ZEROS OF THE WRONSKIAN AND RENORMALIZED OSCILLATION THEORY

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Abstract. For general Sturm-Liouville operators with separated boundary conditions, we prove the following: If $E_{1,2} \in \mathbb{R}$ and if $u_{1,2}$ solve the differential equation $Hu_j = E_j u_j$, $j = 1, 2$ and respectively satisfy the boundary condition on the left/right, then the dimension of the spectral projection $P_{(E_1,E_2)}(H)$ of H equals the number of zeros of the Wronskian of u_1 and u_2.

1. Introduction

For over a hundred and fifty years, oscillation theorems for second-order differential equations have fascinated mathematicians. Originating with Sturm's celebrated memoir [20], extended in a variety of ways by Bôcher [2] and others, a large body of material has been accumulated since then (thorough treatments can be found, e.g., in [4],[13],[18],[19], and the references therein). In this paper we'll add a new wrinkle to oscillation theory by showing that zeros of Wronskians can be used to count eigenvalues in situations where a naive use of oscillation theory would give $\infty - \infty$.

To set the stage, we'll consider operators on $L^2((a, b); r\,dx)$ with $a < b$ in $[-\infty, \infty]$ of the form

$$(\tau u)(x) = r(x)^{-1}[-(p(x)u'(x))' + q(x)u(x)],$$

where

$$r, p^{-1}, q \in L^1_{\text{loc}}((a, b); dx) \text{ are real-valued and } r, p > 0 \text{ a.e. on } (a, b). \tag{1.1}$$

We'll use τ to describe the formal differentiation expression and H the operator given by τ with separated boundary conditions at a and/or b.

If a (resp. b) is finite and q, p^{-1}, r are in addition integrable near a (resp. b), we'll say a (resp. b) is a *regular* end point. We'll say τ respectively H is *regular* if both a and b are regular. As is usual, ([6], Section XIII.2; [15], Section 17; [22], Chapter 3), we consider the local domain

$$D_{\text{loc}} = \{u \in AC_{\text{loc}}((a, b)) \mid pu' \in AC_{\text{loc}}((a, b)),\ \tau u \in L^2_{\text{loc}}((a, b); r\,dx)\}, \tag{1.2}$$

where $AC_{\text{loc}}((a, b))$ is the set of integrals of $L^1_{\text{loc}}((a, b); dx)$-functions (i.e., the set of locally absolutely continuous functions) on (a, b). General ODE theory shows that for any $E \in \mathbb{C}$, $x_0 \in (a, b)$, and $(\alpha, \beta) \in \mathbb{C}^2$, there is a unique $u \in D_{\text{loc}}$ such that $-(pu')' + qu - Eru = 0$ for a.e. $x \in (a, b)$ and $(u(x_0), (pu')(x_0)) = (\alpha, \beta)$.

The maximal and minimal operators are defined by taking

$$D(T_{\max}) = \{u \in L^2((a, b); r\,dx) \cap D_{\text{loc}} \mid \tau u \in L^2((a, b); r\,dx)\},$$

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with

$$T_{\max} u = \tau u. \tag{1.3}$$

$T_{\min}$ is the operator closure of $T_{\max} \restriction D_{\mathrm{loc}} \cap \{u \text{ has compact support in } (a,b)\}$. Then $T_{\min}$ is symmetric and $T_{\min}^* = T_{\max}$.

According to the Weyl theory of self-adjoint extensions ([6], Section XIII.6; [15], Section 18; [17], Appendix to X.1; [21], Section 8.4; [22], Chapters 4 and 5), the deficiency indices of $T_{\min}$ are $(0,0)$ or $(1,1)$ or $(2,2)$ depending on whether it is limit point at both, one or neither end point. Moreover, the self-adjoint extensions can be described in terms of Wronskians ([6], Section XIII.2; [15], Sections 17 and 18; [21], Section 8.4; [22], Chapter 3). Define

$$W(u_1, u_2)(x) = u_1(x)(pu_2')(x) - (pu_1')(x)u_2(x). \tag{1.4}$$

Then if $T_{\min}$ is limit point at both ends, $T_{\min} = T_{\max} = H$. If $T_{\min}$ is limit point at b but not at a, for H any self-adjoint extension of $T_{\min}$, if φ_- is any function in $D(H)\backslash D(T_{\min})$, then

$$D(H) = \{u \in D(T_{\max}) \mid W(u, \varphi_-)(x) \to 0 \text{ as } x \downarrow a\}.$$

Finally, if u_1 is limit circle at both ends, the operators H with separated boundary conditions are those for which we can find $\varphi_\pm \in D(H)$, $\varphi_+ \equiv 0$ near a, $\varphi_- \equiv 0$ near b, and $\varphi_\pm \in D(H)\backslash D(T_{\min})$. In that case,

$$D(H) = \{u \in D(T_{\max}) \mid W(u, \varphi_-)(x) \to 0 \text{ as } x \downarrow a,\ W(u, \varphi_+)(x) \to 0 \text{ as } x \uparrow b\}.$$

Of course, if H is regular, we can just specify the boundary conditions by taking values at a, b since by regularity any $u \in D(T_{\max})$ has u, pu' continuous on $[a,b]$ (cf. (A.4)). It follows from this analysis that

Proposition 1.1. *If $u_{1,2} \in D(H)$, then $W(u_1, u_2)(x) \to 0$ as $x \to a$ or b.*

We'll call such operators SL operators (for Sturm-Liouville, but SL includes separated boundary conditions (if necessary)).

It will be convenient to write $\ell_- = a$, $\ell_+ = b$.

Throughout this paper we will denote by $\psi_\pm(z,x) \in D_{\mathrm{loc}}$ solutions of $\tau\psi = z\psi$ so that $\psi_\pm(z,\,.\,)$ is $L^2(\cdot; r\,dx)$ at $\ell_\pm$ and $\psi_\pm(z,\,.\,)$ satisfies the appropriate boundary condition at $\ell_\pm$ in the sense that for any $u \in D(H)$, $\lim_{x\to\ell_\pm} W(\psi_\pm(z), u)(x) = 0$. If $\psi_\pm(z,\,.\,)$ exist, they are unique up to constant multiples. In particular, $\psi_\pm(z,\,.\,)$ exist for z not in the essential spectrum of H and we can assume them to be holomorphic with respect to z in $\mathbb{C}\backslash\mathrm{spec}(H)$ and real for $z \in \mathbb{R}$. One can choose

$$\psi_\pm(z,x) = ((H-z)^{-1}\chi_{(c,d)})(x) \quad \text{for } x\,{}^{>d}_{<c}, \quad a < c < d < b$$

and uniquely continue $\psi_\pm(z,x)$ for $x\,{}^{<d}_{>c}$. Here $(H-z)^{-1}$ denotes the resolvent of H and χ_Ω the characteristic function of the set $\Omega \subseteq \mathbb{R}$. Clearly we can include a finite number of isolated eigenvalues in the domain of holomorphy of $\psi_\pm$ by removing the corresponding poles. Moreover, to simplify notations, all solutions u of $\tau u = Eu$ are understood to be not identically vanishing and solutions associated with real values of the spectral parameter E are assumed to be real-valued in this paper. Thus if E is real and in the resolvent set for H or an isolated eigenvalue, we are guaranteed there are solutions that obey the boundary conditions at a or b. It can happen if E is in the essential spectrum that such solutions do not exist or it may happen that they do. In Theorems 1.3, 1.4 below, we'll explicitly assume such solutions exist for the energies of interest. If these energies are not in the essential spectrum, that is automatically fulfilled.

With these preliminaries out of the way, we can describe a theorem Hartman proves in [10] which gives an eigenvalue count in some cases where oscillation theory would naively give $\infty - \infty$ (see Weidmann [22], Chapter 14 for some results when

τ is limit circle at b). In fact, we have slightly generalized the theorem in order to include, for instance, certain singular cases like radial Schrödinger operators on $(0, \infty)$ with potentials singular near 0 (we shall give a proof in Section 7).

Theorem 1.2. *Let H be an SL operator on (a, b) which is non-oscillatory at E_2 near a and limit point at b and suppose $E_1 < E_2$. Let u_1 (resp. u_2) be $\psi_-(E_1)$ (resp. $\psi_-(E_2)$). Let $N(c)$, $c \in (a, b)$ denote the number of zeros of u_1 in (a, c) minus the number of zeros of u_2 in (a, c). Let $P_\Omega(H)$ be the spectral projection of H corresponding to the Borel set $\Omega \subseteq \mathbb{R}$. Then, if τ is oscillatory at E_2 near b,*

$$\dim \operatorname{Ran} P_{(E_1,E_2)}(H) = \varliminf_{c\uparrow b} N(c), \tag{1.5a}$$

and if τ is non-oscillatory at E_2 near b,

$$\dim \operatorname{Ran} P_{[E_1,E_2)}(H) = \lim_{c\uparrow b} N(c). \tag{1.5b}$$

Theorem 1.2 is a bit more general than Hartman's result in [10] (see also [9], [11]) since we assume H to be non-oscillatory at E_2 near a while Hartman assumes H to be regular at a. If τ is oscillatory at E_2 near b (i.e., u_2 has infinitely many zeros near b), $N(c)$ is not constant for large c but instead varies between N_0 and $N_0 + 1$. This result leaves several questions open: What happens if H is limit circle at b or in the case where H is not regular at either end (e.g., the important case of the real line $(a, b) = (-\infty, \infty)$)? Moreover, it isn't clear when c is so large that $\varliminf_{c\uparrow b} N(c)$ has been reached. It would be better if we could actually count something analogous to the zero count in ordinary oscillation theory. Our goal in this paper is to prove such theorems.

