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DISSERTATION / DOCTORAL THESIS

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«The modified Camassa–Holm equation with nonvanishing boundary conditions by a Riemann–Hilbert approach»

verfasst von / submitted by

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Abstract

This Thesis aims at the development of the Inverse Scattering Transform (IST), in the form of a Riemann–Hilbert problem, for the modified Camassa–Holm (mCH) equation

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \qquad\qquad m := u - u_{xx}$$

on the line with non-zero boundary conditions.

In the first part (Chapter 2 and 3), we develop the Riemann–Hilbert (RH) formalism to the Cauchy problem on the whole x-line in the case when the solution is assumed to approach a non-zero constant as $|x| \to \infty$. In this case, the spectral problem for the associated Lax pair has a continuous spectrum, which allows formulating the inverse spectral problem as a Riemann–Hilbert factorization problem with jump conditions across the real axis. We obtain a representation for the solution of the Cauchy problem for the mCH equation and also a description of certain soliton-type solutions, both regular and non-regular. Moreover, we apply the nonlinear steepest descent method to study the large-time asymptotics of the solution of this Cauchy problem.

In the second part (Chapter 4), we develop the Riemann–Hilbert formalism for the Cauchy problem on the whole x-line in the case when the solution is assumed to approach two different constants as $x \to +\infty$ and $x \to -\infty$. We present detailed properties of spectral functions associated with the initial data for the Cauchy problem for the mCH equation and obtain a representation for the solution of this problem in terms of the solution of an associated RH problem.

Zusammenfassung

Das Ziel der Dissertation ist die inverse Streutransformation (IST) in der Form eines Riemann–Hilbert-Problems (RHP) für die modifizierte Camassa–Holm (mCH) Gleichung

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \qquad m := u - u_{xx}$$

auf der Achse mit nichttrivialen Randverhalten zu entwickeln.

Im ersten Teil (Kapitel 2 und 3) entwickeln wir den Riemann–Hilbert-Formalismus für das Cauchy-Problem auf der ganzen x-Achse für den Fall, dass die Lösung zu einer von Null verschiedenen Konstante für $|x| \to \infty$ konvergiert. In diesem Fall hat das Spektralproblem für das zugehörige Lax-Paar ein kontinuierliches Spektrum. Das erlaubt das inverse Spektralproblem als ein Riemann–Hilbert-Faktorisierungsproblem mit Sprungbedingung über die reelle Achse zu formulieren. Wir erhalten eine Darstellung für die Lösung des Cauchy-Problems für die mCH-Gleichung und auch eine Beschreibung bestimmter Solitonen-Typ-Lösungen, sowohl regulärer als auch nicht regulärer. Darüber hinaus verwenden wir die nichtlineare Methode des steilsten Abstiegs, um die Langzeit-Asymptotik zu untersuchen.

Im zweiten Teil (Kapitel 4) entwickeln wir den Riemann–Hilbert-Formalismus für das Cauchy-Problem auf der ganzen x-Achse für den Fall, dass die Lösung zu zwei verschiedenen Konstanten für $x \to +\infty$ und $x \to -\infty$ konvergiert. Wir präsentieren detaillierte Eigenschaften der Spektralfunktionen, die mit den Anfangsdaten für das Cauchy-Problem der mCH-Gleichung assoziiert sind, und erhalten eine Darstellung für die Lösung dieses Problems in Bezug auf die Lösung eines zugehörigen RHPs.

Extended Abstract

I. Karpenko, "The modified Camassa-Holm equation with nonvanishing boundary conditions by a Riemann-Hilbert approach," – Scholarly manuscript.

PhD Thesis in Mathematics (specialty code: 111). B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine and University of Vienna.

This Thesis aims at the development of the inverse scattering transform (IST), in the form of a Riemann-Hilbert problem, for the modified Camassa-Holm (mCH) equation:

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \qquad\qquad m := u - u_{xx},$$

in order to study the long-time behavior of solutions.

Two main problem settings are as follows:

(i) The Cauchy problem on the whole x-line in the case when the solution is assumed to approach a non-zero constant as $|x| \to \infty$.

(ii) The Cauchy problem on the whole x-line in the case when the solution is assumed to approach two different constants as $x \to +\infty$ and $x \to -\infty$.

In **Chapter 2**, we consider the Cauchy problem for the modified Camassa–Holm equation on the line:

$$\begin{aligned} &m_t + \left((u^2 - u_x^2)m\right)_x = 0, \qquad m := u - u_{xx}, \qquad t > 0, \qquad -\infty < x < +\infty, \\ &u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\infty < x < +\infty, \end{aligned}$$

assuming that

$$u_0(x) \to 1 \quad \text{as} \ \ x \to \pm\infty$$

and that the time evolution preserves this behavior: $u(x,t) \to 1$ as $x \to \pm\infty$ for all $t > 0$. A non-zero background provides that the spectral problem in the associated Lax pair equations has a continuous spectrum, which allows us to formulate the inverse spectral problem as a Riemann-Hilbert factorization problem with jump conditions across the real axis (constituting the continuous spectrum).

Our development of the Riemann–Hilbert problem formalism is based on the adaptation of a general idea — the use of dedicated (Jost) solutions of the associated Lax pair equations as "building blocks" for a matrix-valued Riemann–Hilbert problem, which is formulated in the complex plane of the spectral parameter and parameterized by the spatial and temporal variable of the nonlinear equation in question — to the case of the mCH equation taking into account particular features of its Lax pair equations.

The Lax pair originally proposed and conventionally used in studies of the mCH equation has the form of 2×2 matrix linear differential equations:

$$\Phi_x(x,t,\lambda) = \mathsf{U}(x,t,\lambda)\Phi(x,t,\lambda), \qquad \Phi_t(x,t,\lambda) = \mathsf{V}(x,t,\lambda)\Phi(x,t,\lambda)$$

where the coefficient matrices U and V are defined in terms of a solution of the mCH equation:

$$\mathsf{U} = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda m & 1 \end{pmatrix}, \ \mathsf{V} = \begin{pmatrix} \lambda^{-2} + \frac{u^2-u_x^2}{2} & -\frac{u-u_x}{\lambda} - \frac{\lambda(u^2-u_x^2)m}{2} \\ \frac{(u+u_x)}{\lambda} + \frac{\lambda(u^2-u_x^2)m}{2} & -\lambda^{-2} - \frac{u^2-u_x^2}{2} \end{pmatrix}.$$

Two specific features of the x-equation associated with the mCH equation (involving U and constituting the spectral problem, with the spectral parameter λ) that affect analytic properties of the Jost solutions are as follows: (a) λ enters U through a product with the "momentum" $m(x,t)$, which, in the framework of the inverse problem, is an unknown function; (b) as $|x| \to \infty$, $m(x,t)$ approaches a non-zero constant. In particular, these features affect the problem of control of the large-λ behavior of the Jost solutions. In our development of the RH formalism, this problem is addressed by (i) transforming (by applying a dedicated gauge transformation) the Lax pair equations to an appropriate form, with selected diagonal parts that dominate, in a certain sense, for large λ; (ii) introducing a new spatial-type variable, in view of having an explicit description of the large-λ behavior of the Jost solutions in terms of space and time parameters; (iii) introducing a new (uniformising) spectral parameter μ (related to λ by $\lambda = -\frac{1}{2}(\mu + \frac{1}{\mu})$), which allows us to avoid non-rational dependence of the coefficients in the Lax pair equations on the spectral parameter.

Moreover, we take advantage of a consequence of property (a) that for $\lambda = 0$, U becomes "solution-independent" (independent of u), which suggests an efficient way for "extracting" the solution of the Cauchy problem from the solution of the RH problem taking the details of the behavior of the latter as $\lambda \to 0$. With this respect, the mCH equation turns out to be remarkably different from other Camassa-Holm-type equations (including the original Camassa-Holm equation): in order to control the Jost solutions at $\lambda = 0$, there is no need of a separate gauge transformation of the original Lax pair, but the required form of the Lax pair comes from regrouping the terms of that appropriate for large λ.

Using the developed formalism, we obtain a parametric representation of the solution of the Cauchy problem for the mCH equation on a constant background in terms of the solution of the associated RH problem, the data for which (the jump matrix and the parameters of the residue conditions, if any) are uniquely determined by the initial data for the Cauchy problem.

Particularly, this formalism allows us to characterize regular as well as non-regular one-soliton solutions associated with the RH problems with trivial jump condition and appropriately prescribed residue conditions. In this way, we specify two families of non-regular soliton solutions of the mCH equation: (i) peakon-type solutions, which are continuous together with their first derivative but having unbounded derivatives of order greater than 2 at the peak points; (ii) loop-shaped, multi-valued solutions, which are conventional, signal-valued solitons in the modified variables that becomes multivalued when going back to the original variables, x and t.

Theorem. *The mCH equation has a family of one-soliton solutions, regular as well as non-regular,* $u(x,t) \equiv u_{\theta,\hat{\delta}}(x,t)$*, parametrized by two parameters,* $\hat{\delta} > 0$ *and* $\theta \in (0, \frac{\pi}{2})$*. These solitons* $u(x,t) \equiv \hat{u}(y(x-t,t),t) + 1$ *are given, in parametric form, by*

$$\begin{aligned}
\hat{u}(y,t) &= 4\tan^2\theta \frac{z^2(y,t) + 2\cos^2\theta \cdot z(y,t) + \cos^2\theta}{(z^2(y,t) + 2z(y,t) + \cos^2\theta)^2} z(y,t), \\
x(y,t) &= y + 2\ln\frac{z(y,t) + 1 + \sin\theta}{z(y,t) + 1 - \sin\theta}, \\
z(y,t) &= 2\hat{\delta}\sin\theta\, \mathrm{e}^{y\sin\theta} \mathrm{e}^{-\frac{2\sin\theta}{\cos^2\theta}t}.
\end{aligned}$$

Depending on the value of the parameter θ*, the solutions have qualitatively different properties:*

(i) *For* $\theta \in (0, \frac{\pi}{3})$*, one-soliton solutions are smooth in the original* $((x,t))$ *variables.*

(ii) *For* $\theta = \frac{\pi}{3}$*, one-soliton solutions have finite smoothness in the* (x,t) *variables.*

(iii) *For* $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$*, one-soliton solutions are regular in the* (y,t) *variables but can be viewed as multivalued and loop-shaped in the* (x,t) *variables.*

In **Chapter 3**, taking the formalism developed in Section 2 as the starting point, we obtain the leading large-t asymptotic terms for the solution of the Cauchy problem for the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach a non-zero constant at the both infinities of the space variable. We focus on the study of the solitonless case assuming that there are no residue conditions (for the soliton case, where the basic RH problem involves residue conditions, one can reduce (using the Blaschke–Potapov factors) this RH problem to that having no residue conditions).

For the sake of the large-t analysis, we reduce the original (singular) RH problem representation for the solution of the mCH equation to the solution of a regular RH problem (i.e., to a RH problem with the jump and normalization conditions only). A notable feature of the modified Camassa-Holm equation is that the associated basic RH problem has two singularity conditions (at $\mu = \pm 1$) with different matrix structures, which does not allow getting rid of them by reducing the matrix RH problem to a vector one, as it can be done in the case of the (original) Camassa-Holm equation. In our approach, we address the reduction problem in two steps. First, we reduce the RH problem with the singularity conditions at $\mu = \pm 1$ to a RH problem which is characterized by the following two conditions: (i) the matrix entries are regular at $\mu = \pm 1$, but the determinant of the (matrix) solution vanishes at $\mu = \pm 1$ (notice that $\det M(\mu) \equiv 1$ for the solution of the original RH problem); (ii) the solution is singular at $\mu = 0$. Then, we represent the solution of the latter RH problem in terms of the solution of a regular one. In turn, the solution of the resulting regular RH problem is analyzed asymptotically, as $t \to +\infty$, using an appropriate adaptation of the nonlinear steepest descent method. This finally allows us to present the leading asymptotic terms for the solution $u(x,t)$ of the Cauchy problem, in two sectors of the (x,t) half-plane, $1 < \frac{x}{t} < 3$ and $\frac{3}{4} < \frac{x}{t} < 1$, where the deviation from the background value is nontrivial (in the remaining sectors $\frac{x}{t} > 3$ and $\frac{x}{t} < \frac{3}{4}$, $u(x,t)$ decays to 1 rapidly).

Theorem. *Let $u_0(x)$ be a smooth function which tends sufficiently fast to 1 as $x \to \pm\infty$ and satisfies $(1-\partial_x^2)u_0(x) > 0$ for all x. Assume the solitonless case, i.e., assume that the appropriate spectral (scattering) function associated with $u_0(x)$ has no zeros in the upper half-plane*

Then the solution $u(x,t)$ of the Cauchy problem has the following large-time asymptotics in two sectors of the (x,t) half-plane specified by $1 < \frac{x}{t} < 3$ and $\frac{3}{4} < \frac{x}{t} < 1$:

(i) For $1 < \zeta := \frac{x}{t} < 3$,

$$u(x,t) = 1 + \frac{C_1(\zeta)}{\sqrt{t}} \cos\left\{C_2(\zeta)t + C_3(\zeta)\ln t + \tilde{C}_4(\zeta)\right\} + \mathrm{o}(t^{-1/2});$$

(ii) For $\frac{3}{4} < \frac{x}{t} < 1$,

$$u(x,t) = 1 + \sum_{j=0,1} \frac{C_1^{(j)}(\zeta)}{\sqrt{t}} \cos\left\{C_2^{(j)}(\zeta)t + C_3^{(j)}(\zeta)\ln t + \tilde{C}_4^{(j)}(\zeta)\right\} + \mathrm{o}(t^{-1/2}),$$

where C_i, $C_i^{(j)}$, $\tilde{C}_4$, $\tilde{C}_4^j$ *are functions of* ζ *that can be specified in terms of the scattering data, which in turn are uniquely determined by the initial data.*

The error term is uniform in any sectors $1+\varepsilon < \zeta < 3-\varepsilon$ and $\frac{3}{4}+\varepsilon < \zeta < 1-\varepsilon$ resp., where ε is a small positive number.

In **Chapter 4**, we consider the Cauchy problem for the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach two different constants at plus and minus infinity of the space variable, namely:

$$\begin{aligned} &m_t + \left((u^2 - u_x^2)m\right)_x = 0, \qquad m := u - u_{xx}, \qquad t > 0, \qquad -\infty < x < +\infty, \\ &u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\infty < x < +\infty, \end{aligned}$$

assuming that

$$u_0(x) \to \begin{cases} A_1, & x \to -\infty \\ A_2, & x \to \infty \end{cases}$$

and that the time evolution preserves this behavior.

We develop the Riemann–Hilbert problem formalism for this Cauchy problem. For this purpose, we introduce appropriate transformations of the Lax pair equations and the associated Jost solutions ("eigenfunctions") and present detailed analytic properties of the eigenfunctions and the corresponding spectral functions (scattering coefficients), including the symmetries and the behavior at the branch points. The construction of the Riemann–Hilbert problem exploits the transformed Lax pair equations involving the functions $k_j(\lambda) := \sqrt{\lambda^2 - \frac{1}{A_j^2}}$,

$j = 1, 2$ specified as having the branch cuts $(-\infty, -\frac{1}{A_j}) \cup (\frac{1}{A_j}, \infty)$. Similarly to the case of the constant background, the solution of the constructed Riemann–Hilbert problem evaluated at $\lambda = 0$ gives a parametric representation of the solution of the Cauchy problem.

Theorem. *Assume that $u(x,t)$ is the solution of the Cauchy problem and let $\hat{M}(y,t,x)$ be the solution of the associated RH problem, whose data are determined by $u_0(x)$. Let*

$$\hat{M}(y,t,\lambda) = \mathrm{i}\begin{pmatrix} 0 & \hat{a}_1(y,t) \\ \hat{a}_1^{-1}(y,t) & 0 \end{pmatrix} + \mathrm{i}\lambda\begin{pmatrix} \hat{a}_2(y,t) & 0 \\ 0 & \hat{a}_3(y,t) \end{pmatrix} + \mathrm{O}(\lambda^2)$$

be the development of $\hat{M}(y,t,\lambda)$ as $\lambda \to 0$. Then the solution $u(x,t)$ of the Cauchy problem can be expressed, in a parametric form, in terms of $\hat{a}_j(y,t)$, $j = 1, 2, 3$ as follows: $u(x,t) = \hat{u}(y(x,t),t)$, where

$$\begin{aligned} \hat{u}(y,t) &= \hat{a}_1(y,t)\hat{a}_2(y,t) + \hat{a}_1^{-1}(y,t)\hat{a}_3(y,t), \\ x(y,t) &= y - 2\ln\hat{a}_1(y,t) + A_2^2 t. \end{aligned}$$

Moreover, $\hat{u}_x(y,t)$ can also be algebraically expressed in terms of $\hat{a}_j(y,t)$, $j = 1, 2, 3$; namely, $u_x(x,t) = \hat{u}_x(y(x,t),t)$, where

$$\hat{u}_x(y,t) = -\hat{a}_1(y,t)\hat{a}_2(y,t) + \hat{a}_1^{-1}(y,t)\hat{a}_3(y,t).$$

Keywords: modified Camassa-Holm equation, Riemann-Hilbert problem, Inverse Scattering Transform method, nonlinear steepest decent method, solitons.

Publications of the candidate related to the thesis topic

Journal papers in which the main results of the thesis are published

1. A. Boutet de Monvel, I. Karpenko, D. Shepelsky, “A Riemann-Hilbert approach to the modified Camassa–Holm equation with nonzero boundary conditions,” J. Math. Phys. **61**, No. 3, 031504, 24 (2020).

   https://doi.org/10.1063/1.5139519

2. I. Karpenko, “Long-time asymptotics for the modified Camassa-Holm equation with nonzero boundary conditions”, Journal of Mathematical Physics, Analysis, Geometry **16**, No.4, 418–453 (2022).

   https://doi.org/10.15407/mag18.2.224

3. I. Karpenko, D. Shepelsky, G. Teschl “A Riemann–Hilbert approach to the modified Camassa–Holm equation with step-like boundary conditions”, Monatshefte für Mathematik (2022).

   https://doi.org/10.1007/s00605-022-01786-y

Approbation of the thesis results

4. I. Karpenko, D. Shepelsky, “A Riemann-Hilbert approach to the modified Camassa-Holm equation with nonzero boundary conditions”, VI International Conference "Analysis and Mathematical Physics", Kharkiv, Ukraine (June 2018).

5. I. Karpenko, D. Shepelsky, “The Riemann-Hilbert approach to the Cauchy problem for the modified Camassa-Holm equation”, 6th Ya. B. Lopatynsky International School-Workshop on Differential Equations and Applications, Vinnytsia, Ukraine (June 2019).

6. I. Karpenko, D. Shepelsky, “The inverse scattering transform, in the form of Riemann-Hilbert problem, for the modified Camassa-Holm equation”, international Conference dedicated to 70th anniversary of Professor A.M.Plichko "Banach Spaces and their Applications", Lviv, Ukraine (June 2019).

7. I. Karpenko, D. Shepelsky, “A Riemann-Hilbert problem approach to the modified Camassa-Holm equation on a nonzero background”, Pidzakharychi, Ukraine (August 2019).

8. I. Karpenko, D. Shepelsky, “The modified Camassa-Holm equation on a nonzero background: large-time asymptotics for the Cauchy problem”, Workshop "New horizons in dispersive hydrodynamics", Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom (June 2021).

9. I. Karpenko, D. Shepelsky, “The large-time asymptotics for the modified Camassa–Holm equation on a non-zero background”, 5-th International Conference “DIFFERENTIAL EQUATIONS and CONTROL THEORY”, V. N. Karazin Kharkiv National University, Kharkiv, Ukraine (September 2021).

10. I. Karpenko, D. Shepelsky, G. Teschl, “A Riemann–Hilbert approach to the modified Camassa–Holm equation with step-like boundary conditions”, The international online conference "CURRENT TRENDS IN ABSTRACT AND APPLIED ANALYSIS", Ivano-Frankivsk, Ukraine (May 2022).

11. I. Karpenko, “The modified Camassa-Holm equation on a step-like background”, Complex Analysis, Spectral Theory and Approximation meet in Linz, Johannes Kepler University, Linz, Austria (July 2022).

12. I. Karpenko, “A Riemann-Hilbert problem approach to the modified Camassa-Holm equation on a step like background”, Workshop From Modeling and Analysis to Approximation and Fast Algorithms, Hasenwinkel, Germany (December 2022).

Анотація

Карпенко І. М., “Метод задачі Рімана-Гільберта для модифікованого рівняння Камаси-Хольма з ненульовими крайовими умовами,” – кваліфікаційна наукова праця на правах рукопису.

Дисертація на здобуття наукового ступеня доктора філософії за спеціальністю 111 «математика» (галузь знань 11 «математика та статистика»). Фізико-технічний інститут низьких температур ім. Б.І. Вєркіна НАН України.

Предметом дослідження дисертаційної роботи є розробка метода оберненої задачі розсіювання (МОЗР) у формі задачі Рімана–Гільберта для модифікованого рівняння Камаси-Хольма (мКХ):

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \qquad m := u - u_{xx},$$

з метою дослідження властивостей розв'язків цього рівняння, зокрема, асимптотики за великим часом.

Основними постановками задачі є наступні:

(i) Задача Коші на x-осі у випадку, коли розв'язок прямує до ненульової сталої при $|x| \to \infty$.

(ii) Задача Коші на x-осі у випадку, коли розв'язок прямує до двох різних сталих при $x \to +\infty$ та $x \to -\infty$.

У **Розділі 2** розглядається задача Коші для модифікованого рівняння Камаси-Хольма на осі:

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m := u - u_{xx}, \quad t > 0, \quad -\infty < x < +\infty,$$

$$u(x,0)=u_0(x), \qquad\qquad -\infty<x<+\infty,$$

за вимоги, що

$$u_0(x)\to 1 \quad \text{коли} \;\; x\to\pm\infty$$

і що еволюція за часом зберігає цю поведінку: $u(x,t)\to 1$ при $x\to\pm\infty$ для всіх $t>0$. Рівняння мКХ є модифікацією з кубічную нелінійністю оригінального рівняння Камаси-Хольма (КХ)

$$m_t+(um)_x+u_x m=0, \quad m:=u-u_{xx}.$$

Рівняння мКХ, як і рівняння КХ, є інтегровними у тому сенсі, що вони є умовами сумісності відповідних пар лінійних диференціальних рівнянь — так званих рівнянь пари Лакса. Завдяки ненульовому фону, x-рівняння з пари Лакса для рівняння мКХ може розглядатися як спектральна задача, яка має неперервний спектр. Це дозволяє сформулювати обернену спектральну задачу (обернену задачу розсіювання) як задачу аналітичної факторизації Рімана-Гільберта у комплексній площині спектрального параметра, з умовою стрибка на дійсній осі (яка є неперервним спектром).

Запропонований формалізм задачі Рімана–Гільберта базується на адаптації загальної ідеї — використання спеціальних розв'язків (Йоста) асоційованих рівняннь пари Лакса як «блоків» для побудови матричної задачі Рімана–Гільберта, що формулюється як задача алалітичної факторизації у комплексній площині спектрального параметра і параметризується просторовою та часовою змінними нелінійного рівняння — до випадку рівняння мКХ, з урахуванням особливостей рівнянь асоційованої пари Лакса.

Стандартна пара Лакса для рівняння мКХ, відома в літературі, має форму 2×2 матричних лінійних диференціальних рівнянь:

$$\Phi_x(x,t,\lambda)=\mathsf{U}(x,t,\lambda)\Phi(x,t,\lambda), \qquad \Phi_t(x,t,\lambda)=\mathsf{V}(x,t,\lambda)\Phi(x,t,\lambda)$$

де матриці-коефіцієнти U та V визначаються у термінах розв'язку рівняння мКХ:

$$\mathsf{U}=\frac{1}{2}\begin{pmatrix}-1 & \lambda m\\ -\lambda m & 1\end{pmatrix}, \; \mathsf{V}=\begin{pmatrix}\lambda^{-2}+\frac{u^2-u_x^2}{2} & -\frac{u-u_x}{\lambda}-\frac{\lambda(u^2-u_x^2)m}{2}\\ \frac{(u+u_x)}{\lambda}+\frac{\lambda(u^2-u_x^2)m}{2} & -\lambda^{-2}-\frac{u^2-u_x^2}{2}\end{pmatrix}.$$

Зазначимо, що x-рівняння з пари Лакса (включає U та є спектральною задачею зі спектральним праметром λ), має дві осообливосі, що суттєво впливають на аналітичні властивості розв'язків Йоста: (а) λ входить у U як добуток з "моментом" $m(x,t)$, який у рамках оберненої задачі є невідомою функцією; (б) коли $|x| \to \infty$, $m(x,t)$ прямує до ненульової сталої. Зокрема, ці особливості впливають на проблему контролю поведінки розв'язків Йоста, коли $\lambda \to \infty$. Ми вирішуємо цю проблему наступним чином: (i) трансформуємо (застосовуючи калібрувальні перетворення) рівняння пари Лакса до зручної форми, у якій діагональні члени домінують, у певному сенсі, коли $\lambda \to \infty$; (ii) вводимо нову просторову змінну, що дозволяє отримати явний опис поведінки розв'язків Йоста коли $\lambda \to \infty$ у термінах (нової) просторової та часової змінних; (iii) вводимо новий (уніформізуючий) спектральний параметр μ (пов'язаний з λ наступним чином: $\lambda = -\frac{1}{2}(\mu + \frac{1}{\mu})$), який дозволяє уникнути нераціональної залежності коефіцієнтів у рівняннях пари Лакса від спектрального параметра.

Крім того, ми використовуємо наслідок властивості (а), який полягає у тому, що при $\lambda = 0$ матриця коефіцієнтів U стає незалежною від u. Ця властивість дозволяє побудувати ефективний алгоритм отримання розв'язку задачі Коші для рівняння мКХ з розв'язку асоційованої задачі РГ, розглядаючи поведінку останнього при $\lambda \to 0$. Зазначимо, що цьому відношенні рівняння мКХ суттєво відрізняється від інших рівнянь типу Камаси-Холма (включно з оригінальним рівнянням КХ): для контролю розв'язків Йоста при $\lambda = 0$ не треба вводити нове калібрувальне перетворення початкової пари Лакса, а достатньо перегрупувати члени у парі Лакса, яка забеспечує єфективний контроль її розв'язків при $\lambda \to \infty$.

Використовуючи розроблений формалізм, ми отримуємо параметричне зображення розв'язку задачі Коші для рівняння мКХ на постійному фоні в термінах розв'язку асоційованої задачі РГ, дані для якої (матриця стрибків і параметри для умов на залишки у сингулярних точках, якщо вони наявні) однозначно визначаються початковими даними для задачі Коші.

Запропонований формалізм дозволяє нам охарактеризувати як регулярні, так і нерегулярні односолітонні розв'язки, що відповідають задачам РГ з тривіальними умовами стрибка та відповідним чином заданими умовами на залишки. Зокрема, можна виділити два типи нерегулярних солітонних розв'язків рівняння мКХ: (i) розв'язки піконного типу, які є функціями неперервними разом із першою похідною, але мають необмежені похідні поряків більших за 2 у точці піку; (ii) петлеподібні багатозначні розв'язки.

Теорема. *Рівняння мКХ має односолітонні розв'язки (серед яких є як регулярні, так і нерегулярні), які параметризуються двома параметрами, $\hat{\delta} > 0$ та $\theta \in (0, \frac{\pi}{2})$, та задаються у параметричній формі формулою $u(x,t) \equiv \hat{u}(y(x-t,t),t) + 1$, де*

$$\begin{aligned}
\hat{u}(y,t) &= 4\tan^2\theta \frac{z^2(y,t) + 2\cos^2\theta \cdot z(y,t) + \cos^2\theta}{(z^2(y,t) + 2z(y,t) + \cos^2\theta)^2} z(y,t), \\
x(y,t) &= y + 2\ln\frac{z(y,t) + 1 + \sin\theta}{z(y,t) + 1 - \sin\theta}, \\
z(y,t) &= 2\hat{\delta}\sin\theta\, \mathrm{e}^{y\sin\theta}\mathrm{e}^{-\frac{2\sin\theta}{\cos^2\theta}t}.
\end{aligned}$$

Залежно від значення параметру θ, розв'язки мають якісно різні властивості:

(i) *При $\theta \in (0, \frac{\pi}{3})$, односолітонні розв'язки є гладкими функціями вихідних фізичних змінних (x,t).*

(ii) *При $\theta = \frac{\pi}{3}$, односолінонні розв'язки мають скінчену гладкість, у змінних (x,t), у точці піку.*

(iii) *При $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, односолітонні розв'язки є регулярними функціями у змінних (y,t), але стають багатозначними (петлеподібними) у змінних (x,t).*

У **Розділі 3**, використовуючи формалізм, розроблений у Розділі 2, ми отримуємо головні члени асимптотики за великим часом t для розв'язку задачі Коші для модифікованого рівняння Камаси–Хольма на сталому ненульовому фоні. Дослідження зосереджене на безсолітонному випадку, тобто у припущенні, що умови на лишки відсутні (нерегулярну задачу

Рімана-Гільберта (яка включає в себе умови на лишки) для загального випадку можна звести регулярної, використовуючи множники Бляшке–Потапова).

Для асимптотичного аналізу розв'язку задачі Коші, коли $t \to \infty$, ми застосовуємо нелінійний метод найскорішого спуску. Попередньо, ми зводимо вихідну задачу РГ, асоційовану з рівнянням мКХ, яка має специфічні сингулярності при $\mu = \pm 1$, до звичайної задачі РГ (тобто такої, що має тільки умову стрибка та умову нормування). Примітною особливістю модифікованого рівняння Камаси-Хольма є те, що асоційована вихідна задача РГ має умови сингулярності у $\mu = 1$ та $\mu = -1$ з різними матричними структурами, що не дозволяє позбутися їх шляхом зведення матричної задачі РГ до векторної (що має місце у випадку звичайного рівняння Камаси-Хольма). Ми вирішуємо цю проблему у два кроки. На першому кроці, задача РГ з умовами сингулярності у $\mu = \pm 1$ зводиться до задачі РГ, що характеризується такими двома умовами: (i) елементи матричного розв'язку регулярні у $\mu = \pm 1$, але його визначник дорівнює нулю у цих точках (зазначимо, що $\det M(\mu) \equiv 1$ для розв'язку вихідної задачі РГ); (ii) розв'язок є сингулярним при $\mu = 0$. Наступним кроком, знаходимо зображення розв'язку цієї задачі РГ через розв'язок відповідної регулярної задачі. Саме розв'язок отриманої регулярної задачі РГ ми аналізуємо асимптотично при $t \to +\infty$, адаптуючи нелінійний метод найскорішого спуску. У підсумку, ми отримуємо головні асимптотичні члени для розв'язку $u(x,t)$ задачі Коші у тих секторах напівплощини (x,t), де відхилення від фону є нетривіальним (у решті секторів, $\frac{x}{t} > 3$ і $\frac{x}{t} < \frac{3}{4}$, $u(x,t)$ швидко спадає до 1).

Теорема. *Нехай $u_0(x)$ є гладка функція така, що (i) вона достатньо швидко прямує до 1, коли $x \to \pm\infty$, і задовольняє нерівність $(1-\partial_x^2)u_0(x) > 0$ для всіх x та (ii) асоційована з нею спектральна функція $a(\mu)$ не має нулів у верхній напівплощині (безсолітонний випадок)*

Тоді розв'язок $u(x,t)$ задачі Коші має наступну асимптотичну поведінку за великим часом у секторах (x,t) напівплощини, що задаються нерівностями $1 < \frac{x}{t} < 3$ та $\frac{3}{4} < \frac{x}{t} < 1$:

(i) Для $1 < \frac{x}{t} < 3$,

$$u(x,t) = 1 + \frac{C_1(\zeta)}{\sqrt{t}} \cos\left\{C_2(\zeta)t + C_3(\zeta)\ln t + \tilde{C}_4(\zeta)\right\} + \mathrm{o}(t^{-1/2});$$

(ii) Для $\frac{3}{4} < \frac{x}{t} < 1$,

$$u(x,t) = 1 + \sum_{j=0,1} \frac{C_1^{(j)}(\zeta)}{\sqrt{t}} \cos\left\{C_2^{(j)}(\zeta)t + C_3^{(j)}(\zeta)\ln t + \tilde{C}_4^{(j)}(\zeta)\right\} + \mathrm{o}(t^{-1/2}),$$

де C_i, $C_i^{(j)}$, $\tilde{C}_4$, $\tilde{C}_4^j$ *є функціями від* $\zeta := \frac{x}{t}$, *що визначаються у термінах спектральних функцій, які, у свою чергу, однозначно визначаються початковими даними* $u_0(x)$.

При цьому оцінки похибки є рівномірними у секторах $1 + \varepsilon < \zeta < 3 - \varepsilon$ та $\frac{3}{4} + \varepsilon < \zeta < 1 - \varepsilon$ відповідно, де ε є довільним додатним числом.

У **Розділі 4** розглядається задача Коші для модифікованого рівняння Камаси–Холма у випадку, коли розв'язок прямує до двох різних констант, коли просторова змінна прямує до різних нескінченностей дійсної осі:

$$\begin{aligned} &m_t + \left((u^2 - u_x^2)m\right)_x = 0, \qquad m := u - u_{xx}, \qquad t > 0, \qquad -\infty < x < +\infty, \\ &u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\infty < x < +\infty, \end{aligned}$$

де

$$u_0(x) \to \begin{cases} A_1, & x \to -\infty \\ A_2, & x \to \infty \end{cases}$$

і еволюція за часом зберігає цю поведінку:

$$u(x,t) \to \begin{cases} A_1, & x \to -\infty \\ A_2, & x \to \infty \end{cases}$$

для всіх t.

Ми розробляємо формалізм задачі Рімана-Гільберта для цієї задачі Коші. Для цього проводяться перетворення рівнянь пари Лакса, які дозволяють детально дослідити аналітичні властивості відповідних розв'язків Йоста та спектральних функцій, зокрема, симетрії та поведінку в точках розгалуження. При побудові задачі Рімана–Гільберта ми використовуємо трансформовані пари Лакса, що включають функції $k_j(\lambda) := \sqrt{\lambda^2 - \frac{1}{A_j^2}}$, $j = 1, 2$, визначені як такі, що мають гілки з розрізами вздовж $(-\infty, -\frac{1}{A_j})$ та $(\frac{1}{A_j}, \infty)$. Подібно до випадку з постійним фоном, аналізуючи поведінку розв'язку побудованої задачі Рімана–Гільберта при $\lambda = 0$, ми отримуємо параметричне зображення розв'язку задачі Коші.

Теорема. *Припустимо, що задача Рімана–Гільберта, асоційована з початковими даними $u_0(x)$, має розв'язок $\hat{M}(y, t, x)$, який має розклад*

$$\hat{M}(y,t,\lambda) = \mathrm{i}\begin{pmatrix} 0 & \hat{a}_1(y,t) \\ \hat{a}_1^{-1}(y,t) & 0 \end{pmatrix} + \mathrm{i}\lambda \begin{pmatrix} \hat{a}_2(y,t) & 0 \\ 0 & \hat{a}_3(y,t) \end{pmatrix} + \mathrm{O}(\lambda^2)$$

коли $\lambda \to 0$. Тоді розв'язок $u(x,t)$ задачі Коші може бути зображений, у параметричній формі, у термінах $\hat{a}_j(y,t)$, $j = 1, 2, 3$ наступним чином: $u(x,t) = \hat{u}(y(x,t),t)$, де

$$\begin{aligned} \hat{u}(y,t) &= \hat{a}_1(y,t)\hat{a}_2(y,t) + \hat{a}_1^{-1}(y,t)\hat{a}_3(y,t), \\ x(y,t) &= y - 2\ln \hat{a}_1(y,t) + A_2^2 t. \end{aligned}$$

Крім того, $\hat{u}_x(y,t)$ також може бути алгебраїчно зображений у термінах $\hat{a}_j(y,t)$, $j = 1, 2, 3$, а саме: $u_x(x,t) = \hat{u}_x(y(x,t),t)$, де

$$\hat{u}_x(y,t) = -\hat{a}_1(y,t)\hat{a}_2(y,t) + \hat{a}_1^{-1}(y,t)\hat{a}_3(y,t).$$

Ключові слова: модифіковане рівняння Камаси-Хольма, задача Рімана-Гільберта, метод обереної задачі розсіювання, нелінійний метод найшвидшого спуска, солітони.

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Contents

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Notation

σ_1 ... first Pauli matrix, $\sigma_1 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

σ_2 ... second Pauli matrix, $\sigma_2 := \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$

σ_3 ... third Pauli matrix, $\sigma_3 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$

$f^*(k)$ ... Schwarz conjugate of the function $f(k)$: for $k \in \mathbb{C}$, $f^*(k) := \overline{f(\bar{k})}$

λ ... spectral parameter

μ ... spectral parameter related to λ by $\lambda = -\frac{1}{2}\left(\mu + \frac{1}{\mu}\right)$

k ... spectral parameter related to λ by $\lambda^2 = 4k^2 + 1$

$\mathbb{C}^+$ ... upper complex half-plane, $\mathbb{C}^+ = \{\lambda \in \mathbb{C} |\, \mathrm{Im}(\lambda) > 0\}$

$\mathbb{C}^-$ ... lower complex half-plane, $\mathbb{C}^- = \{\lambda \in \mathbb{C} |\, \mathrm{Im}(\lambda) < 0\}$

M^i ... i-th column of the matrix M

Σ_j ... closed interval, $\Sigma_j = (-\infty, -\frac{1}{A_j}] \cup [\frac{1}{A_j}, \infty)$

$\dot{\Sigma}_j$ ... open interval, $\Sigma_j = (-\infty, -\frac{1}{A_j}) \cup (\frac{1}{A_j}, \infty)$

Σ_0 ... closed interval, $[-\frac{1}{A_1}, -\frac{1}{A_2}] \cup [\frac{1}{A_2}, \frac{1}{A_1}]$

$\dot{\Sigma}_0$ ... open interval, $\dot{\Sigma}_0 = (-\frac{1}{A_1}, -\frac{1}{A_2}) \cup (\frac{1}{A_2}, \frac{1}{A_1})$

λ_+ ... point on the upper side of Σ_j, $\lambda_+ = \lim_{\epsilon \downarrow 0} \lambda + \mathrm{i}\epsilon$

λ_- ... point on the lower side of Σ_j, $\lambda_- = \lim_{\epsilon \downarrow 0} \lambda - \mathrm{i}\epsilon$

$k_j(\lambda)$ ... branch of the square root $k_j(\lambda) := \sqrt{\lambda^2 - \frac{1}{A_j^2}}$, $j = 1, 2$ with the branch cut Σ_j, fixed by the condition $k_j(0) = \frac{\mathrm{i}}{A_j}$

Introduction

Rationale for the choice of the research topic

Plenty of scientists have been studying intensively the integrable nonlinear equations for over the last 50 years, since they realised that the Inverse Scattering Transform method, which was invented for the integration of a particular nonlinear equation - the Korteweg–de Vries equation [74], was not just an accidental pretty mathematical trick, but could be effectively applied to the study of a wide class of equations that are important models of nonlinear phenomena in many branches of physics.

One of such equations is the Camassa–Holm (CH) equation [35, 36]

$$m_t + (um)_x + u_x m = 0, \quad m := u - u_{xx}.$$

It has been studied intensively over the last 28 years, due to its reach mathematical structure. It is a model for the unidirectional propagation of shallow water waves over a flat bottom [84, 46], is bi-Hamiltonian [35], and is completely integrable with algebro-geometric solutions [107]. The CH equation has both globally strong solutions and blow-up solutions at finite time [41, 43, 44, 45], and also it has globally weak solutions in $H^1(\mathbb{R})$ [33, 47, 118]. The soliton-type solutions of the CH equation vanishing at infinity [36] are weak solutions, having the form of peaked waves ($u(x,t)$ and $u_x(x,t)$ are bounded but $u_x(x,t)$ is discontinuous), which are orbitally stable [48].

Interesting mathematical and physical properties of the CH equation raised the question of studying its various modifications and generalizations, see, e.g., [120]. Novikov [105] applied the perturbative symmetry approach in order to classify integrable equations of the form

$$\left(1-\partial_x^2\right)u_t = F(u,u_x,u_{xx},u_{xxx},\dots),\qquad u=u(x,t),\quad \partial_x=\partial/\partial x,$$

assuming that F is a homogeneous differential polynomial over $\mathbb{C}$, quadratic or cubic in u and its x-derivatives. In the list of equations presented in [105], equation (32), which was the second equation with *cubic* nonlinearity, had the form

$$m_t+\left((u^2-u_x^2)m\right)_x=0,\quad m:=u-u_{xx}.$$

This equation (in an equivalent form) was firstly introduced independently by Fokas in [69] and Olver and Rosenau [106] in 1996 as a new integrable system. Physically, it models unidirectional propagation of shallow water waves over a flat bottom, and has a rich mathematical structure (in particular, there are bi-Hamiltonian system and has a representation in the form of a Lax pair). An Lax pair for this equation was given by Qiao [108], so it is sometimes referred to as the Fokas–Olver–Rosenau–Qiao (FORQ) equation [79], but is mostly known as the modified Camassa–Holm (mCH) equation.

In the Thesis, we develop the IST machinery to the mCH equation. The specificity of our study is that we consider this equation in the case of with non-vanishing boundary conditions at infinity, in particular, step-like ones. Such problems are of particular interest because they can be used as models for studying expanding, oscillatory dispersive shock waves [12].

The obtained results are interesting from theoretical point of view, as well as for potential applications.

Aim and tasks of the research

The Thesis aims at the development of the inverse scattering transform approach to the modified Camassa–Holm equation in view of its further application for studying properties of solutions of the Cauchy problem for this equation with various boundary conditions, particularly, their long-time behavior. Namely, we consider a non-zero constant background (Section 2 and Section 3) and step-like background (Section 4). We also compare the implementation of the RH approach to the mCH and CH equations.

The object of study is the modified Camassa–Holm equation, which is a nonlinear partial differential equation, and initial value problems for it.

The subject of study is the solutions of the initial value problems on the axis for the modified Camassa–Holm equation in the case when these solutions tend to non-zero constants when the spatial variable tends to one or another infinity.

Research methods

In order to investigate our tasks we apply the Inverse Scattering Transform (IST) method in form of the Riemann–Hilbert (RH) problem.

The scheme of the IST method is the following: (i) starting from a given initial data $u_0(x)$, we obtain the scattering data by solving the direct problem; (ii) then we obtain the evolution of this scattering data by solving a certain number of linear problems; (iii) finally, we obtain the solution of the Cauchy problem for the nonlinear equation by solving the inverse scattering problem.

The last step in this procedure, the inverse scattering problem, can be effectively solved by reformulating it as a Riemann–Hilbert (RH) factorization problem. A starting point here is a Lax pair representation. It is a pair of linear differential equations that depend on additional spectral parameter and whose compatibility condition is exactly the nonlinear differential equation. The RH problem method boils down in choosing the solutions of the Lax pair equations in a right way and then constructing a RH problem from these chosen solutions.

Then we analyze the long time asymptotics by using the so-called nonlinear steepest descent method [54]. This method consists in successive transformations of the original RH problem, in order to reduce it to an explicitly solvable problem. The consecutive steps include (i) appropriate triangular factorizations of the jump matrix; (ii) "absorption" of the triangular factors with good large-time behavior; (iii) reduction, after rescaling, to a RH problem which is solvable in terms of certain special functions.

Novelty of the results

All results presented in the works [18], [87] and [88] and included in the dissertation were obtained by the author independently. The results belonging to other scientists are mentioned as necessary for the completeness of the presentation and are accompanied by appropriate references., we develop the inverse scattering transform method in the form of the Riemann–Hilbert problem for the Cauchy problems for modified Camassa–Holm equation with various boundary conditions. In particularly, we obtain the following results:

(i) the representation of the solution of this Cauchy problem for the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach a non-zero constant at the both infinities of the space variable in the form of the solution associated with it Riemann–Hilbert problems. This result was obtained for the first time. (Section 2)

(ii) the leading asymptotic terms for the solution of the Cauchy problem for the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach a non-zero constant at the both infinities of the space variable. This result was obtained for the first time. (Section 3)

(iii) the representation of the solution of this Cauchy problem for the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach two different constants at plus and minus infinity of the space variable in the form of the solution associated with it Riemann–Hilbert problems. This result was obtained for the first time. (Section 4)

Personal contribution

The setting of the problem considered in Section 2 belongs to the scientific advisor Dmitry Shepelsky, the setting of the problem considered in Section 3 belongs to the scientific advisor Dmitry Shepelsky and Anne Boutet de Monvel, the setting of the problem considered in Section 4 belongs to the scientific advisors Dmitry Shepelsky and Gerald Teschl. All results presented in the works [18], [87] and [88] and included in the Thesis were obtained by the author independently. The results that belongs to other scientists are mentioned for the completeness and are accompanied by appropriate references.

Approbation of the thesis results

The thesis results were discussed at the scientific seminar of the Department of Mathematical Physics of B.Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine and the SE Mathematical Physics of University of Vienna, and presented at **nine** international conferences:

1. I. Karpenko, D. Shepelsky, "A Riemann–Hilbert approach to the modified Camassa–Holm equation with nonzero boundary conditions", VI International Conference "Analysis and Mathematical Physics", Kharkiv, Ukraine (June 2018).

2. I. Karpenko, D. Shepelsky, "The Riemann–Hilbert approach to the Cauchy problem for the modified Camassa–Holm equation", 6th Ya. B. Lopatynsky International School-Workshop on Differential Equations and Applications, Vinnytsia, Ukraine (June 2019).

3. I. Karpenko, D. Shepelsky, "The inverse scattering transform, in the form of Riemann–Hilbert problem, for the modified Camassa–Holm equation", international Conference dedicated to 70th anniversary of Professor A.M.Plichko "Banach Spaces and their Applications", Lviv, Ukraine (June 2019).

4. I. Karpenko, D. Shepelsky, "A Riemann–Hilbert problem approach to the modified Camassa–Holm equation on a nonzero background", Pidzakharychi, Ukraine (August 2019).

5. I. Karpenko, D. Shepelsky, "The modified Camassa–Holm equation on a nonzero background: large-time asymptotics for the Cauchy problem", Workshop "New horizons in dispersive hydrodynamics", Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom (June 2021).

6. I. Karpenko, D. Shepelsky, “The large-time asymptotics for the modified Camassa–Holm equation on a non-zero background”, 5-th International Conference “DIFFERENTIAL EQUATIONS and CONTROL THEORY”, V. N. Karazin Kharkiv National University, Kharkiv, Ukraine (September 2021).

7. I. Karpenko, D. Shepelsky, G. Teschl, “A Riemann–Hilbert approach to the modified Camassa–Holm equation with step-like boundary conditions”, Ivano-Frankivsk, Ukraine (May 2022).

8. I. Karpenko, “The modified Camassa–Holm equation on a step-like background”, Complex Analysis, Spectral Theory and Approximation meet in Linz, Johannes Kepler University, Linz, Austria (July 2022).

9. I. Karpenko, “A Riemann–Hilbert problem approach to the modified Camassa–Holm equation on a step like background”, Workshop From Modeling and Analysis to Approximation and Fast Algorithms, Hasenwinkel, Germany (December 2022).

Structure and scope of the thesis

The thesis consists of a table of contents, an acknowledgment, a glossary of notation, an introduction, four chapters, a conclusion and references, which contains 120 items. The total volume of the dissertation is 185 pages. The volume of the main part of the work is 151 pages.

Section 1 is devoted to the review of the literature on the topic of the dissertation. In Section 2, the initial problem for the modified Camassa–Holm equation on non-zero constant background is studied. Section 3 studies the long time asymptotics for this problem. Section 4 deals with the initial problem for the modified Camassa–Holm equation with step-type initial data.

Practical significance of the obtained results

The Thesis is purely theoretical. The obtained results and the proposed methods can be used in further research of initial and initial boundary value problems for Camassa–Holm type equations, which can be promising models of physical processes of various nature.

Publications

The main results of the Thesis are published in 3 scientific articles indexed in international reference and citation databases Scopus and Web of Science.

According to the classification of SCImago Journal and Country Rank, papers [18] and [88] are published in journals from Quartile Q2; paper [87] is published in a journal from Quartile Q3.

In addition, the results of the Thesis are reflected in the publication [19] and 7 theses of conferences.

Chapter 1

The Inverse Scattering Transform method and Camassa–Holm type equations (literature review)

1.1 Integrable equations and the Inverse Scattering Transform method

In 1965 Zabusky and Kruskal discovered that the pulse-like solitary wave solution to the Korteweg-de Vries (KdV) equation had a property which was previously unknown: namely, that this solution interacted "elastically" with another such solution. They called these solutions *solitons*. Shortly after this discovery, Gardner, Greene, Kruskal and Miura (1967), (1974) pioneered a new method of mathematical physics (see [74, 75]). Specifically, they gave a method of solution for the KdV equation by making use of the ideas of direct and inverse scattering. In 1968 Lax considerably generalized these ideas [91]. At that time and shortly thereafter it was not clear if the method would apply to other physically significant nonlinear evolution equations. However, in 1972 Zakharov and Shabat showed that the method was not a fluke. Applying the direct and inverse scattering ideas they solved the initial value problem for the nonlinear Schrodinger equation [121]. In 1973, using these ideas Ablowitz, Kaup, Newell and Segur did the same for the sin–Gordon equation [2]. And then they developed a method to find a rather wide class of nonlinear evolution equations solvable by these techniques [3, 4]. They called the procedure the

Inverse Scattering Transform (IST).

In the most broad terms, the equation $F(u_t, u, u_x, u_{xx}, ...) = 0$ is called integrable, if it is a compatibility condition of the system of *linear* equations (the so-called *Lax pair*):

$$\begin{cases} \Phi_x(x,t,k) = U(x,t,k)\Phi(x,t,k) \\ \Phi_t(x,t,k) = V(x,t,k)\Phi(x,t,k) \end{cases}, \tag{1.1}$$

where U and V are known in terms of the solution of the equation and $k \in \mathbb{C}$ is an auxiliary (spectral) parameter. Exactly the Lax pair is a starting point for studying various problems for integrable equations such as finding the different types of exact solutions (via the so-called Backlund-Darboux type transformations) and solving the initial and initial-boundary value problems.

