \ne 0, \quad b(0) = b_0 + \tilde{b}_0. \tag{2.10}$$

Then this solution is of the form

$$a(t) = a_0 + \tilde{a}(t), \quad b(t) = b_0 + \tilde{b}(t), \quad \textit{where} \quad \|(\tilde{a}(t), \tilde{b}(t))\|_{w,p} < \infty \tag{2.11}$$

for all $t \in \mathbb{R}$.

Proof. The Toda equation (1.1) implies the differential equation

$$\begin{aligned} \frac{d}{dt}\tilde{a}(n,t) =& a(n,t)\Big(\tilde{b}(n+1,t) - \tilde{b}(n,t) + b_0(n+1) - b_0(n)\Big), \\ \frac{d}{dt}\tilde{b}(n,t) =& 2\Big(\big(a(n,t) + a_0(n)\big)\tilde{a}(n,t) - \big(a(n-1,t) + a_0(n-1)\big)\tilde{a}(n-1,t) \\ & + (a_0(n) + a_0(n-1))(a_0(n) - a_0(n-1))\Big), \qquad n \in \mathbb{Z} \end{aligned} \tag{2.12}$$

for $(\tilde{a}, \tilde{b})$. Since our requirement for $w(n)$ implies that the shift operators are continuous with respect to the norm $\|.\|_{w,p}$ and the same is true for the multiplication operator with a bounded sequence, this is an inhomogeneous linear differential equation in our Banach space which has a unique global solution in this Banach space (e.g., [4, Sect. 1.4]). Moreover, since $w(n) \geq 1$ this solution is bounded and the corresponding coefficients (a, b) coincide with the solution of the Toda equation from Theorem 2.3. □

Note that using Gronwall's inequality one can easily obtain an explicit bound

$$\|(\tilde{a}(t), \tilde{b}(t))\|_{w,p} \leq \|(\tilde{a}_0(t), \tilde{b}_0(t))\|_{w,p} \mathrm{e}^{Ct} + \|(a_0^+ - a_0, b_0^+ - b_0)\|_{w,p} \frac{1}{C}(\mathrm{e}^{Ct} - 1), \tag{2.13}$$

where $C = 4(\|H\| + \|a_0\|_\infty)$ (since $\|a(t)\|_\infty \leq \|H\|$ by (2.5)).

To see the claim (1.2) from the introduction, let

$$a_0(n) = \frac{1}{2} + \frac{\alpha}{n^\delta}, \quad b_0(n) = \frac{\beta}{n^\delta}, \quad \alpha, \beta \in \mathbb{R}, \delta > 0, \tag{2.14}$$

for $n > 0$ and $a_0(n) = b_0(n) = 0$ for $n \leq 0$. Now choose $p = \infty$ with

$$w(n) = \begin{cases} (1 + |n|)^{\delta + \varepsilon}, & n > 0, \\ 1, & n \leq 0. \end{cases} \tag{2.15}$$

and apply the previous theorem. To see Lemma 2.4, just choose $a_0(n) = \frac{1}{2}$, $b_0(n) = 0$ and $p = 1$.

Finally, let us remark that the requirement that $w(n)$ does not grow faster than exponentially is important. If it were not present, our result would imply that a compact perturbation of the free solution $a(n,t) = \frac{1}{2}$, $b(n,t) = 0$ remains compact for all time. However, this is wrong except for the free solution. This is well-known for the KdV equation [24], but we are not aware of a reference for the Toda equation.

Theorem 2.6. *Let* $a(n,t)$, $b(n,t)$ *be a bounded solution of the Toda lattice* (1.1). *If the sequences* $a(n,t) - \frac{1}{2}$, $b(n,t)$ *are zero for all except for a finite number of* $n \in \mathbb{Z}$ *for two different times* $t_0 \neq t_1$, *then they vanish identically.*

Proof. Without loss we can choose $t_0 = 0$ and suppose that the sequences $a(n,0) - \frac{1}{2}$, $b(n,0)$ are zero for all except for a finite number of n. Then the associated reflection coefficients $R_\pm(k,0)$ (see [21] Chapter 10) are rational functions with respect to k and by the inverse scattering transform ([21] Theorem 13.8) we have $R_\pm(k,t) = R_\pm(k,0)\exp(\pm(k - k^{-1})t)$, which is not rational for any $t \neq 0$ unless $R_\pm(k,t) \equiv 0$. Hence it must be a pure N soliton solution, which has compact support if and only if it is trivial, $N = 0$. □

For related unique continuation results for the Toda equation see Krüger and Teschl [15].

3. Extension to the Toda and Kac–van Moerbeke hierarchy

In this section we show that our main result extends to the entire Toda hierarchy (which will cover the Kac–van Moerbeke hierarchy as well). To this end, we introduce the Toda hierarchy using the standard Lax formalism following [3] (see also [10], [21]).