The key is to look at zeros of the Wronskian. That zeros of the Wronskian are related to oscillation theory is indicated by an old paper of Leighton [14], who noted that if $u_j, pu_j' \in AC_{\mathrm{loc}}((a, b))$, $j = 1, 2$ and u_1 and u_2 have a non-vanishing Wronskian $W(u_1, u_2)$ in (a, b), then their zeros must intertwine each other. (In fact, pu_1' must have opposite signs at consecutive zeros of u_1, so by non-vanishing of W, u_2 must have opposite signs at consecutive zeros of u_1 as well. Interchanging the role of u_1 and u_2 yields strict interlacing of their zeros.) Moreover, let $E_1 < E_2$ and $\tau u_j = E_j u_j$, $j = 1, 2$. Ifx_0, x_1 are two consecutive zeros of u_1, then the number of zeros of u_2 inside (x_0, x_1) is equal to the number of zeros of the Wronskian $W(u_1, u_2)$ plus one (cf. Theorem 7.4). Hence the Wronskian comes with a built-in renormalization counting the additional zeros of u_2 in comparison to u_1. In particular, this avoids taking limits of the type (1.5a).

We'll let $W_0(u_1, u_2)$ be the number of zeros of the Wronskian in the open interval (a, b) not counting multiplicities of zeros. Given $E_1 < E_2$, we let $N_0(E_1, E_2) = \dim \operatorname{Ran} P_{(E_1,E_2)}(H)$ be the dimension of the spectral projection $P_{(E_1,E_2)}(H)$ of H. Our main results are the following two theorems:

Theorem 1.3. *Suppose $E_1 < E_2$. Let $u_1 = \psi_-(E_1)$ and $u_2 = \psi_+(E_2)$. Then*

$$W_0(u_1, u_2) = N_0(E_1, E_2).$$

Theorem 1.4. *Suppose $E_1 < E_2$. Let $u_1 = \psi_-(E_1)$ and $u_2 = \psi_-(E_2)$. Then either*

$$W_0(u_1, u_2) = N_0(E_1, E_2) \tag{1.6}$$

or

$$W_0(u_1, u_2) = N_0(E_1, E_2) - 1. \tag{1.7}$$

If either $N_0 = 0$ or H is limit point at b, then (1.6) holds.

We'll see that if b is a regular point and $E_2 > e > E_1$ with e an eigenvalue and $|E_2 - E_1|$ is small, then (1.7) holds rather than (1.6). We'll also see that if $u_{1,2}$ are arbitrary solutions of $\tau u_j = E_j u_j$, $j = 1, 2$, then, in general, $|W_0 - N_0| \leq 2$ (this means that if one of the quantities is infinite, the other is as well) and any of $0, \pm 1, \pm 2$ can occur for $W_0 - N_0$. Especially, if either E_1 or E_2 is in the interior of the essential spectrum of H (or $\dim \operatorname{Ran} P_{(E_1,E_2)}(H) = \infty$), then $W_0(u_1, u_2) = \infty$ for any u_1 and u_2 satisfying $\tau u_j = E_j u_j$, $j = 1, 2$ (cf. Theorem 7.3).

Zeros of the Wronskians have two properties that are critical to these results: First, zeros are precisely points where the Prüfer angles for u_1 and u_2 are equal $(\operatorname{mod} \pi)$. Second, if $\psi_- \in D_{\mathrm{loc}}$ and $\psi_+ \in D_{\mathrm{loc}}$ satisfy the boundary conditions at a, b, respectively, and $W(\psi_-, \psi_+)(x_0) = 0$ and if $(\psi_+(x_0), (p\psi_+')(x_0)) \neq (0, 0)$, then there is a γ such that

$$\eta(x) = \begin{cases} \psi_-(x), & x \leq x_0 \\ \gamma\psi_+(x), & x \geq x_0 \end{cases}$$

satisfies $\eta \in D(H)$ and

$$H\eta(x) = \begin{cases} (\tau\psi_-)(x), & x \leq x_0 \\ \gamma(\tau\psi_+)(x), & x \geq x_0. \end{cases}$$

We'll explore these properties further in Propositions 3.1 and 3.2.

Section 2 provides a short proof of the ordinary oscillation theorem in the regular case following the method in Courant-Hilbert ([5], page 454). Even though this result is well-known (see, e.g., [1], Theorem 8.4.5 and [22], Theorem 14.10 which describes the singular case as well) we include it here since our overall strategy in this paper is patterned after this proof: A variational argument will show $N_0 \geq W_0$ in Section 6 and a comparison-type argument in Sections 4 and 5 will prove $N_0 \leq W_0$. Explicitly, in Section 5 we'll show

Theorem 1.5. *Let $E_1 < E_2$. If $u_1 = \psi_-(E_1)$ and either $u_2 = \psi_+(E_2)$ or $\tau u_2 = E_2 u_2$ and H is limit point at b, then*

$$W_0(u_1, u_2) \geq \dim \operatorname{Ran} P_{(E_1,E_2)}(H).$$

In Section 6, we'll prove that

Theorem 1.6. *Let $E_1 < E_2$. Let either $u_1 = \psi_+(E_1)$ or $u_1 = \psi_-(E_1)$ and either $u_2 = \psi_+(E_2)$ or $u_2 = \psi_-(E_2)$. Then*

$$W_0(u_1, u_2) \leq \dim \operatorname{Ran} P_{(E_1,E_2)}(H). \tag{1.8}$$

Remark. *Of course, by reflecting about a point $c \in (a, b)$, Theorems 1.3–1.5 hold for $u_1 = \psi_+(E_1)$ and $u_2 = \psi_-(E_2)$ (and either $N_0 = 0$ or H is limit point at a in the corresponding analog of Theorem 1.4 yields* (1.6) *and similarly, $\tau u_2 = E_2 u_2$ and H is limit point at a yields the conclusion in the corresponding analog of Theorem 1.5).*

In Section 7, we provide a number of comments, examples, and extensions including:

Theorem 1.7. *Let $E_{1,2} \in \mathbb{R}$, $E_1 \neq E_2$, $\tau u_j = E_j u_j$, $j = 1, 2$, $\tau v_2 = E_2 v_2$. Then $|W_0(u_1, u_2) - W_0(u_1, v_2)| \leq 1$.*

In addition, Theorem 7.5 addresses the problem of finite versus infinite total number of eigenvalues in essential spectral gaps of H.

It is easy to see that Theorems 1.5, 1.6, and 1.7 imply Theorems 1.3 and 1.4.

Some facts on quadratic forms are collected in the appendix.

Our interest in this subject originated in attempts to provide a general construction of isospectral potentials for one-dimensional Schrödinger operators (see [8]) following previous work by Finkel, Isaacson, and Trubowitz [7] (see also [3]) in the case of periodic potentials. In fact, in the special case of periodic Schrödinger operators H_p, the non-vanishing of $W(u_1, u_2)(x)$ for Floquet solutions $u_1 = \psi_{\varepsilon_1}(E_1)$, $u_2 = \psi_{\varepsilon_2}(E_2)$, $\varepsilon_{1,2} \in \{+, -\}$ of H_p, for E_1 and E_2 in the same spectral gap of H_p, is proven in [7].

2. Oscillation Theory

For background, we recall the following:

Theorem 2.1 ([22], Theorem 14.10)**.** *Let H be an SL operator which is bounded from below. If $e_1 < \cdots < e_n < \cdots$ are its eigenvalues below the essential spectrum and $\psi_1, \dots, \psi_n, \dots$ its eigenfunctions, then ψ_n has $n-1$ zeros in (a, b). All eigenvalues of H are simple.*

Remark. *(i) Those used to thinking of the Dirichlet boundary condition case need to be warned that it is not in general true that if E is not an eigenvalue of H, then the number of zeros, Z, of $\psi_\pm(E)$ is the number, $N(E)$, of eigenvalues less than E. In general, all one can say is $N = Z$ or $N = Z + 1$.*

(ii) In the special case where τ is regular at a and b, any associated SL operator H is well-known to be bounded from below with compact resolvent (see, e.g., [1]*, Theorem 8.4.5;* [22]*, Theorem 13.2). Thus Theorem 2.1 applies to the regular case (to be used in our proof of Proposition 4.1).*

The first part of the proposition below is a simple integration by parts and the second follows from the first.

Proposition 2.2. *Let $E_1 \le E_2$ and $\tau u_j = E_j u_j$, $j = 1, 2$. Then for $a < c < d < b$,*

$$W(u_1, u_2)(d) - W(u_1, u_2)(c) = (E_1 - E_2) \int_c^d u_1(x) u_2(x) r(x)\, dx.$$

In particular, $W(u_1, u_2) \in AC_{\mathrm{loc}}((a, b))$ and

$$\frac{dW(u_1, u_2)}{dx}(x) = (E_1 - E_2) r(x) u_1(x) u_2(x) \quad a.e.$$

If the problem is regular at a (resp. b), we can take c (resp. d) equal to a (resp. b). In the general case we can take the limit $c \downarrow a$ (resp. $d \uparrow b$) in (2.2) *if u_1 and u_2 are $L^2(\cdot; r\,dx)$ near a (resp. b).*

Corollary 2.3. *Let $E_1 < E_2$ and $\tau u_j = E_j u_j$, $j = 1, 2$. Suppose at each end of $[c, d]$, $a < c < d < b$ either $W(u_1, u_2) = 0$ or $u_1 = 0$. If $\lim_{x \downarrow a} W(u_1, u_2)(x) = 0$ (resp. $\lim_{x \uparrow b} W(u_1, u_2)(x) = 0$), we also consider $c = a$ (resp. $d = b$). Then u_2 must vanish at least once in (c, d).*

Proof. By decreasing d to the first zero of u_1 in $(c, d]$ (and perhaps flipping signs), we can suppose $u_1 > 0$ on (c, d). If u_2 has no zeros in (c, d), we can suppose $u_2 > 0$ on (c, d) again by perhaps flipping signs. At each end point, $W(u_1, u_2)$ vanishes or else $u_1 = 0$, $u_2 > 0$, and $u_1'(c) > 0$ (or $u_1'(d) < 0$). Thus, $W(u_1, u_2)(c) \le 0$, $W(u_1, u_2)(d) \ge 0$. Since the right side of (2.2) is negative, this is inconsistent with (2.2). □

Proof of Theorem 2.1. We first prove that ψ_n has at least $n-1$ zeros and then that if ψ_n has m zeros, then $(-\infty, e_n]$ has at least $(m+1)$ eigenvalues. If ψ_n has m zeros at $x_1, x_2, \dots, x_m$ and we let $x_0 = a$, $x_{m+1} = b$, then by Corollary 2.3, ψ_{n+1} must have at least one zero in each of $(x_0, x_1), (x_1, x_2), \dots, (x_m, x_{m+1})$, that is, ψ_{n+1} has at least $m + 1$ zeros. It follows by induction that ψ_n has at least $n - 1$ zeros.