Integrable nonlinear PDEs with non-vanishing boundary conditions at infinity have received plenty of attention in the literature, see e.g. [11, 15]. Particularly, initial value problems with initial data approaching different "backgrounds" at different spatial infinities (so-called step-like initial data) have attracted considerable attention because they can be used as models for studying expanding, oscillatory dispersive shock waves (DSW), which are large scale, coherent excitation in dispersive systems [12]. The large-time evolution of step-like initial data has been studied or models of uni-directional (Korteweg—de Vries equation) wave propagation [63] as well as bi-directional (nonlinear Schrödinger equation) wave propagation [23, 24].

In general, the IST method for solving initial value problems for integrable nonlinear equations written as the compatibility conditions for linear equations consists in the following (see Figure 1.1): starting from a given data, solve the direct problem, that is determine appropriate eigenfunction (solutions of the differential x-equation in the Lax pair) having well-controlled analytic properties as functions of the spectral parameter λ and the associated spectral functions of λ; then, by virtue of the t-equations in the Lax pair, the associated functions evolve in a simple, explicit way. Finally, using the explicit evolution of the spectral functions, solve the inverse problem of finding the associated coefficient in

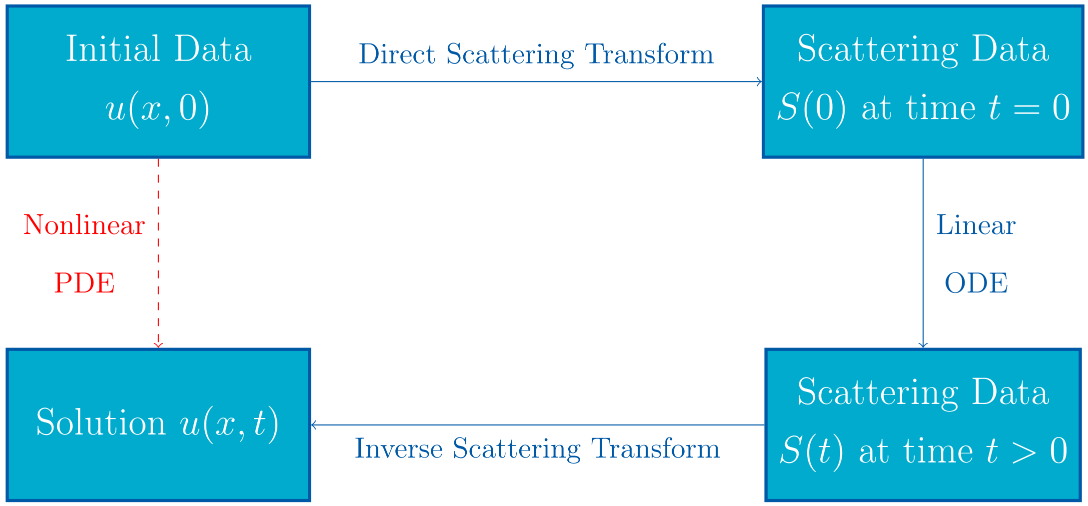

Figure 1.1: The scheme of the IST method

the x-equation, which, by the Lax pair equations, evolve according to the given nonlinear equation and thus solve the Cauchy problem of this equation.

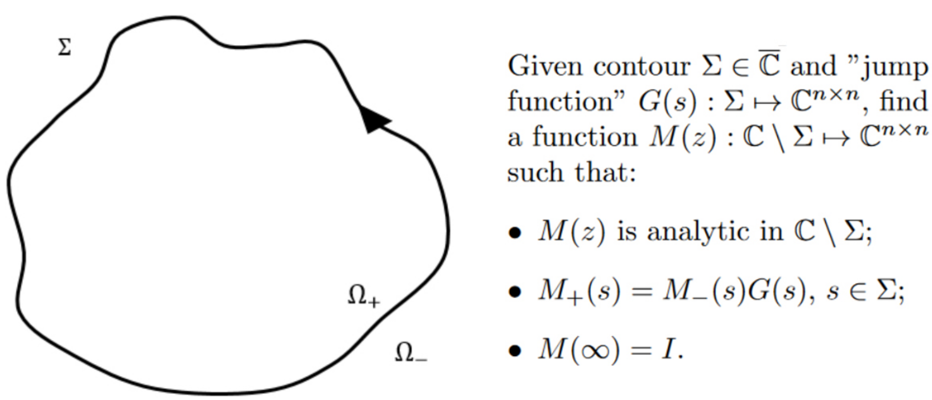

Figure 1.2: Riemann–Hilbert problem: boundary value problem in complex analysis

The last step in this procedure, the inverse scattering problem, can be effectively solved by reformulating it as a Riemann–Hilbert (RH) factorizations problem (see Figure 1.2): giving a contour in the complex plane and a matrix-valued function defined on the contour, find a piecewise (relative to the contour) analytic, matrix valued function, whose limiting values on the contour are related with the help of the given function. In applications to nonlinear equations, the given (jump) matrix depends also on parameters (which are physical variables in the nonlinear equation in question (e.g., space x and time t), and thus the solution of the RH problem also depend on these parameters. If the jump matrix is constructed, in an appropriate way, using the initial data for the (nonlinear) partial differential equation (PDE), then, evaluating the solution of the RH problem at a particular value of the spectral variable, it is possible to obtain the solution of the original Cauchy problem for this PDE.

In a certain sense, a Riemann–Hilbert problem representation for (integrable) nonlinear PDEs play the role of an integral representation in the case of linear PDEs, via Fourier series or Green's functions. For linear PDEs, integral representations allow:

- obtaining existence and uniqueness results directly from the well-understood integration theory;
- studying asymptotics via the method of stationary phase or the method of steepest descent;
- evaluating solutions numerically via simple quadrature.

In the case of integrable nonlinear PDEs, all these goals are achievable, to some extent, through the development of the RH formalism [114]. Particularly, the existence results can be obtained establishing a solution of the associated RH problem and controlling its behaviour w.r.t. the spatial parameter.

The long time asymptotics can be efficiently analyzed by using the so-called nonlinear steepest descent method [54]. This method consists in successive transformations of the original RH problem, in order to reduce it to an explicitly solvable problem. The consecutive steps include (i) appropriate triangular factorizations of the jump matrix; (ii) "absorption" of the triangular factors with good large-time behavior; (iii) reduction, after rescaling, to a RH problem which is solvable in terms of certain special functions.

Despite the fact that both the IST method and the nonlinear steepest descent method are in a certain sense algorithmic, their adaptation to a particular nonlinear equation can be a difficult task that requires a lot of analytical work. For example, the application of the IST method for initial problems with zero background is very different from the application of the IST method for initial problems with nonzero background. In particular, the properties of the corresponding spectral functions, the associated Riemann–Hilbert problem, and the adaptation of the nonlinear steepest descent method are significantly different.

On the other hand, the study of a specific problem can lead not only to obtaining the results for that problem, but can also inspire the development of new analytical methods and approaches that can be effectively applied to a wide class of problems from other areas of mathematics (as it has already happened, in particular, in the theory of orthogonal polynomials and random matrices of large size).

1.2 The Camassa–Holm equation

The Camassa–Holm (CH) equation [35, 36]

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.2}$$

which can also be written in terms of the momentum variable

$$m_t + (um)_x + u_x m = 0, \quad m := u - u_{xx}, \tag{1.3}$$

has been studied intensively over the last 30 years, due to its rich mathematical structure. It is a model for the unidirectional propagation of shallow water waves over a flat bottom [84, 46], is bi-Hamiltonian [35], and is completely integrable with algebro-geometric solutions [107]. The local and global well-posedness of the Cauchy problem for the CH equation have been studied extensively [43, 44, 49]. In particular, it has both globally strong solutions and blow-up solutions at finite time [41, 43, 44, 45], and also it has globally weak solutions in $H^1(\mathbb{R})$ [33, 47, 118].

The soliton-type solutions of the CH equation vanishing at infinity [36] are weak solutions, having the form of peaked waves ($u(x,t)$ and $u_x(x,t)$ are bounded but $u_x(x,t)$ is discontinuous), which are orbitally stable [48]. They can be expressed by $u(x,t) = c\mathrm{e}^{-|x-ct|}$, $c \in \mathbb{R}$. Such solutions are known as peakons (peaked solutions).

On the other hand, adding to (1.2) a linear dispersion term bu_x with $b > 0$ leads to a form of the CH equation

$$u_t - u_{xxt} + bu_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.4}$$

which supports conventional smooth solitons [42, 25, 26].

In the case of the Camassa–Holm equation, the inverse scattering transform method (particularly, in the form of a Riemann–Hilbert factorization problem) works for the version of this equation (considered for functions decaying at spatial infinity) that includes an additional linear dispersion term [21, 25, 26, 27]. Equivalently, this problem can be rewritten as a Cauchy problem for equation (1.3) considered on a constant, nonzero background. Indeed, the inverse scattering transform method requires that the spatial equation from the Lax pair associated to the CH equation have continuous spectrum. On the other hand, the asymptotic analysis of the dispersionless CH equation (1.3) on zero background (where the spectrum is purely discrete) requires a different tool (although having a certain analogy with the Riemann–Hilbert method), namely, the analysis of a coupling problem for entire functions [60, 61, 62].

1.3 Generalizations of the Camassa–Holm equation

Over the last few years various modifications and generalizations of the CH equation have been introduced, see, e.g., [120] and references therein. Novikov [105] applied the perturbative symmetry approach in order to classify integrable equations of the form

$$\left(1 - \partial_x^2\right) u_t = F(u, u_x, u_{xx}, u_{xxx}, \dots), \qquad u = u(x,t), \quad \partial_x = \partial/\partial x,$$

assuming that F is a homogeneous differential polynomial over $\mathbb{C}$, quadratic or cubic in u and its x-derivatives (see also [101]). Such equations are known as *Camassa–Holm type* equations.

The adaptation of the inverse scattering transform method in the form of a Riemann–Hilbert problem for CH type equations has its own characteristic features. Particularly, the initial steps involve gauge transformations $\Phi \mapsto P\tilde{\Phi}$ transforming the original Lax pair to the form

$$\begin{cases} \tilde{\Phi}_x(x,t,\lambda) = \tilde{Q}_x(x,t,\lambda)\tilde{\Phi}(x,t,\lambda) + \tilde{U}(x,t,\lambda)\tilde{\Phi}(x,t,\lambda) \\ \tilde{\Phi}_t(x,t,\lambda) = \tilde{Q}_t(x,t,\lambda)\tilde{\Phi}(x,t,\lambda) + \tilde{V}(x,t,\lambda)\tilde{\Phi}(x,t,\lambda) \end{cases},$$

where

- ▷ near the singular points (w.r.t. the apectral parameter λ), the dominating terms have the form $\tilde{Q}_x$ and $\tilde{Q}_t$ with some diagonal matrix $\tilde{Q}$ depending, in general, on t and x through the solution of the nonlinear equation in question;
- ▷ the remaining terms $\tilde{U}$ and $\tilde{V}$ tend to zero as $x \to \pm\infty$.

Then, $\tilde{Q}$ dictates a change of variables, such that the jump matrix in the master RH problem associated to the Cauchy problem for the nonlinear equation depends on new variables in an explicit way.

In the list of equations presented in [105], equation (32), which was the second equation with *cubic* nonlinearity, had the form

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m := u - u_{xx}. \tag{1.5}$$

In an equivalent form, this equation was given by Fokas in [69] (see also [106] and [73]). Shiff [110] considered equation (1.5) as a dual to the modified Korteweg–de Vries equation (mKdV) and introduced the Lax pair for (1.5) by rescaling the entries of the spatial part of the Lax pair for the mKdV equation. An alternative (in fact, gauge equivalent) Lax pair for (1.5) was given by Qiao [108], so it is sometimes referred to as the Fokas–Olver–Rosenau–Qiao (FORQ) equation [79], but is mostly known as the *modified Camassa–Holm* (mCH) equation.

Equation (1.5) has a bi-Hamiltonian structure [106, 78]. In [85], a Liouville-type transformation was presented relating the isospectral problems for the mKdV equation and the mCH equation, and a Miura-type map form the mCH equation to the CH equation was introduced.

Equation (1.5) belongs to the class of peakon equations: it has solutions in the form of localized, peaked traveling waves – peakons [78]. The solitary, single peaked, wave solutions (peakons) of the mCH equation have the form [78]

$$u(x,t) = \frac{p}{2}\mathrm{e}^{-|x-x(t)|}, \quad m(x,t) = p\delta(x-x(t)) \quad \text{with } x(t) = \frac{1}{6}p^2 t.$$

The dynamical stability of peakons is discussed in [109] (see also [96] for the stability of peakons of a generalized mCH equation). The local well-posedness and wave-breaking mechanisms for the mCH equation and its generalizations, particularly, the mCH equation with linear dispersion, are discussed in [78, 72, 97, 40, 39]. Algebro-geometric quasiperiodic solutions are studied in [79]. The local well-posedness for classical solutions and global weak solutions to (1.5) in Lagrangian coordinates are discussed in [76]. In [38] the authors discuss multipeakon solutions developing the inverse spectral method for the associated peakon system of ordinary differential equations. The Hamilton structure and Liouville integrability of peakon systems are discussed in [8] and [37]. The Bäcklund transformation for the mCH equation and the related nonlinear superposition formula are presented in [117].

1.4 Other peakon equations

The peak-shaped solutions (peaked solutions or peakons) are particular solutions admitted by certain nonlinear PDEs (so called "peakon equations"). These solutios take the form of a train of peak-shaped waves and interact like a particle.

The peakons first appeared as solutions of the Camassa–Holm (CH) equation, and later in many other related PDEs, e.g. Degasperis-Procesi (DP), Novikov (N) and modified Camassa–Holm (mCH) equations.

The Degasperis-Procesi equation

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}$$

was discovered aroud 1998 by Degasperis and Procesi [59]. It arises as a model equation describing the shallow-water approximation in inviscid hydrodynamics in the so-called "moderate amplitude regime". The DP equation ($b = 3$) and the CH ($b = 2$) equation arise as the only integrable cases in the "b-family"

$$m_t + m_x u + bmu_x = 0, \quad m = u - u_{xx}$$

and both have quadratic nonlinearity. It possesses peakon solutions of the same form as CH equation. Despite being similar in appearance to the CH equation, the DP equation has a different underlying integrability structure, and its peakon solutions are connected to approximation theory via the concepts of the discrete cubic string, mixed Hermite–Padé approximations and Cauchy biorthogonal polynomials [98], [99]. Another difference is that the DP equation admits weak solutions that are not continuous (with jump singularities in $u(x,t)$ rather than in $u_x(x,t)$).

The Novikov equation

$$m_t + (um_x + 3u_x m)u = 0, \quad m = u - u_{xx}$$

was obtained by Novikov [105] in the search for a classification of integrable generalized Camassa–Holm equations of the form

$$\left(1 - \partial_x^2\right) u_t = F(u, u_x, u_{xx}, u_{xxx}, \dots), \qquad u = u(x,t), \quad \partial_x = \partial/\partial x.$$

It differs in appearance from the DP equation only by the extra factor u, so that the nonlinearity is cubic (like for mCH). The Novikov equation can exhibit the phenomenon of wave-breaking and possesses peakon solutions with $u(x,t)$ ramaining continuous and discontinuous $u_x(x,t)$. Another remarkable feature of the Novikov equation is that it can exhibit the phenomenon of wave-breaking [10].

Due to the rich mathematical structure and interesting properties of these equation, it is natural to study their modifications and generalisation. In pariticular, many researchers consider their short wave limits (the evolution is involved in $m_t = -u_{txx}$; in the case of original equations the evolution is involved in $m_t = u_t - u_{txx}$) and so-called μ-equations (the evolution is involved in $m_t = \mu(u)_t - u_{txx}$ with $\mu(u) = \int_{\mathbb{S}} u(x,t)dx$). In some sense μ equations can be consideered as midway equations between original equations and their short-wave limits.

The short-wave model for the Camassa–Holm equation

$$m_t + 2u_x - 2u_x m - u m_x = 0, \quad m = -u_{xx}$$

is a model for short capillary waves propagating under the action of gravity [16]. This equation is also known as the modified Hunter-Saxton (mHS) equation. A remarkable feature of mHS is that it possesses cuspon solutions (solutions that take the form of a train of cusp-shaped waves, i.e. both left and right derivatives are infinities).

The short-wave model for the Degasperis-Procesi equation

$$m_t + 3u_x - u m_x - 3u_x m = 0, \quad m = -u_{xx}$$

is a model describing the unidirectional propagation of nonlinear shallow water waves. This equation is also called the Ostrovsky–Vakhnenko equation (OV) equation. The exact soliton-type solutions of OV equation we constructed by using the Riemann–Hilbert formalism in [30]. These solutions are multi-valued functions having the form of a loop (1-soliton) or many loops (multi-solitons).

The μ-CH equation

$$m_t + (um)_x + u_x m = 0, \quad m := \mu(u) - u_{xx},$$

was first introduced in [89] by Khesin, Lenells and Misiołek. It is interesting to note that this equation is integrable in the sense that it admits Lax-pair and bi-Hamiltonian structure, and also describes a geodesic flow on diffeomorphism group of S with certain metric. Its integrability, well-posedness, blow-up and peakons were discussed in [89].

The μ-DP equation

$$m_t + um_x + 3u_x m = 0, \quad m = \mu(u) - u_{xx}$$

was introduced by Lenells, Misiolek and Tiglay in [95]. Its integrability, well-posedness, blow-up and existence of peakons were also investigated in [95].

Another natural extension of the mCH equation is a two-component integrable modified CH (2-mCH) equation:

$$m_t + \left((u - u_x)(v + v_x)m\right)_x = 0, \quad m := u - u_{xx}, \tag{1.6a}$$

$$n_t + \left((u - u_x)(v + v_x)n\right)_x = 0, \quad m := v - v_{xx}. \tag{1.6b}$$

It was proposed by Song, Qu and Qiao in [113].

In [113], it is shown that the equation (1.6) arises from non-streching invariant curve flows in the n-dimensional unit sphere $S^n(1)$. This system of equations is known to possess infinitely many conservation laws as well as a Lax formulation:

$$\Phi_x(x,t,\lambda) = \mathsf{U}(x,t,\lambda)\Phi(x,t,\lambda), \qquad \Phi_t(x,t,\lambda) = \mathsf{V}(x,t,\lambda)\Phi(x,t,\lambda)$$

where the coefficient matrices U and V are defined in terms of a solution of the 2-mCH equation:

$$\mathsf{U} = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix},$$

$$\mathsf{V} = \begin{pmatrix} 4\lambda^{-2} + (u - u_x)(v + v_x) & -\frac{2(u-u_x)}{\lambda} - \lambda m(u - u_x)(v + v_x) \\ \frac{2(v+v_x)}{\lambda} + \lambda n(u - u_x)(v + v_x) & -(u - u_x)(v + v_x) \end{pmatrix}.$$

The local well-posedness for the associated Cauchy problem in the Besov spaces, explicit expresses of its single peakon and two peakon solutions and blow-up scenario were studied in [119].

All the above described equations belong to the class of integrable equations. It is important to emphasise that there exist also non-integrable peakon equations. The prototypical example was first introduced in the work of Degasperis, Holm and Hone [58], who defined a family of equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad b \in \mathbb{R},$$

that reduce to CH and DP equations for $b = 2$, $b = 3$ respectively, while for other values of b are non-integrable which was shown in [59].

The literature review demonstrates a great interest of scientists that work in various fields of mathematics and physics to integrable nonlinear equations, in particularly, to Camassa–Holm equation and its generalizations. This confirms the relevance of the topic chosen in the Thesis.

Chapter 2

The Riemann–Hilbert approach to the modified Camassa–Holm equation with nonzero boundary conditions

The results of this Chapter are published in [18].

We consider the initial value problem for the mCH equation (1.5):

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m := u - u_{xx}, \quad t > 0, \ -\infty < x < +\infty, \quad (2.1a)$$

$$u(x,0) = u_0(x), \qquad -\infty < x < +\infty, \quad (2.1b)$$

assuming that $u_0(x) \to 1$ as $x \to \pm\infty$, and we search for a solution that preserves this behavior: $u(x,t) \to 1$ as $x \to \pm\infty$ for all $t > 0$. Then, in analogy with the CH equation and other CH-type equations, one can expect that the Cauchy problem (2.1) supports smooth soliton solutions.

Introducing a new function $\tilde{u}$ by

$$u(x,t) = \tilde{u}(x - t, t) + 1, \quad (2.2)$$

the mCH equation reduces to

$$\tilde{m}_t + (\tilde{\omega}\tilde{m})_x = 0, \quad (2.3a)$$

$$\tilde{m} := \tilde{u} - \tilde{u}_{xx} + 1, \quad (2.3b)$$

$$\tilde{\omega} := \tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u}. \quad (2.3c)$$

In what follows we will study equation (2.3) on zero background: $\tilde{u} \to 0$ as $x \to \pm\infty$. More precisely, we develop the Riemann–Hilbert (RH) problem approach to equation (2.3a) on zero background, aiming at obtaining a representation of the solution of the Cauchy problem for (2.3) in terms of the solution of an associated RH problem formulated in the complex plane of a spectral parameter.

In Subsection 2.1 we introduce the Jost solutions of the Lax pair equations written in a form appropriate for controlling their analytical properties as function of the spectral parameter. In Subsection 2.2 we formulate the Riemann–Hilbert problem in two settings: (i) in the original setting, it (implicitly) depends on the physical variables (x, t) as parameters and (ii) in a transformed setting, introducing new variables (y, t) in terms of which the RH problem has an explicit parameter dependence. The data for the later RH problem are uniquely determined by the initial data for the mCH equation, which gives rise to a procedure for solving the Cauchy problem (2.1). In Subsection 2.3 we show that starting with the solution of a RH problem with appropriate dependence on the parameters, we always arrive at a solution to the mCH equation, even if the data for this RH problem are not associated with some particular initial data for the mCH equation. Finally, in Subsection 2.4, using the RH problem formalism, we construct smooth as well as non-smooth soliton solutions to the mCH equation. Throughout the text, we emphasize the differences in the implementation of the RH approach to the CH and mCH equations.

2.1 Lax pairs and eigenfunctions

2.1.1 Lax pairs

In order to deduce the Lax pair for equation (2.3a), we take as starting point the Lax pair for the mCH equation (1.5) [108]

$$\Phi_x = \mathsf{U}\Phi, \qquad \Phi_t = \mathsf{V}\Phi$$

where $\Phi \equiv \Phi(x,t,\lambda)$, $\mathsf{U} \equiv \mathsf{U}(x,t,\lambda)$, and $\mathsf{V} \equiv \mathsf{V}(x,t,\lambda)$, the coefficients U and V being defined by

$$\mathsf{U} = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda m & 1 \end{pmatrix},$$

$$\mathsf{V} = \begin{pmatrix} \lambda^{-2} + \frac{u^2-u_x^2}{2} & -\lambda^{-1}(u-u_x) - \frac{\lambda(u^2-u_x^2)m}{2} \\ \lambda^{-1}(u+u_x) + \frac{\lambda(u^2-u_x^2)m}{2} & -\lambda^{-2} - \frac{u^2-u_x^2}{2} \end{pmatrix},$$

with $m := u - u_{xx}$. This leads us to the pair of equations

$$\Phi_x = U\Phi, \tag{2.4a}$$

$$\Phi_t = V\Phi, \tag{2.4b}$$

where the coefficients $U \equiv U(x,t,\lambda)$ and $V \equiv V(x,t,\lambda)$ are now defined by

$$U = \frac{1}{2}\begin{pmatrix} -1 & \lambda\tilde{m} \\ -\lambda\tilde{m} & 1 \end{pmatrix}, \tag{2.5a}$$

$$V = \begin{pmatrix} \lambda^{-2} + \frac{\tilde{\omega}}{2} & -\lambda^{-1}(\tilde{u} - \tilde{u}_x + 1) - \frac{\lambda\tilde{\omega}\tilde{m}}{2} \\ \lambda^{-1}(\tilde{u} + \tilde{u}_x + 1) + \frac{\lambda\tilde{\omega}\tilde{m}}{2} & -\lambda^{-2} - \frac{\tilde{\omega}}{2} \end{pmatrix}. \tag{2.5b}$$

Here, $\tilde{m} := \tilde{u} - \tilde{u}_{xx} + 1$ and $\tilde{\omega} := \tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u}$ as in (2.3b) and (2.3c), with $\tilde{u}$ as in (2.2). It can be directly verified that (2.3a) is the compatibility condition for the system (2.4)-(2.5). Thus, this system (2.4)-(2.5) constitutes a Lax pair for (2.3a).

The RH formalism for integrable nonlinear equations is based on using appropriately defined eigenfunctions, i.e., solutions of the Lax pair, whose behavior as functions of the spectral parameter is well-controlled in the extended complex plane. Notice that the coefficient matrices U and V are traceless, which provides that the determinant of a matrix solution to (2.4) (composed from two vector solutions) is independent of x and t.

Also notice that U and V have singularities (in the extended complex λ-plane) at $\lambda = 0$ and $\lambda = \infty$. In order to control the behavior of solutions to (2.4) as functions of the spectral parameter λ (which is crucial for the Riemann–Hilbert method), we follow a strategy similar to that adopted for the CH equation [25, 26].

Namely, in order to control the large λ behavior of solutions of (2.4), we will transform this Lax pair into an appropriate form (see [9, 25, 26]).

Proposition 2.1.1. *Equation* (2.3a) *admits a Lax pair of the form*

$$\hat{\Phi}_x + Q_x \hat{\Phi} = \hat{U}\hat{\Phi}, \tag{2.6a}$$

$$\hat{\Phi}_t + Q_t \hat{\Phi} = \hat{V}\hat{\Phi}, \tag{2.6b}$$

whose coefficients $Q \equiv Q(x,t,\lambda)$, $\hat{U} \equiv \hat{U}(x,t,\lambda)$, *and* $\hat{V} \equiv \hat{V}(x,t,\lambda)$ *are* 2×2 *matrices having the following properties:*

(i) Q *is diagonal and is unbounded as* $\lambda \to \infty$.

(ii) $\hat{U} = \mathrm{O}(1)$ *and* $\hat{V} = \mathrm{O}(1)$ *as* $\lambda \to \infty$.

(iii) *The diagonal parts of* $\hat{U}$ *and* $\hat{V}$ *decay as* $\lambda \to \infty$.

(iv) $\hat{U} \to 0$ *and* $\hat{V} \to 0$ *as* $x \to \pm\infty$.

Proof. We first note that U in (2.5a) can be written as

$$U(x,t,\lambda) = \frac{\tilde{m}(x,t)}{2}\begin{pmatrix} -1 & \lambda \\ -\lambda & 1 \end{pmatrix} + \frac{\tilde{m}(x,t)-1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.7}$$

where $\tilde{m}(x,t) - 1 \to 0$ as $x \to \pm\infty$. The first (non-decaying, as $x \to \pm\infty$) term in (2.7) can be diagonalized by introducing

$$\hat{\Phi}(x,t,\lambda) := D(\lambda)\Phi(x,t,\lambda),$$

where

$$D(\lambda) := \begin{pmatrix} 1 & -\frac{\lambda}{1+\sqrt{1-\lambda^2}} \\ -\frac{\lambda}{1+\sqrt{1-\lambda^2}} & 1 \end{pmatrix},$$

where the square root is chosen so that $\sqrt{1-\lambda^2} \sim \mathrm{i}\lambda$ as $\lambda \to \infty$. This transforms (2.4a) into

$$\hat{\Phi}_x + \frac{\tilde{m}\sqrt{1-\lambda^2}}{2}\sigma_3\hat{\Phi} = \hat{U}\hat{\Phi}, \tag{2.8a}$$

where $\hat{U} \equiv \hat{U}(x,t,\lambda)$ is given by

$$\hat{U} = \frac{\lambda(\tilde{m}-1)}{2\sqrt{1-\lambda^2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{\tilde{m}-1}{2\sqrt{1-\lambda^2}}\sigma_3. \tag{2.8b}$$

Similarly, the t-equation (2.4b) of the Lax pair is transformed into

$$\hat{\Phi}_t + \sqrt{1-\lambda^2}\left(-\frac{1}{2}\tilde{m}\tilde{\omega} - \frac{1}{\lambda^2}\right)\sigma_3\hat{\Phi} = \hat{V}\hat{\Phi}, \tag{2.8c}$$

where $\hat{V} \equiv \hat{V}(x,t,\lambda)$ is given by

$$\begin{aligned} \hat{V} = {} & \frac{1}{2\sqrt{1-\lambda^2}}\left(\lambda\tilde{\omega}(\tilde{m}-1) + \frac{2\tilde{u}}{\lambda}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\tilde{u}_x}{\lambda}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ & - \frac{1}{\sqrt{1-\lambda^2}}\left(\tilde{u} + \frac{1}{2}(\tilde{m}-1)\tilde{\omega}\right)\sigma_3. \end{aligned} \tag{2.8d}$$

Now notice that equations (2.8a) and (2.8c) have the desired form (2.6), if we define Q by

$$Q(x,t,\lambda) := p(x,t,\lambda)\sigma_3, \tag{2.9a}$$

with

$$p(x,t,\lambda) := -\frac{1}{2}\sqrt{1-\lambda^2}\int_x^{+\infty}(\tilde{m}(\xi,t)-1)\mathrm{d}\xi + \frac{\sqrt{1-\lambda^2}}{2}x - \frac{\sqrt{1-\lambda^2}}{\lambda^2}t. \tag{2.9b}$$

Indeed, p has derivatives

$$\begin{aligned} p_x &= \frac{\tilde{m}\sqrt{1-\lambda^2}}{2}, \\ p_t &= \sqrt{1-\lambda^2}\left(-\frac{1}{2}\tilde{m}\tilde{\omega} - \frac{1}{\lambda^2}\right). \end{aligned}$$

The first formula is clear, while the second follows from (2.3a). □

2.1.2 Eigenfunctions

The Lax pair in the form (2.8) allows us to determine dedicated solutions having a well-controlled behavior as functions of the spectral parameter λ for large values of λ via associated integral equations. Indeed, introducing

$$\widetilde{\Phi} = \hat{\Phi}\mathrm{e}^{Q} \tag{2.10}$$

(understanding $\widetilde{\Phi}$ as a 2×2 matrix), equations (2.8a) and (2.8c) can be rewritten as

$$\begin{cases} \widetilde{\Phi}_x + [Q_x, \widetilde{\Phi}] = \hat{U}\widetilde{\Phi}, \\ \widetilde{\Phi}_t + [Q_t, \widetilde{\Phi}] = \hat{V}\widetilde{\Phi}, \end{cases} \tag{2.11}$$

where $[\,\cdot\,,\,\cdot\,]$ stands for the commutator. We now determine particular (Jost) solutions $\widetilde{\Phi}_\pm \equiv \widetilde{\Phi}_\pm(x,t,\lambda)$ of (2.11) as solutions of the associated Volterra integral equations:

$$\widetilde{\Phi}_\pm(x,t,\lambda) = I + \int_{\pm\infty}^{x} \mathrm{e}^{Q(\xi,t,\lambda)-Q(x,t,\lambda)}\hat{U}(\xi,t,\lambda)\widetilde{\Phi}_\pm(\xi,t,\lambda)\mathrm{e}^{Q(x,t,\lambda)-Q(\xi,t,\lambda)}\mathrm{d}\xi, \tag{2.12}$$

that is, taking into account the definition (2.9) of Q,

$$\begin{aligned} \widetilde{\Phi}_+(x,t,\lambda) &= I - \int_{x}^{+\infty} \mathrm{e}^{\frac{\sqrt{1-\lambda^2}}{2}\int_x^\xi \tilde{m}(\eta,t)\mathrm{d}\eta\,\sigma_3}\hat{U}(\xi,t,\lambda)\widetilde{\Phi}_+(\xi,t,\lambda)\mathrm{e}^{-\frac{\sqrt{1-\lambda^2}}{2}\int_x^\xi \tilde{m}(\eta,t)\mathrm{d}\eta\,\sigma_3}\mathrm{d}\xi, \\ \widetilde{\Phi}_-(x,t,\lambda) &= I + \int_{-\infty}^{x} \mathrm{e}^{\frac{\sqrt{1-\lambda^2}}{2}\int_x^\xi \tilde{m}(\eta,t)\mathrm{d}\eta\,\sigma_3}\hat{U}(\xi,t,\lambda)\widetilde{\Phi}_-(\xi,t,\lambda)\mathrm{e}^{-\frac{\sqrt{1-\lambda^2}}{2}\int_x^\xi \tilde{m}(\eta,t)\mathrm{d}\eta\,\sigma_3}\mathrm{d}\xi \end{aligned} \tag{2.13}$$

(I is the identity matrix). Hereafter, let $\hat{\Phi}_\pm := \widetilde{\Phi}_\pm \mathrm{e}^{-Q}$ denote the corresponding Jost solutions of (2.8).

Introducing a new spectral parameter k by

$$\lambda^2 = 4k^2 + 1,$$

the exponentials in (2.13) become $\mathrm{e}^{\pm\mathrm{i}k\int_x^\xi \tilde{m}(\xi,t)\mathrm{d}\xi\,\sigma_3}$. Moreover, introducing the new space variable

$$y(x,t) := x - \int_x^{+\infty} (\tilde{m}(\xi,t)-1)\mathrm{d}\xi, \tag{2.14}$$

Q takes (by a slight abuse of notations) the form $Q(y,t,k) = -\mathrm{i}k\left(y - \frac{2t}{4k^2+1}\right)\sigma_3$, which coincides with that in the case of the Camassa–Holm equation [25, 26].

Remark 2.1.2. Recall that the pair of renowned integrable equations — the Korteweg–de Vries (KdV) equation and the modified Korteweg–de Vries (mKdV) equation — shares the same Q, which, in those cases, has the form $Q(x,t,k) =$

$(\mathrm{i}kx + 4\mathrm{i}k^3t)\sigma_3$. Therefore, the above consideration gives an additional reason to naming equation (1.5) as the *modified* Camassa–Holm (mCH) equation.

Remark 2.1.3. The change of variables (2.14) is, in fact, a part of the Liouville transformation [85] relating the spatial equations from the Lax pairs for the mKdV equation and the mCH equation and thus establishing the correspondence between the flows in the mCH hierarchy and the mKdV hierarchy. Being combined with the Liouville transformation relating the CH hierarchy and the Korteweg-de Vries (KdV) hierarchy [92], it allows establishing a Miura-type map form the mCH equation to the CH equation [85]. However, since this map is not univalent and involves nonlinear manipulations with dependent variables, it is difficult to use it when studying various properties of solutions of particular problems for the mCH equation (for instance, the long time behavior of solutions of the Cauchy problem with particular boundary conditions). This motivated us to introduce, in the present paper, a more direct approach to our Cauchy problem for the mCH equation, which doesn't rely on a map to the CH equation but deals directly with the Lax pair equations for mCH.

An important difference between the Lax pairs for the CH equation and the mCH equation is that in the latter case, the dependence of the associated coefficient matrix $\hat{U}(x,t,k)$ (by a slight abuse of notations we keep the same notation $\hat{U}$) on the spectral parameter k is not rational (because of $\lambda(k)$):

$$\hat{U}(x,t,k) = \frac{\tilde{m}-1}{2}\left(\frac{1}{2\mathrm{i}k}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\lambda(k)}{2\mathrm{i}k}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right),$$

which would complicate the construction of the RH problem, requiring either the introduction of a branch cut in the k plane or the formulation of the RH problem on the Riemann sphere associated with $\lambda^2 = 4k^2 + 1$.

In order to avoid these complications, we introduce a new (uniformizing) spectral parameter μ such that both λ and k are rational w.r.t. μ:

$$\lambda = -\frac{1}{2}\left(\mu + \frac{1}{\mu}\right), \qquad k = \frac{1}{4}\left(\mu - \frac{1}{\mu}\right). \tag{2.15}$$

More precisely, we define $\mu = -\lambda - \mathrm{i}\sqrt{1-\lambda^2}$, so that $k = -\frac{\mathrm{i}}{2}\sqrt{1-\lambda^2}$ and $\sqrt{1-\lambda^2} = \frac{\mathrm{i}}{2}\frac{\mu^2-1}{\mu} = 2\mathrm{i}k$. In terms of μ we have

$$p(x,t,\mu) = -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\left(\int_x^{+\infty}(\tilde{m}(\xi,t)-1)\mathrm{d}\xi - x + \frac{8\mu^2}{(\mu^2+1)^2}t\right), \tag{2.16}$$

$$\hat{U}(x,t,\mu) = \frac{\mathrm{i}(\mu^2+1)(\tilde{m}-1)}{2(\mu^2-1)}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \frac{\mathrm{i}\mu(\tilde{m}-1)}{\mu^2-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.17}$$

and, accordingly, equations (2.13) become

$$\widetilde{\Phi}_{\pm}(x,t,\mu) = I + \int_{\pm\infty}^{x} \mathrm{e}^{\frac{\mathrm{i}(\mu^2-1)}{4\mu}\int_x^\xi \tilde{m}(\eta,t)\mathrm{d}\eta\,\sigma_3}\hat{U}(\xi,t,\mu)\widetilde{\Phi}_{\pm}(\xi,t,\mu)\mathrm{e}^{-\frac{\mathrm{i}(\mu^2-1)}{4\mu}\int_x^\xi \tilde{m}(\eta,t)\mathrm{d}\eta\,\sigma_3}\mathrm{d}\xi. \tag{2.18}$$

We are now able, by analogy with the case of the CH equation [25, 26], to analyze the analytic and asymptotic properties of the solutions $\widetilde{\Phi}_{\pm}$ of (2.18) as functions of μ, using Neumann series expansions. Let $A^{(1)}$ and $A^{(2)}$ denote the columns of a 2×2 matrix $A = \begin{pmatrix}A^{(1)} & A^{(2)}\end{pmatrix}$. Using these notations we have the following properties:

- $\widetilde{\Phi}_{-}^{(1)}$ and $\widetilde{\Phi}_{+}^{(2)}$ are analytic in $\mathbb{C}^+ = \{\mu \in \mathbb{C} \mid \operatorname{Im}\mu > 0\}$;
- $\widetilde{\Phi}_{+}^{(1)}$ and $\widetilde{\Phi}_{-}^{(2)}$ are analytic in $\mathbb{C}^- = \{\mu \in \mathbb{C} \mid \operatorname{Im}\mu < 0\}$;
- $\widetilde{\Phi}_{-}^{(1)}$, $\widetilde{\Phi}_{+}^{(2)}$, $\widetilde{\Phi}_{+}^{(1)}$, and $\widetilde{\Phi}_{-}^{(2)}$ are continuous up to the real line except at $\mu = \pm 1$.

Further, we first observe that $\hat{U}(\mu) \equiv \hat{U}(x,t,\mu)$, $\hat{V}(\mu) \equiv \hat{V}(x,t,\mu)$ satisfy the same symmetries:

$$\hat{U}(\bar{\mu}) = \sigma_1\overline{\hat{U}(\mu)}\sigma_1, \qquad \hat{U}(-\mu) = \sigma_2\hat{U}(\mu)\sigma_2, \qquad \hat{U}(\mu^{-1}) = \sigma_1\hat{U}(\mu)\sigma_1, \tag{2.19a}$$

$$\hat{V}(\bar{\mu}) = \sigma_1\overline{\hat{V}(\mu)}\sigma_1, \qquad \hat{V}(-\mu) = \sigma_2\hat{V}(\mu)\sigma_2, \qquad \hat{V}(\mu^{-1}) = \sigma_1\hat{V}(\mu)\sigma_1, \tag{2.19b}$$

with $\mu \neq \pm 1$, and also $\mu \neq 0$ for the symmetry $\mu \mapsto \mu^{-1}$. Moreover, $p(\mu) \equiv p(x,t,\mu)$ satisfies the following symmetries:

$$p^*(\mu) = -p(\mu) = p(-\mu) = p(\mu^{-1}). \tag{2.20}$$

It follows that

- $\widetilde{\Phi}_\pm$ also satisfy the same symmetries as in (2.19a):

$$\widetilde{\Phi}_\pm(\bar{\mu}) = \sigma_1 \overline{\widetilde{\Phi}_\pm(\mu)} \sigma_1, \quad \widetilde{\Phi}_\pm(-\mu) = \sigma_2 \widetilde{\Phi}_\pm(\mu) \sigma_2, \quad \widetilde{\Phi}_\pm(\mu^{-1}) = \sigma_1 \widetilde{\Phi}_\pm(\mu) \sigma_1. \tag{2.21}$$

  That means $\widetilde{\Phi}_\pm^{(1)}(\mu) = \sigma_1 \widetilde{\Phi}_\pm^{(2)*}(\mu) = \sigma_3 \sigma_1 \widetilde{\Phi}_\pm^{(2)}(-\mu) = \sigma_1 \widetilde{\Phi}_\pm^{(2)}(\mu^{-1})$ for $\pm \operatorname{Im} \mu \leq 0$, $\mu \neq \pm 1$.

In (2.11) the coefficients are traceless matrices, from which it follows that

- $\det \widetilde{\Phi}_\pm \equiv 1$.

Regarding the values of $\widetilde{\Phi}_\pm$ at particular points in the μ-plane, (2.18) implies the following:

- $\left(\widetilde{\Phi}_-^{(1)} \; \widetilde{\Phi}_+^{(2)} \right) \to I$ as $\mu \to \infty$ with $\operatorname{Im} \mu \geq 0$, and also for $\mu = 0$ (by the symmetry (2.21)).

- $\left(\widetilde{\Phi}_+^{(1)} \; \widetilde{\Phi}_-^{(2)} \right) \to I$ as $\mu \to \infty$ with $\operatorname{Im} \mu \leq 0$, and also for $\mu = 0$.

- As $\mu \to 1$, $\widetilde{\Phi}_\pm(x,t,\mu) = \frac{\mathrm{i}}{2(\mu-1)} \alpha_\pm(x,t) \left(\begin{smallmatrix} -1 & 1 \\ -1 & 1 \end{smallmatrix} \right) + \mathrm{O}(1)$ with $\alpha_\pm(x,t) \in \mathbb{R}$ (understood column-wise, in the corresponding half-planes).

- As $\mu \to -1$, $\widetilde{\Phi}_\pm(x,t,\mu) = -\frac{\mathrm{i}}{2(\mu+1)} \alpha_\pm(x,t) \left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix} \right) + \mathrm{O}(1)$ with the same $\alpha_\pm(x,t)$ as the previous ones (by symmetry (2.21)).

2.1.3 Spectral data

Introduce the scattering matrix $s(\mu)$ as a matrix relating $\widetilde{\Phi}_+$ and $\widetilde{\Phi}_-$ on the real line:

$$\widetilde{\Phi}_+(x,t,\mu) = \widetilde{\Phi}_-(x,t,\mu) \mathrm{e}^{-p(x,t,\mu)\sigma_3} s(\mu) \mathrm{e}^{p(x,t,\mu)\sigma_3}, \qquad \mu \in \mathbb{R}, \; \mu \neq \pm 1. \tag{2.22}$$

By (2.21), $s(\mu)$ can be written in terms of two scalar spectral functions, $a(\mu)$ and $b(\mu)$:

$$s(\mu) = \begin{pmatrix} \overline{a(\mu)} & b(\mu) \\ \overline{b(\mu)} & a(\mu) \end{pmatrix}, \qquad \mu \in \mathbb{R}, \tag{2.23}$$

satisfying the symmetries $\overline{a(\mu)} = a(-\mu) = a(\mu^{-1})$ and $\overline{b(\mu)} = -b(-\mu) = b(\mu^{-1})$ for $\mu \in \mathbb{R}$.

The spectral functions $a(\mu)$ and $b(\mu)$ are uniquely determined by $u(x,0)$ through the solutions $\widetilde{\Phi}_{\pm}(x,0,\mu)$ of equations (2.18). On the other hand, using the representations

$$a(\mu) = \det\begin{pmatrix}\widetilde{\Phi}_{-}^{(1)} & \widetilde{\Phi}_{+}^{(2)}\end{pmatrix}, \quad b(\mu) = \mathrm{e}^{2p}\det\begin{pmatrix}\widetilde{\Phi}_{+}^{(2)} & \widetilde{\Phi}_{-}^{(2)}\end{pmatrix},$$

the analytic properties of $\widetilde{\Phi}_{\pm}$ stated above imply corresponding properties of $a(\mu)$ and $b(\mu)$:

- $a(\mu)$ can be analytically continued into $\mathbb{C}^+$, being continuous up to the real line, except at $\mu = \pm 1$. Moreover, $a(0) = 1$, $a(\mu) \to 1$ as $\mu \to \infty$, and $a(\mu)$ satisfies the symmetries

$$a(\mu) = \overline{a(-\bar{\mu})} = a(-\mu^{-1}) \text{ for } \operatorname{Im}\mu \geq 0.$$

- $b(\mu)$ is continuous for $\mu \in \mathbb{R} \setminus \{-1, 1\}$. Moreover, $b(0) = 0$ and $b(\mu) \to 0$ as $\mu \to \pm\infty$.

- As $\mu \to 1$, $a(\mu) = \gamma\frac{\mathrm{i}}{2(\mu-1)} + \mathrm{O}(1)$ and $b(\mu) = \gamma\frac{\mathrm{i}}{2(\mu-1)} + \mathrm{O}(1)$ with the same $\gamma \in \mathbb{R}$, as follows from (2.22).

- As $\mu \to -1$, $a(\mu) = \gamma\frac{\mathrm{i}}{2(\mu+1)} + \mathrm{O}(1)$ and $b(\mu) = -\gamma\frac{\mathrm{i}}{2(\mu+1)} + \mathrm{O}(1)$ with the same γ as the previous one, by symmetry.

- $|a(\mu)|^2 - |b(\mu)|^2 = 1$ for $\mu \in \mathbb{R}$, $\mu \neq \pm 1$.

Remark 2.1.4. The case $\gamma \neq 0$ is generic. On the other hand, in the non-generic case $\gamma = 0$, we then have $a(\pm 1) = a_1$ and $b(\pm 1) = \pm b_1$ with some $a_1 \in \mathbb{R}$ and $b_1 \in \mathbb{R}$ such that $a_1^2 = 1 + b_1^2$. It then follows from (2.22) that the coefficients $\alpha_+(x,t)$ and $\alpha_-(x,t)$ appearing in the expansions of $\widetilde{\Phi}$ at $\mu = \pm 1$ are related by

$$\alpha_+(x,t) = (a_1 - b_1)\alpha_-(x,t). \tag{2.24}$$

2.2 Riemann–Hilbert problem

2.2.1 RH problem parameterized by (x, t)

The analytic properties of $\widetilde{\Phi}_\pm$ stated above allow rewriting the scattering relation (2.22) as a jump relation for a piece-wise meromorphic (w.r.t. μ), 2×2-matrix valued function (depending on x and t as parameters). Indeed, define $M \equiv M(x,t,\mu)$ by

$$M(x,t,\mu) = \begin{cases} \left(\frac{\widetilde{\Phi}_-^{(1)}(x,t,\mu)}{a(\mu)} \quad \widetilde{\Phi}_+^{(2)}(x,t,\mu) \right), & \operatorname{Im}\mu > 0, \\ \left(\widetilde{\Phi}_+^{(1)}(x,t,\mu) \quad \frac{\widetilde{\Phi}_-^{(2)}(x,t,\mu)}{\overline{a(\bar{\mu})}} \right), & \operatorname{Im}\mu < 0. \end{cases} \tag{2.25}$$

Define also

$$r(\mu) := \frac{b(\mu)}{a^*(\mu)}, \quad \mu \in \mathbb{R}. \tag{2.26}$$

Then the limiting values $M_\pm(x,t,\mu)$, $\mu \in \mathbb{R}$ of M as μ is approached from $\mathbb{C}^\pm$ are related by

$$M_-(x,t,\mu) = M_+(x,t,\mu)J(x,t,\mu), \quad \mu \in \mathbb{R},\ \mu \neq \pm 1, \tag{2.27a}$$

where

$$J(x,t,\mu) = \mathrm{e}^{-p(x,t,\mu)\sigma_3} J_0(\mu) \mathrm{e}^{p(x,t,\mu)\sigma_3} \tag{2.27b}$$

with

$$J_0(\mu) = \begin{pmatrix} 1 & -r(\mu) \\ r^*(\mu) & 1 - r(\mu)r^*(\mu) \end{pmatrix}. \tag{2.27c}$$

Taking into account the properties of $\widetilde{\Phi}_\pm$ and $s(\mu)$ we check that $M(x,t,\mu)$ satisfies the following conditions:

- The *jump* condition (2.27) across $\mathbb{R}$.
- The *determinant* condition $\det M \equiv 1$.
- The *normalization* condition:

$$M \to I \quad \text{as } \mu \to \infty \tag{2.28}$$

  (and also $M(0) = I$ by symmetry, see (2.31)).

- *Singularity* conditions:

$$M(x,t,\mu) = \begin{cases} \frac{\mathrm{i}\alpha_+(x,t)}{2(\mu-1)} \begin{pmatrix} -c & 1 \\ -c & 1 \end{pmatrix} + \mathrm{O}(1), & \mu \to 1, \quad \operatorname{Im}\mu > 0, \\ -\frac{\mathrm{i}\alpha_+(x,t)}{2(\mu+1)} \begin{pmatrix} c & 1 \\ -c & -1 \end{pmatrix} + \mathrm{O}(1), & \mu \to -1, \quad \operatorname{Im}\mu > 0, \end{cases} \tag{2.29}$$

with some $\alpha_+(x,t) \in \mathbb{R}$ and (see Remark 2.1.4)

$$c := \begin{cases} 0, & \text{if } \gamma \neq 0, \\ \frac{a_1+b_1}{a_1}, & \text{if } \gamma = 0, \end{cases} \tag{2.30a}$$

where $a_1 = a(1)$, $b_1 = b(1)$, and $\gamma := -2\mathrm{i}\lim\limits_{\mu\to 1}(\mu-1)a(\mu)$. Notice that in terms of $r(\pm 1)$, the generic case $\gamma \neq 0$ corresponds to $r(1) = -r(-1) = -1$ whereas in the non-generic case, $|r(\pm 1)| < 1$ (see the case of the one-dimensional Schrödinger operator [53], which constitutes the spectral problem for the Korteweg–de Vries equation). Therefore, (2.30a) can be written as

$$c := \begin{cases} 0, & \text{if } r(1) = -1, \\ 1 + r(1) = 1 - r(-1), & \text{if } |r(1)| < 1. \end{cases} \tag{2.30b}$$

Both conditions in (2.29) are actually equivalent by the symmetries (2.31).