Choose constants $c_0 = 1$, c_j, $1 \le j \le r$, $c_{r+1} = 0$, and set

$$\begin{aligned} g_j(n,t) &= \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_n, H(t)^\ell \delta_n \rangle, \\ h_j(n,t) &= 2a(n,t) \sum_{\ell=0}^{j} c_{j-\ell} \langle \delta_{n+1}, H(t)^\ell \delta_n \rangle + c_{j+1}. \end{aligned} \tag{3.1}$$

The sequences g_j, h_j satisfy the recursion relations

$$\begin{aligned} g_0 = 1,\ h_0 &= c_1, \\ 2g_{j+1} - h_j - h_j^- - 2b g_j &= 0, \quad 0 \le j \le r, \\ h_{j+1} - h_{j+1}^- - 2(a^2 g_j^+ - (a^-)^2 g_j^-) - b(h_j - h_j^-) &= 0, \quad 0 \le j < r. \end{aligned} \tag{3.2}$$

Introducing

$$P_{2r+2}(t) = -H(t)^{r+1} + \sum_{j=0}^{r} (2a(t) g_j(t) S^+ - h_j(t)) H(t)^{r-j} + g_{r+1}(t), \tag{3.3}$$

a straightforward computation shows that the Lax equation

$$\frac{d}{dt} H(t) - [P_{2r+2}(t), H(t)] = 0, \qquad t \in \mathbb{R}, \tag{3.4}$$

is equivalent to

$$\mathrm{TL}_r(a(t), b(t)) = \begin{pmatrix} \dot{a}(t) - a(t)\Big(g_{r+1}^+(t) - g_{r+1}(t)\Big) \\ \dot{b}(t) - \Big(h_{r+1}(t) - h_{r+1}^-(t)\Big) \end{pmatrix} = 0, \tag{3.5}$$

where the dot denotes a derivative with respect to t. Varying $r \in \mathbb{N}_0$ yields the Toda hierarchy $\mathrm{TL}_r(a,b) = 0$.

All results mentioned in the previous section, Theorem 2.2, Theorem 2.3, and Lemma 2.4 remain valid for the entire Toda hierarchy (see [21]) and so does our main result:

Theorem 3.1. *Let $w(n) \ge 1$ be some weight with $\sup_n(|\frac{w(n+1)}{w(n)}| + |\frac{w(n)}{w(n+1)}|) < \infty$ and fix some $1 \le p \le \infty$. Suppose a_0, b_0 and $\tilde{a}_0$, $\tilde{b}_0$ are bounded sequences such that*

$$\|(a_0^+ - a_0, b_0^+ - b_0)\|_{w,p} < \infty \quad \text{and} \quad \|(\tilde{a}_0, \tilde{b}_0)\|_{w,p} < \infty. \tag{3.6}$$

Suppose $a(t), b(t)$ is the unique solution of some equation of the Toda hierarchy, $\mathrm{TL}_r(a,b) = 0$, corresponding to the initial conditions

$$a(0) = a_0 + \tilde{a}_0 > 0, \quad b(0) = b_0 + \tilde{b}_0. \tag{3.7}$$

Then this solution is of the form

$$a(t) = a_0 + \tilde{a}(t), \quad b(t) = b_0 + \tilde{b}(t), \quad \text{where} \quad \|(\tilde{a}(t), \tilde{b}(t))\|_{w,p} < \infty \tag{3.8}$$

for all $t \in \mathbb{R}$.

Proof. The proof is almost identical to the one of Theorem 2.5. From $\mathrm{TL}_r(a,b) = 0$ one obtains an inhomogeneous differential equation for $(\tilde{a}, \tilde{b})$. The homogenous part is a finite sum over shifts of $(\tilde{a}, \tilde{b})$ and the inhomogeneous part is $\big(a_0(g^+_{0,r+1} - g_{0,r+1}(t)), h_{0,r+1} - h^-_{0,r+1}\big)$, where $g_{0,r+1}$, $h_{0,r+1}$ are formed from (a_0, b_0). Finally, it is straightforward to show that the $\|.\|_{w,p}$ norm of the inhomogeneous part is finite by induction using the recursive definition of $g_{r+1}(t)$ and $h_{r+1}(t)$. $\square$

Similarly we also obtain

Theorem 3.2. *Let* $a(n,t)$, $b(n,t)$ *be a bounded solution of the of some equation of the Toda hierarchy,* $\mathrm{TL}_r(a,b) = 0$. *If the sequences* $a(n,t) - \frac{1}{2}$, $b(n,t)$ *are zero for all except for a finite number of* $n \in \mathbb{Z}$ *for two different times* $t_0 \neq t_1$, *then they vanish identically.*

Finally since the Kac–van Moerbeke hierarchy can be obtained by setting $b = 0$ in the odd equations of the Toda hierarchy, $\mathrm{KM}_r(a) = \mathrm{TL}_{2r+1}(a, 0)$ (see [18]), this last result also coveres the Kac–van Moerbeke hierarchy.

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Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria

E-mail address: `Gerald.Teschl@univie.ac.at`

URL: `http://www.mat.univie.ac.at/~gerald/`