On the other hand, if an eigenfunction ψ_n has m zeros, define for $j = 0, \dots, m$, $x_0 = a, x_{m+1} = b$,

$$\eta_j(x) = \begin{cases} \psi_n(x), & x_j \le x \le x_{j+1} \\ 0, & \text{otherwise} \end{cases}, \quad 0 \le j \le m.$$

Then η_j is absolutely continuous with $p\eta_j'$ piecewise continuous so η_j is in the form domain of H (see (A.6)) and $\langle |H|^{1/2}\eta_j, \operatorname{sgn}(H)|H|^{1/2}\eta_j\rangle_r = e_n \, \|\eta_j\|_{2,r}$ (where $\langle \,\cdot\,, \cdot\,\rangle_r$ and $\|\cdot\|_{2,r}$ denote the scalar product and norm in $L^2((a,b); r\,dx)$. Thus if $\eta = \sum_{j=0}^{m} c_j \eta_j$, then $\langle |H|^{1/2}\eta, \operatorname{sgn}(H)|H|^{1/2}\eta\rangle_r = e_n \, \|\eta\|_{2,r}$. It follows by the spectral theorem that there are at least $m+1$ eigenvalues in $(-\infty, e_n]$. Since H has separated boundary conditions, its point spectrum is simple. □

The second part of the proof of Theorem 2.1 also shows:

Corollary 2.4. *Let H be an SL operator bounded from below. If $\psi_+(E, \,.\,)$ (resp. $\psi_-(E, \,.\,)$) has m zeros, then there are at least m eigenvalues below E. In particular, E below the spectrum of H implies that $\psi_\pm(E, \,.\,)$ have no zeros.*

3. Zeros of the Wronskian

Here we'll present the two aspects of zeros of the Wronskian which are critical for the two halves of our proofs (i.e., for showing $N_0 \ge W_0$ and that $N_0 \le W_0$). First, the vanishing of the Wronskian lets us patch solutions together:

Proposition 3.1. *Suppose that $\psi_{+,j}, \psi_- \in D_{\mathrm{loc}}$ and that $\psi_{+,j}$ and $\tau\psi_{+,j}$, $j = 1, 2$ are in $L^2((c,b); r\,dx)$ and that ψ_- and $\tau\psi_-$ are in $L^2((a,c); r\,dx)$ for all $c \in (a,b)$. Suppose, in addition, that $\psi_{+,j}$, $j = 1, 2$ satisfy the boundary condition defining H at b (i.e., $W(u, \psi_{+,j})(c) \to 0$ as $c \uparrow b$ for all $u \in D(H)$) and similarly, that ψ_- satisfies the boundary condition at a. Then*

(i) If $W(\psi_{+,1}, \psi_{+,2})(c) = 0$ and $(\psi_{+,2}(c), (p\psi_{+,2}')(c)) \ne (0,0)$, then there exists a γ such that

$$\eta = \chi_{[c,b)}(\psi_{+,1} - \gamma\psi_{+,2}) \in D(H)$$

and

$$H\eta = \chi_{[c,b)}(\tau\psi_{+,1} - \gamma\tau\psi_{+,2}). \tag{3.1}$$

(ii) If $W(\psi_{+,1}, \psi_-)(c) = 0$ and $(\psi_-(c), (p\psi_-')(c)) \ne (0,0)$, then there is a γ such that

$$\eta = \gamma\chi_{(a,c]}\psi_- + \chi_{(c,b)}\psi_{+,1} \in D(H)$$

and

$$H\eta = \gamma\chi_{(a,c]}\tau\psi_- + \chi_{(c,b)}\tau\psi_{+,1}. \tag{3.2}$$

Proof. Clearly, η and the right-hand-sides of (3.1)/(3.2) lie in $L^2((a,b); r\,dx)$ and satisfy the boundary condition at a and b, so it suffices to prove that η and $p\eta'$ are locally absolutely continuous on (a,b).

In case (i), if $\psi_{+,2}(c) \ne 0$, take $\gamma = -\psi_{+,1}(c)/\psi_{+,2}(c)$ and otherwise (i.e., if $\psi_{+,2}(c) = 0$) take $\gamma = -(p\psi_{+,1}')(c)/(p\psi_{+,2}')(c)$. In either case, η and $p\eta'$ are continuous at c. Case (ii) is similar. □

The second aspect connects zeros of the Wronskian to Prüfer variables ρ_u, θ_u (for u, pu' continuous) defined by

$$u(x) = \rho_u(x)\sin(\theta_u(x)), \qquad (pu')(x) = \rho_u(x)\cos(\theta_u(x)).$$

If $(u(x), (pu')(x))$ is never $(0,0)$, then ρ_u can be chosen positive and θ_u is uniquely determined once a value of $\theta_u(x_0)$ is chosen subject to the requirement θ_u continuous in x.

Notice that

$$W(u_1,u_2)(x)=\rho_{u_1}(x)\rho_{u_2}(x)\sin(\theta_{u_1}(x)-\theta_{u_2}(x)).$$

Thus,

Proposition 3.2. *Suppose (u_j, pu_j'), $j=1,2$ are never $(0,0)$. Then $W(u_1,u_2)(x_0)$ is zero if and only if $\theta_{u_1}(x_0)\equiv\theta_{u_2}(x_0) \pmod{\pi}$.*

In linking Prüfer variables to rotation numbers, an important role is played by the observation that because of

$$u(x)=\int\limits_{x_0}^{x}\frac{\rho_u(t)\cos(\theta_u(t))}{p(t)}\,dt,$$

$\theta_u(x_0)\equiv 0 \pmod{\pi}$ implies $[\theta_u(x)-\theta_u(x_0)]/(x-x_0)>0$ for $0<|x-x_0|$ sufficiently small and hence for all $0<|x-x_0|$ if $(u,pu')\neq(0,0)$. (In fact, suppose $x_1\neq x_0$ is the closest x such that $\theta_u(x_1)=\theta_u(x_0)$ then apply the local result at x_1 to obtain a contradiction.) We summarize:

Proposition 3.3. *If $(u,pu')\neq(0,0)$ then $\theta_u(x_0)\equiv 0 \pmod{\pi}$ implies*

$$[\theta_u(x)-\theta_u(x_0)]/(x-x_0)>0$$

for $x\neq x_0$. In particular, if $\theta_u(c)\in[0,\pi)$ and u has n zeros in (c,d), then $\theta_u(d-\epsilon)\in(n\pi,(n+1)\pi)$ for sufficiently small $\epsilon>0$.

In exactly the same way, we have

Proposition 3.4. *Let $E_1<E_2$ and assume that $u_{1,2}$ solve $\tau u_j=E_ju_j$, $j=1,2$. Let $\Delta(x)=\theta_{u_2}(x)-\theta_{u_1}(x)$. Then $\Delta(x_0)\equiv 0 \pmod{\pi}$ implies $(\Delta(x)-\Delta(x_0))/(x-x_0)>0$ for $0<|x-x_0|$.*

Proof. If $\Delta(x_0)\equiv 0 \pmod{2\pi}$ and $\theta_{u_2}(x_0)\not\equiv 0 \pmod{\pi}$, then $\sin(\theta_{u_2}(x_0))\sin(\theta_{u_1}(x_0))>0$ so $u_1(x_0)u_2(x_0)>0$ for $0<|x-x_0|$ sufficiently small, and thus by (2.2), $\frac{dW}{dx}(x_0)>0$ for a.e. x near x_0 and so $\Delta(x)$ is increasing. The same holds for $\Delta(x_0)\equiv\pi \pmod{2\pi}$ and $\theta_{u_2}(x_0)\not\equiv 0 \pmod{\pi}$.

If $\Delta(x_0)\equiv 0 \pmod{2\pi}$ and $\theta_{u_1}(x_0)\equiv\theta_{u_2}(x_0)\equiv 0 \pmod{\pi}$, then $(pu_1')(x_0)(pu_2')(x_0)>0$ and so since $u(x_0)=v(x_0)=0$, we see that it is still true that $\frac{dW}{dx}(x)>0$ a.e. for $0<|x-x_0|$ sufficiently small. □

Remark. *(i) Suppose r,p are continuous on (a,b). If $\theta_{u_1}(x_0)\equiv 0 \pmod{\pi}$ then $\theta_{u_1}(x)-\theta_{u_1}(x_0)=c_0(x-x_0)+o(x-x_0)$ with $c_0>0$. If $\Delta(x_0)\equiv 0 \pmod{\pi}$ and $\theta_{u_1}(x_0)\not\equiv 0 \pmod{\pi}$, then $\Delta(x)-\Delta(x_0)=c_1(x-x_0)+o(x-x_0)$ with $c_1>0$. If $\theta_{u_1}(x_0)\equiv 0\equiv\Delta(x_0) \pmod{\pi}$, then $\Delta(x)-\Delta(x_0)=c_2(x-x_0)^3+o(x-x_0)^3)$ with $c_2>0$. Either way, Δ increases through x_0. (In fact, $c_0=p(x_0)^{-1}$, $c_1=(E_2-E_1)r(x_0)\sin^2(\theta_{u_1}(x_0))$ and $c_2=\frac{1}{3}r(x_0)p(x_0)^{-2}(E_2-E_1))$.*

(ii) In other words, Propositions 3.3 and 3.4 say that the integer parts of θ_u/π and $\Delta_{u,v}/\pi$ are increasing with respect to $x\in(a,b)$ (even though θ_u and $\Delta_{u,v}$ themselves are not necessarily monotone in x).