- *Symmetries* (which result from (2.21)):

$$M(\bar{\mu}) = \sigma_1\overline{M(\mu)}\sigma_1, \qquad M(-\mu) = \sigma_2 M(\mu)\sigma_2, \qquad M(\mu^{-1}) = \sigma_1 M(\mu)\sigma_1, \tag{2.31}$$

where $M(\mu) \equiv M(x,t,\mu)$. The first symmetry can also be written as $\sigma_1 M^{(1)*} = M^{(2)}$. Moreover, (2.31) implies the symmetries $\overline{M(-\bar{\mu})} = M(-\mu^{-1}) = \sigma_3 M(\mu)\sigma_3$.

If $a(\mu)$ is allowed to have zeros in $\mathbb{C}^+$, the above conditions must be supplemented by residue conditions at these zeros. Assume that $a(\mu)$ has a finite number of simple zeros $\{\mu_j\}_1^N$ in $\mathbb{C}^+$. Symmetries $a(\mu) = \overline{a(-\bar{\mu})} = a(-\mu^{-1})$

imply that this set of zeros is invariant under the transformations $\mu \mapsto -\bar{\mu}$ and $\mu \mapsto -\mu^{-1}$: for each j there exist j' and j'' such that $-\bar{\mu}_j = \mu_{j'}$ and $-\mu_j^{-1} = \mu_{j''}$.

- *Residue* conditions: $M^{(1)}(x,t,\mu)$ has simple poles at $\{\mu_j\}_1^N$ and $M^{(2)}(x,t,\mu)$ has simple poles at $\{\bar{\mu}_j\}_1^N$. Moreover

$$\operatorname{Res}_{\mu_j} M^{(1)}(x,t,\mu) = \frac{1}{\varkappa_j(x,t)} M^{(2)}(x,t,\mu_j), \tag{2.32a}$$

$$\operatorname{Res}_{\bar{\mu}_j} M^{(2)}(x,t,\mu) = \frac{1}{\overline{\varkappa_j}(x,t)} M^{(1)}(x,t,\bar{\mu}_j). \tag{2.32b}$$

  Here $\varkappa_j(x,t) = \dot{a}(\mu_j)\delta_j \mathrm{e}^{-2p(x,t,\mu_j)}$ with some constants $\delta_j \neq 0$. By symmetries (2.31) both conditions in (2.32) are equivalent. Note also how the residue changes under the transformations $\mu \mapsto -\bar{\mu}$ and $\mu \mapsto -\mu^{-1}$: if $-\bar{\mu}_j = \mu_{j'}$ and $-\mu_j^{-1} = \mu_{j''}$ then $\varkappa_j = \overline{\varkappa_{j'}} = -\mu_j^{-2}\varkappa_{j''}$.

Proof of (2.32). Indeed, let μ_j be a simple root of $a(\mu)$, that is, $a(\mu_j) = 0$ with $\dot{a}(\mu_j) \neq 0$. Then, using $a(\mu) = \det\begin{pmatrix}\widetilde{\Phi}_-^{(1)} & \widetilde{\Phi}_+^{(2)}\end{pmatrix} = \det\begin{pmatrix}\hat{\Phi}_-^{(1)} & \hat{\Phi}_+^{(2)}\end{pmatrix}$, we have

$$\hat{\Phi}_+^{(2)}(x,t,\mu_j) = \delta_j \hat{\Phi}_-^{(1)}(x,t,\mu_j), \tag{2.33a}$$

$$\widetilde{\Phi}_+^{(2)}(x,t,\mu_j) = \delta_j \mathrm{e}^{-2p(x,t,\mu_j)} \widetilde{\Phi}_-^{(1)}(x,t,\mu_j) \tag{2.33b}$$

with some constant $\delta_j \neq 0$. Hence,

$$\operatorname{Res}_{\mu_j} M^{(1)}(x,t,\mu) = \operatorname{Res}_{\mu_j} \frac{\widetilde{\Phi}_-^{(1)}(x,t,\mu)}{a(\mu)} = \frac{\widetilde{\Phi}_-^{(1)}(x,t,\mu_j)}{\dot{a}(\mu_j)} = \frac{\widetilde{\Phi}_+^{(2)}(x,t,\mu_j)}{\dot{a}(\mu_j)\delta_j \mathrm{e}^{-2p(x,t,\mu_j)}}.$$

Denoting $\varkappa_j(x,t) := \dot{a}(\mu_j)\delta_j \mathrm{e}^{-2p(x,t,\mu_j)}$ we get (2.32a). The residue relation (2.32b) then follows by the symmetry $\mu \mapsto \mu^* = \bar{\mu}$. Indeed, applying this symmetry to (2.32a) and multiplying by σ_1 we get

$$\operatorname{Res}_{\bar{\mu}_j} \sigma_1 M^{(1)*}(x,t,\mu) = \frac{1}{\overline{\varkappa_j}(x,t)} \sigma_1 M^{(2)*}(x,t,\bar{\mu}_j),$$

which reduces to (2.32b) in view of the relation $\sigma_1 M^{(1)*} = M^{(2)}$ (see (2.31)). $\square$

In the framework of the Riemann–Hilbert approach to nonlinear evolution equations, we interpret the jump relation (2.27a), normalization condition (2.28), singularity conditions (2.29), and residue conditions (2.32) as a Riemann–Hilbert problem, with the jump matrix and residue parameters determined by the initial data for the nonlinear problem. We proceed as in the case of the Camassa–Holm equation:

1) In order to have the data for the RH problem to depend explicitly on the parameters, we use the space variable $y(x,t) := x - \int_x^{+\infty}(\tilde{m}(\xi,t)-1)\mathrm{d}\xi$ we have introduced in (2.14).

2) In order to determine an efficient way for retrieving the solution of the mCH equation from the solution of the RH problem, we pay a special attention to the behavior of the Jost solutions of the Lax pair equations at $\mu = \pm\mathrm{i}$, i.e., at those values of μ that correspond to $\lambda = 0$, when the x-equation (2.4a), (2.5a) of the Lax pair becomes trivial (independent of the solution of the nonlinear equation in question).

2.2.2 Eigenfunction near $\mu = \mathrm{i}$

In the case of the Camassa–Holm equation [26] as well as other CH-type nonlinear integrable equations studied so far, see, e.g., [29, 30], the analysis of the behavior of the respective Jost solutions at dedicated points in the complex plane of the spectral parameter (see Item 2) above) requires a dedicated gauge transformation of the Lax pair equations.

It is remarkable that in the case of the mCH equation, we don't need to use such a transformation; all we need is to regroup the terms in the Lax pair (2.8a), (2.8c).

Namely, let us rewrite (2.8a) in terms of μ (keeping the same notation $\hat{\Phi}$ for the solution):

$$\hat{\Phi}_x + \frac{\mathrm{i}(\mu^2-1)}{4\mu}\sigma_3\hat{\Phi} = \hat{U}_0\hat{\Phi}, \tag{2.34a}$$

where

$$\hat{U}_0(x,t,\mu) := \frac{\mathrm{i}(\mu^2+1)(\tilde{m}-1)}{2(\mu^2-1)}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \left(\frac{\mathrm{i}\mu(\tilde{m}-1)}{\mu^2-1} + \frac{\mathrm{i}(\mu^2-1)\tilde{m}}{4\mu} - \frac{\mathrm{i}(\mu^2-1)}{4\mu}\right)\sigma_3, \tag{2.34b}$$

so that $\hat{U}_0(x,t,\pm\mathrm{i}) \equiv 0$. Accordingly, rewrite (2.8c) as

$$\hat{\Phi}_t - \frac{2\mathrm{i}(\mu^2-1)\mu}{(\mu^2+1)^2}\sigma_3\hat{\Phi} = \hat{V}_0\hat{\Phi}, \tag{2.34c}$$

where

$$\hat{V}_0(x,t,\mu) := \frac{\mathrm{i}(\mu^2-1)}{4\mu}(\tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u})\tilde{m}\sigma_3 + \hat{V}(x,t,\mu). \tag{2.34d}$$

Further, introduce (compare with (2.16))

$$p_0(x,t,\mu) := \frac{\mathrm{i}(\mu^2-1)}{4\mu}x - \frac{2\mathrm{i}(\mu^2-1)\mu}{(\mu^2+1)^2}t, \tag{2.35}$$

then $Q_0 := p_0\sigma_3$, and $\widetilde{\Phi}_0 := \hat{\Phi}\mathrm{e}^{Q_0}$ so that equations (2.34a) and (2.34c) become

$$\begin{cases} \widetilde{\Phi}_{0x} + [Q_{0x}, \widetilde{\Phi}_0] = \hat{U}_0\widetilde{\Phi}_0, \\ \widetilde{\Phi}_{0t} + [Q_{0t}, \widetilde{\Phi}_0] = \hat{V}_0\widetilde{\Phi}_0. \end{cases} \tag{2.36}$$

Define the Jost solutions $\widetilde{\Phi}_{0\pm}$ of (2.36) as the solutions of the integral equations

$$\widetilde{\Phi}_{0\pm}(x,t,\mu) = I + \int_{\pm\infty}^{x} \mathrm{e}^{-\frac{\mathrm{i}(\mu^2-1)}{4\mu}(x-\xi)\sigma_3}\hat{U}_0(\xi,t,\mu)\widetilde{\Phi}_{0\pm}(\xi,t,\mu)\mathrm{e}^{\frac{\mathrm{i}(\mu^2-1)}{4\mu}(x-\xi)\sigma_3}\mathrm{d}\xi. \tag{2.37}$$

If $\hat{\Phi}_{0\pm} := \widetilde{\Phi}_{0\pm}\mathrm{e}^{-p_0\sigma_3}$ we observe that $\hat{\Phi}_{0\pm}(x,t,\mu)$ and $\hat{\Phi}_{\pm}(x,t,\mu)$ satisfy the same differential equations (2.34) and thus they are related by matrices $C_\pm(\mu)$ independent of x and t:

$$\hat{\Phi}_\pm = \hat{\Phi}_{0\pm}C_\pm(\mu).$$

It follows that

$$\widetilde{\Phi}_\pm(x,t,\mu) = \widetilde{\Phi}_{0\pm}(x,t,\mu)\mathrm{e}^{-p_0(x,t,\mu)\sigma_3}C_\pm(\mu)\mathrm{e}^{p(x,t,\mu)\sigma_3}. \tag{2.38}$$

Thus, $C_\pm(\mu) = \mathrm{e}^{(p_0(\pm\infty,t,\mu)-p(\pm\infty,t,\mu))\sigma_3}$.

Since $p(x,t,\mu) - p_0(x,t,\mu) = -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\int_x^{+\infty}(\tilde m(\xi,t)-1)\mathrm{d}\xi$ we find that $C_+(\mu)\equiv I$ whereas $C_-(\mu) = \mathrm{e}^{\frac{\mathrm{i}(\mu^2-1)}{4\mu}\int_{-\infty}^{+\infty}(\tilde m(\xi,t))-1)\mathrm{d}\xi\,\sigma_3}$.

Since $\hat U_0(x,t,\mathrm{i})\equiv 0$, it follows from (2.37) that $\widetilde\Phi_{0\pm}(x,t,\mathrm{i})\equiv I$ and thus

$$\widetilde\Phi_+(x,t,\mathrm{i}) = \mathrm{e}^{\frac12\int_x^{+\infty}(\tilde m(\xi,t)-1)\mathrm{d}\xi\,\sigma_3} \text{ and } \widetilde\Phi_-(x,t,\mathrm{i}) = \mathrm{e}^{-\frac12\int_{-\infty}^{x}(\tilde m(\xi,t)-1)\mathrm{d}\xi\,\sigma_3}.$$

Consequently,

$$a(\mathrm{i}) = \mathrm{e}^{-\frac12\int_{-\infty}^{+\infty}(\tilde m(\xi,t)-1)\mathrm{d}\xi}$$

and

$$M(x,t,\mathrm{i}) = \begin{pmatrix} \mathrm{e}^{\frac12\int_x^{+\infty}(\tilde m(\xi,t)-1)\mathrm{d}\xi} & 0 \\ 0 & \mathrm{e}^{-\frac12\int_x^{+\infty}(\tilde m(\xi,t)-1)\mathrm{d}\xi} \end{pmatrix}. \tag{2.39a}$$

Then, by symmetry,

$$M(x,t,-\mathrm{i}) = \begin{pmatrix} \mathrm{e}^{-\frac12\int_x^{+\infty}(\tilde m(\xi,t)-1)\mathrm{d}\xi} & 0 \\ 0 & \mathrm{e}^{\frac12\int_x^{+\infty}(\tilde m(\xi,t)-1)\mathrm{d}\xi} \end{pmatrix}. \tag{2.39b}$$

Remark 2.2.1. The symmetries (2.31) imply that $\overline{M(\mathrm{i})} = M(\mathrm{i}) = \sigma_3 M(\mathrm{i})\sigma_3$ where $M(\mathrm{i})\equiv M(x,t,\mathrm{i})$, and thus $M(\mathrm{i})$ is a diagonal matrix with real entries which, due to the determinant equality $\det M\equiv 1$, has the form

$$M(x,t,\mathrm{i}) = \begin{pmatrix} \varphi(x,t) & 0 \\ 0 & \varphi^{-1}(x,t) \end{pmatrix} \tag{2.40a}$$

with some $\varphi(x,t)\in\mathbb{R}$. Then, referring again to (2.31), it follows that

$$M(x,t,-\mathrm{i}) = \begin{pmatrix} \varphi^{-1}(x,t) & 0 \\ 0 & \varphi(x,t) \end{pmatrix} \tag{2.40b}$$

with the same $\varphi(x,t)$. Therefore, the matrix structure of $M(x,t,\pm\mathrm{i})$ as in (2.39) follows from the general properties of the solution of a Riemann–Hilbert problem (specified by jump, normalization, residue, singularity, and symmetry conditions). This is in contrast with the case of the Camassa–Holm equation [25, 26], where a specific matrix structure of the solution of the associated RH problem, evaluated at a dedicated point ($k = \frac{\mathrm{i}}{2}$ for the CH equation), constitutes an additional requirement for the solution. In that case, the proof of the uniqueness of the solution of the RH problem relies essentially on this additional property.

In what follows we will use (2.39) in order to extract the solution of the mCH equation from the solution of the associated RH problem.

2.2.3 RH problem in the (y,t) scale

Introducing the new space variable $y(x,t)$ by (2.14), $\hat{M}(y,t,\mu)$ so that $M(x,t,\mu) = \hat{M}(y(x,t),t,\mu)$, the jump condition (2.27a) becomes

$$\hat{M}_-(y,t,\mu) = \hat{M}_+(y,t,\mu)\hat{J}(y,t,\mu), \qquad \mu \in \mathbb{R}, \quad \mu \neq \pm 1, \tag{2.41a}$$

where

$$\hat{J}(y,t,\mu) := \mathrm{e}^{-\hat{p}(y,t,\mu)\sigma_3} J_0(\mu) \mathrm{e}^{\hat{p}(y,t,\mu)\sigma_3} \tag{2.41b}$$

with $J_0(\mu)$ defined by (2.27c) and

$$\hat{p}(y,t,\mu) := -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\left(-y + \frac{8\mu^2}{(\mu^2+1)^2}t\right). \tag{2.41c}$$

so that $J(x,t,\mu) = \hat{J}(y(x,t),t,\mu)$ and $p(x,t,\mu) = \hat{p}(y(x,t),t,\mu)$, where the jump $J(x,t,\mu)$ and the phase $p(x,t,\mu)$ are defined in (2.27b) and (2.16), respectively.

Accordingly, in this scale, the residue conditions (2.32) become explicit as well:

$$\begin{aligned}
\operatorname{Res}_{\mu_j} \hat{M}^{(1)}(y,t,\mu) &= \frac{1}{\hat{\varkappa}_j(y,t)} \hat{M}^{(2)}(y,t,\mu_j), \\
\operatorname{Res}_{\bar{\mu}_j} \hat{M}^{(2)}(y,t,\mu) &= \frac{1}{\overline{\hat{\varkappa}_j(y,t)}} \hat{M}^{(1)}(y,t,\overline{\mu_j}),
\end{aligned} \tag{2.42}$$

with $\hat{\varkappa}_j(y,t) = \dot{a}(\mu_j)\delta_j \mathrm{e}^{-2\hat{p}(y,t,\mu_j)}$. Further we denote $\rho_j := \dot{a}(\mu_j)\delta_j$.

Noticing that the normalization condition (2.28), the symmetries (2.31), and the singularity conditions (2.29) at $\mu = \pm 1$ hold when using the new scale (y,t), we arrive at the basic RH problem.

Basic RH problem. Given $r(\mu)$ for $\mu \in \mathbb{R}$, $c \in \mathbb{R}$, and $\{\mu_j, \rho_j\}_1^N$ a set of points $\mu_j \in \mathbb{C}^+$ and complex numbers $\rho_j \neq 0$ invariant by $\mu \mapsto -\bar{\mu}$ and

$\mu \mapsto -\mu^{-1}$ (that is, $-\overline{\mu_j} = \mu_{j'}$ and $-\mu_j^{-1} = \mu_{j''}$ with $\rho_j = \overline{\rho_{j'}} = -\mu_j^{-2}\rho_{j''}$), find a piece-wise (w.r.t. $\mathbb{R}$) meromorphic, 2×2-matrix valued function $\hat{M}(y,t,\mu)$ satisfying the following conditions:

- The jump condition (2.41) across $\mathbb{R}$ (with $J_0(\mu)$ defined by (2.27c)).
- The residue conditions (2.42) with $\hat{\varkappa}_j(y,t) = \rho_j \mathrm{e}^{-2\hat{p}(y,t,\mu_j)}$.
- The normalization condition $\hat{M}(y,t,\mu) \to I$ as $\mu \to \infty$.
- The symmetries

$$\hat{M}(\bar{\mu}) = \sigma_1 \overline{\hat{M}(\mu)} \sigma_1, \quad \hat{M}(-\mu) = \sigma_2 \hat{M}(\mu) \sigma_2, \quad \hat{M}(\mu^{-1}) = \sigma_1 \hat{M}(\mu) \sigma_1 \tag{2.43}$$

  where $\hat{M}(\mu) \equiv \hat{M}(y,t,\mu)$. These symmetries imply that $\hat{M}(-\mu^{-1}) = \sigma_3 \hat{M}(\mu) \sigma_3 = \overline{\hat{M}(-\bar{\mu})}$.
- The singularity conditions

$$\hat{M}(y,t,\mu) = \frac{\mathrm{i}\hat{\alpha}_+(y,t)}{2(\mu-1)} \begin{pmatrix} -c & 1 \\ -c & 1 \end{pmatrix} + \mathrm{O}(1) \qquad \text{as } \mu \to 1, \quad \operatorname{Im}\mu > 0, \tag{2.44a}$$

$$\hat{M}(y,t,\mu) = -\frac{\mathrm{i}\hat{\alpha}_+(y,t)}{2(\mu+1)} \begin{pmatrix} c & 1 \\ -c & -1 \end{pmatrix} + \mathrm{O}(1) \quad \text{as } \mu \to -1, \ \operatorname{Im}\mu > 0, \tag{2.44b}$$

  where $\hat{\alpha}_+(y,t) \in \mathbb{R}$ is not specified. These two singularity conditions are actually equivalent by symmetries (2.43).

Data of this RH problem associated with $\boldsymbol{u_0(x)}$. Specific data for this RH problem can be derived from initial data of the Cauchy problem (2.1) satisfying $u_0(x) \to 1$ as $x \to \pm\infty$.

- We first get $s(\mu)$ through (2.22) at $t = 0$ (using the solutions of (2.18) taken at $t = 0$).

- Spectral data $a(\mu)$, $b(\mu)$, and $r(\mu)$ follow through (2.23) and (2.26).
- Then $\{\mu_j\}_1^N$ are the zeros of $a(\mu)$ in $\mathbb{C}^+$.
- The real constant c is defined through (2.30).
- The constants $\{\delta_j\}_1^N$ are defined by (2.33b) at $t = 0$ (using the solutions of (2.12) at $t = 0$).
- Finally, the $\{\rho_j\}_1^N$ are defined by $\rho_j = \dot{a}(\mu_j)\delta_j$.

Further, the basic RH problem associated with the Cauchy problem (2.1) for the mCH equation is the basic RH problem with data associated with initial data satisfying $u_0(x) \to 1$, as we just specified.

Remark 2.2.2. An important difference between the cases of the CH and mCH equations is that in the former case, there is a possibility to reduce the matrix RH problems to vector ones which have no singularity at a point on the contour: this can be done by multiplying the respective $\hat{M}$ by the vector $(1, 1)$ from the left. This trick will obviously not work in our current case, since the matrix structure (see (2.44)) of the singularity at $\mu = 1$ is different from that at $\mu = -1$.

2.2.4 Uniqueness of the solution of the basic RH problem

Assume that the RH problem (2.41)–(2.44) has a solution $\hat{M}$. In order to prove that this solution is unique, we first observe that $\det \hat{M} \equiv 1$.

Indeed, the conditions for $\hat{M}$ imply that $\det \hat{M}$ has neither a jump across $\mathbb{R}$ no singularities at μ_j. Moreover, $\det \hat{M}$ tends to 1 as $\mu \to \infty$, and the only possible singularities of $\det \hat{M}$ are simple poles at $\mu = \pm 1$. Then, by Liouville's theorem, $\det \hat{M} \equiv 1 + \frac{\phi_1}{\mu-1} + \frac{\phi_2}{\mu+1}$ with some ϕ_j. But then, the symmetry $\hat{M}(\mu^{-1}) = \sigma_1 \hat{M}(\mu)\sigma_1$ from (2.43) implies that $\phi_1 = \phi_2 = 0$ and thus $\det \hat{M} \equiv 1$.

Now suppose that $\hat{M}_1$ and $\hat{M}_2$ are two solutions of the RH problem, and consider $P := \hat{M}_1(\hat{M}_2)^{-1}$. Obviously, P has neither a jump across $\mathbb{R}$ no singularities at μ_j. Moreover, P tends to I as $\mu \to \infty$, and the only possible singularities of P are simple poles at $\mu = \pm 1$.

Consider, for example, the development of $\hat{M}_j$, $j = 1, 2$ as $\mu \to -1$ with $\operatorname{Im} \mu > 0$:

$$\hat{M}_j(y,t,\mu) = -\frac{\mathrm{i}\beta_j(y,t)}{2(\mu+1)} \begin{pmatrix} c & 1 \\ -c & -1 \end{pmatrix} + \begin{pmatrix} n_j(y,t) & m_j(y,t) \\ f_j(y,t) & g_j(y,t) \end{pmatrix} + \mathrm{O}(\mu+1),\ \mu \in \mathbb{C}^+.$$

By $\det \hat{M}_j \equiv 1$ it follows that

$$(\hat{M}_j(y,t,\mu))^{-1} = -\frac{\mathrm{i}\beta_j(y,t)}{2(\mu+1)} \begin{pmatrix} -1 & -1 \\ c & c \end{pmatrix} + \begin{pmatrix} g_j(y,t) & -m_j(y,t) \\ -f_j(y,t) & n_j(y,t) \end{pmatrix} + \mathrm{O}(\mu+1).$$

Moreover, using these expressions to calculate the expansion of $\hat{M}_j \hat{M}_j^{-1}$ as $\mu \to -1$ the vanishing of the term of order $(\mu+1)^{-1}$ reads as

$$n_j(y,t) + f_j(y,t) = c(m_j(y,t) + g_j(y,t)), \quad j = 1, 2. \tag{2.45}$$

Hence, (2.45) implies that

$$P(y,t,\mu) = -\frac{\mathrm{i}\psi(y,t)}{2(\mu+1)} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \mathrm{O}(1) \text{ as } \mu \to -1,\ \mu \in \mathbb{C}^+,$$

for some $\psi(y,t)$. Then, by the symmetry $P(\mu^{-1}) = \sigma_3 P(\mu) \sigma_3$, we have

$$P(y,t,\mu) = -\frac{\mathrm{i}\psi(y,t)}{2(\mu-1)} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \mathrm{O}(1) \text{ as } \mu \to 1,\ \mu \in \mathbb{C}^+,$$

and, according to the Liouville theorem and the normalization condition,

$$P = -\frac{\mathrm{i}}{2}\psi(y,t) \left(\frac{1}{\mu-1} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \frac{1}{\mu+1} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right) + I.$$

Evaluating this at $\mu = \mathrm{i}$ we have

$$P(y,t,\mathrm{i}) = -\frac{\mathrm{i}}{2}\psi(y,t) \begin{pmatrix} -\mathrm{i} & 1 \\ -1 & \mathrm{i} \end{pmatrix} + I. \tag{2.46}$$

But, according to (2.40a), both matrices $\hat{M}_1(\mathrm{i})$ and $\hat{M}_2(\mathrm{i})$ are diagonal. Hence $P(y,t,\mathrm{i})$ is also diagonal and (2.46) implies that $\psi(y,t) \equiv 0$. Consequently, $P(y,t,\mu) \equiv I$ so that $\hat{M}_1 \equiv \hat{M}_2$.

2.2.5 Recovering $u(x,t)$ from the solution of the RH problem

We will show how to recover the solution of the Cauchy problem (2.1) from the solution of the basic RH problem whose data are associated with the initial data $u_0(x)$. We begin with some preliminary observations.

Going back to the construction of $M(x,t,\mu)$ from the Jost solutions, see Section 2.2.2, we can use (2.39a) in order to express the solution $u(x,t)$ of the mCH equation in terms of $M(x,t,\mu)$ evaluated at $\mu = \mathrm{i}$. Indeed, introduce (compare with the case of the CH equation [26])

$$
\begin{aligned}
\tilde{\mu}_1(x,t) &:= M_{11}(x,t,\mathrm{i}) + M_{21}(x,t,\mathrm{i}) = \mathrm{e}^{\frac{1}{2}\int_x^{+\infty}(\tilde{m}(\xi,t)-1)\mathrm{d}\xi},\\
\tilde{\mu}_2(x,t) &:= M_{12}(x,t,\mathrm{i}) + M_{22}(x,t,\mathrm{i}) = \mathrm{e}^{-\frac{1}{2}\int_x^{+\infty}(\tilde{m}(\xi,t)-1)\mathrm{d}\xi}.
\end{aligned}
$$

Using the new space variable $y(x,t) := x - \int_x^{+\infty}(\tilde{m}(\xi,t)-1)\mathrm{d}\xi$ we have introduced in (2.14), the above equations yield

$$
\frac{\tilde{\mu}_1(x,t)}{\tilde{\mu}_2(x,t)} = \mathrm{e}^{\int_x^{+\infty}(\tilde{m}(\xi,t)-1)\mathrm{d}\xi} = \mathrm{e}^{x-y(x,t)} \tag{2.47}
$$

and thus

$$
x = y(x,t) + \ln\frac{\tilde{\mu}_1(x,t)}{\tilde{\mu}_2(x,t)}. \tag{2.48}
$$

Also notice that

$$
\tilde{\mu}_1(x,t)\tilde{\mu}_2(x,t) = 1. \tag{2.49}
$$

Proposition 2.2.3. *Let $\hat{M}(y,t,\mu)$ be the solution of the RH problem* (2.41)–(2.44) *whose data are associated with the initial data $u_0(x)$. Define $\hat{\mu}_1(y,t) := \hat{M}_{11}(y,t,\mathrm{i}) + \hat{M}_{21}(y,t,\mathrm{i})$ and $\hat{\mu}_2(y,t) := \hat{M}_{12}(y,t,\mathrm{i}) + \hat{M}_{22}(y,t,\mathrm{i})$. The solution $u(x,t)$ of the Cauchy problem* (2.1) *has x-derivative given by the parametric representation*

$$
u_x(x+t,t) = \frac{1}{2}\partial_t \ln\frac{\hat{\mu}_1(y,t)}{\hat{\mu}_2(y,t)}, \tag{2.50a}
$$

$$
x(y,t) = y + \ln\frac{\hat{\mu}_1(y,t)}{\hat{\mu}_2(y,t)}. \tag{2.50b}
$$

Proof. In what follows we will express $\tilde{u}_x$ in the variables (y,t). To express a function $\tilde{f}(x,t)$ in (y,t) we will use the notation $\hat{f}(y,t) := \tilde{f}(x(y,t),t)$, e.g.,

$$\hat{u}(y,t) := \tilde{u}(x(y,t),t),\ \hat{u}_x(y,t) := \tilde{u}_x(x(y,t),t),$$

$$\hat{m}(y,t) := \tilde{m}(x(y,t),t),\ \hat{\omega}(y,t) := \tilde{\omega}(x(y,t),t).$$

Differentiation of the identity $x(y(x,t),t) = x$ w.r.t. t gives

$$\partial_t\left(x(y(x,t),t)\right) = x_y(y,t)y_t(x,t) + x_t(y,t) = 0. \tag{2.51}$$

From (2.14) it follows that

$$x_y(y,t) = \frac{1}{\hat{m}(y,t)} \tag{2.52}$$

and $y_t(x,t) = -\int_x^{+\infty} \tilde{m}_t(\xi,t)\mathrm{d}\xi$. By (2.3a), the latter equality becomes

$$y_t(x,t) = \int_x^{+\infty} \left(\tilde{\omega}\tilde{m}\right)_\xi(\xi,t)\mathrm{d}\xi = -\tilde{\omega}\tilde{m}(x,t).$$

Substituting this and (2.52) into (2.51) we obtain

$$x_t(y,t) = \hat{\omega}(y,t). \tag{2.53}$$

Further, differentiating (2.53) w.r.t. y we get

$$x_{ty}(y,t) = \hat{\omega}_x x_y(y,t) = 2\hat{u}_x(\hat{u} - \hat{u}_{xx} + 1)\frac{1}{\hat{m}}(y,t) = 2\hat{u}_x(y,t). \tag{2.54}$$

Therefore, we arrive at a parametric representation of $\tilde{u}_x(x,t)$:

$$\begin{aligned}\tilde{u}_x(x(y,t),t) \equiv \hat{u}_x(y,t) = \frac{1}{2}\partial_t x(y,t),\\ x(y,t) = y + \frac{\ln\hat{\mu}_1(y,t)}{\ln\hat{\mu}_2(y,t)},\end{aligned}$$

which yields (2.50). For the direct determination of u from the solution of the RH problem, see Remark 2.3.8 below. □

Remark 2.2.4. In the case of the Camassa–Holm equation, the relation between the new and original space variables (2.48) is the same whereas the derivative (2.53) gives directly the solution u of the nonlinear equation (in the (y,t) variables) in question.

2.3 From a solution of the RH problem to a solution of the mCH equation

Henceforth we consider a RH problem (2.41)–(2.44) with data not necessarily related to initial data for the mCH equation. This section aims to show that starting from the solution $\hat{M}(y,t,\mu)$ of such a RH problem one can construct a solution (at least, locally) of the mCH equation by manipulations similar to those of Section 2.2.5. For this purpose, we will show that starting from $\hat{M}(y,t,\mu)$ one can define 2×2-matrix valued functions $\hat{\Psi}(y,t,\mu)$ satisfying Lax pair equations

$$\begin{aligned}\hat{\Psi}_y &= \hat{\hat{U}}\hat{\Psi},\\ \hat{\Psi}_t &= \hat{\hat{V}}\hat{\Psi},\end{aligned}$$

whose coefficients $\hat{\hat{U}}$ and $\hat{\hat{V}}$ are obtained from $\hat{M}(y,t,\mu)$, and whose compatibility condition is the mCH equation (written in the (y,t) variables).

First, let us reformulate the original Lax pair equations (2.8) in the (y,t) variables. Introducing $\hat{\Psi}(y,t)=\hat{\Phi}(x(y,t),t)$ and taking into account (2.53) and (2.52), the Lax pair (2.8) in the variables (y,t) takes the form:

$$\begin{aligned}\hat{\Psi}_y + \mathrm{i}k\sigma_3\hat{\Psi} &= \frac{\tilde{m}-1}{\tilde{m}}\frac{\lambda}{4\mathrm{i}k}\begin{pmatrix}\frac{1}{\lambda} & 1\\ -1 & -\frac{1}{\lambda}\end{pmatrix}\hat{\Psi},\\ \hat{\Psi}_t - \frac{2\mathrm{i}k}{\lambda^2}\sigma_3\hat{\Psi} &= \left(\frac{\tilde{u}}{2\mathrm{i}k}\begin{pmatrix}-1 & -\frac{1}{\lambda}\\ \frac{1}{\lambda} & 1\end{pmatrix} + \frac{\tilde{u}_x}{\lambda}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)\hat{\Psi},\end{aligned}$$

where $k := -\frac{\mathrm{i}}{2}\sqrt{1-\lambda^2}$.

Consequently, using μ as spectral parameter (see (2.15)), we have

Proposition 2.3.1. *The Lax pair* (2.8) *has the following form in the variables* (y,t,μ)*:*

$$\begin{aligned}\hat{\Psi}_y + \frac{\mathrm{i}(\mu^2-1)}{4\mu}\sigma_3\hat{\Psi} &= \widetilde{U}\hat{\Psi},\\ \hat{\Psi}_t - \frac{2\mathrm{i}(\mu^2-1)\mu}{(\mu^2+1)^2}\sigma_3\hat{\Psi} &= \widetilde{V}\hat{\Psi},\end{aligned} \tag{2.55}$$

where

$$\widetilde{U}(y,t,\mu) = \frac{\mathrm{i}f(y,t)}{\mu-1}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} + \frac{\mathrm{i}f(y,t)}{\mu+1}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} + \mathrm{i}f(y,t)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{2.56a}$$

$$\begin{aligned}\widetilde{V}(y,t,\mu) = &\frac{\mathrm{i}q(y,t)}{\mu-1}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} + \frac{\mathrm{i}q(y,t)}{\mu+1}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\\ &+ \frac{1}{\mu-\mathrm{i}}\begin{pmatrix}0 & g_1(y,t)\\ g_2(y,t) & 0\end{pmatrix} + \frac{1}{\mu+\mathrm{i}}\begin{pmatrix}0 & g_2(y,t)\\ g_1(y,t) & 0\end{pmatrix},\end{aligned} \tag{2.56b}$$

with f, q, g_1, and g_2 as follows:

$$f = -\frac{\hat{m}-1}{2\hat{m}}, \quad q = \hat{u}, \quad g_1 = -\hat{u} - \hat{u}_x, \quad g_2 = \hat{u} - \hat{u}_x. \tag{2.57}$$

Our goal in this section is to show that giving a solution $\hat{M}(y,t,\mu)$ to the RH problem (2.41)–(2.44), where the data $r(\mu)$ for $\mu \in \mathbb{R}$, $c \in \mathbb{R}$, and $\{\mu_j, \rho_j\}_1^N$ are not a priori associated with some initial data $u_0(x)$, one can "extract" from $\hat{M}(y,t,\mu)$ a solution to the mCH equation. The idea is as follows:

(a) Starting from $\hat{M}(y,t,\mu)$, define $\hat{\Psi}(y,t,\mu) = \hat{M}(y,t,\mu)\mathrm{e}^{-\hat{p}(y,t,\mu)\sigma_3}$ and show that $\hat{\Psi}(y,t,\mu)$ satisfies the system of differential equations:

$$\begin{aligned}\hat{\Psi}_y &= \hat{\hat{U}}\hat{\Psi},\\ \hat{\Psi}_t &= \hat{\hat{V}}\hat{\Psi},\end{aligned} \tag{2.58}$$

where $\hat{\hat{U}}$ and $\hat{\hat{V}}$ have the same (rational) dependence on μ as in (2.55) and (2.56), with coefficients given in terms of $\hat{M}(y,t,\mu)$ evaluated at appropriate values of μ.

(b) Show that the compatibility condition for (2.58), which is the equality $\hat{\hat{U}}_t - \hat{\hat{V}}_y + [\hat{\hat{U}}, \hat{\hat{V}}] = 0$, reduces to the mCH equation.

Proposition 2.3.2. *Let $\hat{M}(y,t,\mu)$ be the solution of the RH problem* (2.41)–(2.44). *Define*

$$\hat{\Psi}(y,t,\mu) := \hat{M}(y,t,\mu)\mathrm{e}^{-\hat{p}(y,t,\mu)\sigma_3}, \tag{2.59}$$

where $\hat{p}(y,t,\mu) := -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\left(-y+\frac{8\mu^2}{(\mu^2+1)^2}t\right)$. *Then $\hat{\Psi}(y,t,\mu)$ satisfies the differential equation*

$$\hat{\Psi}_y = \hat{\hat{U}}\hat{\Psi}$$

with $\hat{\hat{U}} = -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\sigma_3 + \widetilde{U}$, *where $\widetilde{U}$ is as in* (2.56a) *with f given by*

$$f(y,t) := -\frac{\eta(y,t)}{2},$$

$\eta(y,t)$ being extracted from the large μ expansion of $\hat{M}(y,t,\mu)$:

$$\hat{M}(y,t,\mu) = I + \frac{1}{\mu}\begin{pmatrix}\xi(y,t) & \eta(y,t)\\ \eta(y,t) & -\xi(y,t)\end{pmatrix} + \mathrm{O}(\mu^{-2}), \qquad \mu\to\infty.$$

Proof. First, notice that $\hat{\Psi}(y,t,\mu)$ satisfies the jump condition

$$\hat{\Psi}_-(y,t,\mu) = \hat{\Psi}_+(y,t,\mu)J_0(\mu)$$

with the jump matrix J_0 independent of y. Hence, $\hat{\Psi}_y(y,t,\mu)$ satisfies the same jump condition. Consequently, $\hat{\Psi}_y\hat{\Psi}^{-1} = \hat{M}_y\hat{M}^{-1} - \hat{p}_y\hat{M}\sigma_3\hat{M}^{-1}$ has no jump and thus it is a meromorphic function, with possible singularities at $\mu=\infty$, $\mu=0$, and $\mu=\pm1$. Let us evaluate $\hat{\Psi}_y\hat{\Psi}^{-1}$ near these points.

(i) As $\mu\to\infty$, we have $\hat{p}_y = \frac{\mathrm{i}\mu}{4} + \mathrm{O}(\mu^{-1})$ and thus

$$\hat{\Psi}_y\hat{\Psi}^{-1} = -\frac{\mathrm{i}\mu}{4}\sigma_3 - \frac{\mathrm{i}}{4}[\hat{M}^{(\infty)},\sigma_3] + \mathrm{O}(\mu^{-1}),$$

where $\hat{M}^{(\infty)} \equiv \hat{M}^{(\infty)}(y,t)$ comes from the large μ asymptotics of $\hat{M}$:

$$\hat{M} = I + \frac{\hat{M}^{(\infty)}}{\mu} + \mathrm{O}(\mu^{-2}), \qquad \mu\to\infty.$$

Symmetries (2.43) imply that $\sigma_2\hat{M}^{(\infty)}\sigma_2 = -\hat{M}^{(\infty)}$ and $\sigma_1\hat{M}^{(\infty)}\sigma_1 = \overline{\hat{M}^{(\infty)}}$, so that

$$\hat{M}^{(\infty)} = \begin{pmatrix}\xi & \eta\\ \eta & -\xi\end{pmatrix}$$

with some $\xi(y,t) \in \mathrm{i}\mathbb{R}$ and $\eta(y,t) \in \mathbb{R}$. Consequently,

$$\hat{\Psi}_y \hat{\Psi}^{-1} = -\frac{\mathrm{i}\mu}{4}\sigma_3 - \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix} + \mathrm{O}(\mu^{-1}), \qquad \mu \to \infty. \tag{2.60}$$

Then, by symmetry,

$$\hat{\Psi}_y \hat{\Psi}^{-1} = \frac{\mathrm{i}}{4\mu}\sigma_3 + \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix} + \mathrm{O}(\mu), \qquad \mu \to 0. \tag{2.61}$$

(ii) Pushing the expansion (2.44a) of $\hat{M}(\mu)$ a step further, and proceeding as in Section 2.2.4 to get (2.45) we have

$$\hat{\Psi}_y \hat{\Psi}^{-1} = \frac{\mathrm{i}\beta_1}{\mu - 1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \mathrm{O}(1), \qquad \mu \to 1, \tag{2.62}$$

with some $\beta_1(y,t) \in \mathbb{R}$. By symmetry,

$$\hat{\Psi}_y \hat{\Psi}^{-1} = \frac{\mathrm{i}\beta_1}{\mu + 1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \mathrm{O}(1), \qquad \mu \to -1. \tag{2.63}$$

Combining (2.60), (2.61), (2.62), and (2.63), we obtain that the function

$$\hat{\Psi}_y \hat{\Psi}^{-1} + \frac{\mathrm{i}(\mu^2 - 1)}{4\mu}\sigma_3 - \frac{\mathrm{i}\beta_1}{\mu - 1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} - \frac{\mathrm{i}\beta_1}{\mu + 1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}$$

is holomorphic in the whole complex μ-plane and, moreover, vanishes as $\mu \to \infty$. Then, by Liouville's theorem, it vanishes identically.

Further, again by symmetry, $\hat{M}(y,t,\mathrm{i})$ is diagonal (see Remark 2.2.1), which implies that the following sum is diagonal as well:

$$\frac{\mathrm{i}\beta_1}{\mathrm{i} - 1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \frac{\mathrm{i}\beta_1}{\mathrm{i} + 1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} - \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}.$$

It follows that $\frac{\eta}{2} = -\beta_1$, and thus we arrive at the equality $\hat{\Psi}_y = \hat{\hat{U}}\hat{\Psi}$ with $\hat{\hat{U}} = -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\sigma_3 + \widetilde{U}$, where $\widetilde{U}$ is as in (2.56a) with $f = \beta_1$. □

Proposition 2.3.3. *The function $\hat{\Psi}(y,t,\mu)$ defined by (2.59) satisfies the differential equation*

$$\hat{\Psi}_t = \hat{\hat{V}}\hat{\Psi} \tag{2.64}$$

with $\hat{\hat{V}} = \frac{2\mathrm{i}(\mu^2-1)\mu}{(\mu^2+1)^2}\sigma_3 + \widetilde{V}$, where $\widetilde{V}$ is as in (2.56b) with coefficients q, g_1, and g_2 determined by evaluating $\hat{M}(y,t,\mu)$ as $\mu \to 1$ and $\mu \to \mathrm{i}$.

Proof. Similarly to Proposition 2.3.2, we notice that $\hat{\Psi}_t\hat{\Psi}^{-1} = \hat{M}_t\hat{M}^{-1} - \hat{p}_t\hat{M}\sigma_3\hat{M}^{-1}$ has no jump and thus it is a meromorphic function, with possible singularities at $\mu = \infty$, $\mu = 0$, $\mu = \pm 1$, and $\mu = \pm\mathrm{i}$, the latter being due to the singularity of $\hat{p}_t$ at $\mu = \pm\mathrm{i}$:

$$\hat{p}_t(\mu) = \pm\frac{1}{(\mu \mp \mathrm{i})^2} - \frac{\mathrm{i}}{\mu \mp \mathrm{i}} + \mathrm{O}(1), \qquad \mu \to \pm\mathrm{i}. \tag{2.65}$$

Evaluating $\hat{\Psi}_t\hat{\Psi}^{-1}$ near these points, we have the following.

(i) As $\mu \to \infty$, we have $\hat{p}_t(\mu) = \mathrm{O}(\mu^{-1})$ and thus

$$\hat{\Psi}_t\hat{\Psi}^{-1}(\mu) = \mathrm{O}(\mu^{-1}), \qquad \mu \to \infty. \tag{2.66}$$

Then, by symmetry,

$$\hat{\Psi}_t\hat{\Psi}^{-1}(\mu) = \mathrm{O}(\mu), \qquad \mu \to 0. \tag{2.67}$$

(ii) Expanding $\hat{M}(\mu)$ at $\mu = 1$, and proceeding as above to get (2.62), we have

$$\hat{\Psi}_t\hat{\Psi}^{-1}(\mu) = \frac{\mathrm{i}\beta_2}{\mu - 1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \mathrm{O}(1), \qquad \mu \to 1, \tag{2.68}$$

with some $\beta_2(y,t) \in \mathbb{R}$. By symmetry,

$$\hat{\Psi}_t\hat{\Psi}^{-1}(\mu) = \frac{\mathrm{i}\beta_2}{\mu + 1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \mathrm{O}(1), \qquad \mu \to -1. \tag{2.69}$$

(iii) Evaluating $\hat{M}(\mu)$ as $\mu \to \mathrm{i}$, we first notice that, due to symmetries,

$$\hat{M}(\mu) = \begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix} + \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}(\mu - \mathrm{i}) + \mathrm{O}((\mu - \mathrm{i})^2), \qquad \mu \to \mathrm{i}, \tag{2.70}$$

with some $a_j \equiv a_j(y,t)$, $j = 1,2,3$. Taking into account (2.65), we have

$$\hat{\Psi}_t\hat{\Psi}^{-1}(\mu) = -\frac{1}{(\mu-\mathrm{i})^2}\sigma_3 + \frac{1}{\mu-\mathrm{i}}\left(\mathrm{i}\sigma_3 + \begin{pmatrix} 0 & 2a_2a_1 \\ -2a_3a_1^{-1} & 0 \end{pmatrix}\right) + \mathrm{O}(1), \qquad \mu \to \mathrm{i}. \tag{2.71}$$

Then, by symmetry,

$$\hat{\Psi}_t\hat{\Psi}^{-1}(\mu) = \frac{1}{(\mu+\mathrm{i})^2}\sigma_3 + \frac{1}{\mu+\mathrm{i}}\left(\mathrm{i}\sigma_3 + \begin{pmatrix} 0 & -2a_3a_1^{-1} \\ 2a_2a_1 & 0 \end{pmatrix}\right) + \mathrm{O}(1), \qquad \mu \to -\mathrm{i}. \tag{2.72}$$

Combining (2.66), (2.68), and (2.69), (2.71), and (2.72), we obtain that the function

$$\hat{\Psi}_t\hat{\Psi}^{-1}(\mu) - \frac{2\mathrm{i}(\mu^2-1)\mu}{(\mu^2+1)^2}\sigma_3 - \frac{1}{\mu-1}\mathrm{i}\beta_2\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} - \frac{1}{\mu+1}\mathrm{i}\beta_2\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$
$$- \frac{1}{\mu-\mathrm{i}}\begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix} - \frac{1}{\mu+\mathrm{i}}\begin{pmatrix} 0 & \gamma_2 \\ \gamma_1 & 0 \end{pmatrix}$$

with $\gamma_1 = 2a_2a_1$ and $\gamma_2 = -2a_3a_1^{-1}$ is holomorphic in the whole complex μ-plane and, moreover, vanishes as $\mu \to \infty$. Then, by Liouville's theorem, it vanishes identically. Thus we arrive at the equality $\hat{\Psi}_t = \hat{\hat{V}}\hat{\Psi}$ with $\hat{\hat{V}}(\mu) = \frac{2\mathrm{i}(\mu^2-1)\mu}{(\mu^2+1)^2}\sigma_3 + \widetilde{V}(\mu)$, where $\widetilde{V}(\mu)$ is as in (2.56b) with $q = \beta_2$, $g_1 = \gamma_1$, and $g_2 = \gamma_2$. □

The next step is to demonstrate that the compatibility condition

$$\hat{\hat{U}}_t - \hat{\hat{V}}_y + [\hat{\hat{U}}, \hat{\hat{V}}] = 0 \tag{2.73}$$

yields the mCH equation in the (y,t) variables, which is as follows:

Proposition 2.3.4. *The mCH equation* (2.3a) *in the* (y,t) *variables reads as follows:*

$$(\hat{m}^{-1})_t(y,t) = 2\hat{u}_x(y,t), \tag{2.74a}$$
$$\hat{m}(y,t) := \hat{u}(y,t) - \hat{u}_{xx}(y,t) + 1, \tag{2.74b}$$

where $\hat{f}(y,t) := \tilde{f}(x(y,t),t)$ *for any function* $\tilde{f}(x,t)$ *and* $x_y(y,t) = \hat{m}^{-1}(y,t)$.

Proof. Substituting $\tilde{m}_t = -(\tilde{\omega}\tilde{m})_x$ from (2.3a) and $x_t = \hat{\omega}$ from (2.53) into the equality

$$\hat{m}_t(y,t) = \tilde{m}_x(x(y,t),t)x_t(y,t) + \tilde{m}_t(x(y,t),t)$$

and using that $\tilde{\omega}_x = 2\tilde{m}\tilde{u}_x$ we get

$$\begin{aligned}\hat{m}_t(y,t) &= \tilde{m}_x(x(y,t),t)\hat{\omega}(y,t) - \tilde{m}_x(x(y,t),t)\hat{\omega}(y,t) - 2\tilde{m}^2(x(y,t),t)\hat{u}_x(y,t)\\ &= -2\hat{u}_x\hat{m}^2(y,t)\end{aligned}$$

and thus (2.74a) follows. □

Remark 2.3.5. Notice that (2.74b) can be written as

$$\hat{m}(y,t) = \hat{u}(y,t) - (\hat{u}_x)_y\,(y,t)\hat{m}(y,t) + 1. \tag{2.75}$$

Now, evaluating the compatibility equation (2.73) at the singular points for $\hat{\hat{U}}$ and $\hat{\hat{V}}$, we get algebraic and differential equations amongst the coefficients of $\hat{\hat{U}}$ and $\hat{\hat{V}}$, i.e., amongst β_1, β_2, γ_1, and γ_2, that can be reduced to (2.74a).