(iii) Let $E\in[E_1,E_2]$ and assume $[E_1,E_2]$ to be outside the essential spectrum of H. Then, for $x\in(a,b)$ fixed,

$$\frac{d\theta_{\psi_\pm}}{dE}(E,x)=-\frac{\int\limits_x^{\ell_\pm}\psi_\pm(E,t)^2\,dt}{\rho_{\psi_\pm}(E,x)} \tag{3.3}$$

proves that $\mp\theta_{\psi_\pm}(E,x)$ *is strictly increasing with respect to* E. *In fact, from Proposition 2.2 one infers*

$$W(\psi_\pm(E),\psi_\pm(\tilde E))(x) = (\tilde E - E)\int_x^{\ell_\pm} \psi_\pm(E,t)\psi_\pm(\tilde E,t)\,dt$$

and using this to evaluate the limit $\lim_{\tilde E\to E} W(\psi_\pm(E),(\psi_\pm(E)-\psi_\pm(\tilde E))/(E-\tilde E))(x)$, *one obtains*

$$W(\psi_\pm(E),\frac{d\psi_\pm}{dE}(E))(x) = \int_x^{\ell_\pm} \psi_\pm(E,t)^2\,dt.$$

Inserting Prüfer variables completes the proof of (3.3).

4. The Hare and the Tortoise ($N_0 \le W_0$ in the Regular Case)

Our goal in this section is to prove Theorem 1.5 in the regular case with opposite boundary conditions, that is,

Proposition 4.1. *Let* H *be a regular SL operator and suppose* $E_1 < E_2$. *Then*

$$W_0(\psi_-(E_1),\psi_+(E_2)) \ge N_0(E_1,E_2). \tag{4.1}$$

The proof will use Prüfer angles. As a warm-up, let us prove equality in the case that H has $u(a) = u(b) = 0$ boundary conditions and that $E_{1,2}$ are not eigenvalues. Let $\theta_{\psi_\pm}(E,x)$ be the Prüfer angles for $\psi_\pm(E,x)$, normalized such that $\theta_{\psi_\pm}(E,a)\in[0,\pi)$. Since $\psi_-(E_1)$ satisfies the boundary condition at a, $\theta_{\psi_-}(E_1,a) = 0$ and since E_2 is not an eigenvalue, $\theta_{\psi_+}(E_2,a) > 0$. If there are m eigenvalues below E_1 and n_0+m below E_2, then, by standard oscillation theory (essentially Proposition 3.3), $\theta_{\psi_-}(E_1,b)\in(m\pi,(m+1)\pi)$ and $\theta_{\psi_+}(E_2,b) = (n_0+m+1)\pi$. Let $\Gamma_\pm(E,x)\equiv\theta_{\psi_\pm}(E,x) \pmod{\pi}$, that is, $\Gamma_\pm(E,x)\in[0,\pi)$ and $\Gamma_\pm - \theta_{\psi_\pm}\in\mathbb{Z}\pi$.

Borrow a leaf from Aesop. Think of $\Gamma_-(E_1)$ as a tortoise and $\Gamma_+(E_2)$ as a hare racing on a track of size π with 0 as the start and π as the finish. Every time either runs through the finish, it starts all over. Neither has to run only in the forward direction (i.e., $\theta_{\psi_\pm}$ may not be monotone w.r.t. x) but they can't run in the wrong direction back through the start (i.e., Proposition 3.3 holds).

What makes $\Gamma_+(E_2)$ the hare to $\Gamma_-(E_1)$'s tortoise is that $\Gamma_+(E_2)$ can only overtake $\Gamma_-(E_1)$, not the other way around (i.e., Proposition 3.4 holds). Since $\Gamma_-(E_1,a) = 0$ and $\Gamma_+(E_2,a) > 0$, the hare starts out ahead of the tortoise. Since $\Gamma_-(E_1,c) < \pi$ but $\Gamma_+(E_1,c)\nearrow\pi$ as $c\nearrow b$, the hare also ends up ahead (unlike in Aesop!).

Clearly, the number of times the hare crosses the finish line is the sum of the number of times the tortoise does, plus the number of times the hare "laps," that is, passes the tortoise. Thus,

$$n_0 + m = m + W_0(\psi_-(E_1),\psi_+(E_2))$$

so $W_0(\psi_-(E_1),\psi_+(E_2)) = n_0$ in the Dirichlet case.

This picture also explains why it can happen that

$$W_0(\psi_-(E_1),\psi_-(E_2)) = n_0 - 1.$$

For in this case, $\theta_{\psi_-}(E_1,a) = \theta_{\psi_-}(E_2,a) = 0$. The hare and tortoise start out together, so for $x = a+\epsilon$, the hare is slightly ahead. If at b, $\Gamma_-(E_1,b) > \Gamma_-(E_2,b)$, then the tortoise à la Aesop wins the races; thus the hare has lapped the tortoise one time too few, that is,

$$n_0 + m - 1 = m + W_0(\psi_-(E_1),\psi_-(E_2))$$

and so

$$W_0 = n_0 - 1. \tag{4.2}$$

Suppose $E_1 < e < E_2$ with e an eigenvalue. As $E_2 \searrow e$, $\Gamma_-(E_2, b) \searrow 0$ as $E_1 \nearrow e$, $\Gamma_-(E_1, b) \nearrow \pi$. Thus for $E_2 - E_1$ sufficiently small, $\Gamma_-(E_2, b) < \Gamma_-(E_1, b)$ and (4.2) holds.

Now we turn to the proof of Proposition 4.1 in the general case (assuming H to be a regular SL operator for the rest of this section).

Lemma 4.2. *Let $u_{1,2}$ be eigenfunctions of H with eigenvalues $E_1 < E_2$. Let ℓ be the number of eigenvalues of H in (E_1, E_2). Then $W(u_1, u_2)(x)$ has exactly ℓ zeros in (a, b).*

Proof. Suppose u_1 is the kth eigenfunction. By Theorem 2.1, u_1 has $k-1$ zeros and u_2 has $k+\ell$ zeros in (a, b). Moreover, $\Gamma_-(E_1, a) = \Gamma_+(E_2, a)$, $\Gamma_-(E_1, b) = \Gamma_+(E_2, b)$ so $\Gamma_-(E_1, a+\epsilon) < \Gamma_+(E_2, a+\epsilon)$, $\Gamma_+(E_2, b-\epsilon) < \Gamma_-(E_1, b-\epsilon)$. So the hare starts slightly ahead and ends slightly behind and so it laps one less time than the difference of the number of zeros, that is, $W_0(u_1, u_2) = (\ell+1) - 1 = \ell$. □

Lemma 4.3. *Let $E_1 \leq E_2$ be eigenvalues of H and suppose $[E_1, E_2]$ has ℓ eigenvalues. Then for $\epsilon \geq 0$ sufficiently small, $W_0(\psi_-(E_1-\epsilon), \psi_+(E_2+\epsilon)) = \ell$.*

Remark. *Since (E_1, E_2) has $\ell-2$ eigenvalues, Lemma 4.2 says that $W(\psi_-(E_1), \psi_+(E_2))(x)$ has $\ell-2$ zeros in (a, b) and clearly it has zeros at a and b. Essentially, Lemma 4.3 says that replacing E_1 by $E_1-\epsilon$ and E_2 by $E_2+\epsilon$ moves the zeros at a, b inside (a, b) to give $\ell-2+2 = \ell$ zeros.*

Proof. Suppose first $E_1 < E_2$. Compare the tortoises associated to $\psi_-(E_1-\epsilon)$ and $\psi_-(E_1)$. The first starts out at $x = a$ in the same position as the second (i.e., $\Gamma_-(E_1-\epsilon, a) = \Gamma_-(E_1, a)$), which means it must end slightly behind, that is, $\Gamma_-(E_1-\epsilon, b) < \Gamma_-(E_1, b)$. Similarly, since the faster hare for energy $E_2+\epsilon$ ends up where the hare of energy E_2 does (i.e, $\Gamma_+(E_2+\epsilon, b) = \Gamma_+(E_2, b)$), it must start out slightly farther back, that is, $\Gamma_+(E_2+\epsilon, a) < \Gamma_+(E_2, a)$. Thus $W(\psi_-(E_1-\epsilon), \psi_+(E_2+\epsilon))(x)$ picks up two zeros over the $\ell-2$ that $W(\psi_-(E_1), \psi_+(E_2))(x)$ has.

If $E_2 = E_1 \equiv E$, the Γ_+ for $\psi_+(E+\epsilon)$ starts out slightly behind the one for $\psi_+(E)$ and ends up slightly ahead of the Γ_- for $\psi_-(E-\epsilon)$, and so there has to be one crossing, that is, $W_0(\psi_-(E-\epsilon), \psi_+(E+\epsilon)) = 1$. □

Lemma 4.4. *If $E_3 < E_4 < E$ and u is any solution of $\tau u = Eu$, then*

$$W_0(\psi_-(E_3), u) \geq W_0(\psi_-(E_4), u). \tag{4.3}$$

Similarly, if $E_3 > E_4 > E$ and u is any solution of $\tau u = Eu$, then (4.3) *holds.*

Proof. In the first case, think of u as defining a hare and $\psi_-(E_j)$, $j = 3, 4$ as defining tortoises. The E_3 and E_4 tortoises start out at the same place and the E_3 tortoise runs "faster" in that it is always ahead after the start. Clearly, the hare will pass the slower tortoise at least as often as the faster one.

In the second case, there are two hares (defined by $\psi_-(E_j)$, $j = 3, 4$), which start out at the same place, and one tortoise (defined by u) and it is clear the faster hare (given by $\psi_-(E_3)$) has to pass the tortoise at least as often as the slower one. □

Lemma 4.5. *Lemma 4.4 remains true if every ψ_- is replaced by a ψ_+.*

Proof. Reflect at some point $c \in (a, b)$ implying an interchange of ψ_+ and ψ_-. □

Proof of Proposition 4.1. If $N_0 = 0$, there is nothing to prove. If $N_0 \geq 1$, let $\operatorname{spec}(H) \cap (E_1, E_2) = \{e_m\}_{m \in M}$ and let $e_s \leq e_\ell$ be the smallest and largest of the e_m's. Thus, N_0 is the number of eigenvalues in $[e_s, e_\ell]$ and so

$$N_0 = W_0(\psi_-(e_s - \epsilon), \psi_+(e_\ell + \epsilon))$$

by Lemma 4.3. By Lemma 4.4,

$$W_0(\psi_-(e_s - \epsilon), \psi_+(e_\ell + \epsilon)) \leq W_0(\psi_-(E_1), \psi_+(e_\ell + \epsilon))$$

and then by Lemma 4.5, this is no larger than $W_0(\psi_-(E_1), \psi_+(E_2))$. □

5. Strong Limits ($N_0 \leq W_0$ in the General Case)

Using the approach of Weidmann ([22], Chapter 14) to control some limits, we prove Theorem 1.5 in this section. Fix functions $u_1, u_2 \in D_{\text{loc}}$. Pick $c_n \downarrow a$, $d_n \uparrow b$. Define $\tilde{H}_n$ on $L^2((c_n, d_n); r\,dx)$ by imposing the following boundary conditions on $\eta \in D(\tilde{H}_n)$

$$W(u_1, \eta)(c_n) = 0 = W(u_2, \eta)(d_n). \tag{5.1}$$

On $L^2((a,b); r\,dx) = L^2((a, c_n); r\,dx) \oplus L^2((c_n, d_n); r\,dx) \oplus L^2((d_n, b); r\,dx)$ take $H_n = \alpha\mathbb{I} \oplus \tilde{H}_n \oplus \alpha\mathbb{I}$ with α a fixed real constant. Then Weidmann proves:

Lemma 5.1. *Suppose that either H is limit point at a or that u_1 is an $\psi_-(E, x)$ for some E and similarly, that either H is limit point at b or u_2 is an $\psi_+(E', x)$ for some E'. Then H_n converges to H in strong resolvent sense as $n \to \infty$.*

The idea of Weidmann's proof is that it suffices to find a core D_0 of H such that for every $\eta \in D_0$ there exists an $n_0 \in \mathbb{N}$ with $\eta \in D_0$ for $n \geq n_0$ and $H_n\eta \to H\eta$ as n tends to infinity (see [21], Theorem 9.16(i)). If H is limit point at both ends, take $\eta \in D_0 \equiv \{u \in D_{\text{loc}} \mid \operatorname{supp}(u) \text{ compact in } (a,b)\}$. Otherwise, pick $\tilde{u}_1, \tilde{u}_2 \in D(H)$ with $\tilde{u}_2 = u_2$ near b and $\tilde{u}_2 = 0$ near a and with $\tilde{u}_1 = u_1$ near a and $\tilde{u}_1 = 0$ near b. Then pick $\eta \in D_0 + \operatorname{span}[\tilde{u}_1, \tilde{u}_2]$ which one can show is a core for H ([22], Chapter 14).

Secondly we note:

Lemma 5.2. *Let $A_n \to A$ in strong resolvent sense as $n \to \infty$. Then*

$$\dim \operatorname{Ran} P_{(E_1,E_2)}(A) \leq \varliminf_{n\to\infty} \dim \operatorname{Ran} P_{(E_1,E_2)}(A_n).$$

Proof. Fix $m \leq \dim \operatorname{Ran} P_{(E_1,E_2)}(A)$ with $m < \infty$. We'll prove $m \leq$ RHS of (5.2). Suppose first that (E_1, E_2) aren't eigenvalues of A. Then by Theorem VIII.24 of [16], $P_{(E_1,E_2)}(A_n) \to P_{(E_1,E_2)}(A)$ strongly as $n \to \infty$. Picking orthonormal $\varphi_1, \dots, \varphi_m$ in $\operatorname{Ran} P_{(E_1,E_2)}(A)$, we see that

$$\varliminf_{n\to\infty} \operatorname{Tr}(P_{(E_1,E_2)}(A_n)) \geq \varliminf_{n\to\infty} \sum_j \langle \varphi_j, P_{(E_1,E_2)}(A_n)\varphi_j \rangle_r = m$$

as required.

If $E_{1,2}$ are arbitrary, we can always find a $\delta > 0$ such that $E_1 + \delta, E_2 - \delta$ are not eigenvalues of A and such that $\dim \operatorname{Ran} P_{(E_1+\delta, E_2-\delta)}(A) \geq m$. Thus,

$$\varliminf_{n\to\infty} \dim \operatorname{Ran} P_{(E_1,E_2)}(A_n) \geq \varliminf_{n\to\infty} \dim \operatorname{Ran} P_{(E_1+\delta,E_2-\delta)}(A_n) \geq m. \quad \square$$

□

Proof of Theorem 1.5. Let $c_n \downarrow a$, $d_n \uparrow b$ and H_n be as in Lemma 5.1 with $\alpha \notin [E_1, E_2]$. Proposition 4.1 implies $W_0(u_1, u_2) \geq \dim \operatorname{Ran} P_{(E_1,E_2)}(\tilde{H}_n) = \dim \operatorname{Ran} P_{(E_1,E_2)}(H_n)$ since $\alpha \notin [E_1, E_2]$. Thus by Lemmas 5.1 and 5.2,

$$W_0(u_1, u_2) \geq \dim \operatorname{Ran} P_{(E_1,E_2)}(H)$$

as was to be proven. □

6. A Variational Argument ($N_0 \geq W_0$)

In this section, we'll prove Theorem 1.6. Let $E_1 < E_2$. Suppose first that $u_1 = \psi_-(E_1)$ and $u_2 = \psi_+(E_2)$. Let $x_1, \dots, x_m$ be zeros of $W(u_1, u_2)(x)$. We'll prove that $\dim P_{(E_1,E_2)}(H) \geq m$. If $W_0(u_1, u_2) = m$, this proves (1.8). If $W_0 = \infty$, we can take m arbitrarily large, and again (1.8) holds. Define

$$\eta_j(x) = \begin{cases} u_1(x), & x \leq x_j \\ \gamma_j u_2(x), & x \geq x_j \end{cases}, \quad 1 \leq j \leq m,$$

where γ_j is defined such that $\eta_j \in D(H)$ by Proposition 3.1. Let

$$\tilde{\eta}_j(x) = \begin{cases} u_1(x), & x \leq x_j \\ -\gamma_j u_2(x), & x > x_j \end{cases}, \quad 1 \leq j \leq m.$$

If E_2 is an eigenvalue of H, we define in addition $\eta_0 = u_2 = -\tilde{\eta}_0$, $x_0 = a$ and if E_1 is an eigenvalue of H, $\eta_{m+1} = u_1 = \tilde{\eta}_{m+1}$, $x_{m+1} = b$.

Lemma 6.1. *$\langle \eta_j, \eta_k \rangle_r = \langle \tilde{\eta}_j, \tilde{\eta}_k \rangle_r$ for all j, k where $\langle \cdot\,, \cdot \rangle_r$ is the $L^2((a,b); r\,dx)$ inner product.*

Proof. Let $j < k$. This just says that $\int_{x_j}^{x_k} u_1(x) u_2(x) r(x)\,dx = 0$. But by (2.2), this integral is $(E_1 - E_2)^{-1}[W(u_1,u_2)(x_k) - W(u_1,u_2)(x_j)] = 0$ since $W(u_1,u_2)(\cdot)$ vanishes at x_ℓ respectively in the limit (if $x_\ell = a, b$) by Proposition 1.1. □

Notice that by (3.2),

$$\left(H - \frac{E_2 + E_1}{2}\right) \eta_j = \left(\frac{E_1 - E_2}{2}\right) \tilde{\eta}_j. \tag{6.1}$$

This result and Lemma 6.1 imply

Lemma 6.2. *If η is in the span of the η_j, then*

$$\left\| \left(H - \frac{E_2 + E_1}{2}\right) \eta \right\|_{2,r} = \frac{|E_2 - E_1|}{2} \, \|\eta\|_{2,r}.$$

Thus, $\dim \operatorname{Ran} P_{[E_1,E_2]}(H) \geq \dim(\operatorname{span}(\{\eta_j\}))$. But u_1 and u_2 are independent on each interval (since their Wronskian is non-constant) and so the η_j are linearly independent. This proves Theorem 1.6 in the $\psi_-(E_1), \psi_+(E_2)$ case.

The case $u_1 = \psi_-(E_1)$, $u_2 = \psi_-(E_2)$ is similar. We define

$$\eta_j(x) = \begin{cases} u_1(x) - \gamma_j u_2(x), & x \leq x_j \\ 0, & x \geq x_j \end{cases}, \quad 1 \leq j \leq m$$

and

$$\tilde{\eta}_j(x) = \begin{cases} u_1(x) + \gamma_j u_2(x), & x \leq x_j \\ 0, & x > x_j \end{cases}, \quad 1 \leq j \leq m.$$

If E_2 is an eigenvalue of H, we define in addition $\eta_0 = u_2 = -\tilde{\eta}_0$, $x_0 = b$ and if E_1 is an eigenvalue of H, $\eta_{m+1} = u_1 = \tilde{\eta}_{m+1}$, $x_{m+1} = b$. Again, η_j's are linearly independent by considering their supports. To prove the analog of Lemma 6.1, we need

$$\int_a^{x_j} u_1(x) u_2(x) r(x)\,dx = 0, \; 1 \leq j \leq m.$$

But by (2.2), this integral is

$$\lim_{c \downarrow a} (E_1 - E_2)^{-1} [W(u_1,u_2)(x_j) - W(u_1,u_2)(c)].$$

By hypothesis, $W(u_1,u_2)(x_j)=0$ and since u_1 and u_2 satisfy the boundary condition at a, $W(u_1,u_2)(c)\to 0$ as $c\downarrow a$ by Proposition 1.1. The cases $u_1=\psi_+(E_1)$, $u_2=\psi_\pm(E_2)$ can be obtained by reflection.