Proposition 2.3.6. *Let $\beta_1(y,t)$, $\beta_2(y,t)$, $\gamma_1(y,t)$, and $\gamma_2(y,t)$ be the functions determined in terms of $\hat{M}(y,t,\mu)$ as in Propositions 2.3.2 and 2.3.3. Then they satisfy the following equations:*

$$\beta_{1t} + \frac{\gamma_1+\gamma_2}{2} = 0; \tag{2.76a}$$

$$\beta_2 - \frac{\gamma_2-\gamma_1}{2} = 0; \tag{2.76b}$$

$$(\gamma_1-\gamma_2)_y - (1+2\beta_1)(\gamma_1+\gamma_2) = 0; \tag{2.76c}$$

$$(\gamma_2+\gamma_1)_y + 4\beta_1 - (1+2\beta_1)(\gamma_1-\gamma_2) = 0. \tag{2.76d}$$

Proof. Recall β_1 and β_2 are given by (2.62) and (2.68), respectively. Moreover, $\gamma_1 := 2a_2a_1$ and $\gamma_2 := -2a_3a_1^{-1}$, where a_1, a_2, and a_3 are defined by (2.70).

(i) Evaluating the l.h.s. of (2.73) as $\mu \to \infty$, the main term (of order O(1)) is

$$\left(\beta_{1t} + \frac{\gamma_1+\gamma_2}{2}\right)\sigma_2,$$

from which (2.76a) follows.

(ii) Evaluating the l.h.s. of (2.73) as $\mu \to 0$, the main term (of order $\mathrm{O}(\mu^{-1})$) is

$$-\frac{1}{\mu}\left(\beta_2 + \frac{\gamma_1 - \gamma_2}{2}\right)\sigma_1,$$

from which (2.76b) follows.

(iii) Evaluating the l.h.s. of (2.73) as $\mu \to 1$, the diagonal part of the main term (of order $\mathrm{O}((\mu-1)^{-1})$) is

$$\frac{\mathrm{i}}{\mu - 1}\left(\beta_{1t} - \beta_{2y} - \beta_1(\gamma_1 + \gamma_2)\right)\sigma_3,$$

from which (2.76c) follows, taking into account (2.76a) and (2.76b).

(iv) Evaluating the l.h.s. of (2.73) as $\mu \to \mathrm{i}$, the main term (of order $\mathrm{O}((\mu - \mathrm{i})^{-1})$) is

$$\frac{1}{\mu - \mathrm{i}}\left[\begin{pmatrix} 0 & -\gamma_{1y} \\ -\gamma_{2y} & 0 \end{pmatrix} + (1 + 2\beta_1)\begin{pmatrix} 0 & \gamma_1 \\ -\gamma_2 & 0 \end{pmatrix} - 2\beta_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right],$$

from which (2.76d) follows. □

Proposition 2.3.7. *Let $\hat{m}(y,t)$, $\hat{u}(y,t)$, and $x(y,t)$ be defined in terms of β_1, β_2, γ_1, and γ_2 as follows:*

$$\hat{m} = (1 + 2\beta_1)^{-1}, \quad \hat{u} = \beta_2 = \frac{\gamma_2 - \gamma_1}{2}, \quad x_y = 1 + 2\beta_1. \tag{2.77}$$

Then the four equations (2.76) *reduce to* (2.74a) *and* (2.75).

Proof. Indeed, defining $\hat{u}$ and $x(y,t)$ as prescribed in (2.77), equation (2.76c) implies that $\hat{u}_x = \hat{u}_y x_y^{-1}$ can be expressed as

$$\hat{u}_x = -\frac{\gamma_1 + \gamma_2}{2}.$$

Then, taking into account the definition of $\hat{m}$ in (2.77), equation (2.76a) takes the form of the equation (2.74a). Finally, using the notations introduced above, equation (2.75) can be written as

$$\frac{1}{1 + 2\beta_1} = \frac{\gamma_2 - \gamma_1}{2} + \frac{(\gamma_1 + \gamma_2)_y}{2}\frac{1}{1 + 2\beta_1} + 1,$$

which is just equation (2.76d). □

Remark 2.3.8. Formulas $\hat{u} = \frac{\gamma_2 - \gamma_1}{2}$ and $\hat{u}_x = -\frac{\gamma_1 + \gamma_2}{2}$ provide an alternative way to obtain $\hat{u}$ as well as $\hat{u}_x$ from the solution $\hat{M}$ of the RH problem. Indeed, according to Proposition 2.3.3, $\hat{u}$ and $\hat{u}_x$ (as functions of (y,t)) can be obtained using the coefficients $a_j(y,t)$ (see (2.70)) of the development of $\hat{M}(y,t,k)$ as $\mu \to \mathrm{i}$ (thus avoiding the differentiations used in Section 2.2.5):

$$\hat{u}(y,t) = -a_2 a_1 - a_3 a_1^{-1}, \qquad \hat{u}_x(y,t) = -a_2 a_1 + a_3 a_1^{-1}, \tag{2.78}$$

where $a_j(y,t)$ are determined by (2.70). Recall also the representation for $\hat{m}$ in terms of $\hat{M}$ evaluated as $\mu \to \infty$, see Proposition 2.3.2:

$$\begin{aligned} \hat{m}(y,t) &= \frac{1}{1 + 2\beta_1(y,t)} = \frac{1}{1 - \eta(y,t)}, \\ \eta(y,t) &:= \lim_{\mu \to \infty} \mu \hat{M}_{12}(y,t,\mu). \end{aligned} \tag{2.79}$$

Considered together with the expression for the change of variables (2.50b), which can be written as (we indeed have $\hat{\mu}_1 = a_1$ and $\hat{\mu}_2 = a_1^{-1}$)

$$x(y,t) = y + 2 \ln a_1(y,t), \tag{2.80}$$

equations (2.78) and (2.79) give a parametric representation of the solution of the mCH equation (2.3a).

2.4 Solitons

In the Riemann–Hilbert variant of the inverse scattering transform method, pure soliton solutions can be obtained from the solutions of the RH problem assuming that the jump is trivial ($J \equiv I$), which reduces the construction to solving a system of linear algebraic equations generated by the residue conditions.

In order to construct the simplest, one-soliton solution, we consider the RH problem (2.41)–(2.44) with specific data, in particular $r(\mu) \equiv 0$, so that $\hat{J} \equiv I$. Regarding the other data, we require that $\hat{M}^{(1)}$ has a simple pole on the unit circle, at $\mu_1 = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in (0, \frac{\pi}{2})$. It follows that $\hat{M}^{(1)}$ has also a simple pole at $\mu_2 = -e^{-i\theta} = -\bar{\mu}_1 = -\mu_1^{-1}$. According to the symmetries (2.43) the coefficients $\hat{\varkappa}_j(y,t) = \rho_j e^{-2\hat{p}(y,t,\mu_j)}$, $j = 1,2$ in the residue conditions (2.42) must satisfy the relations $\hat{\varkappa}_1 = \overline{\hat{\varkappa}_2} = -\mu_1^{-2}\hat{\varkappa}_2$, that is, $\rho_1 = \overline{\rho_2} = -\mu_1^{-2}\rho_2$ which imply $\rho_1 = ie^{-i\theta}\hat{\delta}$ for some $\hat{\delta} \in \mathbb{R}$. Further we denote $\hat{\varkappa}(y,t) := \hat{\varkappa}_1(y,t)$ and $\rho := \rho_1 \in \mathbb{C}$. So ρ satisfies

$$\bar{\rho} = -e^{2i\theta}\rho. \tag{2.81}$$

Thus we arrive at the following Riemann–Hilbert problem:

Soliton RH problem. Given $\theta \in (0, \frac{\pi}{2})$ and $\hat{\delta} \neq 0$ two real parameters, together with $c \in \mathbb{R}$, find a piece-wise (w.r.t. $\mathbb{R}$) meromorphic, 2×2-matrix valued function $\hat{M}(y,t,\mu)$ satisfying the following conditions:

- The jump condition $\hat{J} \equiv I$ across $\mathbb{R}$.
- The residue conditions (2.42) at $\mu_1 = e^{i\theta}$ and $\bar{\mu}_1 = e^{-i\theta}$:

$$\operatorname{Res}_{e^{i\theta}} \hat{M}^{(1)}(y,t,\mu) = \frac{1}{\hat{\varkappa}(y,t)}\hat{M}^{(2)}(y,t,e^{i\theta}), \tag{2.82a}$$

$$\operatorname{Res}_{e^{-i\theta}} \hat{M}^{(2)}(y,t,\mu) = \frac{1}{\overline{\hat{\varkappa}(y,t)}}\hat{M}^{(1)}(y,t,e^{-i\theta}), \tag{2.82b}$$

  where $\hat{\varkappa}(y,t) = ie^{-i\theta}\hat{\delta}e^{-2\hat{p}(y,t,e^{i\theta})}$ with $\hat{p}(y,t,e^{i\theta}) = \frac{\sin\theta}{2}(-y + \frac{2}{\cos^2\theta}t)$, and $\overline{\hat{\varkappa}} = -e^{2i\theta}\hat{\varkappa}$.

- The normalization condition $\hat{M}(y,t,\infty) = I$.
- The symmetries (2.43).
- The singularity conditions (2.44) at $\mu = \pm 1$.

The residue conditions at μ_2 and $\bar{\mu}_2$ follow from (2.82) using the symmetries (2.43):

$$\operatorname{Res}_{-e^{-i\theta}} \hat{M}^{(1)}(y,t,\mu) = \frac{1}{\overline{\hat{\varkappa}(y,t)}}\hat{M}^{(2)}(y,t,-e^{-i\theta}), \tag{2.83a}$$

$$\operatorname{Res}_{-e^{i\theta}} \hat{M}^{(2)}(y,t,\mu) = \frac{1}{\hat{\varkappa}(y,t)}\hat{M}^{(1)}(y,t,-e^{i\theta}). \tag{2.83b}$$

To summarize, the soliton RH problem of parameters $(\theta,\hat\delta)$ is the RH problem (2.41)–(2.44) with trivial jump condition and residue conditions data $\{\mu_j,\rho_j\}_1^2$ where $\mu_1=-\bar\mu_2=\mathrm{e}^{\mathrm{i}\theta}$ and $\rho_1=\bar\rho_2=\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\hat\delta$.

Remark 2.4.1. Assume that the data of the soliton RH problem are associated with the spectral data corresponding to some initial data $u_0(x)$, see Section 2.1.3. In particular, $b(\mu)\equiv 0$ and $a(\mu)$ has two zeros in $\mathbb{C}^+$, each of multiplicity one, $\mu_1=\mathrm{e}^{\mathrm{i}\theta}$ and $\mu_2=-\mathrm{e}^{-\mathrm{i}\theta}$, both on the unit circle. The coefficient $\hat\varkappa$ in the residue condition for $M^{(1)}$ at μ_1 is given by $\hat\varkappa=\rho\,\mathrm{e}^{-2\hat p(y,t,\mathrm{e}^{\mathrm{i}\theta})}$ with $\rho=\dot a(\mathrm{e}^{\mathrm{i}\theta})\delta$, where the constant δ relates two Jost functions: $\hat\Phi_+^{(2)}(x,t,\mu_1)=\delta\hat\Phi_-^{(1)}(x,t,\mu_1)$. Using the symmetries (2.21) and the relation $\bar\mu_1=\mu_1^{-1}$ we find that $\sigma_1\hat\Phi_\pm(\mathrm{e}^{-\mathrm{i}\theta})\sigma_1=\overline{\hat\Phi_\pm(\mathrm{e}^{\mathrm{i}\theta})}=\hat\Phi_\pm(\mathrm{e}^{\mathrm{i}\theta})$ and thus δ is real. Moreover, from the symmetry relation $a(\mu^{-1})=\overline{a(\bar\mu)}$ it follows that $\overline{\dot a(\mathrm{e}^{\mathrm{i}\theta})}=-\mathrm{e}^{2\mathrm{i}\theta}\dot a(\mathrm{e}^{\mathrm{i}\theta})$, and thus $\rho=\dot a(\mathrm{e}^{\mathrm{i}\theta})\delta$ satisfies (2.81). To conclude, in that case, $\hat\delta=-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\dot a(\mathrm{e}^{\mathrm{i}\theta})\delta$.

Proposition 2.4.2. *Let $\theta\in(0,\frac{\pi}{2})$ and $\hat\delta\neq 0$ be two real parameters. Then, the soliton RH problem of parameters $(\theta,\hat\delta)$ has a solution $\hat M\equiv\hat M_{\theta,\hat\delta}$ provided that $c=1$:*

$$\hat M(y,t,\mu)=I+\frac{\mathrm{i}}{2}\frac{\hat\alpha_+(y,t)}{\mu-1}\begin{pmatrix}-1&1\\-1&1\end{pmatrix}-\frac{\mathrm{i}}{2}\frac{\hat\alpha_+(y,t)}{\mu+1}\begin{pmatrix}1&1\\-1&-1\end{pmatrix}$$
$$+\begin{pmatrix}\frac{\mathrm{i}\hat\kappa_1(y,t)\mathrm{e}^{\mathrm{i}\theta}}{\mu-\mathrm{e}^{\mathrm{i}\theta}}+\frac{\mathrm{i}\hat\kappa_1(y,t)\mathrm{e}^{-\mathrm{i}\theta}}{\mu+\mathrm{e}^{-\mathrm{i}\theta}} & \frac{-\mathrm{i}\hat\kappa_2(y,t)\mathrm{e}^{-\mathrm{i}\theta}}{\mu-\mathrm{e}^{-\mathrm{i}\theta}}+\frac{\mathrm{i}\hat\kappa_2(y,t)\mathrm{e}^{\mathrm{i}\theta}}{\mu+\mathrm{e}^{\mathrm{i}\theta}}\\ \frac{\mathrm{i}\hat\kappa_2(y,t)\mathrm{e}^{\mathrm{i}\theta}}{\mu-\mathrm{e}^{\mathrm{i}\theta}}+\frac{-\mathrm{i}\hat\kappa_2(y,t)\mathrm{e}^{-\mathrm{i}\theta}}{\mu+\mathrm{e}^{-\mathrm{i}\theta}} & \frac{-\mathrm{i}\hat\kappa_1(y,t)\mathrm{e}^{-\mathrm{i}\theta}}{\mu-\mathrm{e}^{-\mathrm{i}\theta}}+\frac{-\mathrm{i}\hat\kappa_1(y,t)\mathrm{e}^{\mathrm{i}\theta}}{\mu+\mathrm{e}^{\mathrm{i}\theta}}\end{pmatrix}, \tag{2.84}$$

where

$$\hat\kappa_2^{-1}(y,t)=-\hat{\hat\varkappa}(y,t)-\frac{\cos^2\theta}{4\hat{\hat\varkappa}(y,t)\sin^2\theta}-\frac{1}{\sin\theta}, \tag{2.85a}$$

$$\hat\kappa_1(y,t)=-\frac{\cos\theta}{2\hat{\hat\varkappa}(y,t)\sin\theta}\hat\kappa_2(y,t), \tag{2.85b}$$

$$\hat\alpha_+(y,t)=2\hat\kappa_2(y,t). \tag{2.85c}$$

Here,

$$\hat{\hat\varkappa}(y,t):=\hat\delta\,\mathrm{e}^{-2\hat p(y,t,\mathrm{e}^{\mathrm{i}\theta})}\quad \textit{with}\quad \hat p(y,t,\mathrm{e}^{\mathrm{i}\theta})=\frac{\sin\theta}{2}\left(-y+\frac{2}{\cos^2\theta}t\right). \tag{2.85d}$$

Proof. Since $\hat{M}(\mu) \equiv \hat{M}(y,t,\mu)$ is solution of the soliton RH problem whose jump condition is trivial, it is a rational function, whose pole structure is specified by the singularity conditions (2.44) at $\mu = \pm 1$ and by the residue conditions (2.82) at $\mu = \pm e^{\pm i\theta}$:

$$\hat{M}(\mu) = I + \frac{i}{2}\frac{\hat{\alpha}_+}{\mu - 1}\begin{pmatrix} -c & 1 \\ -c & 1 \end{pmatrix} - \frac{i}{2}\frac{\hat{\alpha}_+}{\mu + 1}\begin{pmatrix} c & 1 \\ -c & -1 \end{pmatrix} + \begin{pmatrix} \frac{c_1}{\mu - e^{i\theta}} + \frac{c_3}{\mu + e^{-i\theta}} & \frac{\tilde{c}_1}{\mu - e^{-i\theta}} + \frac{\tilde{c}_3}{\mu + e^{i\theta}} \\ \frac{c_2}{\mu - e^{i\theta}} + \frac{c_4}{\mu + e^{-i\theta}} & \frac{\tilde{c}_2}{\mu - e^{-i\theta}} + \frac{\tilde{c}_4}{\mu + e^{i\theta}} \end{pmatrix} \tag{2.86}$$

with some $\hat{\alpha}_+(y,t)$, $c_j(y,t)$, $\tilde{c}_j(y,t)$, and c. We will specify the coefficients using the symmetries (2.43). The symmetry $\hat{M}^{(1)}(-\mu) = \sigma_3\sigma_1\hat{M}^{(2)}(\mu)$ shows that $c = 1$, $\tilde{c}_1 = c_4$, $\tilde{c}_2 = -c_3$, $\tilde{c}_3 = c_2$, and $\tilde{c}_4 = -c_1$. On the other hand, the symmetry $\hat{M}^{(1)}(-\bar{\mu}) = \sigma_3\overline{\hat{M}^{(1)}(\mu)}$ shows that $c_3 = -\bar{c}_1$ and $c_4 = \bar{c}_2$. Thus (2.86) takes the form

$$\hat{M}(\mu) = I + \frac{i}{2}\frac{\hat{\alpha}_+}{\mu - 1}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} - \frac{i}{2}\frac{\hat{\alpha}_+}{\mu + 1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} \frac{c_1}{\mu - e^{i\theta}} + \frac{-\bar{c}_1}{\mu + e^{-i\theta}} & \frac{\bar{c}_2}{\mu - e^{-i\theta}} + \frac{c_2}{\mu + e^{i\theta}} \\ \frac{c_2}{\mu - e^{i\theta}} + \frac{\bar{c}_2}{\mu + e^{-i\theta}} & \frac{\bar{c}_1}{\mu - e^{-i\theta}} + \frac{-c_1}{\mu + e^{i\theta}} \end{pmatrix}.$$

The symmetry $\hat{M}^{(1)}(-\mu^{-1}) = \sigma_3\hat{M}^{(1)}(\mu)$ shows that $c_3 = c_1 e^{-2i\theta}$ and $c_4 = -c_2 e^{-2i\theta}$, so that $\bar{c}_j = -c_j e^{-2i\theta}$ for $j = 1, 2$, that is, $c_j(y,t) = ie^{i\theta}\hat{\kappa}_j(y,t)$ with $\hat{\kappa}_j(y,t) \in \mathbb{R}$. Thus we get (2.84).

Then, using $\hat{M}(0) = \sigma_1\hat{M}(\infty)\sigma_1 = I$, it follows that $\hat{\alpha}_+ = 2\hat{\kappa}_2$, that is, (2.85c). Introducing $\hat{\hat{\varkappa}}(y,t) := \hat{\delta}\, e^{-2\hat{p}(y,t,e^{i\theta})}$ so that $\hat{\varkappa}(y,t) = ie^{-i\theta}\hat{\hat{\varkappa}}(y,t)$ and substituting (2.84) into the residue condition (2.82a) at $e^{i\theta}$, we find (2.85b) on the first row and then (2.85a) on the second one. □

Remark 2.4.3. Assume that the data of our soliton RH problem are derived from the spectral data corresponding to some initial data $u_0(x)$, as in Remark 2.4.1. Then, it directly follows that $c = 1$. Since $b(\mu) \equiv 0$ we indeed have (see Remark 2.1.4 and (2.30)) $\rho = 0$, $b_1 = 0$, and $a_1^2 = 1$; thus $c = 1$.

According to Section 2.3, a solution of the soliton RH problem gives rise to a solution (at least, locally, in the (y,t) variables) of the mCH equation. Thus, Proposition 2.4.2 provides a family of one-soliton solutions parameterized by two real parameters $\theta \in (0, \frac{\pi}{2})$ and $\hat{\delta} \neq 0$.

Proposition 2.4.4. *The one-soliton solution* $\hat{u} \equiv \hat{u}_{\theta,\hat{\delta}}$ *of parameters* $(\theta, \hat{\delta})$ *has the following form in the* (y,t)*-scale:*

$$\hat{u}(y,t) = 4\tan^2\theta \, \frac{z^2(y,t) + 2\cos^2\theta \cdot z(y,t) + \cos^2\theta}{(z^2(y,t) + 2z(y,t) + \cos^2\theta)^2} z(y,t), \tag{2.87a}$$

where

$$z(y,t) = 2\hat{\delta}\sin\theta \, \mathrm{e}^{\sin\theta\left(y - \frac{2}{\cos^2\theta}t\right)}. \tag{2.87b}$$

Proof. Let $z(y,t)$ be defined by

$$z(y,t) := 2\hat{\varkappa}(y,t)\sin\theta. \tag{2.88}$$

Then, $z(y,t) = 2\hat{\delta}\sin\theta \, \mathrm{e}^{\sin\theta\left(y - \frac{2}{\cos^2\theta}t\right)}$. Thus, z is real-valued. Moreover, $z(y,t) > 0$ if $\hat{\delta} > 0$ and $z(y,t) < 0$ if $\hat{\delta} < 0$. Using (2.85a), (2.85b), and (2.88) we get the following expressions of $\hat{\kappa}_2$ and $\hat{\kappa}_1$:

$$\hat{\kappa}_2 = -\frac{2z\sin\theta}{z^2 + 2z + \cos^2\theta} \quad \text{and} \quad \hat{\kappa}_1 = -\frac{\cos\theta}{z}\hat{\kappa}_2 = \frac{2\sin\theta\cos\theta}{z^2 + 2z + \cos^2\theta}. \tag{2.89}$$

In order to obtain the formula for the soliton solution $\hat{u} \equiv \hat{u}(y,t)$, we use the relation

$$\hat{u} = -a_2 a_1 - a_3 a_1^{-1} \tag{2.90}$$

from (2.78). To compute $a_1 \equiv a_1(y,t)$ we observe that $a_1 = \hat{M}_{11}(\mathrm{i})$. We thus obtain

$$a_1 = 1 - \frac{\hat{\alpha}_+}{2} - \mathrm{i}\kappa_1 \frac{1 + \mathrm{e}^{2\mathrm{i}\theta}}{2(1 - \sin\theta)} = 1 - \hat{\kappa}_2 + \hat{\kappa}_1 \frac{\cos\theta}{1 - \sin\theta},$$

using the relation $\frac{\hat{\alpha}_+}{2} = \hat{\kappa}_2$ from (2.85c). Using the expressions of $\hat{\kappa}_1$ and $\hat{\kappa}_2$ from (2.89) we get

$$a_1 = \frac{z + 1 + \sin\theta}{z + 1 - \sin\theta}. \tag{2.91a}$$

To compute $a_2 \equiv a_2(y,t)$ and $a_3 \equiv a_3(y,t)$ we observe that $a_2 = \partial_\mu \hat{M}_{12}(\mathrm{i})$ and $a_3 = \partial_\mu \hat{M}_{21}(\mathrm{i})$. Using in addition the expression of $\hat{\kappa}_2$ from (2.89) we obtain

$$a_2 = \frac{\sin\theta}{1 + \sin\theta}\hat{\kappa}_2 = -\frac{2z\sin^2\theta}{(1 + \sin\theta)(z^2 + 2z + \cos^2\theta)}, \tag{2.91b}$$

$$a_3 = \frac{\sin\theta}{1 - \sin\theta}\hat{\kappa}_2 = -\frac{2z\sin^2\theta}{(1 - \sin\theta)(z^2 + 2z + \cos^2\theta)}. \tag{2.91c}$$

Then, substituting (2.91) into (2.90), we arrive at (2.87a). □

It follows from (2.87a) that if $\hat{\delta} > 0$, then for any $t \geq 0$, $\hat{u}(y,t)$ is a smooth function of y having a single peak and (exponentially) approaching 0 as $y \to \pm\infty$. On the other hand, if $\hat{\delta} < 0$, then $\tilde{u}$ has two singular points corresponding to $z = -1 \pm \sin\theta$.

Now let us discuss the change of variable $(y,t) \mapsto (x,t)$, which can be specified explicitly. This change of variable is associated with $\tilde{u}_{\theta,\hat{\delta}}$, that is, it is given by (2.50b) where $\hat{\mu}_1$ and $\hat{\mu}_2$ are defined in terms of $\hat{M} \equiv \hat{M}_{\theta,\hat{\delta}}$.

Proposition 2.4.5. *The change of variable $x(y,t)$ associated with the soliton $\tilde{u}_{\theta,\hat{\delta}}$ takes the following form:*

$$x(y,t) = y + 2\ln\frac{z(y,t) + 1 + \sin\theta}{z(y,t) + 1 - \sin\theta}. \tag{2.92}$$

Proof. As we have shown in Section 2.3, $x(y,t)$ can be given by (2.80):

$$x(y,t) = y + 2\ln a_1(y,t), \tag{2.93}$$

where $a_1(y,t) = \hat{M}_{11}(y,t,\mathrm{i})$. Substituting (2.91a) into (2.93), we obtain (2.92). □

Corollary 2.4.6. *Let $x(y,t)$ be the change of variable associated with $\tilde{u}_{\theta,\hat{\delta}}$. Its regularity properties are as follows.*

(a) *If $\hat{\delta} < 0$, then $x(\,\cdot\,,t)$ is singular: there exist values of y at which $x(y,t)$ is infinite.*

(b) *If $\hat{\delta} > 0$, then $x(\,\cdot\,,t)\colon \mathbb{R} \to \mathbb{R}$ is a regular map. Moreover, it has the following additional properties:*

   (i) *If $\theta \in (0, \frac{\pi}{3})$, then $x(\,\cdot\,,t)\colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism for any $t \geq 0$.*

   (ii) *If $\theta = \frac{\pi}{3}$, then $x(\,\cdot\,,t)\colon \mathbb{R} \to \mathbb{R}$ is a bijection, but the derivative of the inverse map has a singularity, and only one.*

   (iii) *If $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, then $x(\,\cdot\,,t)$ is not monotonous. More precisely, there are three intervals of monotonicity.*

The possible singularities of $x(y,t)$ are those for $\hat{u}(y,t)$: they correspond to $z = -1 \pm \sin\theta$. Therefore, if $\hat{\delta} > 0$, then $z(y,t) > 0$ and thus there are no singularities, whereas if $\hat{\delta} < 0$, then $x(y,t)$ is singular at those y where $z = -1 \pm \sin\theta$.

We now consider the case $\hat{\delta} > 0$ (and thus $z(y,t) > 0$). The derivative $\partial_y x(y,t) \equiv x_y(y,t)$ is given by

$$x_y(y,t) = R(z(y,t)), \text{ where } R(z) = \frac{z^2 + 2z\cos 2\theta + \cos^2\theta}{z^2 + 2z + \cos^2\theta}. \tag{2.94}$$

It follows that $R(0) = R(\infty) = 1$. Moreover, we have the following:

1) If $\theta \in (0, \frac{\pi}{3})$, then $R(z) > 0$ for all $z \geq 0$.

2) If $\theta = \frac{\pi}{3}$, then $z = \frac{1}{2}$ is a double zero of $R(z)$.

3) If $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, then

   a) $R(z) > 0$ for $z \in [0, -\cos 2\theta - \sqrt{-\sin\theta\cdot\sin 3\theta}) \cup (-\cos 2\theta + \sqrt{-\sin\theta\cdot\sin 3\theta}, +\infty)$,

   b) $R(z) < 0$ for $z \in (-\cos 2\theta - \sqrt{-\sin\theta\cdot\sin 3\theta}, -\cos 2\theta + \sqrt{-\sin\theta\cdot\sin 3\theta})$.

It follows that for $\theta \in (0, \frac{\pi}{3})$ the solution is smooth (both in the (y,t) and the (x,t) variables). On the other hand, for $\theta = \frac{\pi}{3}$ the solution $\tilde{u}(x,t) = \hat{u}(y(x,t),t)$ is given in parametric form by

$$\hat{u}(y,t) = 48z(y,t)\frac{4z^2(y,t) + 2z(y,t) + 1}{(4z^2(y,t) + 8z(y,t) + 1)^2}, \tag{2.95a}$$

$$z(y,t) = \hat{\delta}\sqrt{3}\,\mathrm{e}^{\frac{\sqrt{3}}{2}y}\mathrm{e}^{-4\sqrt{3}t}, \tag{2.95b}$$

$$x(y,t) = y + 2\ln\frac{\hat{\delta}\sqrt{3}\,\mathrm{e}^{\frac{\sqrt{3}}{2}y}\mathrm{e}^{-4\sqrt{3}t} + 1 + \frac{\sqrt{3}}{2}}{\hat{\delta}\sqrt{3}\,\mathrm{e}^{\frac{\sqrt{3}}{2}y}\mathrm{e}^{-4\sqrt{3}t} + 1 - \frac{\sqrt{3}}{2}}. \tag{2.95c}$$

In particular, in the latter case (2.94) and (2.95a) give

$$x_y = \frac{2\hat{z}^2}{2\hat{z}^2 + 6\hat{z} + 3} \quad \text{and} \quad \hat{u}_y = -24\sqrt{3}\,\frac{\hat{z}^3(\hat{z}+1)(2\hat{z}+1)}{(2\hat{z}^2 + 6\hat{z} + 3)^3},$$

where $\hat{z} := z - \frac{1}{2}$. Thus x_y has a double zero at $\hat{z} = 0$, which corresponds to the crest of the solution, whereas, at the same point, $\hat{u}_y$ has a triple zero, so that $\tilde{u}_x = \hat{u}_y/x_y = 0$. Consequently, $\tilde{u}(x,t)$ is still continuous, with a continuous first derivative $\tilde{u}_x$ that vanish at the crest, but the higher order derivatives become unbounded at this point, e.g., $\tilde{u}_{xx} \sim -\frac{3}{2}\,\hat{z}^{-2}$ as $\hat{z} \to 0$. This unusual (finite) smoothness property of the soliton corresponding to the parameters separating (infinitely) smooth solitons from multivalued solutions (associated with the breaking of bijectivity of $x(\,\cdot\,,t)\colon \mathbb{R} \to \mathbb{R}$) was first reported by Matsuno [100], where the soliton solutions were constructed using a direct method.

Thus we arrive at the following description of the one-soliton solutions (consistent with [100]*see (3.4) and (3.14)):

Theorem 2.4.7. *The mCH equation in the form* (2.3) *has a family of one-soliton solutions, regular as well as non-regular,* $\tilde{u}(x,t) \equiv \tilde{u}_{\theta,\hat{\delta}}(x,t)$*, parameterized by two parameters,* $\hat{\delta} > 0$ *and* $\theta \in (0, \frac{\pi}{2})$*. These solitons* $\tilde{u}(x,t) \equiv \hat{u}(y(x,t),t)$ *are given, in parametric form, by*

$$\hat{u}(y,t) = 4\tan^2\theta \frac{z^2(y,t) + 2\cos^2\theta \cdot z(y,t) + \cos^2\theta}{(z^2(y,t) + 2z(y,t) + \cos^2\theta)^2} z(y,t), \tag{2.96a}$$

$$x(y,t) = y + 2\ln\frac{z(y,t) + 1 + \sin\theta}{z(y,t) + 1 - \sin\theta}, \tag{2.96b}$$

$$z(y,t) = 2\hat{\delta}\sin\theta\, \mathrm{e}^{y\sin\theta} \mathrm{e}^{-\frac{2\sin\theta}{\cos^2\theta}t}. \tag{2.96c}$$

They have different properties depending on the value of the parameter θ*:*

(i) *For* $\theta \in (0, \frac{\pi}{3})$*, the one-soliton solution* $\tilde{u}(x,t)$ *is smooth in the* (x,t) *variables.*

(ii) *For* $\theta = \frac{\pi}{3}$*, then* $\tilde{u}(x,t)$ *is given by* (2.95) *and has finite smoothness:* u *and* u_x *are continuous with* $\tilde{u}_x(x,t) = 0$ *at the crest when* $z(y(x,t),t) = \frac{1}{2}$*, but near the crest the higher derivatives become unbounded as* $z \to \frac{1}{2}$*.*

(iii) *If* $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$*, then* $\tilde{u}(x,t) = \hat{u}(y,t)$ *is regular in* (y,t)*, multivalued in* (x,t)*, and loop-shaped.*

2.5 Conclusions to Chapter 2

In this Section, we have considered the Cauchy problem for the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach a non-zero constant at the both infinities of the space variable. A non-zero background provides that the spectral problem in the associated Lax pair equations has a continuous spectrum, which allows us to formulate the inverse spectral problem as a Riemann–Hilbert factorization problem with jump conditions across the real axis.

We have developed the Riemann–Hilbert approach to this problem, which is based on the Jost solutions of the Lax pair and the scattering relations between them. Two specific features of the x-equation associated with the mCH equation that affect analytic properties of the Jost solutions are as follows: (a) λ enters U through a product with the "momentum" $m(x,t)$, which, in the framework of the inverse problem, is an unknown function; (b) as $|x| \to \infty$, $m(x,t)$ approaches a non-zero constant. In particular, these features affect the problem of control of the large-λ behavior of the Jost solutions. In our development of the RH formalism, this problem is addressed by (i) transforming the Lax pair equations to an appropriate form, with selected diagonal parts that dominate, in a certain sense, for large λ; (ii) introducing a new spatial-type variable, in view of having an explicit description of the large-λ behavior of the Jost solutions in terms of space and time parameters; (iii) introducing a new spectral parameter μ (related to λ by $\lambda = -\frac{1}{2}(\mu + \frac{1}{\mu})$), which allows us to avoid non-rational dependence of the coefficients in the Lax pair equations on the spectral parameter. Moreover, we take advantage of a consequence of property (a) that for $\lambda = 0$, U becomes independent of u, which suggests an efficient way for "extracting" the solution of the Cauchy problem from the solution of the RH problem taking the details of the behavior of the latter as $\lambda \to 0$.

Using this approach, we have obtained (i) a representation for the solution of the Cauchy problem for the mCH equation in terms of the solution of the associated Riemann–Hilbert factorization problem and (ii) a description of certain soliton-type solutions, both regular and non-regular. In particular, we have obtained the peakon type solution which has a different behaviour in comparison with the (original) Camass-Holm equation near the "peak": the solution itself and its first spatial derivative are continuous bounded functions, and the older derivatives become unbounded.

Chapter 3

The modified Camassa–Holm equation on a nonzero background: large-time asymptotics for the Cauchy problem

The results of this Chapter are published in [87].

We study the large-time behavior of the solution of the Cauchy problem for the mCH equation on a nonzero background (2.1), taking the formalism developed in Chapter 2 as the starting point. Focusing on the solitonless case, in Subection 3.1 we reduce the original (singular) RH problem representation for the solution of (2.1) to the resolution of a regular RH problem. Then, in Subsection 3.2, the latter problem is analyzed asymptotically, as $t \to +\infty$. We finally obtain the leading asymptotic terms for the solution of the Cauchy problem (2.1), in the two sectors of the (x,t) half-plane, $1 < \frac{x}{t} < 3$ and $\frac{3}{4} < \frac{x}{t} < 1$ where the deviation from the background value is nontrivial. In those sectors this deviation exhibits slowly decaying (of order $t^{-1/2}$), modulated (by $\frac{x}{t}$) oscillations (Theorems 3.2.2 and 3.2.4), while in the remaining sectors $\frac{x}{t} > 3$ and $\frac{x}{t} < \frac{3}{4}$ it decays rapidly to 1.

3.1 Reduction to a regular RH problem

Introducing a new function $\tilde{u}$ by

$$u(x,t) = \tilde{u}(x-t,t) + 1, \tag{3.1}$$

the mCH equation (2.1a) reduces to

$$\tilde{m}_t + (\tilde{\omega}\tilde{m})_x = 0, \tag{3.2a}$$

$$\tilde{m} := \tilde{u} - \tilde{u}_{xx} + 1, \tag{3.2b}$$

$$\tilde{\omega} := \tilde{u}^2 - \tilde{u}_x^2 + 2\tilde{u}, \tag{3.2c}$$

where the solution $\tilde{u}$ is considered on zero background: $\tilde{u}(x,t) \to 0$ as $x \to \pm\infty$ for all $t \geq 0$. In accordance with (2.1), it is assumed that $\tilde{m}_0(x) := (1-\partial_x^2)u_0(x) = (1-\partial_x^2)\tilde{u}_0(x) + 1 > 0$ for all $x > 0$. The Riemann–Hilbert (RH) approach for the Cauchy problem for equation (3.2) has been developed in Chapter 2. This resulted in a parametric representation for $\tilde{u}(x,t)$ in terms of the solution of an appropriate RH problem proposed in Chapter 2, according to the following algorithm:

(a) Given $u_0(x)$, construct the "reflection coefficient" $r(\mu)$, $\mu \in \mathbb{R}$ and, if applicable, the "discrete spectrum data" $\{\mu_j, \rho_j\}_{j=1}^N$, by solving the Lax pair equations associated with (3.2), whose coefficients are determined in terms of $u_0(x)$.

(b) Construct the jump matrix $J(y,t,\mu)$, $\mu \in \mathbb{R}$ by

$$J(y,t,\mu) := \mathrm{e}^{-p(y,t,\mu)\sigma_3} J_0(\mu) \mathrm{e}^{p(y,t,\mu)\sigma_3} \tag{3.3}$$

where

$$p(y,t,\mu) := -\frac{\mathrm{i}(\mu^2-1)}{4\mu}\left(-y + \frac{8\mu^2}{(\mu^2+1)^2}t\right) \tag{3.4}$$

and $J_0(\mu)$ is defined by

$$J_0(\mu) := \begin{pmatrix} 1 - r(\mu)r^*(\mu) & r(\mu) \\ -r^*(\mu) & 1 \end{pmatrix}. \tag{3.5}$$

(c) Solve the following **RH problem** (parametrized by y and t): Find a piecewise (w.r.t. $\mathbb{R}$) meromorphic (in the complex variable μ), 2×2-matrix valued function $M(y,t,\mu)$ satisfying the following conditions:

- The jump condition

$$M_+(y,t,\mu) = M_-(y,t,\mu)J(y,t,\mu), \qquad \mu \in \mathbb{R}, \quad \mu \neq \pm 1. \tag{3.6}$$

- The residue conditions

$$\begin{aligned} \operatorname{Res}_{\mu_j} M^{(1)}(y,t,\mu) &= \frac{1}{\varkappa_j(y,t)} M^{(2)}(y,t,\mu_j), \\ \operatorname{Res}_{\bar{\mu}_j} M^{(2)}(y,t,\mu) &= \frac{1}{\overline{\varkappa_j}(y,t)} M^{(1)}(y,t,\overline{\mu_j}), \end{aligned} \tag{3.7}$$

  with $\varkappa_j(y,t) := \rho_j \mathrm{e}^{-2p(y,t,\mu_j)}$.

- The normalization condition

$$M(y,t,\mu) \to I \text{ as } \mu \to \infty. \tag{3.8}$$

- The symmetries

$$M(\mu) = \overline{M(\bar{\mu}^{-1})} = \sigma_3 \overline{M(-\bar{\mu})} \sigma_3 = \sigma_1 \overline{M(\bar{\mu})} \sigma_1, \tag{3.9}$$

  where $M(\mu) \equiv M(y,t,\mu)$.

- The singularity conditions

$$M(y,t,\mu) = \frac{\mathrm{i}\alpha_+(y,t)}{2(\mu-1)} \begin{pmatrix} -c & 1 \\ -c & 1 \end{pmatrix} + \mathrm{O}(1) \qquad \text{as } \mu \to 1, \quad \operatorname{Im}\mu > 0, \tag{3.10a}$$

$$M(y,t,\mu) = -\frac{\mathrm{i}\alpha_+(y,t)}{2(\mu+1)} \begin{pmatrix} c & 1 \\ -c & -1 \end{pmatrix} + \mathrm{O}(1) \quad \text{as } \mu \to -1, \ \operatorname{Im}\mu > 0, \tag{3.10b}$$

  where $c = 1 + r(1)$ (generically, $c = 0$) whereas $\alpha_+(y,t) \in \mathbb{R}$ is not specified.

(d) Having found the solution $M(y,t,\mu)$ of this RH problem (which is unique, if it exists), extract the real-valued functions $a_j(y,t)$, $j = 1,2,3$ from the expansion of $M(y,t,\mu)$ at $\mu = \mathrm{i}$:

$$M(y,t,\mu) = \begin{pmatrix} a_1(y,t) & 0 \\ 0 & a_1^{-1}(y,t) \end{pmatrix} + \begin{pmatrix} 0 & a_2(y,t) \\ a_3(y,t) & 0 \end{pmatrix} (\mu - \mathrm{i}) \tag{3.11}$$

$$+\,\mathrm{O}((\mu-\mathrm{i})^2), \qquad \mu\to\mathrm{i}.$$

(e) Obtain $\tilde u(x,t)$ in parametric form as follows:

$$\tilde u(x,t) = \hat u(y(x,t),t),$$

where

$$\begin{aligned} \hat u(y,t) &:= -a_2(y,t)a_1(y,t) - a_3(y,t)a_1^{-1}(y,t), \\ x(y,t) &:= y + 2\ln a_1(y,t). \end{aligned} \tag{3.12}$$

Remark 3.1.1. To simplify notations in this paper, compared to Chapter 2, we have removed the symbol "hat" over many functions (e.g., $M(y,t,\mu)$, $\alpha_+(y,t)$, etc.). Another difference is that M_+ and M_- are exchanged in the jump relation (3.6) so that here the jump is the inverse of that in Chapter 2: $J_0 = \hat J_0^{-1}$ and $J = \hat J^{-1}$.

Remark 3.1.2. The symmetries (3.9) are consistent with the symmetries of $r(\mu)$, namely

$$r(\mu) = -\overline{r(-\mu)} = \overline{r(\mu^{-1})}, \tag{3.13}$$

and with the invariance of the set $\{\mu_j,\rho_j\}_{j=1}^N$:

$$-\overline{\mu_j} = \mu_{j'} \text{ and } -\mu_j^{-1} = \mu_{j''} \text{ with } \rho_j = \overline{\rho_{j'}} = -\mu_j^{-2}\rho_{j''}.$$

These symmetries and invariances follow from the construction of the RH problem above in terms of the dedicated (Jost) solutions of the Lax pair equations associated with the mCH equation. Moreover, the symmetries (3.9) imply the particular structure of the matrices in (3.11).

Remark 3.1.3. In the case of the Camassa–Holm (CH) equation, the condition $m_0(x) := (1-\partial_x^2)u_0(x) > 0$ for all x provides the existence of a global solution to the corresponding initial value problem (see, e.g., [42]). In the case of the modified Camassa–Holm (mCH) equation, the situation is different: even if the initial potential m_0 does not change sign the solution $u(x,t)$ may blow-up in finite time [78]. We believe that the Riemann–Hilbert approach for constructing solutions of PDEs, being intrinsically local in the corresponding variables (in the case of the mCH equation, these variables are y and t), is best suited to present solutions (particularly, of initial value problems) that overcome finite time blow-up (see, e.g., [70]*Chapter 3, Section 1, Corollary 3.1) and thus allow us to study the large time behavior of solutions (namely here $\hat{u}(y,t)$) in sectors of the (y,t) half-plane. Then, as the following asymptotic analysis will show (see (3.57b) and (3.69b)), in the solitonless case the correspondence between x and y is one-to-one for any large enough t, and therefore the solutions $u(x,t)$ are also well-defined for any large t in the (x,t) half-plane.

On the other hand, it is the breaking of this one-to-one correspondence $x \leftrightarrow y$ that provides a mechanism of wave breaking of the solution $u(x,t)$ in situations where, however, the solution $M(y,t,\mu)$ of the RH problem (3.6)-(3.10) exists for all y and t. In particular, if the initial data are such that some of the associated discrete spectral points $\{\mu_j\}$ have the form $\mu_j = \mathrm{e}^{\mathrm{i}\theta_j}$ with $\frac{\pi}{3} < \theta_j < \frac{\pi}{2}$, then the correspondence between x and y is no longer one-to-one for any large enough t, see Chapter 2*Corollary 5.6.

In the general context of nonlinear integrable equations, the RH problem formalism (i.e., the representation of the solution of the original problem — the Cauchy problem for a nonlinear integrable PDE — in terms of the solution of an associated RH problem) allows reducing the problem of the large time analysis of the solution of the nonlinear PDE to that of the RH problem. Residue conditions (if any) involved in the RH problem formulation generate a soliton-type, non-decaying contribution to the asymptotics whereas the jump conditions are responsible for the dispersive (decaying) part, details of which can be retrieved applying an appropriate modification of the nonlinear steepest descent method to the asymptotic analysis of a preliminarily regularized RH problem (i.e., a RH problem involving the jump and normalization conditions only).

With this respect we notice that the residue conditions (3.7) can be handled in a standard way:

(i) either adding to the contour small circles around each μ_j and $\bar{\mu}_j$ and reducing the residue conditions to associated jump conditions across the circles

(ii) or using the Blaschke–Potapov factors (see, e.g., [21]).

In both approaches, the original RH problem is reduced to a RH problem without residue conditions.

As for the singularity conditions, we notice that in the case of the Camassa–Holm equation, where such a condition is also involved in the matrix RH problem formalism, an efficient way to handle it is to reduce the matrix RH problem to a vector one, multiplying from the left by the constant vector $(1, 1)$. Indeed, the singularity condition for the CH equation has the form of (3.10b), and thus this multiplication "kill" the singularity, reducing the RH problem to a regular one. With this respect, we notice that the matrix RH problem for the modified Camassa–Holm equation is different: it also involves the singularity condition (3.10a), which, obviously, cannot be removed using the same trick.

In the present paper, we focus on the study of the *dispersive* part of the large-time asymptotics of solutions of Cauchy problems for the mCH equation. Accordingly, we proceed with the solitonless case assuming that there are no residue conditions (the consideration of a possible discrete spectrum can then be done according to an already well developed technique, see, e.g., [21]).

In this section we reduce the original RH problem (which is still singular due to conditions (3.10)) to a regular one, proceeding in two steps.

In Step 1, we reduce the RH problem with the singularity conditions (3.10) at $\mu = \pm 1$ to a RH problem which is characterized by the following two conditions:

(i) the matrix entries are regular at $\mu = \pm 1$, but the determinant of the (matrix) solution vanishes at $\mu = \pm 1$ (note that $\det M(\mu) \equiv 1$ for the solution of the original RH problem);

(ii) the solution is singular at $\mu = 0$.

Then, in Step 2, the latter RH problem is reduced to a regular one, i.e., to a RH problem with the jump and normalization conditions only.

Proposition 3.1.4. *Let $M(y,t,\mu)$ be a solution of the RH problem* (3.6), (3.8)–(3.10). *Define $\tilde{M}$ by*

$$\tilde{M}(y,t,\mu) := \left(I - \frac{1}{\mu}\sigma_1\right) M(y,t,\mu). \tag{3.14}$$

Then $\tilde{M}(\mu) \equiv \tilde{M}(y,t,\mu)$ is the unique solution of the following RH problem:

(C1) *$\tilde{M}(\mu)$ is analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous up to $\mathbb{R} \setminus \{0\}$.*

(C2) *$\tilde{M}(\mu)$ satisfies the jump condition* (3.6) *with the jump defined by* (3.3)–(3.5).

(C3) *$\tilde{M}(\mu) \to I$ as $\mu \to \infty$.*

(C4) *$\tilde{M}(\mu) = -\frac{1}{\mu}\sigma_1 + \mathrm{O}(1)$ as $\mu \to 0$.*

(C5) $\det \tilde{M}(\pm 1) = 0$.

(C6) *$\tilde{M}(\mu^{-1}) = -\mu \tilde{M}(\mu)\sigma_1$.*

Proof. First, let's check that $\tilde{M}(y,t,\mu)$ constructed from $M(y,t,\mu)$ satisfies the conditions above. The limiting properties (C3) and (C4) as $\mu \to \infty$ and as $\mu \to 0$ are obviously satisfied (by construction) whereas (C2) results from the fact that a multiplication from the left does not change the jump conditions. Further, since $\det M(y,t,\mu) \equiv 1$, it follows that $\det \tilde{M}(y,t,\mu) = 1 - \frac{1}{\mu^2}$ and thus $\det \tilde{M}(y,t,\pm 1) = 0$. Moreover, as $\mu \to 1$ we have

$$\begin{aligned}\left(\tilde{M}_{11}(\mu), \tilde{M}_{12}(\mu)\right) &= (M_{11}(\mu), M_{12}(\mu)) - \frac{1}{\mu}(M_{21}(\mu), M_{22}(\mu)) \\ &= (M_{11}(\mu) - M_{21}(\mu), M_{12}(\mu) - M_{22}(\mu)) + \mathrm{O}(1) = \mathrm{O}(1)\end{aligned}$$

due to (3.10a). Similarly, as $\mu \to -1$ we have

$$\left(\tilde{M}_{11}(\mu), \tilde{M}_{12}(\mu)\right) = (M_{11}(\mu) + M_{21}(\mu), M_{12}(\mu) + M_{22}(\mu)) + \mathrm{O}(1) = \mathrm{O}(1)$$

due to (3.10b). Similarly for $\left(\tilde{M}_{21}(\mu), \tilde{M}_{22}(\mu)\right)$. Thus $\tilde{M}(y,t,\mu)$ is non-singular at $\mu = \pm 1$. Finally, (C6) follows from the symmetry relation $M(\mu^{-1}) = \sigma_1 M(\mu)\sigma_1$ from (3.9).

Now, let's prove that the solution of the RH problem (C1)–(C6) above is unique (if it exists). First, we notice that if $\tilde{M}(y,t,\mu)$ solves the RH problem (C1)–(C6), then

$$\det \tilde{M}(y,t,\mu) = 1 - \frac{1}{\mu^2}. \tag{3.15}$$

Indeed, since $\det J(y,t,\mu) \equiv 1$ and $\det M(y,t,\mu)$ is bounded at $\mu = \infty$, it follows that $\det M(\mu)$ is a rational function. Moreover, from (C4) we have that $\det M(\mu) = -\frac{1}{\mu^2} + \frac{c}{\mu} + \mathrm{O}(1)$ as $\mu \to 0$, with some $c \equiv c(y,t)$. Taking into account (C3) we have that $\zeta(y,t,\mu) := \det M(y,t,\mu) - 1 + \frac{1}{\mu^2} - \frac{c}{\mu}$ is a bounded entire function of μ, which, by Liouville's theorem and (C3), vanishes for all (y,t). Finally, evaluating $\zeta(y,t,\mu)$ at $\mu = \pm 1$ and using (C5), it follows that $c(y,t) \equiv 0$ and thus (3.15) follows.