7. Extensions, Comments, and Examples

The following includes Theorem 1.7:

Theorem 7.1. *Let $E_1\neq E_2$. Let $\tau u_j=E_ju_j$, $j=1,2$, $\tau v_2=E_2v_2$ with u_2 linearly independent of v_2. Then the zeros of $W(u_1,u_2)$ interlace the zeros of $W(u_1,v_2)$ and vice versa (in the sense that there is exactly one zero of one function in between two zeros of the other). In particular, $|W_0(u_1,u_2)-W_0(u_1,v_2)|\leq 1$.*

Proof. We'll suppose $E_1<E_2$. A similar argument works if $E_2<E_1$. In the language of Section 4, Γ_{u_1} represents the tortoise and $\Gamma_{u_2},\Gamma_{v_2}$ are two hares. Since $W(u_2,v_2)$ is a non-zero constant, one hare always stays ahead of the other. It follows that if the hare Γ_{u_1} crosses the tortoise Γ_{u_2} at x_1 and x_2, $x_1<x_2$, the hare Γ_{v_2} must cross it at some point in (x_1,x_2). ☐

By applying this theorem twice, we conclude

Theorem 7.2. *Let $E_1\neq E_2$. Let u_1,u_2,v_1,v_2 be the linearly independent functions with $\tau u_j=E_ju_j$ and $\tau v_j=E_jv_j$. Then*

$$|W_0(u_1,u_2)-W_0(v_1,v_2)|\leq 2.$$

Theorem 7.3. *If $\dim\operatorname{Ran}P_{(E_1,E_2)}(H)=\infty$, then $W_0(u_1,u_2)=\infty$ for any u_1 and u_2 satisfying $\tau u_j=E_ju_j$, $j=1,2$.*

Proof. Firstly, if $W_0(u_1,u_2)=\infty$ for one pair $u_{1,2}$ this is true for any pair by Theorem 7.2. Secondly, pick $u_{1,2}$ such that the operator H_n of Lemma 5.1 converges to H in strong resolvent sense as $n\to\infty$. Hence by Theorem 1.3 (applied to $\tilde H_n$ defined before Lemma 5.1) and Lemma 5.2 the number of zeros of the Wronskian in (c_n,d_n) must go to infinity as $n\to\infty$. ☐

Example 1. *Let us take $p=r=1$, $q=0$ with $[a,b]=[0,1]$ and Neumann boundary conditions $u'(0)=u'(1)=0$. Let $E_1=-k_1^2$, $E_2=k_2^2$, and $u_1(x)=\psi_-(E_1,x)$, $u_2(x)=\psi_-(E_2,x)$. Then $u_1(x)=\cosh(k_1x)$, $u_2(x)=\cos(k_2x)$, and*

$$W(u_1,u_2)(x)=-k_2\cosh(k_1x)\sin(k_2x)-k_1\sinh(k_1x)\cos(k_2x)$$

has no zero in $[0,1]$ if $0<k_1$, $0<k_2<\frac{\pi}{2}$. Thus, while $N_0=1$, $W_0=0$ so we see that $W_0=N_0-1$, that is, (1.7) in Theorem 1.4 can happen if the boundary conditions hold on the same side (note that the problem is limit circle at $b=1$ as it must be, given Theorem 1.4). This result is not surprising since $W(u_1,u_2)$ contains no information about the boundary condition at b.

Example 2. *Again $p=r=1$, $q=0$. Take $[a,b]=[-1,1]$. Consider the two sets of boundary conditions*

$$\text{(B1)}\qquad u(\pm1)=0,$$
$$\text{(B2)}\qquad u(\pm1)=\pm u'(\pm1),$$

with corresponding operators H_1 and H_2. The lowest eigenvalue of H_1 is $\frac14\pi^2$. H_2 has 0 as an eigenvalue with eigenvector $\varphi(x)=x$. H_2 has the lowest eigenvalue at α where α satisfies $\sqrt{-\alpha}\tanh\sqrt{-\alpha}=1$ (i.e., $\alpha\sim-1.44$). Let $E_1=-2$, $E_2=0.5$, and $-u_j''=E_ju_j$, $-v_j''=E_jv_j$, $j=1,2$, with $u_2(1)=v_1(-1)=0$ and $v_2(1)-v_2'(1)=u_1(0)+u_1'(0)=0$. Since H_2 has two eigenvalues in (E_1,E_2) and H_1 has none, we see by Theorem 1.3 that $W_0(v_1,u_2)=0$, $W_0(u_1,v_2)=2$, and thus any of $0,\pm1,\pm2$ can occur in Theorem 7.2.

Theorem 7.4. *Let $E_1 < E_2$. Let $\tau u_j = E_j u_j$, $j = 1, 2$. If $a < x_0 < x_1 < b$ are zeros of u_1 or of $W(u_1, u_2)(\,.\,)$, then the number of zeros of u_2 inside (x_0, x_1) equals the number of zeros of $W(u_1, u_2)(\,.\,)$ inside (x_0, x_1) plus the number of zeros of u_1 inside (x_0, x_1) plus one.*

Proof. Let Γ_{u_1} be the tortoise and Γ_{u_2} the hare. Γ_{u_2} starts out ahead or equal and the number of times Γ_{u_2} laps (inside (x_0, x_1)) is equal to $W_0(u_1, u_2)$. Since Γ_{u_2} ends up slightly ahead (i.e., $\Gamma_{u_2}(E_2, b-\epsilon) > \Gamma_{u_1}(E_1, b-\epsilon)$), the number of zeros of u_2 equals the number of laps plus the number of zeros of u_1 plus one. □

The following result is of special interest in connection with the problem of whether the total number of eigenvalues of H in one of its essential spectral gaps is finite or infinite. In particular, the energies E_1, E_2 in Theorem 7.5 below may lie in the essential spectrum of H. For this purpose we consider an auxiliary Dirichlet operator $H^D_{x_0}$, $x_0 \in (a, b)$ associated with H. $H^D_{x_0}$ is obtained by taking the direct sum of the restrictions $H^D_{x_0,\pm}$ of H to (a, x_0), respectively (x_0, b), with a Dirichlet boundary condition at x_0. We emphasize that the Dirichlet boundary conditions can be replaced by boundary conditions of the type $\lim_{\epsilon\downarrow 0}[u'(x_0 \pm \epsilon) + \beta u(x_0 \pm \epsilon)] = 0$, $\beta \in \mathbb{R}$.

Theorem 7.5. *Let $E_1 < E_2$. Let $\tau u_j = E_j u_j$, $\tau s_j = E_j s_j$, and $s_j(E_j, x_0) = 0$, $j = 1, 2$. Then we have*

(i) $\dim \operatorname{Ran} P_{(E_1,E_2)}(H) < \infty$ *if and only if* $W_0(u_1, u_2) < \infty$.

(ii) $\dim \operatorname{Ran} P_{(E_1,E_2)}(H) - 1 \le \dim \operatorname{Ran} P_{(E_1,E_2)}(H^D_{x_0}) \le \dim \operatorname{Ran} P_{(E_1,E_2)}(H) + 2$.

(iii) $W_0(s_1, s_2) - 1 \le \dim \operatorname{Ran} P_{(E_1,E_2)}(H^D_{x_0}) \le W_0(s_1, s_2) + 1$.

Proof. Items (ii), (iii), and Theorem 7.2 imply (i). (ii) is clear if the essential spectrum of H and (E_1, E_2) are not disjoint. Otherwise, if the essential spectrum of H and (E_1, E_2) are disjoint, a standard rank-one perturbation argument, combined with the strict monotonicity of the Green's function $G(E, x_0, x_0)$ of H with respect to E in essential spectral gaps of H, applies. For (iii) it suffices to prove

$$W_{0,\pm}(s_1, s_2) \le \dim \operatorname{Ran} P_{(E_1,E_2)}(H^D_{x_0,\pm}) \le W_{0,\pm}(s_1, s_2) + 1,$$

where $W_{0,\pm}(s_1, s_2)$ abbreviates the number of zeros of the Wronskian $W(s_1, s_2)$ inside (x_0, b), respectively (a, x_0). But this is immediate from Theorems 1.5 and 1.6. □

Next we want to see how Theorem 1.2 (and hence Hartman's theorem [10]) follows from Theorem 1.4. We start by assuming τ to be oscillatory at E_2 near b. By Theorem 1.4, $W_0(u_1, u_2) = N_0$ since H in Theorem 1.2 is assumed to be limit point at b, so we need only show that $W_0(u_1, u_2) = \varliminf_{c\uparrow b} N(c)$ in order to prove (1.5a). Suppose first that $W_0(u_1, u_2) = m < \infty$. Since τ is non-oscillatory at E_2 near a we can pick x_0 such that u_2 and $W(u_1, u_2)$ have no zeros in $(a, x_0]$. Hence we can assume without loss of generality that $\theta_{u_1}(x_0) = \theta_0 \in (0, \pi)$ and $\theta_{u_2}(x_0) \in (\theta_0, \pi)$. Let x_m be the last zero of $W(u_1, u_2)(x)$ (set $m = 0$ and skip equation (7.1) if there are no zeros). At x_m,

$$\theta_{u_2}(x_m) = \theta_{u_1}(x_m) + m\pi \tag{7.1}$$

and then

$$\Gamma_{u_2}(x_m + \epsilon) > \Gamma_{u_1}(x_m + \epsilon). \tag{7.2}$$

Let $N_{u_j}(x)$ be the number of zeros of u_j, $j = 1, 2$ in (a, x). By (7.1) and Proposition 3.3,

$$N_{u_2}(x_m) = N_{u_1}(x_m) + m.$$

As x increases, (7.2) says that the next zero is of u_2 and then since W has no zeros, zeros of u_1 and u_2 must alternate. So for $c > x_m$, $N(c) \equiv N_{u_2}(c) - N_{u_1}(c)$ alternates between m and $m+1$ and since τ is assumed to be oscillatory at E_2 near b, we immediately get $\varliminf_{c\uparrow b} N(c) = m$.

If $W_0(u_1, u_2) = \infty$, let x_m be the mth zero. Then (7.1) and (7.2) still hold so $N(x_m) = m$. Since u_2 has zeros between any pair of zeros of u_1, $N(x) \geq m$ for any $x \geq x_m$, so $\varliminf_{c\uparrow b} N(x) = \infty$, as required.

If τ is non-oscillatory at E_2 near b, we first assume that $E_{1,2}$ are not eigenvalues. We need to show that the hare ends up further along than the tortoise. Without loss we assume $u_{1,2}(x) > 0$ for x near b and claim in addition that $u_1 u_2$ is not L_1 near b. If $u_1 < u_2$ or $u_2 < u_1$ eventually near b, we are done since $u_1 u_2 > u_1^2$ or $u_1 u_2 > u_2^2$ for x near b and $u_j \notin L^2(\cdot; r\,dx)$ near b. In fact, by hypothesis, $u_j \in L^2(\cdot; r\,dx)$ near a and since E_j are not eigenvalues and τ is limit point at b, u_j cannot be $L^2(\cdot; r\,dx)$ near b. Otherwise we can find two points x_0 and x_1 close to b such that $W(u_1, u_2)(x_0) \geq 0$ and $W(u_1, u_2)(x_1) \leq 0$, contradicting (2.2). But $u_1 u_2$ not $L^1(\cdot; r\,dx)$ near b together with (2.2) implies that $u_2'/u_2 > u_1'/u_1$ for x near b which, by monotonicity of $\cot(\,.\,)$, yields that the hare ends up ahead.

It remains to treat the case where $E_{1,2}$ could be eigenvalues. Choose $E' < E''$ with $u(E')$ (resp. $u(E'')$) equal to $\psi_-(E')$ (resp. $\psi_-(E'')$) the corresponding wave functions. Next, choosing E' below the spectrum of H (implying that $u(E')$ has no zeros by Corollary 2.4) shows that the number of zeros of $u(E'')$ equals the number of eigenvalues below E'' (compare Corollary 2.4), that is, equals $\dim \operatorname{Ran} P_{(-\infty, E'')}(H)$ if E'' is not an eigenvalue. Theorem 2.1 then covers the case where E'' is an eigenvalue. Applying this to $E'' = E_1$ and $E'' = E_2$ proves (1.5b) since

$$\dim \operatorname{Ran} P_{(-\infty, E'')}(H) - \dim \operatorname{Ran} P_{(-\infty, E')}(H) = \dim \operatorname{Ran} P_{[E', E'')}(H).$$

Finally, we want to consider the relation to the density of states. Given an SL operator H, let $H_{(L)}^D$ be the operator on $[-L, L]$ with Dirichlet boundary conditions. If the limit exists, we define the integrated density of states (ids), $k(E)$, by the limit:

$$k(E) = \lim_{L\to\infty} (2L)^{-1} \dim \operatorname{Ran} P_{(-\infty, E)}(H_{(L)}^D).$$

Theorem 7.6. *Suppose H is such that the ids exists for all E. Let $E_1 < E_2$ and suppose $\tau u = E_1 u$, $\tau v = E_2 v$. Let $W_{(L)}$ be the number of zeros of $W(u_1, u_2)$ in $[-L, L]$. Then*

$$\lim_{L\to\infty} (2L)^{-1} W_{(L)} = k(E_2) - k(E_1).$$

Proof. By Theorem 7.2 and Theorem 1.3, $|W_{(L)} - \dim \operatorname{Ran} P_{(E_1, E_2)}(H_{(L)}^D)| \leq 2$, so the result follows from

$$\lim_{L\to\infty} (2L)^{-1} \dim \operatorname{Ran} P_{(E_1, E_2)}(H_{(L)}^D) = k(E_2) - k(E_1).$$

□

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Appendix A. Associated Quadratic Forms

The purpose of this appendix is to clarify some form domain questions which arise due to our general conditions on the local behavior on r, p, and q in (1.1).

We'll consider regular SL operators and hence assume $(a,b) \subset \mathbb{R}$ to be a finite interval with

$$r, p^{-1}, q \in L^1((a,b); dx) \quad \text{real-valued and } r, p > 0 \text{ a.e. on } (a,b). \tag{A.1}$$

Next, define in $L^2((a,b); r\,dx)$ the following linear operators

$$\begin{aligned}
&(H^0_{\alpha,\beta} u)(x) = -r(x)^{-1}(p(x)u'(x))', \\
&D(H^0_{\alpha,\beta}) = \{u \in L^2((a,b); r\,dx) \mid u, pu' \in AC([a,b]), r^{-1}(pu')' \in L^2((a,b); r\,dx), \\
&\qquad (pu')(a) + \alpha u(a) = (pu')(b) + \beta u(b) = 0\}, \\
&\qquad \alpha, \beta \in \mathbb{R} \cup \{\infty\}
\end{aligned}$$

(here $\alpha = \infty$ denotes a Dirichlet boundary condition $u(a) = 0$ and similarly at b),

$$\begin{aligned}
&S_{\alpha,\beta} u = s\,u, \quad (s\,u)(x) = (p(x)/r(x))^{1/2} u'(x), \quad \alpha, \beta \in \{0, \infty\}, \\
&D(S_{\alpha,\beta}) = \{u \in L^2((a,b); r\,dx) \mid u \in AC([a,b]), s\,u \in L^2((a,b); r\,dx), \\
&\qquad u(a) = 0 \text{ if } \alpha = \infty,\ u(b) = 0 \text{ if } \beta = \infty\}, \\
&S^+_{\alpha,\beta} u = s^+ u, \quad (s^+ u)(x) = -r(x)^{-1}[(p(x)r(x))^{1/2} u(x)]', \quad \alpha, \beta \in \{0, \infty\}, \\
&D(S^+_{\alpha,\beta}) = \{u \in L^2((a,b); r\,dx) \mid (pr)^{1/2} u \in AC([a,b]), s^+ u \in L^2((a,b); r\,dx), \\
&\qquad ((pr)^{1/2} u)(a) = 0 \text{ if } \alpha = 0, ((pr)^{1/2} u)(b) = 0 \text{ if } \beta = 0\},
\end{aligned}$$

and the form

$$R^0_{\alpha,\beta}(u,v) = \langle S_{\alpha,\beta} u, S_{\alpha,\beta} v \rangle_r, \quad D(R^0_{\alpha,\beta}) = D(S_{\alpha,\beta}), \quad \alpha, \beta \in \{0, \infty\}$$

($\langle\,.\,,\,.\,\rangle_r$ the scalar product in $L^2((a,b); r\,dx)$).

Lemma A.1. *(i)* $S_{\alpha,\beta} = (S^+_{\alpha,\beta})^*$ *and* $S^+_{\alpha,\beta} = S^*_{\alpha,\beta}$ *for all* $\alpha, \beta \in \{0, \infty\}$.
(ii) $H^0_{\alpha,\beta} = S^*_{\alpha,\beta} S_{\alpha,\beta}$, $\alpha, \beta \in \{0, \infty\}$.

Proof. Define

$$\begin{aligned}
&K : L^2((a,b); r\,dx) \to D(S_{\infty,0}), && \widehat{K} : L^2((a,b); r\,dx) \to D(S^+_{0,\infty}), \\
&g \mapsto \int_a^x \frac{g(y) r(y)\,dy}{(p(y)r(y))^{1/2}}, && g \mapsto -(pr)(x)^{-1/2} \int_a^x g(y) r(y)\,dy.
\end{aligned}$$

A straightforward calculation verifies $(K\,g)(a) = 0$, $sK\,g = g$ and $\big((pr)^{1/2} \widehat{K}\,g\big)(a) = 0$, $s^+ \widehat{K}\,g = g$.

We only show $S^*_{\alpha,\beta} = S^+_{\alpha,\beta}$, the case $(S^+_{\alpha,\beta})^* = S_{\alpha,\beta}$ being analogous. Moreover, since $S_{\infty,\infty} \subseteq S_{\alpha,\beta}$ implies $S^*_{\alpha,\beta} \subseteq S^*_{\infty,\infty}$ we only concentrate on proving $S^*_{\infty,\infty} = S^+_{\infty,\infty}$, the rest following from an additional integration by parts.

An integration by parts proves $S^+_{\infty,\infty} \subseteq S^*_{\infty,\infty}$. Conversely, let $f \in D(S^*_{\infty,\infty})$ and set $g = \widehat{K} S^*_{\infty,\infty} f$. Then

$$\int_a^b (\bar{f} - \bar{g})(S_{\infty,\infty} h) r\,dx = \int_a^b (S^*_{\infty,\infty} \bar{f} - s^+ \bar{g}) h r\,dx = 0$$

for all $h \in D(S_{\infty,\infty})$. Thus, $\mathrm{Ran}(S_{\infty,\infty})$ is a subset of the kernel of the linear functional $k \mapsto \langle f - g, k \rangle_r$. But $\mathrm{Ran}(S_{\infty,\infty}) = \{(pr)^{-1/2}\}^\perp$ (since $g \in \mathrm{Ran}(S_{\infty,\infty})$ is equivalent to $(Kg)(b) = 0$) and hence $f = g + c(pr)^{-1/2} \in D(S^+_{\infty,\infty})$ for some constant c proving $S^*_{\infty,\infty} \subseteq S^+_{\infty,\infty}$ and hence (i).

By inspection, we obtain $D(S^+_{\alpha,\beta} S_{\alpha,\beta}) = \{u \in D(S_{\alpha,\beta}) \mid S_{\alpha,\beta} u \in D(S^+_{\alpha,\beta})\} = D(H^0_{\alpha,\beta})$ since $pu' \in AC([a,b])$ implies $(p/r)^{1/2} u' = (pr)^{-1/2}(pu') \in L^2((a,b); r\,dx)$ and $S^+_{\alpha,\beta} S_{\alpha,\beta} u = H^0_{\alpha,\beta} u$. This fact together with (i) proves (ii). $\square$

Furthermore, we introduce the forms

$$Q_{q/r}(u,v) = \int_a^b q(x)r(x)^{-1}\,\overline{u(x)}\,v(x)r(x)\,dx,$$
$$D(Q_{q/r}) = \{u \in L^2((a,b); r\,dx) \mid (|q|/r)^{1/2}u \in L^2((a,b); r\,dx)\},$$

and

$$q_{\alpha,\beta}(u,v) = \tilde\beta\,\overline{u(b)}\,v(b) - \tilde\alpha\,\overline{u(a)}\,v(a), \quad D(q_{\alpha,\beta}) = AC([a,b]),$$
$$\tilde\alpha = \begin{cases} \alpha, & \alpha \in \mathbb{R} \\ 0, & \alpha = \infty \end{cases}, \quad \tilde\beta = \begin{cases} \beta, & \beta \in \mathbb{R} \\ 0, & \beta = \infty \end{cases}, \quad \alpha,\beta \in \mathbb{R}\cup\{\infty\}.$$

Finally, we set

$$Q^0_{\alpha,\beta} = R^0_{\widehat\alpha,\widehat\beta} + q_{\alpha,\beta}, \quad D(Q^0_{\alpha,\beta}) = D(S_{\widehat\alpha,\widehat\beta}),$$
$$\widehat\alpha = \begin{cases} 0, & \alpha \in \mathbb{R} \\ \infty, & \alpha = \infty \end{cases}, \quad \widehat\beta = \begin{cases} 0, & \beta \in \mathbb{R} \\ \infty, & \beta = \infty \end{cases}, \quad \alpha,\beta \in \mathbb{R}\cup\{\infty\}$$

and

$$Q_{\alpha,\beta} = Q^0_{\alpha,\beta} + Q_{q/r}, \quad D(Q_{\alpha,\beta}) = D(S_{\widehat\alpha,\widehat\beta}), \quad \alpha,\beta \in \mathbb{R}\cup\{\infty\}. \tag{A.2}$$

Lemma A.2. *(i) $q_{\alpha,\beta}$ is infinitesimally form bounded with respect to $Q^0_{0,0}$.*
(ii) $Q_{q/r}$ is relatively form compact with respect to $Q^0_{\alpha,\beta}$, $\alpha,\beta \in \mathbb{R}\cup\{\infty\}$.