Now let's assume that $\tilde{\tilde{M}}$ is another solution of the RH problem (C1)–(C6) and define $N(\mu) := \tilde{M}(\mu)\tilde{\tilde{M}}^{-1}(\mu)$. Since $\tilde{M}$ and $\tilde{\tilde{M}}$ satisfy the same jump conditions, $N(\mu)$ is a rational function, with possible singularities at $\mu = 0, -1, 1$. In view of (3.15) and (C3), $\tilde{\tilde{M}}^{-1}(\mu) = \frac{\mu^2}{\mu^2-1}(\frac{1}{\mu}\sigma_1 + \mathrm{O}(1)) = \mathrm{O}(\mu)$ as $\mu \to 0$ and thus $N(\mu)$ is non-singular at $\mu = 0$. In order to prove that $N(\mu)$ is non-singular at $\mu = \pm 1$, we use relation (C6). In particular, we have $\tilde{M}(1) = -\tilde{M}(1)\sigma_1$ and thus $\tilde{M}(\mu) = \left(\begin{smallmatrix} g_1 & -g_1 \\ g_2 & -g_2 \end{smallmatrix}\right) + \mathrm{O}(\mu - 1)$ as $\mu \to 1$, with some g_j, $j = 1,2$. Consequently, $\tilde{\tilde{M}}^{-1}(\mu) = \frac{\mu^2}{\mu^2-1}\left(\left(\begin{smallmatrix} -\tilde{g}_2 & \tilde{g}_1 \\ -\tilde{g}_2 & \tilde{g}_1 \end{smallmatrix}\right) + \mathrm{O}(\mu-1)\right)$ as $\mu \to 1$, with some $\tilde{g}_j$, $j = 1,2$, which implies that $N(\mu)$ is bounded as $\mu \to 1$. Similarly for $\mu \to -1$. Therefore, $N(\mu)$ is an entire function such that $N(\infty) = I$ and thus $N(\mu) \equiv I$ by Liouville's theorem. □

Remark 3.1.5. Assuming $r(\mu) = -\overline{r(-\mu)}$ (see (3.13)), we have that $J(\mu)$ satisfies the symmetries

$$J(\mu) = \sigma_3 \overline{J(-\mu)} \sigma_3 = \sigma_1 \overline{J^{-1}(\mu)} \sigma_1,$$

which, due to uniqueness, imply for $\tilde{M}$ the same symmetries as for M:

$$\tilde{M}(\mu) = \sigma_3 \overline{\tilde{M}(-\bar{\mu})} \sigma_3 = \sigma_1 \overline{\tilde{M}(\bar{\mu})} \sigma_1 \tag{3.16}$$

(taking also into account that the symmetries (3.16) are consistent with all conditions in the RH problem in Proposition 3.1.4).

Step 2 in the reduction of the RH problem is formulated in the following proposition (see [82, 115, 116] for the case of the nonlinear Schrödinger equation with "finite density" boundary conditions).

Proposition 3.1.6 (regular RH problem). *The solution $\tilde{M}$ of the RH problem from Proposition 3.1.4 can be represented in terms of the solution of a regular RH problem as follows:*

$$\tilde{M}(y,t,\mu) = \left(I - \frac{1}{\mu}\Delta(y,t)\right) M^R(y,t,\mu), \tag{3.17}$$

where $M^R(\mu) \equiv M^R(y,t,\mu)$ is the solution of the following RH problem:

Find $M^R(\mu)$ such that

(R1) *$M^R(\mu)$ is analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ and continuous up to the real axis.*

(R2) *$M^R(\mu)$ satisfies the jump condition* (3.3)–(3.6)*.*

(R3) *$M^R(\mu) \to I$ as $\mu \to \infty$.*

Here Δ in (3.17) *is expressed in terms of the solution M^R of the RH problem above by:*

$$\Delta(y,t) = \sigma_1 [M^R(y,t,0)]^{-1}.$$

Proof. Let $M^R(\mu)$ be the solution of the regular RH problem (R1)-(R3) above. Then $\tilde{M}(y,t,\mu)$ defined by (3.17) obviously (by construction) satisfies conditions (C1)-(C4) of the RH problem from Proposition 3.1.4. In order to check conditions (C5) and (C6), we use the matrix structure of Δ that follows from the symmetries of $M^R(\mu)$.

(i) Since $M^R(\mu)$ and $M(\mu)$ satisfy the same jump condition, the uniqueness of the solution of the regular RH problem implies that $M^R(\mu)$ satisfies the same symmetries (see (3.9)) (generated by the symmetry $r(\mu) = -\overline{r(-\mu)}$):

$$M^R(\mu) = \sigma_3 \overline{M^R(-\bar{\mu})} \sigma_3 = \sigma_1 \overline{M^R(\bar{\mu})} \sigma_1. \tag{3.18}$$

Considering this for $\mu = 0$ it follows that $M^R(y,t,0) = \begin{pmatrix} \alpha(y,t) & \mathrm{i}\beta(y,t) \\ -\mathrm{i}\beta(y,t) & \alpha(y,t) \end{pmatrix}$ with some $\alpha(y,t) \in \mathbb{R}$ and $\beta(y,t) \in \mathbb{R}$. Moreover, $\alpha^2(y,t) - \beta^2(y,t) \equiv 1$ since $\det M^R(\mu) \equiv 1$. Consequently, $\Delta(y,t)$ has the structure

$$\Delta = \begin{pmatrix} \mathrm{i}\beta & \alpha \\ \alpha & -\mathrm{i}\beta \end{pmatrix} \text{ with } \alpha^2 - \beta^2 = 1 \tag{3.19}$$

and thus $\det(I - \mu^{-1}\Delta(y,t)) = 1 - \frac{\alpha^2-\beta^2}{\mu^2} = 1 - \frac{1}{\mu^2}$, which implies (C5). Notice that $\Delta^2 \equiv I$.

(ii) Now consider the symmetry $\mu \mapsto \mu^{-1}$. From $r(\mu) = \overline{r(\mu^{-1})}$ it follows that $J(\mu) = \sigma_1 J^{-1}(\mu^{-1})\sigma_1$ and thus $\check{M}(\mu) := \sigma_1 M^R(\mu^{-1})\sigma_1$ satisfies the same jump condition as $M^R(\mu)$ does. Taking into account that $\check{M}(\infty) = \sigma_1 M^R(0)\sigma_1$, Liouville's theorem implies that $\check{M}^{-1}(\infty)\check{M}(\mu) \equiv \sigma_1[M^R(0)]^{-1}M^R(\mu^{-1})\sigma_1 = M(\mu)$, or, in terms of Δ,

$$M^R(\mu^{-1}) = \Delta M^R(\mu)\sigma_1. \tag{3.20}$$

Now, combining (3.17) with (3.20) we can express $\tilde{M}(\mu^{-1})$ in terms of $\tilde{M}(\mu)$ as follows:

$$\tilde{M}(\mu^{-1}) = (I - \Delta\mu)M^R(\mu^{-1}) = (I - \Delta\mu)\Delta M^R(\mu)\sigma_1 = Q(\mu)\tilde{M}(\mu)\sigma_1 \tag{3.21}$$

with

$$Q(\mu) = (I - \Delta\mu)\Delta\left(I - \Delta\mu^{-1}\right)^{-1}.$$

Using (3.19), direct calculations give $Q(\mu) = -\mu I$ and thus the symmetry (3.20) takes the form of (C6) in Proposition 3.1.4. □

From M^R back to $\tilde{u}$

Now, we can obtain a parametric representation of the solution $\tilde{u}(x,t)$ of the Cauchy problem (3.2) in terms of the solution $M^R(y,t,\mu)$ of the regular RH problem from Proposition 3.1.6. First, using (3.14) and (3.17), we get M from M^R:

$$M(\mu) = \left(I - \frac{1}{\mu}\sigma_1\right)^{-1}\left(I - \frac{1}{\mu}\Delta\right)M^R(\mu). \tag{3.22}$$

Then, by (3.11) and (3.12) we find

$$M(y,t,\mu) \rightsquigarrow \{a_1(y,t), a_2(y,t), a_3(y,t)\} \rightsquigarrow \{\hat{u}(y,t), x(y,t)\},$$

and finally $\tilde{u}(x,t) = \hat{u}(y(x,t),t)$.

3.2 Large-time asymptotics of the regular RH problem

In this section, we study the large-time asymptotics of the solution $M^R(y,t,\mu)$ of the regular RH problem from Proposition 3.1.6 using the ideas and tools of the nonlinear steepest descent method [54]. The method consists in successive transformations of the original RH problem, in order to reduce it to an explicitly solvable RH problem. The different steps include

(a) appropriate triangular factorizations of the jump matrix;

(b) "absorption" of the triangular factors with good large-time behavior;

(c) reduction, after rescaling, to a RH problem which is solvable in terms of certain special functions;

(d) analysis of the approximation errors.

The information on L^p-RH problems and their applications to the asymptotics can be found in [50, 55, 70, 122]. Here we focus on deriving the leading terms of the large-time asymptotics, while for error estimates we refer to [93].

3.2.1 Transformations of the regular RH problem

Introduce

$$\theta(\mu,\xi) := \hat{\theta}(k(\mu),\xi),$$

where

$$\xi := \frac{y}{t}, \quad k(\mu) := \frac{1}{4}\left(\mu - \frac{1}{\mu}\right), \quad \hat{\theta}(k,\xi) := k\xi - \frac{2k}{1+4k^2}. \tag{3.23}$$

Hence, $p(y,t,\mu) = \mathrm{i}t\theta(\mu,\xi)$. The jump matrix $J(y,t,\mu)$ in (3.6) which is defined by (3.3)–(3.5) allows two triangular factorizations:

$$J(y,t,\mu) = \begin{pmatrix} 1 & r(\mu)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r^*(\mu)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, \tag{3.24a}$$

$$J(y,t,\mu) = \tag{3.24b}$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{r^*(\mu)}{1-r(\mu)r^*(\mu)}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix} \begin{pmatrix} 1-r(\mu)r^*(\mu) & 0 \\ 0 & \frac{1}{1-r(\mu)r^*(\mu)} \end{pmatrix} \begin{pmatrix} 1 & \frac{r(\mu)}{1-r(\mu)r^*(\mu)}\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}.$$

Following the basic idea of the nonlinear steepest descent method [54], the factorizations (3.24) can be used in such a way that the (oscillating) jump matrix on $\mathbb{R}$ for a modified RH problem reduces (see the RH problem for M_2 below) to the identity matrix whereas the arising jumps outside $\mathbb{R}$ are exponentially small as $t \to +\infty$. The use of one or another form of the factorization is dictated by the "signature table" for θ, i.e., the distribution of signs of $\operatorname{Im}\theta(\mu,\xi)$ (that depends on ξ) in the μ-complex plane.

a) The factorization (3.24a) is appropriate for the (open) intervals of $\mathbb{R}$ for which $\operatorname{Im}\theta(\mu)$ is positive for $\mu \in \mathbb{C}^+$ close to these intervals (and negative for $\mu \in \mathbb{C}^-$ close to the same intervals). We denote by $\Sigma_a \equiv \Sigma_a(\xi)$ the union of these intervals.

b) On the other hand the factorization (3.24b) is appropriate for the (open) intervals of $\mathbb{R}$ for which $\operatorname{Im}\theta(\mu)$ is negative for $\mu \in \mathbb{C}^+$ close to these intervals. We denote their union by $\Sigma_b(\xi) = \mathbb{R} \setminus \overline{\Sigma_a(\xi)}$.

In turn, one can get rid of the diagonal factor in (3.24b) using the solution of the following *scalar RH problem*: Find a scalar function $\delta(\mu,\xi)$ (ξ being a parameter) analytic in $\mu \in \mathbb{C} \setminus \overline{\Sigma_b(\xi)}$ and such that

$$\delta_+(\mu,\xi) = \delta_-(\mu,\xi)(1-|r(\mu)|^2), \quad \mu \in \Sigma_b(\xi), \tag{3.25a}$$

$$\delta(\mu,\xi) \to 1, \quad \mu \to \infty. \tag{3.25b}$$

The solution of the RH problem (3.25) is given by the Cauchy integral:

$$\delta(\mu,\xi) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_b(\xi)} \frac{\ln(1-|r(s)|^2)}{s-\mu}\mathrm{d}s\right\}. \tag{3.26}$$

Define $M_1(y,t,\mu) := M^R(y,t,\mu)\delta^{-\sigma_3}(\mu,\xi)$. Then M_1 can be characterized as the solution of the RH problem including the standard normalization condition $M_1(\mu) \to I$ as $\mu \to \infty$ and the jump condition

$$M_{1+}(y,t,\mu) = M_{1-}(y,t,\mu)J_1(y,t,\mu), \quad \mu \in \mathbb{R}, \tag{3.27}$$

where the jump matrix is factorized as

$$J_1(y,t,\mu) = \begin{pmatrix} 1 & r(\mu)\delta^2(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r^*(\mu)\delta^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, \quad \mu \in \Sigma_a(\xi) \tag{3.28a}$$

$$J_1(y,t,\mu) = \tag{3.28b}$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{r^*(\mu)}{1-r(\mu)r^*(\mu)}\delta_-^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{r(\mu)}{1-r(\mu)r^*(\mu)}\delta_+^2(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, \quad \mu \in \Sigma_b(\xi).$$

Now let us discuss the structure of $\Sigma_a(\xi)$ and $\Sigma_b(\xi)$. First, we notice that $\hat{\theta}(\xi,k)$ is exactly the same as in the case of the CH equation [27]. Taking into account the relation between μ and k (see (3.23)), the "signature table" for the CH equation near the real axis suggests that for the mCH equation (the latter being, additionally, symmetric w.r.t. $\mu \mapsto 1/\mu$) while the ranges of values of ξ for which the "signature table" keeps the same structure are the same. Namely, one can distinguish four ranges of values of ξ for which $\Sigma_a(\xi)$ and $\Sigma_b(\xi)$ have qualitatively different structures (which, consequently, implies four qualitatively different types of large-time asymptotics):

(I) $\xi > 2$,

(II) $0 < \xi < 2$,

(III) $-\frac{1}{4} < \xi < 0$,

(IV) $\xi < -\frac{1}{4}$.

Each range of values of ξ is characterized by the structure of $\Sigma_a(\xi)$ (or $\Sigma_b(\xi)$): $\Sigma_a(\xi)$ is the union of disjoint intervals whose (finite) end points are (real) stationary points of $\theta(\mu,\xi)$, i.e., points $\mu \in \mathbb{R}$ where $\frac{\mathrm{d}\theta}{\mathrm{d}\mu}(\mu,\xi) = 0$, and similarly for $\Sigma_b(\xi)$. More precisely,

$$\Sigma_b(\xi) = \begin{cases} \emptyset, & \xi > 2, \\ (-\mu_0, -\frac{1}{\mu_0}) \cup (\frac{1}{\mu_0}, \mu_0), & 0 < \xi < 2, \\ (-\infty, -\mu_1) \cup (-\mu_0, -\frac{1}{\mu_0}) \cup (-\frac{1}{\mu_1}, \frac{1}{\mu_1}) \cup (\frac{1}{\mu_0}, \mu_0) \cup (\mu_1, +\infty), & -\frac{1}{4} < \xi < 0, \\ (-\infty, +\infty), & \xi < -\frac{1}{4}. \end{cases} \tag{3.29}$$

Here the values of $\mu_0(\xi) > 1$ and $\mu_1(\xi) > 1$ are those associated (via $\kappa_j = \frac{1}{4}(\mu_j - \frac{1}{\mu_j})$, $j = 0, 1$) with the (real) stationary points $\kappa_0(\xi)$ and $\kappa_1(\xi)$ of $\hat{\theta}(k)$, i.e., the end points in the case of the CH equation. They are determined by the relation $\xi = \frac{2-8\kappa^2}{(1+4\kappa^2)^2}$ (see [27]):

$$\kappa_0^2(\xi) = \frac{\sqrt{1+4\xi} - 1 - \xi}{4\xi}, \qquad \kappa_1^2(\xi) = -\frac{\sqrt{1+4\xi} + 1 + \xi}{4\xi}$$

($\kappa_0(\xi)$ is relevant for ranges II and III whereas $\kappa_1(\xi)$ is relevant for range III only). In analogy with the case of the CH equation, for ξ in ranges I and IV, the solution M_2 of the RH problem (see below) decays rapidly (as $t \to +\infty$) to the identity matrix, which corresponds (in the case without discrete spectrum) to rapid decay of the resulting $\hat{u}(y, t)$. On the other hand, ranges II and III are those where the large-time asymptotics in the case of the CH equation are of Zakharov–Manakov type (trigonometric oscillations decaying as $t^{-1/2}$), see [21, 27]. Our main goal in the present paper is the derivation of analogous asymptotic formulas, for ranges II and III, in the case of the mCH equation.

The next step in the transformation of the RH problem is the "absorption" of the triangular factors in (3.28a) and (3.28b) into the solution of a deformed RH problem, with an enhanced jump contour (having parts outside $\mathbb{R}$). This absorption requires the triangular factors in (3.28a) and (3.28b) to have analytic continuation at least into a band surrounding $\mathbb{R}$. With this respect we notice that, as in the case of other integrable equations (in particular, the CH equation), the reflection coefficient $r(\mu)$ is defined, in general, for $\mu \in \mathbb{R}$ only.

However, one can approximate $r(\mu)$ and $\frac{r(\mu)}{1-r(\mu)r^*(\mu)}$ by some rational functions with well-controlled errors (see, e.g., [93]). Alternatively, if we *assume* that the initial data $\tilde{u}(x,0)$ decays exponentially to 0 as $x \to \pm\infty$ (or that $\tilde{u}(x,0)$ has finite support in $\mathbb{R}$), then $r(\mu)$ turns out to be analytic in a band containing the real axis (or analytic in the whole plane) and thus there is no need to use rational approximations in order to be able to perform this absorption (see the transformation $M_1 \rightsquigarrow M_2$ below). Henceforth, in order to avoid technicalities and to keep the presentation of our main result as simple as possible, we *assume* that $r(\mu)$ (and thus $1-r(\mu)r^*(\mu)$) is analytic in a domain of the complex plane containing the contours of the successive RH problems (and refer to [93] for details related to the rational approximations).

For $0 < \xi < 2$ and for $-\frac{1}{4} < \xi < 0$, we define a contour $\Sigma \equiv \Sigma(\xi)$ consistent with the signature table for $\theta(\mu, \xi)$, see Figures 3.1 and 3.2, respectively.

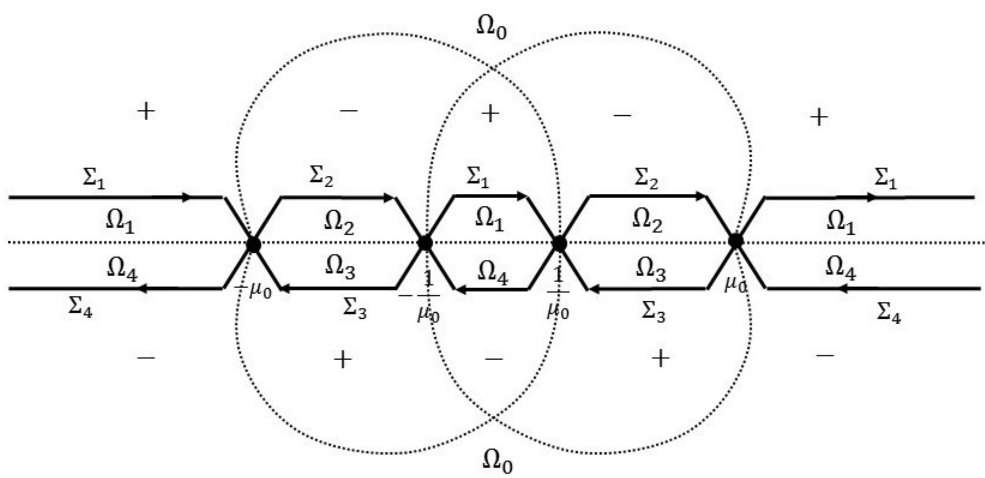

Figure 3.1: Signature table (dotted lines), contour $\Sigma(\xi) = \cup_{j=1}^{4}\Sigma_j$ (solid lines) and domains $\Omega_j(\xi)$ for $0 < \xi < 2$.

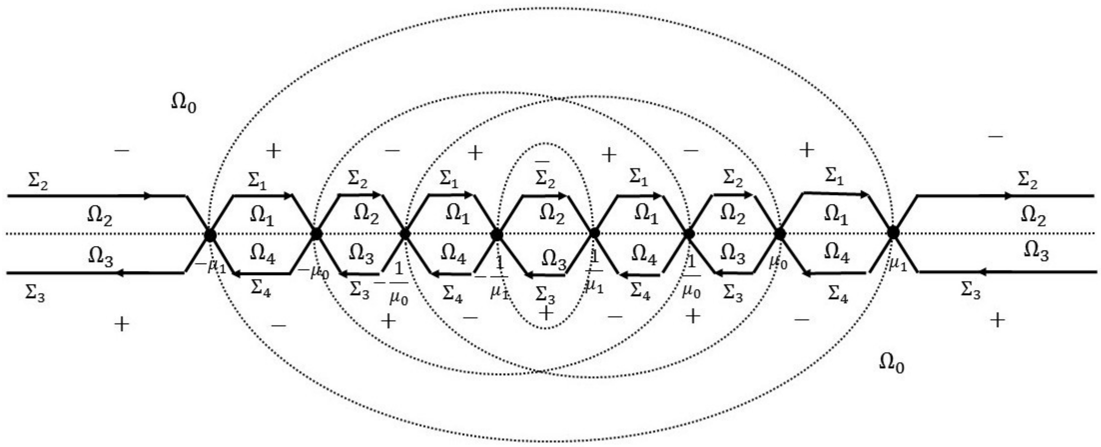

Figure 3.2: Signature table (dotted lines), contour $\Sigma(\xi) = \cup_{j=1}^4 \Sigma_j$ (solid lines) and domains $\Omega_j(\xi)$ for $-\frac{1}{4} < \xi < 0$.

Further, define M_2 by $M_2(y,t,\mu) := M_1(y,t,\mu)P(y,t,\mu)$, where

$$P(y,t,\mu) = \begin{cases} I, & \mu \in \Omega_0, \\ \begin{pmatrix} 1 & 0 \\ r^*(\mu)\delta^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & \mu \in \Omega_1, \\ \begin{pmatrix} 1 & -\frac{r(\mu)}{1-r(\mu)r^*(\mu)}\delta^{2}(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & \mu \in \Omega_2, \\ \begin{pmatrix} 1 & 0 \\ -\frac{r^*(\mu)}{1-r(\mu)r^*(\mu)}\delta^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & \mu \in \Omega_3, \\ \begin{pmatrix} 1 & r(\mu)\delta^{2}(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & \mu \in \Omega_4. \end{cases} \tag{3.30}$$

Then $M_2(y,t,\mu)$ can be characterized as the solution of the RH problem with the standard normalization condition $M_2(\mu) \to I$ as $\mu \to \infty$ and the jump condition

$$M_{2+}(y,t,\mu) = M_{2-}(y,t,\mu)J_2(y,t,\mu), \quad \mu \in \Sigma := \cup_{j=1}^4 \Sigma_j, \tag{3.31}$$

where $\Sigma_j := \overline{\Omega_0} \cap \overline{\Omega_j}$ and

$$J_2(y,t,\mu) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r^*(\mu)\delta^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & \mu \in \Sigma_1, \\ \begin{pmatrix} 1 & \frac{r(\mu)}{1-r(\mu)r^*(\mu)}\delta^{2}(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & \mu \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ \frac{r^*(\mu)}{1-r(\mu)r^*(\mu)}\delta^{-2}(\mu,\xi)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & \mu \in \Sigma_3, \\ \begin{pmatrix} 1 & -r(\mu)\delta^{2}(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & \mu \in \Sigma_4. \end{cases} \tag{3.32}$$

The RH problem for M_2 is such that uniform decay (as $t \to +\infty$) of the jump matrix is violated only near the stationary phase points of $\theta(\mu)$. The large-time analysis, with appropriate estimates, of such problems involves the "comparison" of the RH problem with that modified in small vicinities of the stationary phase points, using rescaled spectral parameters as well as approximations of the jump matrices in these vicinities [54].

In our large-time analysis for M_2, we follow the strategy presented in [93].

Step (i). Add to Σ small circles γ_j $(j = 0, 1)$ surrounding μ_j, together with their images $-\gamma_j$ (surrounding $-\mu_j$) and $\pm\gamma_j^{-1}$ (surrounding $\pm 1/\mu_j$) under the mappings $\mu \mapsto -\mu$ and $\mu \mapsto 1/\mu$, respectively.

Step (ii). Inside the circles around μ_0 and μ_1, define (explicitly) a function $m_0(y,t,\mu)$ which exactly satisfies the jump condition across Σ obtained from (3.32) by replacing $r(\mu)$ with $r(\mu_0)$ and $r(\mu_1)$, respectively, and by replacing $\delta^2(\mu,\xi)\mathrm{e}^{-2\mathrm{i}t\theta(\mu,\xi)}$ with its large-time approximation.

Step (iii). Define $m_0(y,t,\mu)$ inside the other small contours using the symmetries $m_0(\mu) = \overline{m_0(1/\bar{\mu})}$ and $m_0(\mu) = \sigma_3\overline{m_0(-\bar{\mu})}\sigma_3$ (which are consistent with the symmetries of $M_2(\mu)$).

Step (iv). Define $\hat{m}(\mu)$ by

$$\hat{m}(y,t,\mu) = \begin{cases} M_2(y,t,\mu)m_0^{-1}(y,t,\mu), & \text{inside } \pm\gamma_j \text{ and } \pm\gamma_j^{-1}, \\ M_2(y,t,\mu), & \text{otherwise}, \end{cases}$$

Then $\hat{m}(\mu)$ satisfies the conditions of the RH problem

$$\begin{cases} \hat{m}_+(y,t,\mu) = \hat{m}_-(y,t,\mu)\hat{J}(y,t,\mu), & \mu \in \hat{\Sigma} := \Sigma \cup_j \{\pm\gamma_j\} \cup_j \{\pm\gamma_j^{-1}\}, \\ \hat{m}(y,t,\mu) \to I, & \mu \to \infty, \end{cases}$$

where

$$\hat{J}(y,t,\mu) = \begin{cases} m_0^{-1}(y,t,\mu), & \mu \in \cup_j\{\pm\gamma_j\} \cup_j \{\pm\gamma_j^{-1}\}, \\ m_{0-}^{-1}(y,t,\mu)J_2(y,t,\mu)m_{0+}(y,t,\mu), & \mu \in \Sigma \cap \{\mu \mid \mu \text{ inside } \cup_j \{\pm\gamma_j^{\pm 1}\}\}, \\ J_2(y,t,\mu), & \text{otherwise}. \end{cases}$$

On the other hand, the unique solution of this problem can be expressed in terms of the solution $\Theta(\mu)$ of the singular integral equation (see [93]*Lemma 2.9):

$$\hat{m}(y,t,\mu) = I + \frac{1}{2\pi \mathrm{i}} \int_{\hat{\Sigma}} \Theta(y,t,s)\hat{w}(y,t,s)\frac{\mathrm{d}s}{s-\mu}. \qquad (3.33)$$

Here $\hat{w}(y,t,s) := \hat{J}(y,t,s) - I$ and $\Theta \in I + L^2(\hat{\Sigma})$ is the solution of the integral equation

$$\Theta(\mu) - \mathcal{C}_{\hat{w}}\Theta(\mu) = I,$$

where $\mathcal{C}_{\hat{w}}\colon L^2(\hat{\Sigma}) + L^\infty(\hat{\Sigma}) \to L^2(\hat{\Sigma})$ is an integral operator defined with the help of the singular Cauchy operator: $\mathcal{C}_{\hat{w}}f := \mathcal{C}_-(f\hat{w})$, where $\mathcal{C}_- = \frac{1}{2}(-I + S_{\hat{\Sigma}})$ and $S_{\hat{\Sigma}}$ is the operator associated with $\hat{\Sigma}$ and defined by the principal value of the Cauchy integral:

$$(S_{\hat{\Sigma}}f)(\mu) = \frac{1}{2\pi \mathrm{i}} \int_{\hat{\Sigma}} \frac{f(s)}{s-\mu} ds, \quad \mu \in \hat{\Sigma}.$$

Here $L^2(\hat{\Sigma}) + L^\infty(\hat{\Sigma})$ denotes the space of all functions that can be written as the sum of a function in $L^2(\hat{\Sigma})$ and a function in $L^\infty(\hat{\Sigma})$.

Step (v). Estimate the large-time behavior of $\hat{m}(y,t,\mu)$ at $\mu=\mathrm{i}$ and $\mu=0$ taking into account the following facts:

- The main contribution to the r.h.s. of (3.33) comes from the integrals over the small contours, where $\hat{w}(y,t,\mu)=m_0^{-1}(y,t,\mu)-I$:

$$\hat{m}(y,t,\mu)=I+\frac{1}{2\pi\mathrm{i}}\int_{\cup_j\{\pm\gamma_j\}\cup_j\{\pm\gamma_j^{-1}\}}\frac{m_0^{-1}(y,t,s)-I}{s-\mu}\mathrm{d}s+\mathrm{o}(t^{-1/2}). \quad (3.34)$$

  Henceforth the error estimates are uniform for $\varepsilon<\xi<2-\varepsilon$ and $-\frac{1}{4}+\varepsilon<\xi<-\varepsilon$, for any small $\varepsilon>0$. For detailed estimates, see [93].

- In turn, the main contribution to $m_0^{-1}(y,t,\mu)-I$ comes from the asymptotics of the RH problem for parabolic cylinder functions (involved in the construction of $m_0(y,t,\mu)$), see [93]*Appendix B, which can be given explicitly.

3.2.2 Range $0<\xi<2$

This range is characterized by the presence of four real critical points: $\pm\mu_0$ and $\pm\mu_0^{-1}$.

Construction of m_0

First, we approximate $\mathrm{i}t\theta(\mu,\xi)$ using (3.23), the relation

$$\kappa_0=\frac{1}{4}\left(\mu_0-\frac{1}{\mu_0}\right) \quad (3.35)$$

between μ_0 and κ_0, and the approximation for $\hat{\theta}(k,\xi)$ near κ_0, see [27]:

$$\hat{\theta}(k,\xi)\approx\hat{\theta}(\kappa_0)+8f_0(\kappa_0)(k-\kappa_0)^2,$$

where

$$f_0(\kappa_0)=\frac{\kappa_0(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3},\qquad \hat{\theta}(\kappa_0)=-\frac{16\kappa_0^3}{(1+4\kappa_0^2)^2}. \quad (3.36)$$

Here and below we use the symbol $\approx$ somewhat loosely to express that the left-hand side is approximated by the right-hand side as a function of the spectral parameter with an error term that we are able to control in the subsequent error estimates (see, e.g., (3.44) and (3.47)–(3.49)). We have $-\mathrm{i}t\theta(\mu,\xi) \approx -\mathrm{i}t\hat{\theta}(\kappa_0) - \frac{\mathrm{i}\hat{\mu}^2}{4}$, where the scaled spectral variable $\hat{\mu}$ is introduced by

$$\mu - \mu_0 = \frac{\hat{\mu}}{(1+\mu_0^{-2})\sqrt{2f_0 t}}. \tag{3.37}$$

Now we approximate $\delta(\mu,\xi)$ near $\mu = \mu_0$. From (3.26) we have

$$\begin{aligned}\delta(\mu,\xi) &= \exp\left\{\frac{1}{2\pi\mathrm{i}}\left(\int_{-\mu_0}^{-1/\mu_0} + \int_{1/\mu_0}^{\mu_0}\right)\frac{\ln(1-|r(s)|^2)}{s-\mu}\mathrm{d}s\right\} \\ &= \left(\frac{\mu-\mu_0}{\mu-1/\mu_0}\right)^{\mathrm{i}h_0}\left(\frac{\mu+1/\mu_0}{\mu+\mu_0}\right)^{\mathrm{i}h_0}\mathrm{e}^{\chi(\mu)},\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}h_0 &= -\frac{1}{2\pi}\ln(1-|r(\mu_0)|^2), \\ \chi(\mu) &= \frac{1}{2\pi\mathrm{i}}\left(\int_{-\mu_0}^{-1/\mu_0} + \int_{1/\mu_0}^{\mu_0}\right)\ln\frac{1-|r(s)|^2}{1-|r(\mu_0)|^2}\frac{\mathrm{d}s}{s-\mu}\end{aligned}$$

(notice that $|r(\mu)| = |r(-\mu)| = |r(1/\mu)|$). Therefore (cf. [27]),

$$\delta(\mu,\xi) \approx (\mu-\mu_0)^{\mathrm{i}h_0}\left(\frac{\mu_0+1/\mu_0}{2\mu_0(\mu_0-1/\mu_0)}\right)^{\mathrm{i}h_0}\mathrm{e}^{\chi(\mu_0)} = \hat{\mu}^{\mathrm{i}h_0}(128 f_0\kappa_0^2 t)^{-\frac{\mathrm{i}h_0}{2}}\mathrm{e}^{\chi(\mu_0)}$$

and thus

$$\delta(\mu,\xi)\mathrm{e}^{-\mathrm{i}t\theta(\mu,\xi)} \approx \delta_{\mu_0}(\xi,t)\hat{\mu}^{\mathrm{i}h_0}\mathrm{e}^{-\frac{\mathrm{i}\hat{\mu}^2}{4}}, \tag{3.39}$$

where

$$\delta_{\mu_0}(\xi,t) = \mathrm{e}^{-\mathrm{i}t\hat{\theta}(\kappa_0(\mu_0))}\mathrm{e}^{\chi(\mu_0)}(128 f_0(\kappa_0(\mu_0))\kappa_0^2(\mu_0)t)^{-\frac{\mathrm{i}h_0}{2}}. \tag{3.40}$$

The approximation (3.39) suggests introducing $m_0(y,t,\mu)$ (near $\mu = \mu_0$) as follows:

$$m_0(y,t,\mu) = D(\xi,t)m^X(\xi,\hat{\mu})D^{-1}(\xi,t), \tag{3.41}$$

where $D(\xi,t) = \delta_{\mu_0}^{\sigma_3}(t)$ and $m^X(\xi,\hat{\mu})$ is the solution of the RH problem, in the $\hat{\mu}$-complex plane, whose solution is given in terms of parabolic cylinder functions [93] (with $q = -\bar{r}(\mu_0)$).

Since (see (3.37)) finite values of μ correspond to growing (with t) values of $\hat{\mu}$, the large-time asymptotics of $m_0(y,t,\mu)$ for μ on the small contours surrounding $\pm\mu_0$ and $\pm\frac{1}{\mu_0}$ involves the large-$\hat{\mu}$ asymptotics of $m^X(\xi,\hat{\mu})$, which is given by (see [93]*Appendix B)

$$m^X(\xi,\hat{\mu}) = I + \frac{\mathrm{i}}{\hat{\mu}}\begin{pmatrix} 0 & -\beta_{\mu_0}(\xi) \\ \bar{\beta}_{\mu_0}(\xi) & 0 \end{pmatrix} + \mathrm{O}(\hat{\mu}^{-2}) \tag{3.42}$$

with

$$\beta_{\mu_0}(\xi) = \sqrt{h_0}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}-\arg(-\bar{r}(\mu_0))+\arg\Gamma(\mathrm{i}h_0)\right)}, \tag{3.43}$$

where Γ is Euler's gamma function. From (3.37), (3.41) and (3.42) we have

$$\begin{aligned} m_0^{-1}(y,t,\mu) &= D(\xi,t)(m^X)^{-1}(\xi,\hat{\mu}(\mu))D^{-1}(\xi,t) \\ &= D(\xi,t)\left(I - \frac{\mathrm{i}}{\hat{\mu}(\mu)}\begin{pmatrix} 0 & -\beta_{\mu_0}(\xi) \\ \bar{\beta}_{\mu_0}(\xi) & 0 \end{pmatrix}\right)D^{-1}(\xi,t) + \mathrm{O}(t^{-1}) \\ &= I + \frac{B(\xi,t)}{\sqrt{t}(\mu-\mu_0)} + \mathrm{O}(t^{-1}), \end{aligned} \tag{3.44}$$

where

$$B(\xi,t) = \begin{pmatrix} 0 & B_0(\xi,t) \\ \bar{B}_0(\xi,t) & 0 \end{pmatrix} \text{ with } B_0(\xi,t) = \frac{\mathrm{i}\delta_{\mu_0}^2(\xi,t)\beta_{\mu_0}(\xi)}{(1+\mu_0^{-2})\sqrt{2f_0(\kappa_0(\mu_0))}}. \tag{3.45}$$

Here the estimate $\mathrm{O}(t^{-1})$ is uniform for ξ and μ such that $\varepsilon_1 < \xi < 2-\varepsilon_1$ and $|\mu-\mu_0| = \varepsilon_2$ for any small positive ε_j, $j=1,2$.

Asymptotics for $\hat{m}$

In view of our algorithm for representing u in terms of the solution of the associated regular RH problem, see (3.22), (3.11), (3.12), and (3.1), we need to know the asymptotics for $\hat{m}(y,t,0)$, $\hat{m}(y,t,\mathrm{i})$, and $\hat{m}_1(y,t)$, where $\hat{m}_1$ is extracted from the expansion $\hat{m}(y,t,\mu) = \hat{m}(y,t,\mathrm{i}) + \hat{m}_1(y,t)(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2)$ as $\mu \to \mathrm{i}$. By (3.44) and the residue theorem, the leading contributions of the integral over γ_0 into (3.34) for these quantities are, respectively,

$$\frac{B}{\mu_0\sqrt{t}}, \quad \frac{B}{(\mu_0-\mathrm{i})\sqrt{t}} \text{ and } \frac{B}{(\mu_0-\mathrm{i})^2\sqrt{t}}. \tag{3.46}$$

In order to take into account the contributions of all small contours, we extend the definition of m_0 by symmetries (as indicated in Step (iii)). This gives

$$\begin{aligned}\hat{m}(y,t,0) &= I + \left(\frac{B}{\mu_0} - \frac{\bar{B}}{\mu_0} - \frac{1}{\mu_0^2}\frac{\bar{B}}{\mu_0^{-1}} + \frac{1}{\mu_0^2}\frac{B}{\mu_0^{-1}}\right)\frac{1}{\sqrt{t}} + \mathrm{o}(t^{-1/2}) \\ &= I + \frac{4i\,\mathrm{Im}\,B_0(\xi,t)}{\mu_0\sqrt{t}}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} + \mathrm{o}(t^{-1/2}),\end{aligned} \tag{3.47}$$

$$\begin{aligned}\hat{m}(y,t,\mathrm{i}) &= I + \left(\frac{B}{\mu_0-\mathrm{i}} + \frac{\bar{B}}{-\mu_0-\mathrm{i}} - \frac{1}{\mu_0^2}\frac{\bar{B}}{\mu_0^{-1}-\mathrm{i}} - \frac{1}{\mu_0^2}\frac{B}{-\mu_0^{-1}-\mathrm{i}}\right)\frac{1}{\sqrt{t}} + \mathrm{o}(t^{-1/2}) \\ &= I + \frac{2\mathrm{i}\,\mathrm{Im}\,B_0(\xi,t)}{\mu_0\sqrt{t}}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} + \mathrm{o}(t^{-1/2}),\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}\hat{m}_1(y,t) &= \left(\frac{B}{(\mu_0-\mathrm{i})^2} + \frac{\bar{B}}{(-\mu_0-\mathrm{i})^2} - \frac{1}{\mu_0^2}\frac{\bar{B}}{(\mu_0^{-1}-\mathrm{i})^2} - \frac{1}{\mu_0^2}\frac{B}{(-\mu_0^{-1}-\mathrm{i})^2}\right)\frac{1}{\sqrt{t}} + \mathrm{o}(t^{-1/2}) \\ &= \frac{4}{\sqrt{t}}\begin{pmatrix} 0 & \mathrm{Re}\,\frac{B_0}{(\mu_0-\mathrm{i})^2} \\ \mathrm{Re}\,\frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2} & 0\end{pmatrix} + \mathrm{o}(t^{-1/2}).\end{aligned} \tag{3.49}$$

From $\hat{m}$ back to M^R

In Section 3.2.2 we presented the large-time asymptotics of $\hat{m}(y,t,\mu)$ (and thus of $M_2(y,t,\mu)$) for the dedicated values of μ. Since $P(y,t,0) = I$ whereas $P(y,t,\mu)$ tends to I exponentially fast, as $t \to +\infty$ for all μ close to i, in order to obtain the leading terms of the asymptotics for $M^R(y,t,\mu) = M_1(y,t,\mu)\delta^{\sigma_3}(\mu,\xi) = M_2(y,t,\mu)P^{-1}(y,t,\mu)\delta^{\sigma_3}(\mu,\xi)$, we need to know $\delta(\mu,\xi)$ (3.26) for $\mu = 0$ and μ near i.

Due to the symmetry $|r(\mu)| = |r(-\mu)|$ we have

$$\delta(0,\xi) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_b(\xi)}\frac{\ln(1-|r(s)|^2)}{s}\mathrm{d}s\right\} \equiv 1. \tag{3.50}$$

Let I_0 and I_1 be such that $\delta(\mu,\xi) = \mathrm{e}^{I_0+I_1(\mu-\mathrm{i})+\dots}$ as $\mu \to \mathrm{i}$. Then, using again the symmetry $|r(\mu)| = |r(-\mu)|$),

$$I_0 = \frac{1}{2\pi \mathrm{i}} \int_{\Sigma_b(\xi)} \frac{\ln(1-|r(s)|^2)}{s-\mathrm{i}} \mathrm{d}s = \frac{1}{\pi} \int_{1/\mu_0}^{\mu_0} \frac{\ln(1-|r(s)|^2)}{s^2+1} \mathrm{d}s.$$

On the other hand,

$$\begin{aligned} I_1 &= \frac{1}{2\pi \mathrm{i}} \int_{1/\mu_0}^{\mu_0} \ln(1-|r(s)|^2) \left(\frac{1}{(s-\mathrm{i})^2} + \frac{1}{(-s-\mathrm{i})^2} \right) \mathrm{d}s \\ &= \frac{1}{\pi \mathrm{i}} \int_{1/\mu_0}^{\mu_0} \ln(1-|r(s)|^2) \frac{s^2-1}{(s^2+1)^2} \mathrm{d}s \equiv 0, \end{aligned}$$

the latter equality being due to the symmetry $|r(\mu)| = |r(\mu^{-1})|$. Thus, as $\mu \to \mathrm{i}$,

$$\delta(\mu, \xi) = \delta(\mathrm{i}, \xi) + \mathrm{O}((\mu-\mathrm{i})^2) \text{ with } \delta(\mathrm{i}, \xi) = \exp \left\{ \frac{1}{\pi} \int_{1/\mu_0}^{\mu_0} \frac{\ln(1-|r(s)|^2)}{s^2+1} \mathrm{d}s \right\}. \tag{3.51}$$

Therefore, if $M^R(y,t,\mu) = M^R(y,t,\mathrm{i}) + M_1^R(y,t)(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2)$ we have the following asymptotics for $M^R(y,t,0)$, $M^R(y,t,\mathrm{i})$, and $M_1^R(y,t)$:

$$M^R(y,t,0) = \hat{m}(y,t,0) = I + \frac{4\mathrm{i} \operatorname{Im} B_0(\xi,t)}{\mu_0 \sqrt{t}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathrm{o}(t^{-1/2}), \tag{3.52a}$$

$$\begin{aligned} M^R(y,t,\mathrm{i}) &= \hat{m}(y,t,\mathrm{i}) \delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{O}(\mathrm{e}^{-\varepsilon t}) \qquad (3.52\mathrm{b}) \\ &= \left(I + \frac{2\mathrm{i} \operatorname{Im} B_0(\xi,t)}{\mu_0 \sqrt{t}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}), \\ M_1^R(y,t) &= \hat{m}_1(y,t) \delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{O}(\mathrm{e}^{-\varepsilon t}) \qquad (3.52\mathrm{c}) \\ &= \frac{4}{\sqrt{t}} \begin{pmatrix} 0 & \operatorname{Re} \frac{B_0}{(\mu_0-\mathrm{i})^2} \\ \operatorname{Re} \frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2} & 0 \end{pmatrix} \delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}), \end{aligned}$$

where $B_0(\xi,t)$ is given by (3.45) and $\delta(\mathrm{i},\xi)$ is given by (3.51).

Large-time asymptotics of u

Combining the asymptotics (3.52) for $M^R(y,t,\mu)$ with (3.11), (3.12), (3.14), and (3.17), we can obtain the leading term of the large-time asymptotics of $u(x,t)$.

Introducing $\eta := \frac{2\operatorname{Im} B_0}{\mu_0\sqrt{t}}$, from (3.52a) we have:

$$\Delta(y,t) = \sigma_1[M^R(y,t,0)]^{-1} = \begin{pmatrix} 2\mathrm{i}\eta & 1 \\ 1 & -2\mathrm{i}\eta \end{pmatrix} + \mathrm{o}(t^{-1/2}). \tag{3.53}$$

Therefore, for

$$M(\mu) = \left(I - \frac{1}{\mu}\sigma_1\right)^{-1}\left(I - \frac{1}{\mu}\Delta\right) M^R(\mu) \tag{3.54}$$

we have $M(\mu) = I_1(\mu)I_2(\mu)M^R(\mu) + \mathrm{o}(t^{-1/2})$, where

$$I_1(\mu) = \begin{pmatrix} \frac{\mu^2}{\mu^2-1} & \frac{\mu}{\mu^2-1} \\ \frac{\mu}{\mu^2-1} & \frac{\mu^2}{\mu^2-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\mathrm{i}}{2} \\ -\frac{\mathrm{i}}{2} & \frac{1}{2} \end{pmatrix} - \frac{\mathrm{i}}{2}I(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2), \tag{3.55a}$$

$$\begin{aligned} I_2(\mu) &= \begin{pmatrix} 1 - \frac{2\mathrm{i}\eta}{\mu} & -\frac{1}{\mu} \\ -\frac{1}{\mu} & 1 + \frac{2\mathrm{i}\eta}{\mu} \end{pmatrix} \\ &= \begin{pmatrix} 1-2\eta & \mathrm{i} \\ \mathrm{i} & 1+2\eta \end{pmatrix} + \begin{pmatrix} -2\mathrm{i}\eta & -1 \\ -1 & 2\mathrm{i}\eta \end{pmatrix}(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2), \end{aligned} \tag{3.55b}$$

$$M^R(\mu) = \begin{pmatrix} 1 & \mathrm{i}\eta \\ -\mathrm{i}\eta & 1 \end{pmatrix}\delta^{\sigma_3}(\mathrm{i}) + \begin{pmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{pmatrix}\delta^{\sigma_3}(\mathrm{i})(\mu-\mathrm{i}) + \mathrm{O}((\mu-\mathrm{i})^2), \tag{3.55c}$$

with

$$\beta_1 = \frac{4}{\sqrt{t}}\operatorname{Re}\frac{B_0}{(\mu_0-\mathrm{i})^2}, \quad \beta_2 = \frac{4}{\sqrt{t}}\operatorname{Re}\frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2}. \tag{3.56}$$

Substituting (3.55) into (3.54) and keeping the terms of order $t^{-1/2}$ we have

$$\begin{aligned} M(\mu) = &\begin{pmatrix} (1-\eta)\delta(\mathrm{i}) & 0 \\ 0 & (1+\eta)\delta^{-1}(\mathrm{i}) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & (\beta_1+\eta)\delta^{-1}(\mathrm{i}) \\ (\beta_2-\eta)\delta(\mathrm{i}) & 0 \end{pmatrix}(\mu-\mathrm{i}) + \mathrm{o}((\mu-\mathrm{i})t^{-1/2}) \end{aligned}$$

and thus (see (3.11))

$$a_1 = (1-\eta)\delta(\mathrm{i}) + \mathrm{o}(t^{-1/2}), \ a_2 = (\beta_1+\eta)\delta^{-1}(\mathrm{i}) + \mathrm{o}(t^{-1/2}), \ a_3 = (\beta_2-\eta)\delta(\mathrm{i}) + \mathrm{o}(t^{-1/2}).$$

It follows (see (3.12)) that

$$\hat{u}(y,t) = -(\beta_1+\beta_2) + \mathrm{o}(t^{-1/2}) = \frac{8(1-\mu_0^2)}{(1+\mu_0^2)^2\sqrt{t}}\operatorname{Re} B_0 + \mathrm{o}(t^{-1/2}), \tag{3.57a}$$

$$x(y,t) = y + 2\ln((1-\eta)\delta(\mathrm{i})) + \mathrm{o}(t^{-1/2}) = y + y_0(\xi) + \mathrm{O}(t^{-1/2}), \quad (3.57\mathrm{b})$$

where (see (3.51)) $y_0(\xi) = \frac{2}{\pi}\int_{1/\mu_0}^{\mu_0} \frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s$.

Recalling the definition (3.45) of B_0 and introducing the real-valued functions $\varphi_\delta(\xi,t)$ and $\varphi_\beta(\xi)$ (see (3.43) and (3.40)) by

$$\beta_{\mu_0}(\xi) = \sqrt{h_0}\mathrm{e}^{\mathrm{i}\varphi_\beta(\xi)}, \quad \delta^2_{\mu_0}(\xi,t) = \mathrm{e}^{\mathrm{i}\varphi_\delta(\xi,t)}$$

we have $B_0 = \frac{\sqrt{h_0}}{(1+\mu_0^{-2})\sqrt{2f_0}}\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+\varphi_\delta(\xi,t))+\varphi_\beta(\xi))}$ and thus

$$\operatorname{Re} B_0(\xi,t) = \frac{\sqrt{h_0}}{(1+\mu_0^{-2})\sqrt{2f_0}}\cos\left\{\frac{\pi}{2} + \varphi_\delta(\xi,t) + \varphi_\beta(\xi)\right\}. \quad (3.58)$$

Substituting (3.58) into (3.57a) gives the asymptotics of the solution of the Cauchy problem for the mCH equation (in the form (3.2)) expressed parametrically, in the (y,t) variables. Recalling the definitions of f_0, φ_δ, φ_β, β_{μ_0} (see (3.36), (3.40), (3.43)) and the relationship (3.35) between μ_0 and κ_0 we obtain the following large-time asymptotics along the rays $\frac{y}{t} = \xi$ for $0 < \xi < 2$:

$$\hat{u}(y,t) = \frac{C_1(\xi)}{\sqrt{t}}\cos\left\{C_2(\xi)t + C_3(\xi)\ln t + C_4(\xi)\right\} + \mathrm{o}(t^{-1/2}), \quad (3.59)$$

where

$$C_1(\xi) = -\left(\frac{8h_0\kappa_0}{3-4\kappa_0^2}\right)^{\frac{1}{2}}, \quad (3.60\mathrm{a})$$

$$C_2(\xi) = \frac{32\kappa_0^3}{(1+4\kappa_0^2)^2}, \quad (3.60\mathrm{b})$$

$$C_3(\xi) = -h_0, \quad (3.60\mathrm{c})$$

$$C_4(\xi) = \frac{3\pi}{4} - \frac{1}{\pi}\left(\int_{-\mu_0}^{-1/\mu_0} + \int_{1/\mu_0}^{\mu_0}\right)\ln\frac{1-|r(s)|^2}{1-|r(\mu_0)|^2}\frac{\mathrm{d}s}{s-\mu_0} \quad (3.60\mathrm{d})$$

$$- h_0\ln\frac{128\kappa_0^3(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3} - \arg(-\bar{r}(\mu_0)) + \arg\Gamma(\mathrm{i}h_0),$$

taking into account that h_0, κ_0, and μ_0 are defined as functions of ξ.

In order to express the asymptotics of $\tilde{u}(x,t) = \hat{u}(y(x,t),t)$ in the (x,t) variables, we notice that (3.57b) reads

$$\frac{y}{t} = \frac{x}{t} - \frac{y_0}{t} + \mathrm{O}(t^{-3/2})$$

and thus introducing $\zeta := \frac{x}{t}$ gives $C_j(\xi) = C_j(\zeta) + \mathrm{O}(t^{-1})$, $j = 1, \ldots, 4$ and

$$C_2(\xi)t = C_2(\zeta)t - \frac{\mathrm{d}C_2}{\mathrm{d}\zeta}(\zeta)y_0(\zeta) + \mathrm{o}(1).$$

It follows that the leading term of the asymptotics for $\tilde{u}(x,t)$ can be obtained from the r.h.s. of (3.59), where

(i) $C_j(\xi)$ is replaced by $C_j(\zeta)$ for $j = 1, 2, 3$, and

(ii) $C_4(\xi)$ is replaced by $\tilde{C}_4(\zeta) := C_4(\zeta) - C_2'(\zeta)y_0(\zeta)$.

In turn, calculating $C_2'(\zeta)$ in terms of $\kappa_0(\zeta)$ and using (3.60b) and $\zeta = \frac{2-8\kappa_0^2}{(1+4\kappa_0^2)^2}$, we get $C_2'(\zeta) = -2\kappa_0$ and thus

$$\tilde{C}_4(\zeta) = C_4(\zeta) + \frac{4\kappa_0(\zeta)}{\pi} \int_{1/\mu_0}^{\mu_0} \frac{\ln(1 - |r(s)|^2)}{s^2 + 1} \mathrm{d}s. \tag{3.61}$$

The asymptotic analysis we have presented above can be summarized in the following

Theorem 3.2.1. *In the solitonless case, the solution $\tilde{u}(x,t)$ of the Cauchy problem for the mCH equation in the form* (3.2) *has the following large-time asymptotics along the rays $\frac{x}{t} =: \zeta$ in the sector of the (x,t) half-plane $0 < \zeta < 2$:*

$$\tilde{u}(x,t) = \frac{C_1(\zeta)}{\sqrt{t}} \cos\left\{C_2(\zeta)t + C_3(\zeta)\ln t + \tilde{C}_4(\zeta)\right\} + \mathrm{o}(t^{-1/2}) \tag{3.62}$$

with C_1, C_2, C_3 defined by (3.60a)-(3.60c), *and $\tilde{C}_4$ defined by* (3.61)-(3.60d). *Moreover, in these definitions $h_0 = -\frac{1}{2\pi}\ln(1-|r(\mu_0)|^2)$, $\kappa_0(\zeta) = \left(\frac{\sqrt{1+4\zeta}-1-\zeta}{4\zeta}\right)^{\frac{1}{2}}$, and $\mu_0(\zeta) > 1$ is characterized by the relation $\kappa_0(\zeta) = \frac{1}{4}(\mu_0(\zeta) - \mu_0(\zeta)^{-1})$.*

By using the relation (3.1) between $\tilde{u}$ and u we immediately obtain, as a corollary, the large-time asymptotics for $u(x,t)$ in the sector $1 < \frac{x}{t} < 3$.

Theorem 3.2.2 (1st oscillatory region)**.** *Let $u_0(x)$ be a smooth function which tends sufficiently fast to* 1 *as $x \to \pm\infty$ and satisfies $(1 - \partial_x^2)u_0(x) > 0$ for all x. Assume we are in the solitonless case,* i.e., *assume that the spectral function* *associated with $u_0(x)$ has no zeros in the upper half-plane and thus the "discrete spectrum" is empty.*

Then the solution $u(x,t)$ of the Cauchy problem (2.1) *for the mCH equation has the following large-time asymptotics in the sector of the* (x,t) *half-plane defined by* $1 < \zeta := \frac{x}{t} < 3$*:*

$$u(x,t) = 1+\frac{C_1(\zeta-1)}{\sqrt{t}}\cos\left\{C_2(\zeta-1)t + C_3(\zeta-1)\ln t + \tilde{C}_4(\zeta-1)\right\}+\mathrm{o}(t^{-1/2}). \tag{3.63}$$

The error term is uniform in any sector $1+\varepsilon < \zeta < 3-\varepsilon$ *where* ε *is a small positive number.*

3.2.3 Range $-\frac{1}{4} < \xi < 0$

This range is characterized by the presence of eight real critical points: $\pm\mu_0$, $\pm\mu_1$, $\pm\mu_0^{-1}$, and $\pm\mu_1^{-1}$, see Figure 3.2. Similarly to the range $0<\xi<2$, we proceed, first, by evaluating the contribution to (3.34) from γ_0 and $-\gamma_1$ and then by using the symmetries $\mu\mapsto-\mu$ and $\mu\mapsto 1/\mu$. Notice that choosing $-\gamma_1$ surrounding $-\mu_1$ is suggested by the structure (3.29) of $\Sigma_b(\xi)$: the parts of $\Sigma_b(\xi)$ ending at μ_0 and at $-\mu_1$ are located to the left of these points. This implies that the construction of the local approximation near $-\mu_1$ follows exactly the same lines as for μ_0, the only difference being in the contributions to the r.h.s. of (3.38) from other critical points.

Namely, from (3.26) we have

$$\delta(\mu,\xi) = \left(\frac{\mu-\mu_0}{\mu-\mu_0^{-1}}\right)^{\mathrm{i}h_0}\left(\frac{\mu+\mu_0^{-1}}{\mu+\mu_0}\right)^{\mathrm{i}h_0}\left(\frac{\mu-\mu_1^{-1}}{\mu+\mu_1^{-1}}\right)^{\mathrm{i}h_1}\left(\frac{\mu+\mu_1}{\mu_1-\mu}\right)^{\mathrm{i}h_1}\mathrm{e}^{\chi(\mu)}, \tag{3.64}$$

where $h_j = -\frac{1}{2\pi}\ln(1-|r(\mu_j)|^2)$, $j=0,1$ and

$$\chi(\mu) = \frac{1}{2\pi\mathrm{i}}\left\{-\int_{-\infty}^{-\mu_1}\ln(\mu-s)\mathrm{d}\ln(1-|r(s)|^2)\right. \tag{3.65}$$
$$+\left(\int_{-\mu_0}^{-\mu_0^{-1}}+\int_{\mu_0^{-1}}^{\mu_0}\right)\ln\frac{1-|r(s)|^2}{1-|r(\mu_0)|^2}\frac{\mathrm{d}s}{s-\mu}$$

$$+\int_{-\mu_1^{-1}}^{\mu_1^{-1}} \ln \frac{1-|r(s)|^2}{1-|r(\mu_1)|^2}\frac{\mathrm{d}s}{s-\mu} - \int_{\mu_1}^{+\infty} \ln(s-\mu)\mathrm{d}\ln(1-|r(s)|^2)\Bigg\}.$$

Thus, using $\kappa_0(\mu_0)$, $f_0(\kappa_0(\mu_0))$, (see (3.35), (3.36)), and similarly for $\kappa_1(\mu_1)$ and $f_1(\kappa_1(\mu_1))$

$$\delta(\mu,\xi) \approx \hat{\mu}^{\mathrm{i}h_0}(128 f_0\kappa_0^2 t)^{-\frac{\mathrm{i}h_0}{2}}\left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_1}\mathrm{e}^{\chi(\mu_0)} \text{ with } \hat{\mu} = (\mu-\mu_0)\left(1+\frac{1}{\mu_0^2}\right)\sqrt{2f_0 t}.$$

for μ near μ_0 and

$$\delta(\mu,\xi) \approx \hat{\mu}^{\mathrm{i}h_1}(-128 f_1\kappa_1^2 t)^{-\frac{\mathrm{i}h_1}{2}}\left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_0}\mathrm{e}^{\chi(-\mu_1)} \text{ with } \hat{\mu} = (\mu+\mu_1)\left(1+\frac{1}{\mu_1^2}\right)\sqrt{-2f_1 t}$$

for μ near $-\mu_1$ (notice that $f_0(\kappa_0) = \frac{\kappa_0(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3} > 0$ whereas $f_1(\kappa_1) = \frac{\kappa_1(3-4\kappa_1^2)}{(1+4\kappa_1^2)^3} < 0$). Consequently, the coefficients $\delta_{\mu_0}(\xi,t)$ and $\delta_{\mu_1}(\xi,t)$ to be used in the construction of m_0 (3.41) for μ near μ_0 and $-\mu_1$, respectively, are as follows:

$$\begin{aligned}
\delta_{\mu_0}(\xi,t) &= \mathrm{e}^{-\mathrm{i}t\hat{\theta}(\kappa_0)}\mathrm{e}^{\chi(\mu_0)}\left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_1}(128 f_0\kappa_0^2(\mu_0)t)^{-\frac{\mathrm{i}h_0}{2}},\\
\delta_{\mu_1}(\xi,t) &= \mathrm{e}^{\mathrm{i}t\hat{\theta}(\kappa_1)}\mathrm{e}^{\chi(-\mu_1)}\left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{\mathrm{i}h_0}(-128 f_1\kappa_1^2(\mu_1)t)^{-\frac{\mathrm{i}h_1}{2}},
\end{aligned} \tag{3.66}$$

which implies (cf. (3.44))

$$\begin{aligned}
m_0^{-1}(y,t,\mu) &= I + \frac{B_{\mu_0}(\xi,t)}{\sqrt{t}(\mu-\mu_0)} + \mathrm{O}(t^{-1}), \quad \text{for } \mu \text{ inside } \gamma_0,\\
m_0^{-1}(y,t,\mu) &= I + \frac{B_{\mu_1}(\xi,t)}{\sqrt{t}(\mu+\mu_1)} + \mathrm{O}(t^{-1}), \quad \text{for } \mu \text{ inside } -\gamma_1,
\end{aligned}$$

where (cf.(3.45))

$$B_{\mu_0}(\xi,t) = \begin{pmatrix} 0 & B_0(\xi,t)\\ \bar{B}_0(\xi,t) & 0\end{pmatrix}, \quad B_{\mu_1}(\xi,t) = \begin{pmatrix} 0 & B_1(\xi,t)\\ \bar{B}_1(\xi,t) & 0\end{pmatrix},$$

with

$$\begin{aligned}
B_0(\xi,t) &= \left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{2\mathrm{i}h_1}\frac{\mathrm{i}\delta_{\mu_0}^2(\xi,t)\beta_{\mu_0}(\xi)}{(1+\mu_0^{-2})\sqrt{2f_0(\kappa_0)}},\\
B_1(\xi,t) &= \left(\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}\right)^{2\mathrm{i}h_0}\frac{\mathrm{i}\delta_{\mu_1}^2(\xi,t)\beta_{\mu_1}(\xi)}{(1+\mu_1^{-2})\sqrt{-2f_1(\kappa_1)}}.
\end{aligned} \tag{3.67}$$

Here $\beta_{\mu_0}(\xi)$ is given by (3.43) and

$$\beta_{\mu_1}(\xi) = \sqrt{h_1}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}-\arg(-\bar{r}(-\mu_1))+\arg\Gamma(\mathrm{i}h_1)\right)}.$$

In turn, due to the symmetries, the asymptotics for $\hat{m}(y,t,0)$, $\hat{m}(y,t,\mathrm{i})$, and $\hat{m}_1(y,t)$ (and thus for $M^R(y,t,0)$, $M^R(y,t,\mathrm{i})$, and $M_1^R(y,t)$) in the present case (cf. (3.47)-(3.49) and (3.52)) involve two terms:

$$M^R(y,t,0) = I + \frac{4\mathrm{i}}{\sqrt{t}}\left(\frac{\operatorname{Im} B_0(\xi,t)}{\mu_0} - \frac{\operatorname{Im} B_1(\xi,t)}{\mu_1}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mathrm{o}(t^{-1/2}),$$

$$M^R(y,t,\mathrm{i}) = \left(I + \frac{2\mathrm{i}}{\sqrt{t}}\left(\frac{\operatorname{Im} B_0(\xi,t)}{\mu_0} - \frac{\operatorname{Im} B_1(\xi,t)}{\mu_1}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}),$$

$$M_1^R(y,t) = \frac{4}{\sqrt{t}}\begin{pmatrix} 0 & \operatorname{Re}\frac{B_0}{(\mu_0-\mathrm{i})^2} + \operatorname{Re}\frac{B_1}{(\mu_1+\mathrm{i})^2} \\ \operatorname{Re}\frac{\bar{B}_0}{(\mu_0-\mathrm{i})^2} + \operatorname{Re}\frac{\bar{B}_1}{(\mu_1+\mathrm{i})^2} & 0 \end{pmatrix}\delta^{\sigma_3}(\mathrm{i},\xi) + \mathrm{o}(t^{-1/2}),$$

where $\delta(\mathrm{i},\xi)$ is now given by

$$\delta(\mathrm{i},\xi) = \exp\left\{\frac{1}{\pi}\left(\int_0^{\mu_1^{-1}} + \int_{\mu_0^{-1}}^{\mu_0} + \int_{\mu_1}^{+\infty}\right)\frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s\right\}. \tag{3.68}$$

It follows that the asymptotics for the parametric representation of $\tilde{u}$, see (3.57a) and (3.57b), takes the form

$$\hat{u}(y,t) = \frac{8}{\sqrt{t}}\left(\frac{(1-\mu_0^2)}{(1+\mu_0^2)^2}\operatorname{Re} B_0 + \frac{(1-\mu_1^2)}{(1+\mu_1^2)^2}\operatorname{Re} B_1\right) + \mathrm{o}(t^{-1/2}), \tag{3.69a}$$

$$x(y,t) = y + y_{01}(\xi) + \mathrm{O}(t^{-1/2}), \tag{3.69b}$$

where $y_{01}(\xi) = \frac{2}{\pi}\left(\int_0^{\mu_1^{-1}} + \int_{\mu_0^{-1}}^{\mu_0} + \int_{\mu_1}^{+\infty}\right)\frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s$.

Recalling the definitions (3.67) of B_j, $j=0,1$, and arguing as in the case $0<\xi<2$, we arrive at the asymptotics of $\hat{u}(y,t)$ (cf. (3.59))

$$\hat{u}(y,t) = \sum_{j=0,1}\frac{C_1^{(j)}(\xi)}{\sqrt{t}}\cos\left\{C_2^{(j)}(\xi)t + C_3^{(j)}(\xi)\ln t + C_4^{(j)}(\xi)\right\} + \mathrm{o}(t^{-1/2}), \tag{3.70}$$

where

$$C_1^{(j)}(\xi) = -\left(\frac{8h_j\kappa_j}{|3-4\kappa_j^2|}\right)^{\frac{1}{2}}, \tag{3.71a}$$

$$C_2^{(j)}(\xi) = \frac{(-1)^j 32\kappa_j^3}{(1+4\kappa_j^2)^2}, \tag{3.71b}$$

$$C_3^{(j)}(\xi) = -h_j, \tag{3.71c}$$

$$C_4^{(j)}(\xi) = \frac{3\pi}{4} - 2\mathrm{i}\chi((-1)^j\mu_j) - h_j \ln \frac{128\kappa_j^3|3-4\kappa_j^2|}{(1+4\kappa_j^2)^3} - \arg(-\bar{r}((-1)^j\mu_j)) + \arg\Gamma(\mathrm{i}h_j) + 2h_{1-j}\ln\frac{\kappa_1+\kappa_0}{\kappa_1-\kappa_0}, \tag{3.71d}$$

and $\chi(\mu)$ is given by (3.65).

Returning to the (x,t) variables, $C_4^{(j)}(\xi)$, $j=0,1$ are to be replaced, similarly to (3.61), by

$$\tilde{C}_4^{(j)}(\zeta) = C_4^{(j)}(\zeta) + \frac{(-1)^j 4\kappa_j(\zeta)}{\pi}\left(\int_0^{\mu_1^{-1}} + \int_{\mu_0^{-1}}^{\mu_0} + \int_{\mu_1}^{+\infty}\right)\frac{\ln(1-|r(s)|^2)}{s^2+1}\mathrm{d}s, \tag{3.72}$$

which finally leads us to

Theorem 3.2.3. *In the solitonless case, the solution $\tilde{u}(x,t)$ of the Cauchy problem for the mCH equation in the form* (3.2) *has the following large-time asymptotics along the rays $\frac{x}{t} =: \zeta$ in the sector of the (x,t) half-plane $-\frac{1}{4} < \zeta < 0$:*

$$\tilde{u}(x,t) = \sum_{j=0,1}\frac{C_1^{(j)}(\zeta)}{\sqrt{t}}\cos\left\{C_2^{(j)}(\zeta)t + C_3^{(j)}(\zeta)\ln t + \tilde{C}_4^{(j)}(\zeta)\right\} + \mathrm{o}(t^{-1/2})$$

with an error term uniform in any sector $-\frac{1}{4}+\varepsilon < \zeta < -\varepsilon$ where ε is a small positive number. The coefficients $C_1^{(j)}, C_2^{(j)}, C_3^{(j)}$ are defined by (3.71a)-(3.71c) *and $\tilde{C}_4^{(j)}$ is defined by* (3.72)-(3.71d). *In these definitions*

$$h_j = -\frac{1}{2\pi}\ln(1-|r(\mu_j)|^2),$$

$$\kappa_0(\zeta) = \left(\frac{\sqrt{1+4\zeta}-1-\zeta}{4\zeta}\right)^{\frac{1}{2}}, \quad \kappa_1(\zeta) = \left(-\frac{\sqrt{1+4\zeta}+1+\zeta}{4\zeta}\right)^{\frac{1}{2}},$$

and $\mu_j(\zeta) > 1$, $j=0,1$ is characterized by the relation $\kappa_j(\zeta) = \frac{1}{4}(\mu_j(\zeta) - \mu_j(\zeta)^{-1})$.

Using again (3.1) we obtain, as a corollary, the large-time asymptotics of $u(x,t)$ in the sector $\frac{3}{4} < \frac{x}{t} < 1$.

Theorem 3.2.4 (2^{nd} oscillatory region)**.** *Let $u_0(x)$ be a smooth function which tends sufficiently fast to 1 as $x \to \pm\infty$ and satisfies $(1-\partial_x^2)u_0(x) > 0$ for all x. Assume we are in the solitonless case,* i.e., *assume that the spectral function associated with $u_0(x)$ has no zeros in the upper half-plane and thus the "discrete spectrum" is empty.*

Then the solution $u(x,t)$ of the Cauchy problem (2.1) for the mCH equation has the following large-time asymptotics along the rays $\frac{x}{t} =: \zeta$ in the sector of the (x,t) half-plane defined by $\frac{3}{4} < \zeta < 1$:

$$u(x,t) = 1 + \sum_{j=0,1} \frac{C_1^{(j)}(\zeta-1)}{\sqrt{t}} \cos\left\{C_2^{(j)}(\zeta-1)t + C_3^{(j)}(\zeta-1)\ln t + \tilde{C}_4^{(j)}(\zeta-1)\right\} + \mathrm{o}(t^{-1/2})$$

The error term is uniform in any sector $\frac{3}{4}+\varepsilon < \zeta < 1-\varepsilon$ where ε is small and positive.

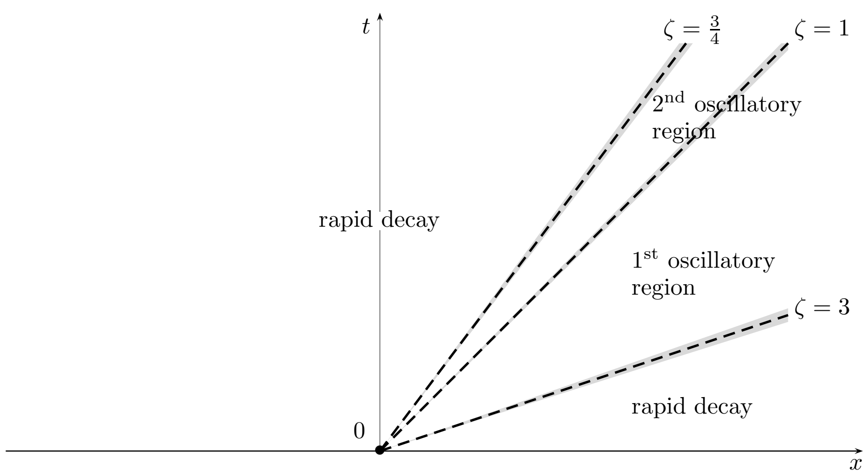

Figure 3.3: Asymptotics for $u(x,t)$ according to $\zeta := \frac{x}{t}$: the four regions.

Remark 3.2.5 (other regions). In the solitonless case, $u(x,t)$ decays rapidly to 1 in the sectors $\frac{x}{t} > 3$ and $\frac{x}{t} < \frac{3}{4}$, cf. [27]. This is due to the fact that for these ranges of values of $\frac{x}{t}$, $\theta(\mu,\xi)$ has no real stationary points (lying on the contour of the original RH problem).

Remark 3.2.6 (transition zones). Transitions between the sectors (i.e., for $\frac{x}{t}$ near $\frac{3}{4}$ and 3) are characterized by the merging of real stationary points of $\theta(\mu,\xi)$, which implies that the error terms in Theorems 3.2.2 and 3.2.4 grow as $\varepsilon \to 0$ and thus the presented asymptotics becomes incorrect. On the other hand, in analogy with the case of the Camassa–Holm equation (see [17]), using a different scaling of the spectral parameter, one can obtain a correct asymptotics in the transition zones in terms of Painlevé transcendents [20].

3.3 Soliton asymptotics

As for other soliton equations, the soliton solutions of the mCH equation are associated with the residue conditions (2.42). Accordingly, these conditions give rise to soliton asymptotics in a dedicated sector of the (x,t) plane. They can be handled by adding to the contour small circles around each μ_j and its symmetry counterparts and thus reducing the residue conditions to associated jump conditions across the circles and then proceeding as in the case without residue conditions [21].

The one-soliton solution $u \equiv u_{\theta,\delta}$ with parameters (θ,δ), where $\theta \in (0,\frac{\pi}{2})$, has the following parametric representation:

$$u(x,t) = \tilde{u}(x-t,t) + 1 = \hat{u}(y(x-t,t),t) + 1, \tag{3.73a}$$

where

$$\hat{u}(y,t) = 4\tan^2\theta\,\frac{z^2(y,t) + 2\cos^2\theta\cdot z(y,t) + \cos^2\theta}{(z^2(y,t) + 2z(y,t) + \cos^2\theta)^2}z(y,t), \tag{3.73b}$$

$$x(y,t) = t + y + 2\ln\frac{z(y,t)+1+\sin\theta}{z(y,t)+1-\sin\theta}, \tag{3.73c}$$

and

$$z(y,t) = 2\delta\sin\theta\,\mathrm{e}^{\sin\theta\left(y-\frac{2}{\cos^2\theta}t\right)}. \tag{3.73d}$$

Notice that if $\theta \in (\frac{\pi}{3},\frac{\pi}{2})$, then the x to y correspondence (3.73c) is not one-to-one and thus in this case (3.73) represent a loop-type multi-valued function of x. On the other hand, if $\theta \in (0,\frac{\pi}{3})$, then (3.73) represent a smooth function, which dominates the long-time behavior of the solution of problem (2.1) in an associated sector. Similarly to [21] the following theorem holds:

Theorem 3.3.1 (soliton asymptotics). *Assume that $a(\mu)$ associated with $u_0(x)$ has $2n$ simple zeros: $\mu_j = \mathrm{e}^{\mathrm{i}\theta_j}$ with $0 < \theta_1 < \cdots < \theta_n < \frac{\pi}{3}$ and $\mu_{n+l} = -\bar{\mu}_l$ for $l = 1, \ldots, n$. Then the asymptotics of u (understood as a global solution of (2.1a) or a solution continued beyond possible blow-ups following the RH formalism) in the sector $3 < \frac{x}{t} < 9$ is given as follows:*

1. *In the sectors $\left|\frac{x}{t} - 1 - \frac{2}{\cos^2\theta_j}\right| < \varepsilon$ with any $\varepsilon > 0$ sufficiently small,*

$$u(x,t) = u_j(x,t) + \mathrm{O}(t^{-l}), \qquad j = 1, \ldots, n$$

with $l \geq 1$ depending on the rate of decay of $u_0(x) - 1$ as $|x| \to \infty$, where u_j is given, parametrically, by (3.73) with θ, δ, and z replaced by θ_j, δ_j, and z_j respectively, where

$$z_j(y,t) = 2\delta_j \sin\theta_j \, \mathrm{e}^{\sin\theta_j\left(y - \frac{2}{\cos^2\theta_j}t + y_j^0\right)}$$

and y_j^0 are constants determined by $\{\theta_m, \delta_m\}_{m=j+1}^n$.

2. *Outside these sectors, $u(x,t) = \mathrm{O}(t^{-l})$.*

Remark 3.3.2. Since it is the RH problem parametrized by y and t that undergoes the asymptotic analysis, and the soliton solutions (3.73b) are smooth in (y,t) variables, the asymptotic results of Theorem 3.3.1 hold true for the mCH equation written in (y,t) variables, see Chapter 2, even if $a(\mu)$ has zeros at some $\mu^* = \mathrm{e}^{\mathrm{i}\theta^*}$ with $\theta^* \in (\frac{\pi}{3}, \frac{\pi}{2})$. On the other hand, this allows deducing a sufficient condition for wave breaking of solutions of problem (2.1a) (in (x,t) variables): If $a(\mu)$ has a zero $\mu^* = \mathrm{e}^{\mathrm{i}\theta^*}$ with $\theta^* \in (\frac{\pi}{3}, \frac{\pi}{2})$, then wave breaking occurs at a certain finite time. In this case, the mechanism of wave breaking consists in breaking the one-to-one correspondence $x \leftrightarrow y$ (cf. [32]).

Remark 3.3.3 (other regions). $u(x,t)$ decays rapidly to 1 in the sectors $\frac{x}{t} > 9$ and $\frac{x}{t} < \frac{3}{4}$, cf. [27]. This is due to the fact that for these ranges of values of $\frac{x}{t}$, $\theta(\mu,\xi)$ has no real stationary points (lying on the contour of the original RH problem).

3.4 Conclusions to Chapter 3

In this Section, we have applied the nonlinear steepest descent method, based on the Riemann–Hilbert formalism, to study the large-time asymptotics of the solution of the Cauchy problem for the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach a non-zero constant at the both infinities of the space variable. We have focused on the study of the solitonless case assuming that there are no residue conditions (for the soliton case, where the basic RH problem involves residue conditions, one can reduce (using the Blaschke–Potapov factors) this RH problem to that having no residue conditions).

For the sake of the large-t analysis, we have reduced the original (singular) RH problem representation for the solution of the mCH equation to the solution of a regular RH problem (i.e., to a RH problem with the jump and normalization conditions only). A notable feature of the modified Camassa–Holm equation is that the associated basic RH problem has two singularity conditions (at $\mu = \pm 1$) with different matrix structures, which does not allow getting rid of them by reducing the matrix RH problem to a vector one, as it can be done in the case of the (original) Camassa–Holm equation. In our approach, we have addressed the reduction problem in two steps. First, we have reduced the RH problem with the singularity conditions at $\mu = \pm 1$ to a RH problem which is characterized by the following two conditions: (i) the matrix entries are regular at $\mu = \pm 1$, but the determinant of the (matrix) solution vanishes at $\mu = \pm 1$ (notice that $\det M(\mu) \equiv 1$ for the solution of the original RH problem); (ii) the solution is singular at $\mu = 0$. Then, we have represented the solution of the latter RH problem in terms of the solution of a regular one. In turn, the solution of the resulting regular RH problem was analyzed asymptotically, as $t \to +\infty$, using an appropriate adaptation of the nonlinear steepest descent method.

In such a way, we have obtained the results of the asymptotic analysis in the solitonless case for the two sectors $\frac{3}{4} < \frac{x}{t} < 1$ and $1 < \frac{x}{t} < 3$ (in the (x, t)

half-plane, $t > 0$), where the leading asymptotic term of the deviation of the solution from the background is nontrivial: this term is given by modulated (with parameters depending on $\frac{x}{t}$), decaying (as $t^{-1/2}$) trigonometric oscillations.

Chapter 4

The Riemann–Hilbert approach to the modified Camassa–Holm equation with step-like boundary conditions

The results of this Chapter are published in [88].

We consider the initial value problem for the mCH equation (4.1a):

$$m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m := u - u_{xx}, \quad t > 0, \ -\infty < x < +\infty, \tag{4.1a}$$

$$u(x, 0) = u_0(x), \qquad -\infty < x < +\infty, \tag{4.1b}$$

assuming that

$$u_0(x) \to \begin{cases} A_1 \text{ as } x \to -\infty \\ A_2 \text{ as } x \to \infty \end{cases}, \tag{4.2}$$

where A_1 and A_2 are some different constants, and that the solution $u(x,t)$ preserves this behavior for all fixed $t > 0$.

We develop the Riemann–Hilbert formalism to problem (4.1) with the step-like initial data (4.2) assuming that $0 < A_1 < A_2$ and that $u(x,t)$ approaches its large-x limits sufficiently fast. We also assume that $m(x,0) = u_0(x) - u_{0xx}(x) > 0$ for all x; then it can be shown that $m(x,t) > 0$ for all t (see Appendix 4.1, for the case of the CH equation, see [41, 43]). In Section 4.2, we introduce appropriate transformations of the Lax pair equations and the associated Jost solutions ("eigenfunctions") and discuss analytic properties of the eigenfunctions and the corresponding spectral functions (scattering coefficients), including the symmetries and the behavior at the branch points. Here the analysis is performed when fixing the branches of the functions $k_j(\lambda) := \sqrt{\lambda^2 - \frac{1}{A_j^2}}$, $j = 1, 2$ involved in the Lax pair transformations as having the branch cuts $(-\infty, -\frac{1}{A_j}) \cup (\frac{1}{A_j}, \infty)$. In Section 4.3, the introduced eigenfunctions are used in the construction of the Riemann–Hilbert problems, whose solutions evaluated at $\lambda = 0$ (where λ is the spectral parameter in the Lax pair equations) give parametric representations of the solution of problem (4.1). The case $0 < A_2 < A_1$ is briefly discussed in Subsection 4.4.

4.1 Sign-preserving property of m

In order to control the analytic properties of the Jost solution the sign-preserving property of m plays a crucial role. The analogous result for the Camassa–Holm equation can be found in [41, 43].

Assume that $u(x,t) - A_1 \in H^3(-\infty, a)$ and $u(x,t) - A_2 \in H^3(a, \infty)$ for any real a and for any $t \in (0, T)$, where $T \leq +\infty$ is the maximal existing time. Then Morrey's inequality implies that $(mu_x)(s, x)$ is uniformly bounded for $0 < s < t < T$, $x \in \mathbb{R}$. Consider the Cauchy problem for $q(t, x)$:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = (u^2 - u_x^2)(q(t,x), t), \quad t \in (0, T), \ x \in \mathbb{R}, \tag{4.3a}$$

$$q(0, x) = x, \quad x \in \mathbb{R}, \tag{4.3b}$$

where $u(x, t)$ solves (4.1). Differentiating (4.3) with respect to x leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} q_x(t,x) = (2mu_x)(q(t,x), t) q_x(t,x), \tag{4.4a}$$

$$q_x(0, x) = 1, \quad x \in \mathbb{R}. \tag{4.4b}$$

It follows that

$$q_x(t,x) = \mathrm{e}^{2\int_0^t (mu_x)(q(s,x),s)\mathrm{d}s} > 0 \tag{4.5}$$

and, moreover,

$$\mathrm{e}^{k(t)} \leq q_x(t,x) \leq \mathrm{e}^{K(t)}, \quad t \in [0, T) \tag{4.6}$$

for some $k(t)$ and $K(t)$.

Now observe that from (4.1a) and (4.3) it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\left[m(q(t,x),t)q_x(t,x)\right] = 0$. Indeed,

$$\begin{aligned}
&\frac{d}{dt}\left[m(q(t,x),t)q_x(t,x)\right] \\
&\quad = \left[m_t(q(t,x),t) + m_x(q(t,x),t)q_t(t,x)\right](q(t,x),t)q_x(t,x) + m(q(t,x),t)q_{tx}(t,x) \\
&\quad = \left[-(u^2-u_x^2)_x m - (u^2-u_x^2)m_x + m_x(u^2-u_x^2)\right](q(t,x),t)q_x(t,x) \\
&\quad + 2(m^2u_x)(q(t,x),t)q_x(t,x) = 0.
\end{aligned}$$

Thus, due to (4.3b) and (4.4b), we have

$$m(t,q(t,x))q_x(t,x) = m(0,q(0,x))q_x(0,x) = m(0,x).$$

Hence, if $m(x,0) > 0$, then $m(q(t,x),t) > 0$ for all $t \in [0,T)$, $x \in \mathbb{R}$. Since $q_x(t,x) > 0$, we have that $q(t,x)$ is strictly increasing function. Moreover, integrating (4.6) w.r.t. x, we also have $\lim_{x\to\pm\infty} q(t,x) = \pm\infty$. Hence $q(x,t)$ is one-to-one from $\mathbb{R}$ onto $\mathbb{R}$ and thus $m(t,x) > 0$ for all $t \in [0,T)$, $x \in \mathbb{R}$.

4.2 Lax pairs and eigenfunctions

4.2.1 Lax pairs

The Lax pair for the mCH equation (4.1a) has the following form [108]:

$$\Phi_x(x,t,\lambda) = U(x,t,\lambda)\Phi(x,t,\lambda), \tag{4.7a}$$

$$\Phi_t(x,t,\lambda) = V(x,t,\lambda)\Phi(x,t,\lambda), \tag{4.7b}$$

where the coefficients U and V are defined by

$$U = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda m & 1 \end{pmatrix}, \tag{4.7c}$$

$$V = \begin{pmatrix} \lambda^{-2} + \frac{u^2-u_x^2}{2} & -\lambda^{-1}(u-u_x) - \frac{\lambda(u^2-u_x^2)m}{2} \\ \lambda^{-1}(u+u_x) + \frac{\lambda(u^2-u_x^2)m}{2} & -\lambda^{-2} - \frac{u^2-u_x^2}{2} \end{pmatrix}, \tag{4.7d}$$

with $m(x,t)=u(x,t)-u_{xx}(x,t)$. The RH formalism for integrable nonlinear equations is based on using appropriately defined eigenfunctions, i.e., solutions of the Lax pair, whose behavior as functions of the spectral parameter is well-controlled in the extended complex plane. Notice that the coefficient matrices U and V are traceless, which provides that the determinant of a matrix solution to (4.7) (composed of two vector solutions) is independent of x and t.

Also notice that U and V have singularities (in the extended complex λ-plane) at $\lambda=0$ and $\lambda=\infty$. In particular, U is singular at $\lambda=\infty$, which necessitates a special care when constructing solutions with controlled behavior as $\lambda\to\infty$. On the other hand, U becomes u-independent at $\lambda=0$ (a property shared by many Camassa–Holm-typed equations, including the CH equation itself), which suggests using the behavior of the constructed solutions as $\lambda\to 0$ in order to "extract" the solution of the nonlinear equation in question from the solution of an associated Riemann–Hilbert problem (whose construction, in the direct problem, involves the dedicated solutions of the Lax pair equations).

Notations

- We introduce the following notations for various intervals of the real axis:
$$\Sigma_j=(-\infty,-\frac{1}{A_j}]\cup[\frac{1}{A_j},\infty),\qquad \dot{\Sigma}_j=(-\infty,-\frac{1}{A_j})\cup(\frac{1}{A_j},\infty),$$
$$\Sigma_0=[-\frac{1}{A_1},-\frac{1}{A_2}]\cup[\frac{1}{A_2},\frac{1}{A_1}],\qquad \dot{\Sigma}_0=(-\frac{1}{A_1},-\frac{1}{A_2})\cup(\frac{1}{A_2},\frac{1}{A_1}).$$
Notice that $\Sigma_1\subset\Sigma_2$ since we assume $A_1<A_2$.

- For $\lambda\in\Sigma_j$ we denote by λ_+ (λ_-) the point of the upper (lower) side of Σ_j (i.e. $\lambda_\pm=\lim_{\epsilon\downarrow 0}\lambda\pm \mathrm{i}\epsilon$). Then we have $-\lambda_+=(-\lambda)_-$ and $\overline{\lambda_+}=\lambda_-$.

- $k_j(\lambda):=\sqrt{\lambda^2-\frac{1}{A_j^2}}$, $j=1,2$ with the branch cut Σ_j and the branch is fixed by the condition $k_j(0)=\frac{\mathrm{i}}{A_j}$.

Observe that $\operatorname{Im}k_j(\lambda)\geq 0$ on $\mathbb{C}$, and $k_j(\lambda)$ is real valued on the both sides of Σ_j. Also notice that $k_j(\lambda)=\omega_j^+(\lambda)\omega_j^-(\lambda)$, where $\omega_j^+(\lambda)=\sqrt{\lambda-\frac{1}{A_j}}$ with the branch cut $[\frac{1}{A_j}, \infty)$ and $\omega_j^+(0) = \frac{\mathrm{i}}{\sqrt{A_j}}$, and $\omega_j^-(\lambda) = \sqrt{\lambda + \frac{1}{A_j}}$ with the branch cut $(-\infty, -\frac{1}{A_j}]$ and $\omega_j^-(0) = \frac{1}{\sqrt{A_j}}$.

Observe the following symmetry relations:

$$k_j(-\lambda) = k_j(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_j, \tag{4.8a}$$

$$k_j(\lambda_+) = -k_j((-\lambda)_+), \quad \lambda \in \Sigma_j, \tag{4.8b}$$

$$\overline{k_j(\overline{\lambda})} = -k_j(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_j, \tag{4.8c}$$

$$\overline{k_j(\lambda_+)} = k_j(\lambda_+), \quad \lambda \in \Sigma_j \tag{4.8d}$$

(here (4.8b) follows from (4.8a) and (4.8c)).

In order to control the large λ behavior of solutions of (4.7), we introduce two gauge transformations associated with $x \to (-1)^j\infty$ and $m \to A_j$ (in a similar way as it was done in the case of the constant background in Chapter 2).

Proposition 4.2.1. *Equation* (4.1a) *admits Lax pairs of the form ($j = 1, 2$)*

$$\hat{\Phi}_{jx} + Q_{jx}\hat{\Phi}_j = \hat{U}_j\hat{\Phi}_j, \tag{4.9a}$$

$$\hat{\Phi}_{jt} + Q_{jt}\hat{\Phi}_j = \hat{V}_j\hat{\Phi}_j, \tag{4.9b}$$

whose coefficients $Q_j \equiv Q_j(x,t,\lambda)$, $\hat{U}_j \equiv \hat{U}_j(x,t,\lambda)$, and $\hat{V}_j \equiv \hat{V}_j(x,t,\lambda)$ are 2×2 matrices given by (4.13) *and* (4.14)*, which are characterized by the following properties:*

(i) *Q_j is diagonal and is unbounded as $\lambda \to \infty$.*

(ii) *$\hat{U}_j = \mathrm{O}(1)$ and $\hat{V}_j = \mathrm{O}(1)$ as $\lambda \to \infty$.*

(iii) *The diagonal parts of $\hat{U}_j$ and $\hat{V}_j$ decay as $\lambda \to \infty$.*

(iv) *$\hat{U}_j \to 0$ and $\hat{V}_j \to 0$ as $x \to (-1)^j\infty$.*

Proof. Notice that U in (4.7c) can be written as

$$U(x,t,\lambda) = \frac{m(x,t)}{2A_j}\begin{pmatrix} -1 & \lambda A_j \\ -\lambda A_j & 1 \end{pmatrix} + \frac{m(x,t) - A_j}{2A_j}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.10}$$

where $m(x,t) - A_j \to 0$ as $x \to (-1)^j\infty$. The first (non-decaying, as $x \to (-1)^j\infty$) term in (4.10) can be diagonalized by introducing

$$\hat{\Phi}_j(x,t,\lambda) := D_j(\lambda)\Phi(x,t,\lambda), \tag{4.11}$$

where

$$D_j(\lambda) := \sqrt{\frac{1}{2}}\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda)} - 1}\begin{pmatrix} \frac{\lambda A_j}{1-\mathrm{i}A_jk_j(\lambda)} & -1 \\ -1 & \frac{\lambda A_j}{1-\mathrm{i}A_jk_j(\lambda)} \end{pmatrix} \tag{4.12}$$

with

$$D_j^{-1}(\lambda) := \sqrt{\frac{1}{2}}\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda)} - 1}\begin{pmatrix} \frac{\lambda A_j}{1-\mathrm{i}A_jk_j(\lambda)} & 1 \\ 1 & \frac{\lambda A_j}{1-\mathrm{i}A_jk_j(\lambda)} \end{pmatrix}.$$

The factor $\sqrt{\frac{1}{2}}\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda)} - 1}$ provides $\det D_j(\lambda) = 1$ for all λ, and the branch of the square root is chosen so that the branch cut is $[0,\infty)$ and $\sqrt{-1} = \mathrm{i}$; then $\overline{\sqrt{w_j}} = -\sqrt{\overline{w_j}}$. Observe that $\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda)} - 1}$ is well defined as a function of λ on $\mathbb{C} \setminus \Sigma_j$ as well as on the sides of Σ_j. Then (4.11) transforms (4.7a) into

$$\hat{\Phi}_{jx} + \frac{\mathrm{i}k_j(\lambda)m}{2}\sigma_3\hat{\Phi}_j = \hat{U}_j\hat{\Phi}_j, \tag{4.13a}$$

where $\hat{U}_j \equiv \hat{U}_j(x,t,\lambda)$ is given by

$$\hat{U}_j = \frac{\lambda(m - A_j)}{2A_jk_j(\lambda)}\sigma_2 + \frac{m - A_j}{2\mathrm{i}A_j^2k_j(\lambda)}\sigma_3. \tag{4.13b}$$

In turn, the t-equation (4.7b) of the Lax pair is transformed into

$$\hat{\Phi}_{jt} + \mathrm{i}A_jk_j(\lambda)\left(-\frac{1}{2A_j}m(u^2 - u_x^2) - \frac{1}{\lambda^2}\right)\sigma_3\hat{\Phi}_j = \hat{V}_j\hat{\Phi}_j, \tag{4.13c}$$

where $\hat{V}_j \equiv \hat{V}_j(x,t,\lambda)$ is given by

$$\begin{aligned} \hat{V}_j = &-\frac{1}{2A_jk_j(\lambda)}\left(\lambda(u^2 - u_x^2)(m - A_j) + \frac{2(u - A_j)}{\lambda}\right)\sigma_2 + \frac{\tilde{u}_x}{\lambda}\sigma_1 \\ &- \frac{1}{\mathrm{i}A_jk_j(\lambda)}\left(A_j(u - A_j) + \frac{1}{2A_j}(u^2 - u_x^2)(m - A_j)\right)\sigma_3. \end{aligned} \tag{4.13d}$$

Now notice that equations (4.13a) and (4.13c) have the desired form (4.9), if we define Q_j by

$$Q_j(x,t,\lambda) := p_j(x,t,\lambda)\sigma_3, \tag{4.14a}$$

with

$$p_j(x,t,\lambda) := \mathrm{i}A_j k_j(\lambda)\left(\frac{1}{2A_j}\int_{(-1)^j\infty}^{x}(m(\xi,t)-A_j)\mathrm{d}\xi+\frac{x}{2}-t\big(\frac{1}{\lambda^2}+\frac{A_j^2}{2}\big)\right). \tag{4.14b}$$

Indeed, we obviously have $p_{jx}=\frac{\mathrm{i}k_j(\lambda)m}{2}$; on the other hand, the equality

$$p_{jt}=\mathrm{i}A_j k_j(\lambda)\left(-\frac{1}{2A_j}m(u^2-u_x^2)-\frac{1}{\lambda^2}\right)$$

follows from (4.1a). □

Remark 4.2.2. In Chapter 2, which deals with the mCH equation on a single background, introducing a uniformizing spectral parameter (such that λ and the respective $k(\lambda)$ are rational with respect to it) allowed getting rid of square roots and thus avoiding the problem of specifying particular branches. In the present case, since we have to deal with two different functions, $k_1(\lambda)$ and $k_2(\lambda)$, associated with two different backgrounds, we keep the original spectral parameter λ as the spectral variable in the RH problem formalism we are going to develop.

4.2.2 Eigenfunctions

The Lax pair in the form (4.13) allows us to determine, via associated integral equations, dedicated solutions having a well-controlled behavior as functions of the spectral parameter λ for large values of λ. Indeed, introducing

$$\widetilde{\Phi}_j=\hat{\Phi}_j\mathrm{e}^{Q_j} \tag{4.15}$$

(understanding $\widetilde{\Phi}_j$ as a 2×2 matrix), equations (4.13a) and (4.13c) can be rewritten as

$$\begin{cases}\widetilde{\Phi}_{jx}+[Q_{jx},\widetilde{\Phi}_j]=\hat{U}_j\widetilde{\Phi}_j,\\ \widetilde{\Phi}_{jt}+[Q_{jt},\widetilde{\Phi}_j]=\hat{V}_j\widetilde{\Phi}_j,\end{cases} \tag{4.16}$$

where $[\,\cdot\,,\,\cdot\,]$ stands for the commutator. We now determine the Jost solutions $\widetilde{\Phi}_j\equiv\widetilde{\Phi}_j(x,t,\lambda)$, $j=1,2$ of (4.16) as the solutions of the associated Volterra integral equations:

$$\widetilde{\Phi}_j(x,t,\lambda) = I + \int_{(-1)^j\infty}^{x} \mathrm{e}^{Q_j(\xi,t,\lambda)-Q_j(x,t,\lambda)} \hat{U}_j(\xi,t,\lambda)\widetilde{\Phi}_j(\xi,t,\lambda)\mathrm{e}^{Q_j(x,t,\lambda)-Q_j(\xi,t,\lambda)}\mathrm{d}\xi, \tag{4.17}$$

or, taking into account the definition (4.14) of Q_j,

$$\widetilde{\Phi}_j(x,t,\lambda) = I + \int_{(-1)^j\infty}^{x} \mathrm{e}^{\frac{\mathrm{i}k_j(\lambda)}{2}\int_x^\xi m(\tau,t)\mathrm{d}\tau\sigma_3} \hat{U}_j(\xi,t,\lambda)\widetilde{\Phi}_j(\xi,t,\lambda)\mathrm{e}^{-\frac{\mathrm{i}k_j(\lambda)}{2}\int_x^\xi m(\tau,t)\mathrm{d}\tau\sigma_3}\mathrm{d}\xi, \tag{4.18}$$

(I is the 2×2 identity matrix).

Hereafter, $\hat{\Phi}_j := \widetilde{\Phi}_j\mathrm{e}^{-Q_j}$, $j = 1, 2$ denote the corresponding Jost solutions of (4.13) whereas $\Phi_j := D_j^{-1}(\lambda)\hat{\Phi}_j$ denote the corresponding Jost solutions of (4.7).

We are now able to analyze the analytic and asymptotic properties of the solutions $\widetilde{\Phi}_j$ of (4.18) as functions of λ, using Neumann series expansions. Let $A^{(1)}$ and $A^{(2)}$ denote the columns of a 2×2 matrix $A = \begin{pmatrix} A^{(1)} & A^{(2)} \end{pmatrix}$. Using these notations we have the following properties:

- $\widetilde{\Phi}_j^{(j)}$ is analytic in $\mathbb{C}\setminus\Sigma_j$ and has a continuous extension on the lower and upper sides of $\dot{\Sigma}_j$.
- $\widetilde{\Phi}_j^{(1)}$ and $\widetilde{\Phi}_j^{(2)}$ are well defined and continuous on the lower and upper sides of $\dot{\Sigma}_j$.

In (4.16) the coefficients are traceless matrices, from which it follows that $\det\tilde{\Phi}_j$ is independent on x and t, and hence

- $\det\widetilde{\Phi}_j \equiv 1$.

Regarding the values of $\widetilde{\Phi}_j$ at particular points in the λ-plane, (4.18) implies the following:

- $\begin{pmatrix} \tilde{\Phi}_1^{(1)} & \tilde{\Phi}_2^{(2)} \end{pmatrix} \to I$ as $\lambda\to\infty$ (since the diagonal part of $\hat{U}_j$ is $\mathrm{O}(\frac{1}{\lambda})$ and the off-diagonal part of $\hat{U}_j$ is bounded).
- $\tilde{\Phi}_j$ has singularities at $\lambda = \pm\frac{1}{A_j}$ of order $\frac{1}{2}$ (this will be discussed below, see Subsection 4.2.8).