Proof. (i) Since for arbitrary $c \in [a,b]$ and $u \in D(S_{0,0})$,

$$|u(c)|^2 = \left| u(x)^2 - 2\int_c^x u(t)u'(t)\,dt \right| \le |u(x)|^2 + 2\int_a^b |u(t)u'(t)|\,dt,$$

one infers (after taking the supremum over all $c \in [a,b]$, multiplying by r, and integrating from a to b) for any $\epsilon > 0$,

$$\begin{aligned}\|u\|^2_{L^\infty((a,b);dx)} &\le \|r\|^{-1}_{L^1((a,b);dx)}\|u\|^2_{2,r} + 2\int_a^b \frac{|u(t)|}{(\epsilon p(t)/2)^{1/2}}\,(\epsilon p(t)/2)^{1/2}|u'(t)|\,dt \\ &\le \|r\|^{-1}_{L^1((a,b);dx)}\|u\|^2_{2,r} + \int_a^b \left(\frac{2}{\epsilon}\,\frac{|u(t)|^2}{p(t)} + \frac{\epsilon}{2}\,p(t)|u'(t)|^2\right) dt. \end{aligned} \tag{A.3}$$

Since $0 < p^{-1} \in L^1((a,b); dx)$, we can find a $\delta_1(\epsilon) > 0$ such that $\int_{I_1(\epsilon)} p(t)^{-1}\,dt \le \frac{\epsilon}{8}$ with $I_1(\epsilon) = \{x \in (a,b) \mid p(x) < \delta_1(\epsilon)\}$. Thus,

$$\int_a^b \frac{|u(t)|^2}{p(t)}\,dt = \int_{I_1(\epsilon)} \frac{|u(t)|^2}{p(t)}\,dt + \int_{(a,b)\backslash I_1(\epsilon)} \frac{|u(t)|^2}{p(t)}\,dt \le \frac{\epsilon}{8}\,\|u\|^2_{L^\infty((a,b);dx)} + \frac{1}{\delta_1(\epsilon)}\int_a^b |u(t)|^2 dt.$$

In addition, since $r > 0$ a.e. on (a,b), we can find a $\delta_2(\epsilon) > 0$ such that $|I_2(\epsilon)| \le \frac{\epsilon\delta_1(\epsilon)}{8}$ with $I_2(\epsilon) = \{x \in (a,b) \mid r(x) < \delta_2(\epsilon)\}$. Thus,

$$\int_a^b |u(t)|^2\,dt = \int_{I_2(\epsilon)} |u(t)|^2\,dt + \int_{(a,b)\backslash I_2(\epsilon)} |u(t)|^2\,dt \le \frac{\epsilon\delta_1(\epsilon)}{8}\,\|u\|^2_{L^\infty((a,b);dx)} + \frac{1}{\delta_2(\epsilon)}\,\|u\|^2_{2,r}$$

and one obtains from (A.3),

$$\|u\|^2_{L^\infty((a,b);dx)} \le 2\big\{\|r\|^{-1}_{L^1((a,b);dx)} + 2[\epsilon\delta_1(\epsilon)\delta_2(\epsilon)]^{-1}\big\}\|u\|^2_{2,r} + \epsilon Q^0_{0,0}(u,u), \tag{A.4}$$

completing the proof of (i).

(ii) Let $G^0_{\alpha,\beta}(z,x,y)$ denote the Green's function of $H^0_{\alpha,\beta}$, $\alpha,\beta \in \mathbb{R}\cup\{\infty\}$, that is,

$$((H^0_{\alpha,\beta}-z)^{-1}u)(x) = \int_a^b G^0_{\alpha,\beta}(z,x,y)u(y)r(y)\,dy, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Then $|q/r|^{1/2}(H^0_{\alpha,\beta}-z)^{-1}|q/r|^{1/2} \in \mathfrak{B}_2(L^2((a,b);r\,dx))$ ($\mathfrak{B}_2(\,.\,)$ the set of Hilbert-Schmidt operators) since

$$\begin{aligned}\||q/r|^{1/2}(H^0_{\alpha,\beta}-z)^{-1}|q/r|^{1/2}\|_2^2 &= \int_a^b\int_a^b \frac{|q(x)|}{r(x)}\,|G^0_{\alpha,\beta}(z,x,y)|^2\frac{|q(y)|}{r(y)}\,r(x)r(y)\,dxdy\\ &\le M(z)\Bigg[\int_a^b |q(x)|\,dx\Bigg]^2\end{aligned}$$

using the fact that $|G^0_{\alpha,\beta}(z,\,.\,,\,.\,)|$ is bounded on $(a,b)\times(a,b)$. □

Thus the forms $Q_{\alpha,\beta}$ in (A.2) are densely defined, closed, and bounded from below ([12], Section VI.1). We denote by $H_{\alpha,\beta}$ the uniquely associated self-adjoint operators bounded from below guaranteed by the KLMN theorem ([12], Theorem VI.2.1; [17], Theorem X.17). The following theorem identifies $H_{\alpha,\beta}$ as the usual regular SL operators (with separated boundary conditions).

Theorem A.3. *$H_{\alpha,\beta}$ associated with $Q_{\alpha,\beta}$ is given by*

$$\begin{aligned}&(H_{\alpha,\beta}u)(x) = r(x)^{-1}[-(p(x)u'(x))' + q(x)u(x)],\\ &D(H_{\alpha,\beta}) = \{u \in L^2((a,b);r\,dx) \mid u, pu' \in AC([a,b]), r^{-1}(-(pu')'+qu) \in L^2((a,b);r\,dx),\\ &\qquad (pu')(a)+\alpha u(a) = (pu')(b)+\beta u(b) = 0\},\\ &\qquad \alpha,\beta \in \mathbb{R}\cup\{\infty\}.\end{aligned} \tag{A.5}$$

Proof. It suffices to consider the Dirichlet case $\alpha=\beta=\infty$, the other cases being similar. Denote by $\widehat{H}_{\infty,\infty}$ the operator defined in (A.5) for $\alpha=\beta=\infty$ and by $H_{\infty,\infty}$ the unique operator associated with $Q_{\infty,\infty}$. Choose $u \in D(Q_{\infty,\infty})$ and $v \in D(\widehat{H}_{\infty,\infty})$. Then an integration by parts yields

$$Q_{\infty,\infty}(u,v) = \langle u, \widehat{H}_{\infty,\infty}v\rangle_r.$$

Thus $\widehat{H}_{\infty,\infty} \subseteq H_{\infty,\infty}$ by Corollary VI.2.4 of [12] and hence $\widehat{H}_{\infty,\infty} = H_{\infty,\infty}$ since $\widehat{H}_{\infty,\infty}$ is self-adjoint. □ □

Remark. *It follows from the above theorem, that for arbitrary SL operators H (not necessarily regular), elements $u \in L^2((a,b);r\,dx)$ which satisfy*

$$u \in AC_{\mathrm{loc}}((a,b)), \quad (p/r)^{1/2}u' \in L^2_{\mathrm{loc}}((a,b);r\,dx) \tag{A.6}$$

and which are in the domain of H near a and b, are in the form domain of H. Moreover, let $u(x), v(x)$ be as in (A.6) *and in $D(H)$ for $x \le c$ and $x \ge d$, then*

$$\begin{aligned}Q_H(u,v) = &\int_{(a,b)\backslash(c,d)} \overline{u(x)}\,(\tau v)(x)\,dx + \overline{u(d)}\,(pv')(d) - \overline{u(c)}\,(pv')(c)\\ &+\int_c^d [\overline{u'(x)}\,(pv')(x) + q(x)\overline{u(x)}\,v(x))]\,dx.\end{aligned} \tag{A.7}$$

In fact, take u as in (A.6) *and in $D(H)$ for $x \le c$ and $x \ge d$. Consider the operator $\tilde{H}_{\alpha,\beta}$ associated with τ and boundary conditions induced by u on the finite interval*

(c,d) (cf. (5.1)). Since $u\restriction_{(c,d)} \in D(\tilde{Q}_{\alpha,\beta})$ ($\tilde{Q}_{\alpha,\beta}$ the form of $\tilde{H}_{\alpha,\beta}$), we can pick a sequence $\tilde{u}_n$ in $D(\tilde{H}_{\alpha,\beta})$ such that $\|\tilde{u}_n - u\restriction_{(c,d)}\|_{2,r} \to 0$ and $\langle(\tilde{u}_n - \tilde{u}_m), \tilde{H}_{\alpha,\beta}(\tilde{u}_n - \tilde{u}_m)\rangle_r \to 0$ (implying $\|\tilde{u}_n - u\restriction_{(c,d)}\|_{L^\infty((a,b);dx)} \to 0$ by (A.4) and Lemma A.2). Extend $\tilde{u}_n$ to a function u_n on (a,b) by patching it with u such that $u_n \in D(H)$ (which is possible since u and $\tilde{u}_n$ satisfy the same boundary conditions at c and d). By construction, u_n satisfies $\|u_n - u\|_{2,r} \to 0$ and $\langle(u_n - u_m), H(u_n - u_m)\rangle_r \to 0$ and hence is in the form domain of H. This proves (A.6) and an integration by parts then proves (A.7).

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