4.2.3 "Background" solution

Introduce $\Phi_{0,j}(x,t,\lambda) := D_j^{-1}(\lambda)\mathrm{e}^{-Q_j(x,t,\lambda)}$. We see that $\Phi_{0,j}$ satisfy the differential equations:

$$\begin{cases} \Phi_{0,jx} = \frac{m(x,t)}{2A_j}\begin{pmatrix} -1 & \lambda A_j \\ -\lambda A_j & 1 \end{pmatrix}\Phi_{0,j}, \\ \Phi_{0,jt} = \left(-\frac{1}{2A_j}m(u^2-u_x^2)-\frac{1}{\lambda^2}\right)\begin{pmatrix} -1 & \lambda A_j \\ -\lambda A_j & 1 \end{pmatrix}\Phi_{0,j}. \end{cases} \tag{4.19}$$

Comparing this with (4.9), $\Phi_j(x,t,\lambda)$ can be characterized as the solutions of the integral equations:

$$\Phi_j(x,t,\lambda) = \Phi_{0,j}(x,t,\lambda)+\int_{(-1)^j\infty}^{x}\Phi_{0,j}(x,t,\lambda)\Phi_{0,j}^{-1}(y,t,\lambda)\frac{m(y,t)-A_j}{2A_j}\sigma_3\Phi_j(y,t,\lambda)dy. \tag{4.20}$$

Observe that $\Phi_{0,j}(x,t,\lambda)\Phi_{0,j}^{-1}(y,t,\lambda)$ is entire w.r.t. λ. Hence the "lack of analyticity" (jumps) of $\Phi_j(x,t,\lambda)$ is generated by the "lack of analyticity" of $\Phi_{0,j}(x,t,\lambda)$. Notice that $\det\Phi_j = \det\Phi_{0,j} = 1$.

4.2.4 Spectral functions

Introduce the scattering matrices $s(\lambda_\pm)$ for $\lambda\in\dot{\Sigma}_1$ as matrices relating Φ_1 and Φ_2:

$$\Phi_1(x,t,\lambda_\pm) = \Phi_2(x,t,\lambda_\pm)s(\lambda_\pm), \qquad \lambda\in\dot{\Sigma}_1 \tag{4.21}$$

with $\det s(\lambda_\pm)=1$. In turn, $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are related by

$$D_1^{-1}(\lambda_\pm)\tilde{\Phi}_1(x,t,\lambda_\pm) = D_2^{-1}(\lambda_\pm)\tilde{\Phi}_2(x,t,\lambda_\pm)\mathrm{e}^{-Q_2(x,t,\lambda_\pm)}s(\lambda_\pm)\mathrm{e}^{Q_1(x,t,\lambda_\pm)},\ \ \lambda\in\dot{\Sigma}_1. \tag{4.22}$$

Introducing

$$\tilde{s}(x,t,\lambda_\pm) := \mathrm{e}^{-Q_2(x,t,\lambda_\pm)}s(\lambda_\pm)\mathrm{e}^{Q_1(x,t,\lambda_\pm)} \tag{4.23}$$

we have

$$(D_1^{-1}\tilde{\Phi}_1)(x,t,\lambda_\pm) = (D_2^{-1}\tilde{\Phi}_2)(x,t,\lambda_\pm)\tilde{s}(x,t,\lambda_\pm), \qquad \lambda\in\dot{\Sigma}_1. \tag{4.24}$$

Notice that the scattering coefficients (s_{ij}) can be expressed as follows:

$$s_{11} = \det(\Phi_1^{(1)}, \Phi_2^{(2)}), \tag{4.25a}$$
$$s_{12} = \det(\Phi_1^{(2)}, \Phi_2^{(2)}), \tag{4.25b}$$
$$s_{21} = \det(\Phi_2^{(1)}, \Phi_1^{(1)}), \tag{4.25c}$$
$$s_{22} = \det(\Phi_2^{(1)}, \Phi_1^{(2)}). \tag{4.25d}$$

Accordingly,

$$\tilde{s}_{1j} = \det((D_1^{-1}\tilde{\Phi}_1)^{(j)}, (D_2^{-1}\tilde{\Phi}_2)^{(2)}), \tag{4.26a}$$
$$\tilde{s}_{2j} = \det((D_2^{-1}\tilde{\Phi}_2)^{(1)}, (D_1^{-1}\tilde{\Phi}_1)^{(j)}). \tag{4.26b}$$

Then (4.25a) implies that $s_{11}(\lambda)$ can be analytically extended to $\mathbb{C} \setminus \Sigma_2$ and defined on the upper and lower sides of $\dot{\Sigma}_2$. On the other hand, since $\Phi_1^{(1)}$ is analytic in $\mathbb{C} \setminus \Sigma_1$ and $\Phi_2^{(1)}$ is defined on the upper and lower sides of Σ_2, $s_{21}(\lambda)$ can be extended by (4.25c) to the lower and upper sides of $\dot{\Sigma}_2$. It follows that (4.21) and (4.22) restricted to the first column hold also on Σ_0, namely,

$$\Phi_1^{(1)}(x,t,\lambda_\pm) = s_{11}(\lambda_\pm)\Phi_2^{(1)}(x,t,\lambda_\pm) + s_{21}(\lambda_\pm)\Phi_2^{(2)}(x,t,\lambda_\pm), \quad \lambda \in \dot{\Sigma}_0, \tag{4.27}$$

and, respectively,

$$(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda_\pm) = \tilde{s}_{11}(\lambda_\pm)(D_2^{-1}\tilde{\Phi}_2^{(1)})(\lambda_\pm) + \tilde{s}_{21}(\lambda_\pm)(D_2^{-1}\tilde{\Phi}_2^{(2)})(\lambda_\pm), \quad \lambda \in \dot{\Sigma}_0. \tag{4.28}$$

4.2.5 Symmetries

Let's analyse the symmetry relations amongst the eigenfunctions and scattering coefficients. In order to simplify the notations, we will omit the dependence on x and t (e.g., $U(\lambda) \equiv U(x,t,\lambda)$).

First symmetry: $\lambda \longleftrightarrow -\lambda$.

Proposition 4.2.3. *The following symmetries hold:*

$$\Phi_1^{(1)}(\lambda) = -\sigma_3\Phi_1^{(1)}(-\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_1, \tag{4.29a}$$
$$\Phi_2^{(2)}(\lambda) = \sigma_3\Phi_2^{(2)}(-\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_2. \tag{4.29b}$$

Proof. Observe that $\sigma_3 U(\lambda)\sigma_3 \equiv U(-\lambda)$ and $\sigma_3 V(\lambda)\sigma_3 \equiv V(-\lambda)$. Hence $\sigma_3\Phi_j^{(j)}(-\lambda)$ solves (4.7) together with $\Phi_j^{(j)}(\lambda)$. Comparing their asymptotic behaviour as $x \to (-1)^j\infty$ and using (4.8a), the symmetries (4.29) follow. □

Corollary 4.2.4. *We have*

1.

$$s_{11}(-\lambda) = s_{11}(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_2. \tag{4.30}$$

2.

$$\tilde{\Phi}_1^{(1)}(\lambda) = \sigma_3\tilde{\Phi}_1^{(1)}(-\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_1, \tag{4.31a}$$

$$\tilde{\Phi}_2^{(2)}(\lambda) = -\sigma_3\tilde{\Phi}_2^{(2)}(-\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_2. \tag{4.31b}$$

3.

$$(D_1^{-1}\tilde{\Phi}_1^{(1)})(-\lambda) = -\sigma_3(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_1, \tag{4.32a}$$

$$(D_2^{-1}\tilde{\Phi}_2^{(2)})(-\lambda) = \sigma_3(D_2^{-1}\tilde{\Phi}_2^{(2)})(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_2. \tag{4.32b}$$

Proof. 1. Substitute (4.29) into (4.25a).

2. Observe that due to (4.8a), we have $D_j^{-1}(-\lambda) = -\sigma_3 D_j^{-1}(\lambda)\sigma_3$ and $Q_j(-\lambda) = Q_j(\lambda)$. Combining this with (4.29) and using the connection between Φ_j and $\tilde{\Phi}_j$, we obtain (4.31).

3. Combine $D_j^{-1}(-\lambda) = -\sigma_3 D_j^{-1}(\lambda)\sigma_3$ and (4.31).

□

Proposition 4.2.5. *The following symmetry holds*

$$\Phi_j(\lambda_+) = -\sigma_3\Phi_j(-\lambda_+)\sigma_3, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.33}$$

Proof. Since $\sigma_3 U(\lambda)\sigma_3 \equiv U(-\lambda)$ and $\sigma_3 V(\lambda)\sigma_3 \equiv V(-\lambda)$ and U and V do not have jumps along Σ_j, it follows that if $\Phi_j(\lambda_+)$ solves (4.7), so does $\sigma_3\Phi_j(-\lambda_+)$. Comparing their asymptotic behaviour as $x \to (-1)^j\infty$ and using (4.8a), the symmetry (4.33) follows. □

Corollary 4.2.6. *We have*

1.
$$s(\lambda_+) = \sigma_3 s(-\lambda_+)\sigma_3, \quad \lambda \in \dot{\Sigma}_1 \tag{4.34}$$

2.
$$\tilde{\Phi}_j(\lambda_+) = \sigma_3 \tilde{\Phi}_j(-\lambda_+)\sigma_3, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.35}$$

3.
$$(D_j^{-1}\tilde{\Phi}_j)((-\lambda)_-) = -\sigma_3 (D_j^{-1}\tilde{\Phi}_j)(\lambda_+)\sigma_3, \quad \lambda_+ \in \dot{\Sigma}_j. \tag{4.36}$$

Proof. 1. Substitute (4.33) into (4.21).

2. Observe that due to (4.8a), we have $D_j^{-1}(-\lambda_+) = -\sigma_3 D_j^{-1}(\lambda_+)\sigma_3$ and $Q_j(-\lambda_+) = Q_j(\lambda_+)$. Combining this with (4.33) and using the connection between Φ_j and $\tilde{\Phi}_j$, we obtain (4.35).

3. Combine $D_j^{-1}(-\lambda_+) = -\sigma_3 D_j^{-1}(\lambda_+)\sigma_3$ and (4.35).

□

Second symmetry: $\lambda \longleftrightarrow -\overline{\lambda}$.

Proposition 4.2.7. *The following symmetry holds*

$$\Phi_j(\lambda_+) = \sigma_3 \Phi_j((-\lambda)_+)\sigma_2, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.37}$$

Proof. Since U and V are single valued functions of λ, we have $\sigma_3 U(\lambda_+)\sigma_3 \equiv U((-\lambda)_+)$ and $\sigma_3 V(\lambda_+)\sigma_3 \equiv V((-\lambda)_+)$ for $\lambda \in \Sigma_j$. Hence, if $\Phi_j(\lambda_+)$ solves (4.7), so does $\sigma_3\Phi_j((-\lambda_+)$. Comparing their asymptotic behaviour as $x \to (-1)^j\infty$ and using (4.8b) and the equality $\sqrt{\frac{1}{\mathrm{i}A_j k_j(\lambda_+)} - 1}\sqrt{-\frac{1}{\mathrm{i}A_j k_j(\lambda_+)} - 1} = -\frac{\lambda_+}{k_j(\lambda_+)}$ for $\lambda_+ \in \dot{\Sigma}_j$, the symmetry (4.37) follows. □

Corollary 4.2.8. *We have*

1.
$$s(\lambda_+) = \sigma_2 s((-\lambda)_+)\sigma_2, \qquad \lambda \in \dot{\Sigma}_1. \tag{4.38}$$

2.

$$s(\lambda_+) = \sigma_1 s(\lambda_-)\sigma_1, \qquad \lambda \in \dot{\Sigma}_1. \tag{4.39}$$

3.

$$\tilde{\Phi}_j(\lambda_+) = \sigma_2 \tilde{\Phi}_j((-\lambda)_+)\sigma_2, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.40}$$

4.

$$(D_j^{-1}\tilde{\Phi}_j)((-\lambda)_+) = \sigma_3 (D_j^{-1}\tilde{\Phi}_j)(\lambda_+)\sigma_2, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.41}$$

Proof. 1. Substitute (4.37) into (4.21).

2. Combine (4.38) with (4.34).

3. Observe that $k_j(\lambda_+) \in \mathbb{R}$ and that due to (4.8b) and $\sqrt{\frac{1}{\mathrm{i}A_j k_j(\lambda_+)} - 1}\sqrt{-\frac{1}{\mathrm{i}A_j k_j(\lambda_+)} - 1} = -\frac{\lambda_+}{k_j(\lambda_+)}$, we have $D_j(\lambda_+)\sigma_3 D_j^{-1}((-\lambda)_+) = \sigma_2$ and $Q_j((-\lambda)_+) = -Q_j(\lambda_+)$ for $\lambda \in \dot{\Sigma}_j$. Combining this with (4.37) and using the connection between Φ_j and $\tilde{\Phi}_j$, we obtain (4.40).

4. Combine $D_j(\lambda_+)\sigma_3 D_j^{-1}((-\lambda)_+) = \sigma_2$ and (4.40).

□

Third symmetry: $\lambda \longleftrightarrow \overline{\lambda}$.

Proposition 4.2.9. *The following symmetries hold*

$$\overline{\Phi_j^{(j)}(\overline{\lambda})} = -\Phi_j^{(j)}(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_j. \tag{4.42}$$

Proof. Since $\overline{U(\overline{\lambda})} \equiv U(\lambda)$ and $\overline{V(\overline{\lambda})} \equiv V(\lambda)$, it follows that $\overline{\Phi_j^{(j)}(\overline{\lambda})}$ solves (4.7a) together with $\Phi_j^{(j)}(\lambda)$. Hence, comparing their asymptotic behaviour as $x \to (-1)^j\infty$ and using (4.8c) and the equality $\overline{\sqrt{\frac{1}{\mathrm{i}A_j k_j(\overline{\lambda})} - 1}} = -\sqrt{\frac{1}{\mathrm{i}A_j k_j(\lambda)} - 1}$, we obtain the symmetries (4.42). □

Corollary 4.2.10. *We have*

1.

$$\overline{s_{11}(\overline{\lambda})} = s_{11}(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_2. \tag{4.43}$$

2.

$$\overline{\tilde{\Phi}_j^{(j)}(\overline{\lambda})} = \tilde{\Phi}_j^{(j)}(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_j. \tag{4.44}$$

3.

$$\overline{(D_j^{-1}\tilde{\Phi}_j^{(j)})(\overline{\lambda})} = -(D_j^{-1}\tilde{\Phi}_j^{(j)})(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_j. \tag{4.45}$$

Proof. 1. Substitute (4.42) into (4.25a).

2. Observe that due to (4.8c) and $\overline{\sqrt{\frac{1}{\mathrm{i}A_j k_j(\overline{\lambda})} - 1}} = -\sqrt{\frac{1}{\mathrm{i}A_j k_j(\lambda)} - 1}$, we have $\overline{D_j^{-1}(\overline{\lambda})} = -D_j^{-1}(\lambda)$ and $\overline{Q_j(\overline{\lambda})} = Q_j(\lambda)$. Hence combining this with (4.42) and using the connection between Φ_j and $\tilde{\Phi}_j$, we obtain (4.44).

3. Combine $\overline{D_j^{-1}(\overline{\lambda})} = -D_j^{-1}(\lambda)$ and (4.44).

□

Proposition 4.2.11. *The following symmetry holds*

$$\overline{\Phi_j(\overline{\lambda_+})} = -\Phi_j(\lambda_+), \qquad \lambda \in \dot{\Sigma}_j. \tag{4.46}$$

Proof. As above, since $\overline{U(\overline{\lambda})} \equiv U(\lambda)$ and $\overline{V(\overline{\lambda})} \equiv V(\lambda)$ and U and V have no jumps along Σ_j, we have $\overline{U(\lambda_-)} \equiv U(\lambda_+)$ and $\overline{V(\lambda_-)} \equiv V(\lambda_+)$. It follows that if $\Phi_j(\lambda_+)$ solves (4.7), so does $\overline{\Phi_j(\overline{\lambda_+})}$. Comparing their asymptotic behaviour as $x \to (-1)^j\infty$ and using (4.8c) and the fact that $\overline{\sqrt{\frac{1}{\mathrm{i}A_j k_j(\overline{\lambda})} - 1}} = -\sqrt{\frac{1}{\mathrm{i}A_j k_j(\lambda)} - 1}$, the symmetry (4.46) follows. □

Corollary 4.2.12. *We have*

1.

$$\overline{s(\overline{\lambda_+})} = s(\lambda_+), \quad \lambda \in \dot{\Sigma}_1. \tag{4.47}$$

2.

$$\overline{\tilde{\Phi}_j(\overline{\lambda_+})} = \tilde{\Phi}_j(\lambda_+), \qquad \lambda \in \dot{\Sigma}_j. \tag{4.48}$$

3.

$$\overline{(D_j^{-1}\tilde{\Phi}_j)(\overline{\lambda_+})} = -(D_j^{-1}\tilde{\Phi}_j)(\lambda_+), \quad \lambda \in \dot{\Sigma}_j. \tag{4.49}$$

Proof. 1. Substitute (4.46) into (4.21).

2. Observe that due to (4.8c) and $\overline{\sqrt{\frac{1}{\mathrm{i}A_jk_j(\overline{\lambda})}-1}} = -\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda)}-1}$, we have $\overline{D_j^{-1}(\lambda_-)} = -D_j^{-1}(\lambda_+)$ and $\overline{Q_j(\lambda_-)} = Q_j(\lambda_+)$ for $\lambda \in \dot{\Sigma}_j$. Combining this with (4.46) and using the connection between Φ_j and $\tilde{\Phi}_j$, we obtain the result.

3. Combine $\overline{D_j^{-1}(\lambda_-)} = -D_j^{-1}(\lambda_+)$ and $\overline{Q_j(\lambda_-)} = Q_j(\lambda_+)$ and (4.48).

□

Fourth symmetry $\lambda_+ \longleftrightarrow \lambda_+$.

Proposition 4.2.13. *The following symmetry holds*

$$\overline{\Phi_j(\lambda_+)} = \mathrm{i}\Phi_j(\lambda_+)\sigma_1, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.50}$$

Proof. Since $\overline{U(\lambda_+)} \equiv U(\lambda_+)$ and $\overline{V(\lambda_+)} \equiv V(\lambda_+)$ for $\lambda \in \Sigma_j$, in follows that if $\Phi_j(\lambda_+)$ solves (4.7), so does $\overline{\Phi_j(\lambda_+)}$. Comparing their asymptotic behaviour as $x \to (-1)^j\infty$ and using (4.8d) and the equalities $\sqrt{-\frac{1}{\mathrm{i}A_jk_j(\lambda_+)}-1}\cdot\frac{\lambda_+A_j}{1+\mathrm{i}A_jk_j(\lambda_+)} = -\mathrm{i}\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda_+)}-1}$ and $\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda_+)}-1}\cdot\frac{\lambda_+A_j}{1-\mathrm{i}A_jk_j(\lambda_+)} = \mathrm{i}\sqrt{-\frac{1}{\mathrm{i}A_jk_j(\lambda_+)}-1}$ for $\lambda \in \dot{\Sigma}_j$, the symmetry (4.50) follows. □

Corollary 4.2.14. *We have*

1. $s(\lambda_+) = \sigma_1\overline{s(\lambda_+)}\sigma_1$, $\lambda \in \dot{\Sigma}_1$, *which, in terms of the matrix entries, reads as follows:*

$$s_{11}(\lambda_+) = \overline{s_{22}(\lambda_+)}, \tag{4.51a}$$

$$s_{12}(\lambda_+) = \overline{s_{21}(\lambda_+)}. \tag{4.51b}$$

2. $|s_{11}(\lambda_+)|^2 - |s_{21}(\lambda_+)|^2 = 1$ *for* $\lambda \in \dot{\Sigma}_1$.

3. $\left|\frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)}\right| \leq 1$ *for* $\lambda \in \dot{\Sigma}_1$.
Notice that $\left|\frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)}\right| = 1$ *for* $\lambda \in \dot{\Sigma}_1$ *iff* $s_{11}(\lambda_+) = \infty$.

4.

$$s_{11}(\lambda_-) = \overline{s_{22}(\lambda_-)}, \qquad \lambda \in \dot{\Sigma}_1, \tag{4.52a}$$

$$s_{12}(\lambda_-) = \overline{s_{21}(\lambda_-)}, \qquad \lambda \in \dot{\Sigma}_1. \tag{4.52b}$$

5.

$$\Phi_j(\lambda_+) = \mathrm{i}\Phi_j(\lambda_-)\sigma_1, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.53}$$

6.

$$\Phi_1^{(1)}(\lambda_+) = \mathrm{i}\Phi_1^{(2)}(\lambda_-), \qquad \lambda \in \dot{\Sigma}_1, \tag{4.54a}$$

$$\Phi_2^{(2)}(\lambda_+) = \mathrm{i}\Phi_2^{(1)}(\lambda_-), \qquad \lambda \in \dot{\Sigma}_2. \tag{4.54b}$$

7.

$$s_{11}(\lambda_+) = s_{22}(\lambda_-), \qquad \lambda \in \dot{\Sigma}_1, \tag{4.55a}$$

$$s_{11}(\lambda_+) = -\mathrm{i}s_{21}(\lambda_-), \quad \lambda \in \dot{\Sigma}_0, \tag{4.55b}$$

$$s_{11}(\lambda_-) = \mathrm{i}s_{21}(\lambda_+), \qquad \lambda \in \dot{\Sigma}_0. \tag{4.55c}$$

8. $\left|\frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)}\right| = 1$ *for* $\lambda \in \dot{\Sigma}_0$.

9.

$$\overline{\tilde{\Phi}_j(\lambda_+)} = \sigma_1\tilde{\Phi}_j(\lambda_+)\sigma_1, \qquad \lambda \in \dot{\Sigma}_j. \tag{4.56}$$

10.

$$\tilde{\Phi}_1^{(1)}(\lambda_-) = \sigma_1\tilde{\Phi}_1^{(2)}(\lambda_+), \quad \lambda \in \Sigma_1, \tag{4.57a}$$

$$\tilde{\Phi}_2^{(2)}(\lambda_-) = \sigma_1\tilde{\Phi}_2^{(1)}(\lambda_+), \quad \lambda \in \Sigma_2. \tag{4.57b}$$

11.

$$\overline{(D_j^{-1}\tilde{\Phi}_j)(\lambda_+)} = \mathrm{i}(D_j^{-1}\tilde{\Phi}_j)(\lambda_+)\sigma_1, \quad \lambda \in \dot{\Sigma}_j. \tag{4.58}$$

12.

$$D_j^{-1}(\lambda_-)\tilde{\Phi}_j^{(j)}(\lambda_-) = (-\mathrm{i}D_j^{-1}(\lambda_+)\tilde{\Phi}_j(\lambda_+)\sigma_1)^{(j)}, \quad \lambda \in \dot{\Sigma}_1, \tag{4.59a}$$

$$D_2^{-1}(\lambda_-)\tilde{\Phi}_2^{(2)}(\lambda_-) = (-\mathrm{i}D_2^{-1}(\lambda_+)\tilde{\Phi}_2(\lambda_+)\sigma_1)^{(2)}, \quad \lambda \in \dot{\Sigma}_0, \tag{4.59b}$$

$$D_1^{-1}(\lambda_-)\tilde{\Phi}_1^{(1)}(\lambda_-) = D_1^{-1}(\lambda_+)\tilde{\Phi}_1^{(1)}(\lambda_+), \qquad \lambda \in \dot{\Sigma}_0. \tag{4.59c}$$

13.

$$(D_1^{-1}\tilde{\Phi}_1)((-\lambda)_+) = \sigma_3\overline{(D_1^{-1}\tilde{\Phi}_1)(\lambda_+)}, \qquad \lambda \in \dot{\Sigma}_1, \tag{4.60a}$$

$$(D_2^{-1}\tilde{\Phi}_2)((-\lambda)_+) = -\sigma_3\overline{(D_2^{-1}\tilde{\Phi}_2)(\lambda_+)}, \quad \lambda \in \dot{\Sigma}_2. \tag{4.60b}$$

14.

$$s_{11}((-\lambda)_+) = \overline{s_{11}(\lambda_+)}, \quad \lambda \in \dot{\Sigma}_1. \tag{4.61}$$

Proof. 1. Substitute (4.50) into (4.21).

2. This follows from the fact that $\det s(\lambda_\pm) = 1$ for all $\lambda \in \Sigma_1$ and (4.51).

3. Dividing the previous equality by $|s_{11}(\lambda_+)|^2$, we obtain $1 - \left|\frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)}\right|^2 = \left|\frac{1}{s_{11}(\lambda_+)}\right|^2 \geq 0$. Hence $\left|\frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)}\right| \leq 1$.

4. Combine (4.51) and (4.47).

5. Combine (4.50) and (4.46).

6. Rewrite (4.53) columnwise.

7. Substituting (4.53) into (4.25a) leads to (4.55). Notice that in proving (4.55b) and (4.55c) we use the fact that $\Phi_1^{(1)}$ is analytic on Σ_0.

8. Using the previous result for the first equality and (4.43) for the second one, we get $\left|\frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)}\right| = \left|\frac{-\mathrm{i}s_{11}(\lambda_-)}{s_{11}(\lambda_+)}\right| = \left|\frac{\overline{s_{11}(\lambda_+)}}{s_{11}(\lambda_+)}\right| = 1$.

9. Observe that $\sqrt{-\frac{1}{\mathrm{i}A_jk_j(\lambda_+)} - 1}\cdot\frac{\lambda_+ A_j}{1+\mathrm{i}A_jk_j(\lambda_+)} = -\mathrm{i}\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda_+)} - 1}$ and $\sqrt{\frac{1}{\mathrm{i}A_jk_j(\lambda_+)} - 1}\cdot\frac{\lambda_+ A_j}{1-\mathrm{i}A_jk_j(\lambda_+)} = \mathrm{i}\sqrt{-\frac{1}{\mathrm{i}A_jk_j(\lambda_+)} - 1}$ imply $\overline{D_j^{-1}(\lambda_+)} = \mathrm{i}D_j^{-1}(\lambda_+)\sigma_1$ and $\overline{D_j(\lambda_+)} = -\mathrm{i}\sigma_1 D_j(\lambda_+)$, and (4.8d) imply $\overline{Q_j(\lambda_+)} = -Q_j(\lambda_+)$ for $\lambda \in \dot{\Sigma}_j$. Combining this with (4.50) and using the connection between Φ_j and $\tilde{\Phi}_j$, we obtain (4.56).

10. Combine (4.56) and (4.48).

11. Combine $\overline{D_j^{-1}(\lambda_+)} = \mathrm{i}D_j^{-1}(\lambda_+)\sigma_1$ and (4.56).

12. Use (4.58) combined with (4.49) for the first two equalities and the fact that $k_1(\lambda)$ is analytic on $\dot{\Sigma}_0$ for the last one.

13. Combine (4.58) and (4.41).

14. (4.38) implies $s_{22}(\lambda_+) = s_{11}((-\lambda)_+)$. Combine this with (4.51a).

□

4.2.6 Limits of the eigenfunctions and scattering coefficients from below and above the branch cut

Recall that $k_j(\lambda)$ is analytic in $\mathbb{C} \setminus \Sigma_j$ and discontinuous across Σ_j.

Notations. It will be useful in what follows to introduce the following notations (for $\lambda \in \Sigma_j$):

$$k_j^+(\lambda) := k_j(\lambda_+) = \lim_{\epsilon \downarrow 0} k_j(\lambda + \mathrm{i}\epsilon), \qquad k_j^-(\lambda) := k_j(\lambda_-) = \lim_{\epsilon \downarrow 0} k_j(\lambda - \mathrm{i}\epsilon).$$

Similarly,

$$\tilde{\Phi}_1^{(1)+}(\lambda) := \tilde{\Phi}_1^{(1)}(\lambda_+) = \lim_{\epsilon \downarrow 0} \tilde{\Phi}_1^{(1)}(\lambda + \mathrm{i}\epsilon), \qquad \tilde{\Phi}_1^{(1)-}(\lambda) := \tilde{\Phi}_1^{(1)}(\lambda_-) = \lim_{\epsilon \downarrow 0} \tilde{\Phi}_1^{(1)}(\lambda - \mathrm{i}\epsilon).$$

Observe that

$$k_j^-(\lambda) = -k_j^+(\lambda), \quad \lambda \in \Sigma_1, \tag{4.62a}$$

$$k_1^-(\lambda) = k_1^+(\lambda) = k_1(\lambda), \quad \lambda \in \Sigma_0, \tag{4.62b}$$

$$k_2^-(\lambda) = -k_2^+(\lambda), \quad \lambda \in \Sigma_0. \tag{4.62c}$$

Combining (4.56) and (4.48) we have

$$\tilde{\Phi}_1^{(1)-}(\lambda) = \sigma_1 \tilde{\Phi}_1^{(2)+}(\lambda), \quad \lambda \in \Sigma_1, \tag{4.63a}$$

$$\tilde{\Phi}_2^{(2)-}(\lambda) = \sigma_1 \tilde{\Phi}_2^{(1)+}(\lambda), \quad \lambda \in \Sigma_2. \tag{4.63b}$$

4.2.7 Discrete spectrum and zeros of scattering coefficients

Multiplying (4.7a) by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we arrive at the spectral problem for a weighted Dirac operator:

$$\frac{2}{m}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Phi_x + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Phi\right) = \lambda\Phi, \quad x \in (-\infty, \infty). \tag{4.64}$$

Since $\lim_{x\to(-1)^j\infty} m(x,t) = A_j \neq 0$, this operator can be viewed as a self-adjoint operator in $L^2(-\infty,\infty)$ and thus its spectrum in real.

Observe that for $\lambda \in \dot{\Sigma}_1$, both $k_j(\lambda)$, $j = 1, 2$ are real-valued and hence the eigenfunctions Φ_j are bounded but not square integrable near $(-1)^j\infty$. Since they are related by a matrix independent on x and t, Φ_j are bounded and not square integrable near $\pm\infty$. Hence $\dot{\Sigma}_1$ comprise the continuous spectrum.

For $\lambda \in (-1/A_2, 1/A_2)$, $\Phi_1^{(1)}$ decays (exponentially fast) as $x \to -\infty$ and $\Phi_2^{(2)}$ decays (exponentially fast) as $x \to +\infty$; hence the the eigenvalues in $(-1/A_2, 1/A_2)$ coincides with the zeros of $s_{11}(\lambda) = \det(\Phi_1^{(1)}, \Phi_2^{(2)})$.

Note that since $|s_{11}(\lambda_+)|^2 - |s_{21}(\lambda_+)|^2 = 1$ for $\lambda \in \dot{\Sigma}_1$ (see Corollary 4.2.14), we have $s_{11}(\lambda_+) \neq 0$ for $\lambda \in \dot{\Sigma}_1$.

Let's show that $s_{11}(\lambda_+) \neq 0$ as well as $s_{21}(\lambda_+) \neq 0$ for $\lambda \in \dot{\Sigma}_0$ (the similar result for λ_- will then follow from the symmetry (4.47)). Indeed, we have $|\frac{s_{21}}{s_{11}}(\lambda_\pm)| = 1$ for $\lambda \in \dot{\Sigma}_0$ (see Corollary 4.2.14). Hence $s_{11}(\lambda_{0+})s_{21}(\lambda_{0+}) = 0$ iff $s_{11}(\lambda_{0+}) = 0$ and $s_{21}(\lambda_{0+}) = 0$ simultaneously. But $s_{11}(\lambda_{0+}) = 0$ implies that $\Phi_1^{(1)}(\lambda_{0+})$ and $\Phi_2^{(2)}(\lambda_{0+})$ are dependent. Silarly, $s_{21}(\lambda_{0+}) = 0$ implies that $\Phi_1^{(1)}(\lambda_{0+})$ and $\Phi_2^{(1)}(\lambda_{0+})$ are dependent. Hence $\Phi_2^{(1)}(\lambda_{0+})$ and $\Phi_2^{(2)}(\lambda_{0+})$ are dependent, which contradicts the fact that $\det\Phi_{0,2} \equiv 1$ (the latter follows from evaluating $\det\Phi_{0,2}(x,t,\lambda)$ as $x \to \infty$ and using the fact that the determinant of a matrix composed by two vector solutions of (4.64) does not depend on x).

Assumption. We will assume that $s_{11}(\lambda)$ has a finite number of zeros on $\mathbb{R} \setminus \Sigma_2$. Since s_{11} is analytic on $\mathbb{C} \setminus \Sigma_2$, the uniqueness theorem implies that the sufficient condition is $s_{11}(\pm\frac{1}{A_2}) \neq 0$.

Let $\{\lambda_k\}_{k=1}^n$ be the zeros of $s_{11}(\lambda)$. For such λ_k we have

$$\Phi_1^{(1)}(\lambda_k) = b_k \Phi_2^{(2)}(\lambda_k), \quad b_k := b(\lambda_k).$$

Proposition 4.2.15. *The zeros of $s_{11}(\lambda)$ are simple.*

Proof. We will denote by $'$ the dirivative w.r.t. λ.

Using the definition of $s_{11}(\lambda)$ we have

$$s_{11}'(\lambda) = \det(\Phi_1^{(1)}, \Phi_2^{(2)})'(\lambda) = \det((\Phi')_1^{(1)}, \Phi_2^{(2)})(\lambda) + \det(\Phi_1^{(1)}, (\Phi')_2^{(2)})(\lambda).$$

Since $\Phi_j^{(j)}$ solves (4.7a), we have

$$(\Phi')_{jx}^{(j)} = U(\Phi')_j^{(j)} + m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_j^{(j)},$$

and, using the fact that $\det(U(\Phi')_1^{(1)}, \Phi_2^{(2)}) = -\det((\Phi')_1^{(1)}, U\Phi_2^{(2)})$, we have

$$\frac{d}{dx} \det((\Phi')_1^{(1)}, \Phi_2^{(2)}) = \det\left(\begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix} \Phi_1^{(1)}, \Phi_2^{(2)}\right),$$

and

$$\frac{d}{dx} \det(\Phi_1^{(1)}, (\Phi')_2^{(2)}) = -\det\left(\begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix} \Phi_2^{(2)}, \Phi_1^{(1)}\right).$$

Evaluating at $\lambda = \lambda_k$ and using $\Phi_1^{(1)}(\lambda_k) = b_k \Phi_2^{(2)}(\lambda_k)$, we get

$$\frac{d}{dx} \det((\Phi')_1^{(1)}, \Phi_2^{(2)})(\lambda_k) = b_k m \det\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_2^{(2)}(\lambda_k), \Phi_2^{(2)}(\lambda_k)\right),$$

$$\frac{d}{dx} \det(\Phi_1^{(1)}, (\Phi')_2^{(2)})(\lambda_k) = -b_k m \det\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_2^{(2)}(\lambda_k), \Phi_2^{(2)}(\lambda_k)\right).$$

Using the symmetry (4.42) and observing that $\lambda_k \in \mathbb{R}$, we have

$$\det(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_2^{(2)}(\lambda_k), \Phi_2^{(2)}(\lambda_k)) = -(|(\Phi_2)_{22}|^2 + |(\Phi_2)_{12}|^2)(\lambda_k)$$

and hence

$$\frac{d}{dx}\det((\Phi')^{(1)}_1, \Phi^{(2)}_2)(\lambda_k) = b_k \int_x^\infty m(|(\Phi_2)_{22}|^2 + |(\Phi_2)_{12}|^2)d\tau,$$

$$\frac{d}{dx}\det(\Phi^{(1)}_1, (\Phi')^{(2)}_2)(\lambda_k) = b_k \int_{-\infty}^x m(|(\Phi_2)_{22}|^2 + |(\Phi_2)_{12}|^2)dx.$$

It follows that

$$s'_{11}(\lambda_k) = b_k \int_{-\infty}^\infty m(|(\Phi_2)_{22}|^2 + |(\Phi_2)_{12}|^2)dx,$$

and thus $s'_{11}(\lambda_k) \neq 0$.

□

Observe that due to the symmetry (4.30), if $s_{11}(\lambda_k) = 0$, then $s_{11}(-\lambda_k) = 0$ as well. Since, according to Proposition 4.2.15, all zeros of s_{11} are simple, it follows that $s_{11}(0) \neq 0$. This fact will also be discussed in Subsection 4.3.2.

4.2.8 Behaviour at the branch points

Observe that $k_j(\pm\frac{1}{A_j}) = 0$.

Proposition 4.2.16. $\tilde{\Phi}_j(x,t,\lambda)$ *has the following behaviour at the branch points*

$$\tilde{\Phi}_j(x,t,\lambda) = \frac{\mathrm{i}\alpha_j(x,t)}{\omega_j^+(\lambda)}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} a_j(x,t) & b_j(x,t) \\ b_j(x,t) & a_j(x,t) \end{pmatrix} + \mathrm{O}(\sqrt{\lambda - \frac{1}{A_j}}), \quad \lambda \to \frac{1}{A_j},$$

$$\tilde{\Phi}_j(x,t,\lambda) = \frac{\alpha_j(x,t)}{\omega_j^-(\lambda)}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} a_j(x,t) & -b_j(x,t) \\ -b_j(x,t) & a_j(x,t) \end{pmatrix} + \mathrm{O}(\sqrt{\lambda + \frac{1}{A_j}}), \quad \lambda \to -\frac{1}{A_j},$$

whith some real-valued $\alpha_j(x,t)$, $a_j(x,t)$, *and* $b_j(x,t)$, $j = 1,2$.

Proof. Recall that $\omega_j^+(\lambda) = \sqrt{\lambda - \frac{1}{A_j}}$ with a branch cut on $[\frac{1}{A_j}, \infty)$ and $\omega_j^+(0) = \frac{\mathrm{i}}{\sqrt{A_j}}$, and $\omega_j^-(\lambda) = \sqrt{\lambda + \frac{1}{A_j}}$ with a branch cut on $(-\infty, -\frac{1}{A_j}]$ and $\omega_j^-(0) = \frac{1}{\sqrt{A_j}}$.

First, consider the behavior of the eigenfunctions near $\frac{1}{A_j}$. Introduce $\tilde{\tilde{\Phi}}_j(x,t,\lambda)$ such that $\tilde{\Phi}_j(x,t,\lambda) = W^+\tilde{\tilde{\Phi}}_j(x,t,\lambda)$ with $W^+ = \begin{pmatrix} 1 & \frac{\mathrm{i}}{\omega_j^+(\lambda)} \\ 1 & -\frac{\mathrm{i}}{\omega_j^+(\lambda)} \end{pmatrix}$. Then $\tilde{\tilde{\Phi}}_j(x,t,\lambda)$ solves the following integral equation:

$$\tilde{\tilde{\Phi}}_j(x,t,\lambda) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i}\omega_j^+(\lambda) & \mathrm{i}\omega_j^+(\lambda) \end{pmatrix} + \int_{(-1)^i\infty}^{x} A^{-1}\mathrm{e}^{\frac{\mathrm{i}}{2}k_j(\lambda)\int_x^\xi md\tau\sigma_3}\hat{U}_j A\tilde{\tilde{\Phi}}_j \mathrm{e}^{-\frac{\mathrm{i}}{2}k_j(\lambda)\int_x^\xi md\tau\sigma_3}.$$

The kernel of this equation and hence $\tilde{\tilde{\Phi}}_j$ has no singularity at $\frac{1}{A_j}$. Hence

$$\tilde{\Phi}_j(x,t,\lambda) = \frac{\mathrm{i}}{\omega_j^+(\lambda)}\begin{pmatrix} \tilde{\tilde{c}}_j & \tilde{\tilde{d}}_j \\ -\tilde{\tilde{c}}_j & -\tilde{\tilde{d}}_j \end{pmatrix} + \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} + \mathrm{O}\left(\sqrt{\lambda - \frac{1}{A_j}}\right), \quad \lambda \to \frac{1}{A_j}.$$

Using (4.48), we get $\tilde{\tilde{c}}_j,\ \tilde{\tilde{d}}_j \in \mathbb{R}$ and $a_j,\ b_j,\ c_j,\ d_j \in \mathbb{R}$. Then, using (4.56), we get $\tilde{\tilde{c}}_j = \tilde{\tilde{d}}_j$ and $a_j = d_j,\ c_j = b_j$; thus

$$\tilde{\Phi}_j(x,t,\lambda) = \frac{\mathrm{i}\alpha_j(x,t)}{\omega_j^+(\lambda)}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} + \mathrm{O}\left(\sqrt{\lambda - \frac{1}{A_j}}\right), \quad \lambda \to \frac{1}{A_j}.$$

In order to get the simiular result for $-\frac{1}{A_j}$, we use $W^- = \begin{pmatrix} \frac{\mathrm{i}}{\omega_j^-(\lambda)} & 1 \\ \frac{\mathrm{i}}{\omega_j^-(\lambda)} & -1 \end{pmatrix}$ instead of W^+, which leads to

$$\tilde{\Phi}_j(x,t,\lambda) = \frac{\beta_j(x,t)}{\omega_j^-(\lambda)}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} \hat{a}_j & \hat{b}_j \\ \hat{b}_j & \hat{a}_j \end{pmatrix} + \mathrm{O}(\sqrt{\lambda + \frac{1}{A_j}}), \quad \lambda \to -\frac{1}{A_j}.$$

Finally, using (4.35) and (4.40), we get $\alpha_j = -\beta_j$ and $a_j = \hat{a}_j$ and $b_j = -\hat{b}_j$. □

Evaluating $D_j^{-1}(\lambda)$ near $\pm\frac{1}{A_j}$ gives

Proposition 4.2.17. *$D_j^{-1}(\lambda)$ has the following behaviour at the branch points:*

$$D_j^{-1}(\lambda) = \frac{\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{(2A_j)^{\frac{1}{4}}\nu_j^+(\lambda)}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{\mathrm{i}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}(2A_j)^{\frac{1}{4}}\nu_j^+(\lambda)}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \mathrm{O}((\lambda - \frac{1}{A_j})^{\frac{3}{4}}), \quad \lambda \to \frac{1}{A_j}$$

and

$$D_j^{-1}(\lambda) = \frac{\mathrm{i}}{(2A_j)^{\frac14}\nu_j^-(\lambda)}\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix} + \frac{\mathrm{i}(2A_j)^{\frac14}\nu_j^-(\lambda)}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \mathrm{O}((\lambda+\frac{1}{A_j})^{\frac34}), \quad \lambda\to -\frac{1}{A_j}.$$

Here $\nu_j^+(\lambda) = (\lambda - \frac{1}{A_j})^{\frac14}$ *with the branch cut* $(\frac{1}{A_j},\infty)$ *and* $\nu_j^+(0) = \frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{(A_j)^{\frac14}}$, *and* $\nu_j^-(\lambda) = (\lambda + \frac{1}{A_j})^{\frac14}$ *with the branch cut* $(-\infty, -\frac{1}{A_j})$ *and* $\nu_j^-(0) = \frac{1}{(A_j)^{\frac14}}$ *(observe that* $(\nu_j^{\pm}(\lambda))^2 = \omega_j^{\pm}(\lambda)$*).*

4.3 Riemann–Hilbert problems

4.3.1 RH problem parametrized by (x,t)

Notations. We denote

$$\rho(\lambda) := \frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)}, \quad \lambda\in\dot\Sigma_1\cup\dot\Sigma_0. \tag{4.65}$$

Observe that Corollary 4.2.14 implies that

$$|\rho(\lambda)|\le 1, \quad \lambda\in\dot\Sigma_1, \tag{4.66a}$$

$$|\rho(\lambda)| = 1, \quad \lambda\in\dot\Sigma_0. \tag{4.66b}$$

Motivated by the analytic properties of eigenfunctions and scattering coefficients, we introduce the matrix-values function

$$M(x,t,\lambda) = \left(\frac{(D_1^{-1}\tilde\Phi_1^{(1)})(x,t,\lambda)}{s_{11}(\lambda)\mathrm{e}^{p_1(x,t,\lambda)-p_2(x,t,\lambda)}}, (D_2^{-1}\tilde\Phi_2^{(2)})(x,t,\lambda)\right), \qquad \lambda\in\mathbb{C}\setminus\Sigma_2, \tag{4.67a}$$

meromorphic in $\mathbb{C}\setminus\Sigma_2$, where p_j, $j=1,2$ are defined in (4.14b).

Observe that $D_j^{-1}(\lambda)\tilde\Phi_j(x,t,\lambda) = \Phi_j(x,t,\lambda)\mathrm{e}^{Q_j(x,t,\lambda)}$ and thus $M(x,t,\lambda)$ can be written as

$$M(x,t,\lambda) = \left(\frac{\Phi_1^{(1)}(x,t,\lambda)}{s_{11}(\lambda)}, \Phi_2^{(2)}(x,t,\lambda)\right)\mathrm{e}^{p_2(x,t,\lambda)\sigma_3}. \tag{4.67b}$$

It follows that $\det M\equiv 1$.

Jump matrix

Since $(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda)$ is analytic in $\mathbb{C}\setminus\Sigma_1$, the limiting values $M^{\pm}$ of M as λ approaches Σ_2 from $\mathbb{C}^{\pm}$ can be expressed as follows:

$$M^{\pm}(x,t,\lambda) := M(x,t,\lambda_{\pm}) = \left(\frac{(D_1^{-1}\tilde{\Phi}_1^{(1)})(x,t,\lambda_{\pm})}{s_{11}(\lambda_{\pm})\mathrm{e}^{p_1(x,t,\lambda_{\pm})-p_2(x,t,\lambda_{\pm})}}, (D_2^{-1}\tilde{\Phi}_2^{(2)})(x,t,\lambda_{\pm})\right), \quad \lambda\in\dot{\Sigma}_1$$

$$M^{\pm}(x,t,\lambda) := M(x,t,\lambda_{\pm}) = \left(\frac{(D_1^{-1}\tilde{\Phi}_1^{(1)})(x,t,\lambda)}{s_{11}(\lambda_{\pm})\mathrm{e}^{p_1(x,t,\lambda)-p_2(x,t,\lambda_{\pm})}}, (D_2^{-1}\tilde{\Phi}_2^{(2)})(x,t,\lambda_{\pm})\right), \quad \lambda\in\dot{\Sigma}_0.$$

Proposition 4.3.1. *M^+ and M^- are related as follows:*

$$M^+(x,t,\lambda) = M^-(x,t,\lambda)J(x,t,\lambda), \quad \lambda\in\dot{\Sigma}_1\cup\dot{\Sigma}_0,$$

where

$$J(x,t,\lambda) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-p_2(x,t,\lambda_+)} & 0 \\ 0 & \mathrm{e}^{p_2(x,t,\lambda_+)} \end{pmatrix} J_0(\lambda) \begin{pmatrix} \mathrm{e}^{p_2(x,t,\lambda_+)} & 0 \\ 0 & \mathrm{e}^{-p_2(x,t,\lambda_+)} \end{pmatrix} \tag{4.68a}$$

with

$$J_0(\lambda) = \begin{cases} \begin{pmatrix} 1-|\rho(\lambda)|^2 & -\overline{\rho(\lambda)} \\ \rho(\lambda) & 1 \end{pmatrix}, & \lambda\in\dot{\Sigma}_1, \\ \begin{pmatrix} 0 & -\frac{1}{\rho(\lambda)} \\ \rho(\lambda) & 1 \end{pmatrix}, & \lambda\in\dot{\Sigma}_0. \end{cases} \tag{4.68b}$$

Proof. (i) $\lambda\in\dot{\Sigma}_1$. Considering (4.24) columnwise, rearranging the columns and using (4.59a) for $\lambda\in\dot{\Sigma}_1$, we obtain

$$M^+(x,t,\lambda) = M^-(x,t,\lambda)\mathrm{i}\begin{pmatrix} \frac{\tilde{s}_{21}(x,t,\lambda_+)\tilde{s}_{11}(x,t,\lambda_-)}{\tilde{s}_{11}(x,t,\lambda_+)\tilde{s}_{22}(x,t,\lambda_+)} & \frac{\tilde{s}_{11}(x,t,\lambda_-)}{\tilde{s}_{22}(x,t,\lambda_+)} \\ 1-\frac{\tilde{s}_{21}(x,t,\lambda_+)\tilde{s}_{12}(x,t,\lambda_+)}{\tilde{s}_{11}(x,t,\lambda_+)\tilde{s}_{22}(x,t,\lambda_+)} & -\frac{\tilde{s}_{12}(x,t,\lambda_+)}{\tilde{s}_{22}(x,t,\lambda_+)} \end{pmatrix}. \tag{4.69}$$

Since $\mathrm{e}^{p_1(x,t,\lambda_-)-p_2(x,t,\lambda_-)} = \mathrm{e}^{p_2(x,t,\lambda_+)-p_1(x,t,\lambda_+)}$, from (4.47) and (4.51a) we have $\frac{\tilde{s}_{11}(\lambda_-)}{\tilde{s}_{22}(\lambda_+)} = \frac{s_{11}(\lambda_-)}{s_{22}(\lambda_+)} = 1$. Moreover, using the definition (4.65) of $\rho(\lambda)$ and (4.51), we have $\overline{\rho(\lambda)} = \frac{s_{12}(\lambda_+)}{s_{22}(\lambda_+)}$. Hence we can rewrite the jump condition (4.69) as (4.68a) with (4.68b).

(ii) $\lambda \in \dot{\Sigma}_0$. Considering (4.28) columnwise, rearranging the columns and using (4.59b) and (4.59c) for $\lambda_+ \in \dot{\Sigma}_0$, we obtain

$$M^+(x,t,\lambda) = M^-(x,t,\lambda)\mathrm{i}\begin{pmatrix} \frac{\tilde{s}_{21}(x,t,\lambda_+)}{\tilde{s}_{11}(x,t,\lambda_+)} & 1 \\ 0 & -\frac{\tilde{s}_{11}(x,t,\lambda_+)}{\tilde{s}_{21}(x,t,\lambda_+)} \end{pmatrix}. \tag{4.70}$$

Then, using the definition of $\rho(\lambda)$ together with (4.55c) and (4.55b), we can rewrite the jump condition (4.70) as (4.68a) with (4.68b).

□

Remark 4.3.2. Notice that

$$\det J \equiv 1 \tag{4.71}$$

and that $J_0(\lambda)$ (and hence J) is continuous at $\pm\frac{1}{A_1}$ if $|\rho(\pm\frac{1}{A_1})| = 1$ and $\rho(\pm\frac{1}{A_1}+0) = \rho(\pm\frac{1}{A_1}-0)$, and discontinuous otherwise.

Normalization condition at $\lambda \to \infty$.

Proposition 4.3.3. *As $\lambda \to \infty$:*

$$M(x,t,\lambda) = \begin{cases} \sqrt{\frac{1}{2}}\begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty,\ \lambda \in \mathbb{C}^+, \\ \sqrt{\frac{1}{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty,\ \lambda \in \mathbb{C}^-. \end{cases} \tag{4.72}$$

Proof. Expanding $D_j^{-1}(\lambda)$ (4.12) as $\lambda \to \infty$, we get

$$D_j^{-1}(\lambda) = \begin{cases} \sqrt{\frac{1}{2}}\begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty,\ \lambda \in \mathbb{C}^+, \\ \sqrt{\frac{1}{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty,\ \lambda \in \mathbb{C}^-. \end{cases}$$

Recalling that $\left(\tilde{\Phi}_1^{(1)}\ \tilde{\Phi}_2^{(2)} \right) \to I$ as $\lambda \to \infty$, we have, for $\lambda \in \mathbb{C}^+$,

$$(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda) = \sqrt{\frac{1}{2}}\begin{pmatrix} -1 \\ \mathrm{i} \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), \quad \lambda \to \infty,$$

$$(D_2^{-1}\tilde{\Phi}_2^{(2)})(\lambda) = \sqrt{\frac{1}{2}}\begin{pmatrix} \mathrm{i} \\ -1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), \quad \lambda \to \infty.$$

Substituting this into (4.26a), we get $\tilde{s}_{11}(\lambda) = 1 + \mathrm{O}(\frac{1}{\lambda}),\ \lambda \to \infty$.

Similarly, for $\lambda \in \mathbb{C}^-$ we have

$$(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda) = \sqrt{\frac{1}{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), \quad \lambda \to \infty,$$

$$(D_2^{-1}\tilde{\Phi}_2^{(2)})(\lambda) = \sqrt{\frac{1}{2}}\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), \quad \lambda \to \infty,$$

and $\tilde{s}_{11}(\lambda) = 1 + \mathrm{O}(\frac{1}{\lambda}),\ \lambda \to \infty$. Then the claim follows. □

Remark 4.3.4. In order to have a standard normalisation as $\lambda \to \infty$, we can introduce

$$\tilde{M}(x,t,\lambda) := \begin{cases} \sqrt{\frac{1}{2}}\begin{pmatrix} -1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} M(x,t,\lambda), & \lambda \in \mathbb{C}^+, \\ \sqrt{\frac{1}{2}}\begin{pmatrix} -1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix} M(x,t,\lambda)\mathrm{i}\sigma_1, & \lambda \in \mathbb{C}^-. \end{cases} \tag{4.73}$$

Then we have $\tilde{M} \to I$ at $\lambda \to \infty$. On the other hand, $\tilde{M}$ acquires an additional jump across $\lambda \in \mathbb{R} \setminus \Sigma_2$:

$$\tilde{M}^+(x,t,\lambda) = \tilde{M}^-(x,t,\lambda)\tilde{J}(x,t,\lambda),\ \lambda \in \mathbb{R} \setminus \left\{ \cup_{j=1,2} \{A_j^{-1}\} \cup \{-A_j^{-1}\}\right\}$$

with

$$\tilde{J}(x,t,\lambda) = \begin{cases} \tilde{J}_{\Sigma_j}(x,t,\lambda), & \lambda \in \dot{\Sigma}_j, \quad j = 0,1 \\ \tilde{J}_{\mathbb{R}\setminus\Sigma_2}(x,t,\lambda), & \lambda \in \mathbb{R} \setminus \Sigma_2, \end{cases}$$

where $\tilde{J}_{\Sigma_j}(x,t,\lambda) = \mathrm{e}^{-p_2(x,t,\lambda_+)\sigma_3} J_0(\lambda)\mathrm{e}^{p_2(x,t,\lambda_+)\sigma_3}$, $j = 0,1$ and $\tilde{J}_{\mathbb{R}\setminus\Sigma_2}(x,t,\lambda) = -\mathrm{i}\sigma_1$.

Remark 4.3.5. Using (4.26b), we obtain $\tilde{s}_{21}(\lambda) = \mathrm{O}(\frac{1}{\lambda})$ as $\lambda \to \infty$. Notice that $\rho(\lambda) = \frac{s_{21}(\lambda_+)}{s_{11}(\lambda_+)} = \frac{\tilde{s}_{21}(\lambda_+)}{\tilde{s}_{11}(\lambda_+)} \mathrm{e}^{-2p_2(x,t,\lambda_+)}$; since $p_2(x,t,\lambda_+)$ is purely imaginary for $\lambda \in \Sigma_2$, $\mathrm{e}^{-2p_2(x,t,\lambda_+)}$ is bounded and thus $\rho(\lambda) = \mathrm{O}(\frac{1}{\lambda})$ as $\lambda \to \infty$. Consequently,

$$J_0(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), \quad \lambda \to \pm\infty$$

and

$$J(x,t,\lambda) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), \quad \lambda \to \pm\infty.$$

Symmetries

From the symmetry properties of the eigenfunctions and scattering functions (4.32), (4.45), (4.36), and (4.49) it follows that

$$M(-\lambda) = -\sigma_3 M(\lambda)\sigma_3, \qquad \overline{M(\overline{\lambda})} = -M(\lambda), \qquad \lambda \in \mathbb{C} \setminus \Sigma_2, \tag{4.74a}$$

$$M((-\lambda)_-) = -\sigma_3 M(\lambda_+)\sigma_3, \qquad \overline{M(\lambda_-)} = -M(\lambda_+), \qquad \lambda \in \dot{\Sigma}_1. \tag{4.74b}$$

where $M(\lambda) \equiv M(x,t,\lambda)$.

Singularities at $\pm\frac{1}{A_j}$.

Let $A^{(ij)}$ denote the elements of a 2×2 matrix $A = \begin{pmatrix} A^{(11)} & A^{(12)} \\ A^{(21)} & A^{(22)} \end{pmatrix}$.

Proposition 4.3.6. *$M(x,t,\lambda)$ has the following behaviour at the branch points*

$$M(x,t,\lambda)=\begin{cases}\frac{e^{\frac{3\pi i}{4}}}{\nu_2^+(\lambda)}\begin{pmatrix}0 & \Upsilon_2\\ 0 & \Lambda_2\end{pmatrix}+O(1), \quad \lambda\to\frac{1}{A_2},\\ \frac{i}{\nu_2^-(\lambda)}\begin{pmatrix}0 & \Upsilon_2\\ 0 & -\Lambda_2\end{pmatrix}+O(1), \quad \lambda\to-\frac{1}{A_2},\\ \frac{c_+e^{\frac{3\pi i}{4}}}{\nu_1^+(\lambda)}\begin{pmatrix}\Upsilon_1 & 0\\ \Lambda_1 & 0\end{pmatrix}+O(1), \quad \lambda\to\frac{1}{A_1},\ \lambda\in\mathbb{C}_+,\\ \frac{\overline{c_+}e^{\frac{3\pi i}{4}}}{\nu_1^+(\lambda)}\begin{pmatrix}\Upsilon_1 & 0\\ \Lambda_1 & 0\end{pmatrix}+O(1), \quad \lambda\to\frac{1}{A_1},\ \lambda\in\mathbb{C}_-,\\ \frac{\overline{c_+}i}{\nu_1^-(\lambda)}\begin{pmatrix}-\Upsilon_1 & 0\\ \Lambda_1 & 0\end{pmatrix}+O(1), \quad \lambda\to-\frac{1}{A_1},\ \lambda\in\mathbb{C}_+,\\ \frac{c_+i}{\nu_1^-(\lambda)}\begin{pmatrix}-\Upsilon_1 & 0\\ \Lambda_1 & 0\end{pmatrix}+O(1), \quad \lambda\to-\frac{1}{A_1},\ \lambda\in\mathbb{C}_-,\end{cases} \tag{4.75}$$

where $\nu_j^\pm(\lambda)$ are defined in Proposition 4.2.17, and $\Upsilon_j=-(2A_j)^{\frac14}\alpha_j(x,t)+\frac{(a_j(x,t)+b_j(x,t))}{(2A_j)^{\frac14}}$, $\Lambda_j=(2A_j)^{\frac14}\alpha_j(x,t)+\frac{(a_j(x,t)+b_j(x,t))}{(2A_j)^{\frac14}}$ with $\alpha_j(x,t),\ a_j(x,t),\ b_j(x,t)\in\mathbb{R}$, $j=1,2$ as in Proposition 4.2.16.

Moreover, $c_+(x,t)=0$ if $\beta_1(x,t)\neq 0$ and $c_+(x,t)=\frac{1}{\tilde{s}_{11}(x,t,\frac{1}{A_2})}$ if $\beta_1(x,t)=0$, where $\beta_1(x,t)$ is defined in (4.76b).

Proof. Combining Proposition 4.2.16 with Proposition 4.2.17 we get

$$D_j^{-1}(\lambda)\tilde{\Phi}_j(x,t,\lambda)=\frac{e^{\frac{3\pi i}{4}}}{\nu_j^+(\lambda)}\left(-(2A_j)^{\frac14}\alpha_j\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}+\frac{a_j+b_j}{(2A_j)^{\frac14}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)$$
$$+O\left((\lambda-\frac{1}{A_j})^{1/4}\right)$$

as $\lambda\to\frac{1}{A_j}$, where $\alpha_j=\alpha_j(x,t)$, $a_j=a_j(x,t)$ and $b_j=b_j(x,t)$.

First, consider the behaviour of M near $\frac{1}{A_2}$. Since $D_1^{-1}(\lambda)\tilde{\Phi}_1^{(1)}(x,t,\lambda)$ is analytic at $\frac{1}{A_2}$, we have

$$D_1^{-1}(\frac{1}{A_2})\tilde{\Phi}_1^{(1)}(x,t,\frac{1}{A_2}) = \mathrm{i}\begin{pmatrix} a(x,t) \\ c(x,t) \end{pmatrix}$$

with

$$a(x,t) = \left|\sqrt{\frac{A_2+|\sqrt{A_2^2-A_1^2}|}{|\sqrt{A_2^2-A_1^2}|}}\right| \left(\frac{A_1}{A_2+|\sqrt{A_2^2-A_1^2}|}\tilde{\Phi}_1^{(11)}(x,t,\frac{1}{A_2}) + \tilde{\Phi}_1^{(21)}(x,t,\frac{1}{A_2})\right)$$

and

$$c(x,t) = \left|\sqrt{\frac{A_2+|\sqrt{A_2^2-A_1^2}|}{|\sqrt{A_2^2-A_1^2}|}}\right| \left(\frac{A_1}{A_2+|\sqrt{A_2^2-A_1^2}|}\tilde{\Phi}_1^{(21)}(x,t,\frac{1}{A_2}) + \tilde{\Phi}_1^{(11)}(x,t,\frac{1}{A_2})\right).$$

Then, using (4.26a), we get the following expansion of $\tilde{s}_{11}(x,t,\lambda)$ at $\frac{1}{A_2}$:

$$\tilde{s}_{11}(x,t,\lambda) = \frac{\mathrm{i}e^{\frac{3\pi \mathrm{i}}{4}}}{\nu_2^+(\lambda)}\beta_2(x,t) + \mathrm{O}(1), \quad \lambda \to \frac{1}{A_2}$$

with $\beta_2(x,t) = \left((2A_2)^{\frac{1}{4}}\alpha_2(x,t)(a(x,t)+c(x,t)) + \frac{(a_2(x,t)+b_2(x,t))(a(x,t)-c(x,t))}{(2A_2)^{\frac{1}{4}}}\right)$.

Notice that the symmetry (4.44) implies that $\tilde{\Phi}_1^{(11)}(x,t,\frac{1}{A_2})$ and $\tilde{\Phi}_1^{(21)}(x,t,\frac{1}{A_2})$ are real-valued and thus $a(x,t) \in \mathbb{R}$ and $c(x,t) \in \mathbb{R}$.

Recall the assumption $s_{11}(\frac{1}{A_2}) \neq 0$, which implies $\tilde{s}_{11}(\frac{1}{A_2}) \neq 0$. Thus there are two possibilities: either $\beta_2(x,t) \neq 0$ or $\beta_2(x,t) = 0$ and $\tilde{s}_{11}(\frac{1}{A_2}) =: \gamma \neq 0$. In the both cases,

$$M(x,t,\lambda) = \frac{e^{\frac{3\pi \mathrm{i}}{4}}}{\nu_2^+(\lambda)}\begin{pmatrix} 0 & -(2A_2)^{\frac{1}{4}}\alpha_2(x,t) + \frac{(a_2(x,t)+b_2(x,t))}{(2A_2)^{\frac{1}{4}}} \\ 0 & (2A_2)^{\frac{1}{4}}\alpha_2(x,t) + \frac{(a_2(x,t)+b_2(x,t))}{(2A_2)^{\frac{1}{4}}} \end{pmatrix} + \mathrm{O}(1), \quad \lambda \to \frac{1}{A_2}.$$

Now consider the behaviour of M as λ approaches $\frac{1}{A_1}$ from the upper half-plane. Since $D_2^{-1}(\lambda)\tilde{\Phi}_2^{(2)}(x,t,\lambda)$ has no singularity at $\frac{1}{A_1}$, we have

$$D_2^{-1}(\frac{1}{A_{1+}})\tilde{\Phi}_2^{(2)}(x,t,\frac{1}{A_{1+}}) = \begin{pmatrix} b_+(x,t) \\ d_+(x,t) \end{pmatrix}$$

with

$$b_+ = \left|\sqrt{\frac{-\mathrm{i}A_1-|\sqrt{A_2^2-A_1^2}|}{|\sqrt{A_2^2-A_1^2}|}}\right| \left(\frac{A_2}{A_1-\mathrm{i}|\sqrt{A_2^2-A_1^2}|}\tilde{\Phi}_2^{(12)}(x,t,\frac{1}{A_{1+}}) + \tilde{\Phi}_2^{(22)}(x,t,\frac{1}{A_{1+}})\right)$$

and

$$d_+ = \left|\sqrt{\frac{-\mathrm{i}A_1 - |\sqrt{A_2^2 - A_1^2}|}{|\sqrt{A_2^2 - A_1^2}|}}\right| \left(\frac{A_2}{A_1 - \mathrm{i}|\sqrt{A_2^2 - A_1^2}|}\tilde{\Phi}_2^{(22)}(x, t, \frac{1}{A_1}_+) + \tilde{\Phi}_2^{(12)}(x, t, \frac{1}{A_1}_+)\right).$$

Then, using (4.26a), we get the following expansion of $\tilde{s}_{11}(x, t, \lambda)$ at $\frac{1}{A_1}$ in the upper half-plane:

$$\tilde{s}_{11}(x, t, \lambda) = \frac{\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\nu_1^+(\lambda)}\beta_1(x, t) + \mathrm{O}(1), \quad \lambda \to \frac{1}{A_1}, \quad \lambda \in \mathbb{C}_+ \tag{4.76a}$$

with

$$\beta_1(x, t) = -(2A_2)^{\frac{1}{4}}\alpha_1(x, t)(b_+(x, t)+d_+(x, t))+\frac{(a_1(x, t) + b_1(x, t))(d_+(x, t) - b_+(x, t))}{(2A_1)^{\frac{1}{4}}}. \tag{4.76b}$$

As above, we have two possibilities: either $\beta_1(x, t) \neq 0$ (generic case) or $\beta_1(x, t) = 0$ and $\tilde{s}_{11}(\frac{1}{A_1}_+) = \gamma_1^+ \neq 0$. This gives

$$M(x, t, \lambda) = \frac{c_+\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\nu_1^+(\lambda)}\begin{pmatrix} -(2A_1)^{\frac{1}{4}}\alpha_1(x, t) + \frac{(a_1(x,t)+b_1(x,t))}{(2A_1)^{\frac{1}{4}}} & 0 \\ (2A_1)^{\frac{1}{4}}\alpha_1(x, t) + \frac{(a_1(x,t)+b_1(x,t))}{(2A_1)^{\frac{1}{4}}} & 0 \end{pmatrix}+\mathrm{O}(1), \quad \lambda \to \frac{1}{A_1}, \lambda \in \mathbb{C}_+,$$

where $c_+ = 0$ if $\beta_1(x, t) \neq 0$, and $c_+ = \frac{1}{\tilde{s}_{11}(\frac{1}{A_1}_+)}$ if $\beta_1(x, t) = 0$.

The other the statements follow from the symmetry considerations. □

Remark 4.3.7.

1. $\rho(\lambda) = \frac{\tilde{s}_{21}(\lambda_+)}{\tilde{s}_{11}(\lambda_+)}\mathrm{e}^{-2p_2(x,t,\lambda_+)} = \mathrm{O}(1)$ as $\lambda \to \frac{1}{A_2}$. Indeed, in the proof of the Proposition 4.3.6, we have seen that $\tilde{s}_{11}(x, t, \lambda) = \frac{\mathrm{i}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\nu_2^+(\lambda)}\beta_2(x, t) + \mathrm{O}(1)$ as $\lambda \to \frac{1}{A_2}$. Analogously, due to (4.26b), we have $\tilde{s}_{21}(x, t, \lambda) = -\frac{\mathrm{i}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\nu_2^+(\lambda)}\beta_2(x, t) + \mathrm{O}(1)$ as $\lambda \to \frac{1}{A_2}$. Moreover, by our assumptions, $\tilde{s}_{11}(\frac{1}{A_2}) \neq 0$, and hence the claim follows.

2. $\rho(\lambda) = \mathrm{O}(1)$ as $\lambda \to \frac{1}{A_1}$. Indeed, we already know that $\tilde{s}_{11}(\lambda) = \frac{\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\nu_1^+(\lambda)}\beta_1(x, t)+\mathrm{O}(1)$ as $\lambda \to \frac{1}{A_1}$, $\lambda \in \mathbb{C}_+$. Analogously, (4.26b) together with (4.58) implies that if $\beta_1 \neq 0$ we have $\tilde{s}_{21}(\lambda) = \frac{\mathrm{i}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}}{\nu_1^+(\lambda)}\overline{\beta_1(x, t)}+\mathrm{O}(1)$, $\lambda \to \frac{1}{A_1}$, $\lambda \in \mathbb{C}_+$. Moreover, by our assumptions, $\tilde{s}_{11}(\frac{1}{A_1}_+) \neq 0$, and hence the claim follows.

Residue conditions.

By (4.23), zeros of $\tilde{s}_{11}(\lambda)$ coincide with zeros $s_{11}(\lambda)$; hence, by Proposition 4.2.15, they are real and simple. Moreover, the symmetry (4.30) implies that $-\lambda_k$ is a zero of $\tilde{s}_{11}(\lambda)$ together with λ_k; we will denote the set of zeros of $s_{11}(\lambda)$ by $\{\lambda_k, -\lambda_k\}_1^n$, where $\lambda_k \in (0, \frac{1}{A_2})$.

Proposition 4.3.8. *$M^{(1)}$ has simple poles at $\{\lambda_k, -\lambda_k\}_1^n$. Moreover,*

$$\operatorname{Res}_{\pm\lambda_k} M^{(1)}(x,t,\lambda) = \frac{b_k}{s_{11}'(\lambda_k)} \mathrm{e}^{2p_2(\lambda_k)} M^{(2)}(x,t,\pm\lambda_k), \tag{4.77}$$

Moreover, $\frac{b_k}{s_{11}'(\lambda_k)}\mathrm{e}^{2p_2(\lambda_k)} \in \mathbb{R}$.

Proof. Recall that $\Phi_1^{(1)}(\lambda_k) = b_k \Phi_2^{(2)}(\lambda_k)$ with $b_k = b(\lambda_k) \in \mathbb{R}$ due the symmetry (4.42). Then $(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda_k)$ and $(D_2^{-1}\tilde{\Phi}_2^{(2)})(\lambda_k)$ are related as

$$\frac{(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda_k)}{s_{11}(\lambda_k)\mathrm{e}^{p_1(\lambda_k)-p_2(\lambda_k)}} = \frac{b_k}{s_{11}(\lambda_k)} \mathrm{e}^{2p_2(\lambda_k)} (D_2^{-1}\tilde{\Phi}_2^{(2)})(\lambda_k),$$

and hence (4.77) follows. Moreover, differentiating (4.43) and using the fact that $\lambda_k \in \mathbb{R}$, we get $s_{11}'(\lambda_k) \in \mathbb{R}$, and thus $\frac{b_k}{s_{11}'(\lambda_k)}\mathrm{e}^{2p_2(\lambda_k)} \in \mathbb{R}$.

Differentiating (4.30), we get $s_{11}'(\lambda_k) = -s_{11}'(-\lambda_k)$. On the other hand, (4.29) implies that $b(-\lambda_k) = -b(\lambda_k)$. Combining these facts, we obtain (4.77) with the minus sign. □

Remark 4.3.9. In terms of $\tilde{M}$ (4.73), the residue conditions take the following form:

$$\tilde{M}^{(1)}(x,t,\lambda) = \frac{1}{\lambda-\lambda_k} \frac{b_k}{s_{11}'(\lambda_k)} \mathrm{e}^{2p_2(\lambda_k)} \tilde{M}^{(2)}(x,t,\lambda_{k+}) + \mathrm{O}(1), \ \lambda \to \lambda_k, \ \lambda \in \mathbb{C}_+, \tag{4.78a}$$

$$\tilde{M}^{(2)}(x,t,\lambda) = \frac{1}{\lambda-\lambda_k} \frac{b_k}{s_{11}'(\lambda_k)} \mathrm{e}^{2p_2(\lambda_k)} \tilde{M}^{(1)}(x,t,\lambda_{k-}) + \mathrm{O}(1), \ \lambda \to \lambda_k, \ \lambda \in \mathbb{C}_-. \tag{4.78b}$$

RH problem parameterized by (x, t).

In the framework of the Riemann–Hilbert approach to nonlinear evolution equations, one interprets the jump relation, normalization condition, singularity conditions, and residue conditions as a Riemann–Hilbert problem, with the jump matrix and residue parameters determined by the initial data for the nonlinear problem in question. The considerations above imply that $M(x,t,\lambda)$ can be characterized as the solution of the following Riemann–Hilbert problem:

Find a 2×2 meromorphic matrix $M(x,t,\lambda)$ that satisfies the following conditions:

- *Jump* condition (4.68).

- *Normalization* condition (4.72).

- *Singularity* conditions: the singularities of $M(x,t,\lambda)$ at $\pm\frac{1}{A_j}$ are of order not bigger than $\frac{1}{4}$.

- *Residue* conditions (if any): given $\{\lambda_k, \kappa_k\}_1^N$ with $\lambda_k \in (0, \frac{1}{A_2})$ and $\kappa_k \in \mathbb{R} \setminus \{0\}$, $M^{(1)}(x,t,\lambda)$ has simple poles at $\{\lambda_k, -\lambda_k\}_1^N$, with the residues satisfying the equations

$$\operatorname{Res}_{\pm\lambda_k} M^{(1)}(x,t,\lambda) = \kappa_k \mathrm{e}^{2p_2(\lambda_k)} M^{(2)}(x,t,\pm\lambda_k). \tag{4.79}$$

Remark 4.3.10. The solution of the RH problem above, if exists, satisfies the following properties:

1. $\det M \equiv 1$ (follows from the fact that $\det J \equiv 1$).

2. *Symmetries*

$$M(-\lambda) = -\sigma_3 M(\lambda)\sigma_3, \qquad \overline{M(\overline{\lambda})} = -M(\lambda), \qquad \lambda \in \mathbb{C} \setminus \Sigma_2, \tag{4.80a}$$

$$M((-\lambda)_-) = -\sigma_3 M(\lambda_+)\sigma_3, \qquad \overline{M(\lambda_-)} = -M(\lambda_+), \qquad \lambda \in \dot{\Sigma}_1. \tag{4.80b}$$

where $M(\lambda) \equiv M(x,t,\lambda)$ (follows from the respective symmetries of the jump matrix and the residue conditions, assuming the uniqueness of the solution).

Remark 4.3.11. We do not need to specify the singularities at the branch points $\pm\frac{1}{A_j}$ in order to formulate RH problem. It is enough to require them to be of order not bigger than $\frac{1}{4}$.

As for other Camassa–Holm-type equations, a principal drawback of the RH formalism presented above is that the jump condition (4.68) involves not only the scattering functions uniquely determined by the initial data for problem (4.1), but the solution itself, via $p_2(x,t,\lambda)$ involving $m(x,t)$ (4.14b). In order to have the data for a RH problem to be explicitly determined by the initial data only, we introduce the space variable $y(x,t) := x - \frac{1}{A_2}\int_x^{+\infty}(m(\xi,t) - A_2)\mathrm{d}\xi - A_2^2 t$, which will play the role of a parameter (together with t) for the RH problem, see Section 4.3.3 below.

In order to determine an efficient way for retrieving the solution of the mCH equation from the solution of the RH problem, we will use the behavior of the Jost solutions of the Lax pair equations evaluated at $\lambda = 0$, for which the x-equation (4.7a) of the Lax pair becomes trivial (independent of the solution of the mCH equation).

4.3.2 Eigenfunctions near 0.

In the case of the Camassa–Holm equation [26] as well as other CH-type nonlinear integrable equations studied so far, see, e.g., [30], the analysis of the behavior of the respective Jost solutions at a dedicated point in the complex plane of the spectral parameter (in our case, at $\lambda = 0$) requires a dedicated gauge transformation of the Lax pair equations.

It is remarkable that in the case of the mCH equation, in order to control the behavior of the eigenfunctions at $\lambda = 0$, we don't need to introduce an additional transformation; all we need is to regroup the terms in the Lax pair (4.13).

Namely, we rewrite (4.13) as follows:

$$\hat{\Phi}_{jx} + \frac{\mathrm{i}A_j k_j(\lambda)}{2}\sigma_3 \hat{\Phi}_j = \hat{U}_j^0 \hat{\Phi}_j, \tag{4.81a}$$

where $\hat{U}_j^0 \equiv \hat{U}_j^0(x,t,\lambda)$ is given by

$$\hat{U}_j^0 = \frac{(m - A_j)}{2} \frac{\lambda}{\mathrm{i}k_j(\lambda)} \begin{pmatrix} \lambda & \frac{1}{A_j} \\ -\frac{1}{A_j} & -\lambda \end{pmatrix}, \tag{4.81b}$$

and

$$\hat{\Phi}_{jt} + \mathrm{i}A_j k_j(\lambda) \left(-\frac{A_j^2}{2} - \frac{1}{\lambda^2} \right) \sigma_3 \hat{\Phi}_j = \hat{V}_j^0 \hat{\Phi}_j, \tag{4.81c}$$

where $\hat{V}_j^0 \equiv \hat{V}_j^0(x,t,\lambda)$ is given by

$$\hat{V}_j^0 = \hat{V}_j + \mathrm{i}A_j k_j(\lambda) \left(\frac{(u^2 - u_x^2)m}{2A_j} - \frac{A_j^2}{2} \right) \sigma_3. \tag{4.81d}$$

Further, introduce (compare with (4.14b))

$$p_j^0(x,t,\lambda) := \frac{\mathrm{i}A_j k_j(\lambda)}{2} \left(x - 2\left(\frac{A_j^2}{2} + \frac{1}{\lambda^2} \right) t \right). \tag{4.82}$$

Then, introducing $Q_j^0 := p_j^0 \sigma_3$ and $\widetilde{\Phi}_j^0 := \hat{\Phi}_j \mathrm{e}^{Q_j^0}$, equations (4.81a) and (4.81c) reduce to

$$\begin{cases} \widetilde{\Phi}_{jx}^0 + [Q_{jx}^0, \widetilde{\Phi}_j^0] = \hat{U}_j^0 \widetilde{\Phi}_j^0, \\ \widetilde{\Phi}_{jt}^0 + [Q_{jt}^0, \widetilde{\Phi}_j^0] = \hat{V}_0^j \widetilde{\Phi}_j^0. \end{cases} \tag{4.83}$$

Define the Jost solutions $\widetilde{\Phi}_j^0$ of (4.83) as the solutions of the integral equations

$$\widetilde{\Phi}_j^0(x,t,\lambda) = I + \int_{(-1)^j\infty}^{x} \mathrm{e}^{\frac{-\mathrm{i}A_j k_j(\lambda)}{2}(x-\xi)\sigma_3} \hat{U}_j^0(\xi,t,\lambda) \widetilde{\Phi}_j^0(\xi,t,\lambda) \mathrm{e}^{\frac{\mathrm{i}A_j k_j(\lambda)}{2}(x-\xi)\sigma_3 3} \mathrm{d}\xi. \tag{4.84}$$

Further, defining $\hat{\Phi}_j^0 := \widetilde{\Phi}_j^0 \mathrm{e}^{-p_j^0 \sigma_3}$, we observe that $\hat{\Phi}_j^0(x,t,\lambda)$ and $\hat{\Phi}_j(x,t,\lambda)$ satisfy the same differential equations (4.13) and thus they are related by matrices $C_j(\lambda)$ independent of x and t:

$$\hat{\Phi}_j = \hat{\Phi}_j^0 C_j(\lambda).$$

Consequently,

$$\widetilde{\Phi}_j(x,t,\lambda) = \widetilde{\Phi}_j^0(x,t,\lambda)\mathrm{e}^{-p_j^0(x,t,\lambda)\sigma_3}C_j(\lambda)\mathrm{e}^{p_j(x,t,\lambda)\sigma_3}. \tag{4.85}$$

Since $p_j(x,t,\lambda) - p_j^0(x,t,\lambda) = \frac{\mathrm{i}k_j(\lambda)}{2}\int_x^{(-1)^j\infty}(m(\xi,t) - A_j)\mathrm{d}\xi$ and

$$\widetilde{\Phi}_j(x,t,\lambda) = \widetilde{\Phi}_j^0(x,t,\lambda)\mathrm{e}^{\frac{\mathrm{i}k_j(\lambda)}{2}\int_{(-1)^j\infty}^x(m(\xi,t)-A_j)\mathrm{d}\xi\sigma_3},$$

passing to the limits $x \to (-1)^j\infty$, we get $C_j(\lambda) = I$.

Noticing that $\hat{U}_j^0(x,t,0) \equiv 0$, it follows from (4.84) that $\widetilde{\Phi}_j^0(x,t,0) \equiv I$ and thus $\widetilde{\Phi}_j(x,t,0) = \mathrm{e}^{-\frac{1}{2A_j}\int_{(-1)^j\infty}^x(m(\xi,t)-A_j)\mathrm{d}\xi\sigma_3}$. Combining this with $D_j^{-1}(0) = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}$ gives

$$(D_j^{-1}\widetilde{\Phi}_j)(x,t,0) = \mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{\frac{1}{2A_j}\int_{(-1)^j\infty}^x(m(\xi,t)-A_j)\mathrm{d}\xi} \\ \mathrm{e}^{-\frac{1}{2A_j}\int_{(-1)^j\infty}^x(m(\xi,t)-A_j)\mathrm{d}\xi} & 0\end{pmatrix}$$

Consequently,

$$\tilde{s}_{11}(0) = \mathrm{e}^{-\frac{1}{2A_1}\int_{-\infty}^x(m(\xi,t)-A_1)\mathrm{d}\xi - \frac{1}{2A_2}\int_x^\infty(m(\xi,t)-A_2)\mathrm{d}\xi}$$

(hence $\tilde{s}_{11}(0) \neq 0$) and

$$M(x,t,0) = \mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{-\frac{1}{2A_2}\int_x^\infty(m(\xi,t)-A_2)\mathrm{d}\xi} \\ \mathrm{e}^{\frac{1}{2A_2}\int_x^\infty(m(\xi,t)-A_2)\mathrm{d}\xi} & 0\end{pmatrix}. \tag{4.86}$$

Remark 4.3.12. Considering $M(x,t,\lambda)$ as the solution of the RH problem in Section 4.3.1, the matrix structure of $M(x,t,0)$ as in (4.86), i.e.,

$$M(x,t,0) = \mathrm{i}\begin{pmatrix} 0 & a_1(x,t) \\ a_1^{-1}(x,t) & 0\end{pmatrix} \tag{4.87}$$

with some $a(x,t) \in \mathbb{R}$, which follows from the symmetry properties (4.80a) of the solution taking into account that $\det M \equiv 1$ (provided the solution is unique).

In order to extract the solution of the mCH equation from the solution of the associated RH problem, it turns to be useful to find the next term in the expansion of $M(x,t,\lambda)$ at $\lambda = 0$.

First, expanding $D_j^{-1}(\lambda)$ near 0, we have

$$D_j^{-1}(\lambda) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \lambda \begin{pmatrix} \mathrm{i}\frac{A_j}{2} & 0 \\ 0 & \mathrm{i}\frac{A_j}{2} \end{pmatrix} + \mathrm{O}(\lambda^2).$$

On the other hand, $\mathrm{e}^{\frac{\mathrm{i}k_j(\lambda)}{2}\int_{(-1)^j\infty}^{x}(m(\xi,t)-A_j)\mathrm{d}\xi\sigma_3} = \mathrm{e}^{-\frac{1}{2A_j}\int_{(-1)^j\infty}^{x}(m(\xi,t)-A_j)\mathrm{d}\xi\sigma_3} + \mathrm{O}(\lambda^2)$, $\lambda \to 0$. Then, expanding $\widetilde{\Phi}_j^0(x,t,\lambda)$ at 0 using the Neumann series, we have

$$\widetilde{\Phi}_j^0(x,t,\lambda) = I + \lambda \begin{pmatrix} 0 & -\int_{(-1)^j\infty}^{x} e^{x-\xi}\frac{m-A_j}{2}\mathrm{d}\xi \\ \int_{(-1)^j\infty}^{x} e^{-(x-\xi)}\frac{m-A_j}{2}\mathrm{d}\xi & 0 \end{pmatrix} + \mathrm{O}(\lambda^2).$$

In particular,

$$\tilde{s}_{11}(\lambda) = \mathrm{e}^{-\frac{1}{2A_1}\int_{-\infty}^{x}(m(\xi,t)-A_1)\mathrm{d}\xi - \frac{1}{2A_2}\int_{x}^{\infty}(m(\xi,t)-A_2)\mathrm{d}\xi} + \mathrm{O}(\lambda^2).$$

Finally, we have

$$M(x,t,\lambda) = \mathrm{i}\begin{pmatrix} 0 & a_1(x,t) \\ a_1^{-1}(x,t) & 0 \end{pmatrix} + \mathrm{i}\lambda \begin{pmatrix} a_2(x,t) & 0 \\ 0 & a_3(x,t) \end{pmatrix} + \mathrm{O}(\lambda^2), \quad (4.88)$$

where

$$a_1(x,t) = \mathrm{e}^{-\frac{1}{2A_2}\int_{x}^{\infty}(m(\xi,t)-A_2)\mathrm{d}\xi}, \quad (4.89a)$$

$$a_2(x,t) = (\int_{-\infty}^{x} e^{-(x-\xi)}\frac{m-A_1}{2}\mathrm{d}\xi + \frac{A_1}{2})\mathrm{e}^{\frac{1}{2A_2}\int_{x}^{\infty}(m(\xi,t)-A_2)\mathrm{d}\xi}, \quad (4.89b)$$

$$a_3(x,t) = (\int_{x}^{\infty} e^{(x-\xi)}\frac{m-A_2}{2}\mathrm{d}\xi + \frac{A_2}{2})\mathrm{e}^{-\frac{1}{2A_2}\int_{x}^{\infty}(m(\xi,t)-A_2)\mathrm{d}\xi}. \quad (4.89c)$$

Notice that the matrix structure of terms in the r.h.s. of (4.88) is consistent with the symmetry properties (4.80a) of M.

Proposition 4.3.13. *$u(x,t)$ and $u_x(x,t)$ can be algebraically expressed in terms of the coefficients $a_j(x,t)$, $j = 1,3$ in the development (4.88) of $M(x,t,\lambda)$ as follows:*

$$u(x,t) = a_1(x,t)a_2(x,t) + a_1^{-1}(x,t)a_3(x,t), \quad (4.90a)$$

$$u_x(x,t) = -a_1(x,t)a_2(x,t) + a_1^{-1}(x,t)a_3(x,t). \quad (4.90b)$$

Proof. Introduce $v(x,t) := a_1(x,t)a_2(x,t) + a_1^{-1}(x,t)a_3(x,t)$. Using (4.89) it follows that

$$v(x,t) = \frac{A_1 + A_2}{2} + \int_{-\infty}^{x} e^{-(x-\xi)}\frac{m - A_1}{2}\mathrm{d}\xi + \int_{x}^{\infty} e^{(x-\xi)}\frac{m - A_2}{2}\mathrm{d}\xi \qquad (4.91)$$

and thus, differentiating w.r.t. x,

$$v_x(x,t) = \frac{A_2 - A_1}{2} - \int_{-\infty}^{x} e^{-(x-\xi)}\frac{m - A_1}{2}\mathrm{d}\xi + \int_{x}^{\infty} e^{(x-\xi)}\frac{m - A_2}{2}\mathrm{d}\xi. \qquad (4.92)$$

Since we assume that $\lim_{x\to(-1)^j\infty} m(x,t) = A_i$, from (4.91) it follows that $v - v_{xx} = m$ and that

$$\lim_{x\to(-1)^j\infty} v(x,t) = A_i, \qquad \lim_{x\to(-1)^i\infty} v_x(x,t) = 0;$$

thus $v \equiv u$. Finally, we notice that the expression in the r.h.s. of (4.92) can be written as the r.h.s. of (4.90b) taking into account (4.89).

□

4.3.3 RH problem in the (y,t) scale

As we already mentioned, the jump condition (4.68) involves not only the scattering functions uniquely determined by the initial data for problem (4.1), but the solution itself, via $m(x,t)$, which enters the definition of $p_2(x,t,\lambda)$ (4.14b). In order to have the data for the RH problem to be explicitly determined by the initial data only, we introduce the new space variable $y(x,t)$ by

$$y(x,t) = x - \frac{1}{A_2}\int_{x}^{+\infty}(m(\xi,t) - A_2)\mathrm{d}\xi - A_2^2 t, \qquad (4.93)$$

Then, introducing $\hat{M}(y,t,\lambda)$ so that $M(x,t,\lambda) = \hat{M}(y(x,t),t,\lambda)$, the dependence of the jump matrix in (4.68) on y and t as parameters becomes explicit: the jump condition for $\hat{M}(y,t,\lambda)$ has the form

$$\hat{M}^+(y,t,\lambda) = \hat{M}^-(y,t,\lambda)\hat{J}(y,t,\lambda), \quad \lambda \in \dot{\Sigma}_1 \cup \dot{\Sigma}_0. \qquad (4.94a)$$

Here

$$\hat{J}(y,t,\lambda) := \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\hat{p}_2(y,t,\lambda_+)} & 0 \\ 0 & \mathrm{e}^{\hat{p}_2(y,t,\lambda_+)} \end{pmatrix} J_0(\lambda) \begin{pmatrix} \mathrm{e}^{\hat{p}_2(y,t,\lambda_+)} & 0 \\ 0 & \mathrm{e}^{-\hat{p}_2(y,t,\lambda_+)} \end{pmatrix}, \qquad (4.94b)$$

where $J_0(\lambda)$ is defined by (4.68b) and p_2 is explicitly given in terms of y and t:

$$\hat{p}_2(y,t,\lambda) := \frac{\mathrm{i}A_2 k_2(\lambda)}{2}\left(y - \frac{2t}{\lambda^2}\right). \tag{4.94c}$$

Similarly, the residue conditions (4.79) become explicit as well:

$$\mathrm{Res}_{\pm\lambda_k}\,\hat{M}^{(1)}(y,t,\lambda) = \kappa_k \mathrm{e}^{2\hat{p}_2(y,t,\lambda_k)}\hat{M}^{(2)}(y,t,\pm\lambda_k), \tag{4.95}$$

with $\kappa_k = \frac{b_k}{s_{11}'(\lambda_k)}$.

Noticing that the normalization condition (4.72) and the singularity conditions at $\lambda = \pm\frac{1}{A_j}$ hold in the new scale (y,t), we arrive at the basic RH problem characterizing problem (4.1a).

Basic RH problem. Given $\rho(\lambda)$ for $\lambda \in \dot{\Sigma}_1 \cup \dot{\Sigma}_0$, and $\{\lambda_k, \kappa_k\}_1^N$ with $\lambda_k \in (0, \frac{1}{A_2})$ and $\kappa_k \in \mathbb{R}\setminus\{0\}$, associated with the initial data $u_0(x)$ in (4.1), find a piece-wise (w.r.t. $\dot{\Sigma}_2$) meromorphic, 2×2-matrix valued function $\hat{M}(y,t,\lambda)$ satisfying the following conditions:

- Jump condition (4.94) across $\dot{\Sigma}_1 \cup \dot{\Sigma}_0$ (with $J_0(\lambda)$ defined by (4.68b)).
- Residue conditions (4.95).
- Normalization condition:

$$\hat{M}(y,t,\lambda) = \begin{cases} \sqrt{\frac{1}{2}}\begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda\to\infty,\ \lambda\in\mathbb{C}^+, \\ \sqrt{\frac{1}{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda\to\infty,\ \lambda\in\mathbb{C}^-. \end{cases} \tag{4.96}$$

- Singularity conditions: the singularities of $\hat{M}(y,t,\lambda)$ at $\pm\frac{1}{A_j}$ are of order not bigger than $\frac{1}{4}$.

Evaluating the solution of this problem as $\lambda\to 0$, we are able to present the solution u to the initial value problem (4.1) in a parametric form, see below. As for the data for the RH problem, the scattering matrix $s(\lambda)$ (and hence $s_{11}(\lambda)$,

$s_{21}(\lambda)$, and $\rho(\lambda)$) as well as the discrete data $\{\lambda_k, \kappa_k\}_1^n$ are determined by $u_0(x)$ via the solutions of (4.17) considered for $t = 0$.

The uniqueness of the solution of the basic RH problem follows using standard arguments based on the application of Liouville's theorem to the ratio $\hat{M}_1(\hat{M}_2)^{-1}$ of two potential solutions, $\hat{M}_1$ and $\hat{M}_2$. Particularly, the singularity condition implies that the possible singularities of $\hat{M}_1(\hat{M}_2)^{-1}$ are of order no bigger that 1/2 and that these singularities, being isolated, are removable.

The uniqueness, in particular, implies the symmetries

$$\hat{M}(-\lambda) = -\sigma_3 \hat{M}(\lambda)\sigma_3, \qquad \overline{\hat{M}(\overline{\lambda})} = -M(\lambda), \qquad \lambda \in \mathbb{C} \setminus \Sigma_2, \quad (4.97a)$$

$$\hat{M}((-\lambda)_-) = -\sigma_3 \hat{M}(\lambda_+)\sigma_3, \qquad \overline{\hat{M}(\lambda_-)} = -\hat{M}(\lambda_+), \qquad \lambda \in \dot{\Sigma}_1. \quad (4.97b)$$

where $\hat{M}(\lambda) \equiv \hat{M}(y,t,\lambda)$, which follows from the corresponding symmetries of $\hat{J}(y,t,\lambda)$.

4.3.4 Recovering $u(x,t)$ from the solution of the basic RH problem

Comparing the RH problem (4.68), (4.72), (4.79) parametrized by x and t with the RH problem (4.94)–(4.96) parametrized by y and t and using (4.89)–(4.93) we arrive at our main representation result.

Theorem 4.3.14. *Assume that $u(x,t)$ is the solution of the Cauchy problem* (4.1) *and let $\hat{M}(y,t,x)$ be the solution of the associated RH problem* (4.94)–(4.96)*, whose data are determined by $u_0(x)$. Let*

$$\hat{M}(y,t,\lambda) = \mathrm{i}\begin{pmatrix} 0 & \hat{a}_1(y,t) \\ \hat{a}_1^{-1}(y,t) & 0 \end{pmatrix} + \mathrm{i}\lambda \begin{pmatrix} \hat{a}_2(y,t) & 0 \\ 0 & \hat{a}_3(y,t) \end{pmatrix} + \mathrm{O}(\lambda^2) \quad (4.98)$$

be the development of $\hat{M}(y,t,x)$ at $\lambda = 0$. Then the solution $u(x,t)$ of the Cauchy problem (4.1) *can be expressed, in a parametric form, in terms of $\hat{a}_j(y,t)$, $j = 1,2,3$: $u(x,t) = \hat{u}(y(x,t),t)$, where*

$$\hat{u}(y,t) = \hat{a}_1(y,t)\hat{a}_2(y,t) + \hat{a}_1^{-1}(y,t)\hat{a}_3(y,t), \quad (4.99a)$$

$$x(y,t) = y - 2\ln \hat{a}_1(y,t) + A_2^2 t. \quad (4.99b)$$

Additionally, $\hat{u}_x(y,t)$ can also be algebraically expressed in terms of $\hat{a}_j(y,t)$, $j = 1, 2, 3$: $u_x(x,t) = \hat{u}_x(y(x,t),t)$, where

$$\hat{u}_x(y,t) = -\hat{a}_1(y,t)\hat{a}_2(y,t) + \hat{a}_1^{-1}(y,t)\hat{a}_3(y,t). \tag{4.99c}$$

Alternatively, one can express $\hat{u}_x(y,t)$ in terms of the first term in (4.98) only. The price to pay is that this expression involves the derivatives of this term.

Proposition 4.3.15. *The x-derivative of the solution $u(x,t)$ of the Cauchy problem* (4.1) *has the parametric representation*

$$\hat{u}_x(y,t) = -\frac{1}{A_2}\partial_{ty} \ln \hat{a}_1(y,t), \tag{4.100a}$$

$$x(y,t) = y - 2\ln \hat{a}_1(y,t) + A_2^2 t. \tag{4.100b}$$

Proof. Differentiating the identity $x(y(x,t),t) = x$ w.r.t. t gives

$$0 = \frac{d}{dt}\left(x(y(x,t),t)\right) = x_y(y,t)y_t(x,t) + x_t(y,t). \tag{4.101}$$

From (4.93) it follows that

$$x_y(y,t) = \frac{A_2}{\hat{m}(y,t)}, \tag{4.102}$$

where $\hat{m}(y,t) = m(x(y,t),t)$, and

$$y_t(x,t) = -\frac{1}{A_2}(u^2 - u_x^2)m.$$

Substituting this and (4.102) into (4.101) we obtain

$$x_t(y,t) = \hat{u}^2(y,t) - \hat{u}_x^2(y,t). \tag{4.103}$$

Further, differentiating (4.103) w.r.t. y we get

$$x_{ty}(y,t) = (\hat{u}^2(y,t) - \hat{u}_x^2(y,t))_x x_y(y,t) = 2A_2\hat{u}_x(y,t) \tag{4.104}$$

and thus

$$u_x(x(y,t),t) \equiv \hat{u}_x(y,t) = \frac{1}{2A_2}\partial_{ty}x(y,t) = -\frac{1}{A_2}\partial_{ty}\ln \hat{a}_1(y,t).$$

□

4.4 The case $A_2 < A_1$

Notice that in this case $\Sigma_2 \subset \Sigma_1$ and $\Sigma_0 = [-\frac{1}{A_2}, -\frac{1}{A_1}] \cup [\frac{1}{A_1}, \frac{1}{A_2}]$.

We define Φ_i and $\tilde{\Phi}_i$ as in (4.20) and (4.18), and introduce the scattering matrices $s(\lambda_\pm)$, this time for $\lambda \in \dot{\Sigma}_2$, as matrices relating Φ_1 and Φ_2 (for brevity we keep for it the same notation s):

$$\Phi_1(x,t,\lambda_\pm) = \Phi_2(x,t,\lambda_\pm)s(\lambda_\pm), \qquad \lambda \in \dot{\Sigma}_2 \tag{4.105a}$$

with $\det s(\lambda_\pm) = 1$. In turn, $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are related by

$$D_1^{-1}(\lambda_\pm)\tilde{\Phi}_1(x,t,\lambda_\pm) = D_2^{-1}(\lambda_\pm)\tilde{\Phi}_2(x,t,\lambda_\pm)\mathrm{e}^{-Q_2(x,t,\lambda_\pm)}s(\lambda_+)\mathrm{e}^{Q_1(x,t,\lambda_\pm)}, \qquad \lambda \in \dot{\Sigma}_2. \tag{4.106a}$$

The scattering coefficients s_{ij} can be expressed as in (4.25). However, in this case, (4.25a) implies that $s_{11}(\lambda)$ can be analytically extended to $\mathbb{C} \setminus \Sigma_1$ and defined on the upper and lower parts of $\dot{\Sigma}_1$, and, since $\Phi_2^{(2)}$ is analytic in $\mathbb{C} \setminus \Sigma_2$ and $\Phi_1^{(2)}$ is defined on the upper and lower sides of Σ_1, $s_{12}(\lambda)$ can be extended by (4.25c) to the lower and upper sides of $\dot{\Sigma}_1$. Thus the following relations hold also on $\dot{\Sigma}_0$:

$$\Phi_2^{(2)}(x,t,\lambda_\pm) = s_{11}(\lambda_\pm)\Phi_1^{(2)}(x,t,\lambda_\pm) - s_{12}(\lambda_\pm)\Phi_1^{(1)}(x,t,\lambda_\pm), \quad \lambda \in \dot{\Sigma}_0. \tag{4.107a}$$

and, respectively,

$$(D_2^{-1}\Phi_2^{(2)})(x,t,\lambda_\pm) = \tilde{s}_{11}(x,t,\lambda_\pm)(D_1^{-1}\Phi_1^{(2)})(x,t,\lambda_\pm) - \tilde{s}_{12}(x,t,\lambda_\pm)(D_1^{-1}\Phi_1^{(1)})(x,t,\lambda_\pm), \ \lambda \in \dot{\Sigma}_0, \tag{4.108a}$$

where $\tilde{s}(x,t,\lambda_\pm) := \mathrm{e}^{-Q_2(x,t,\lambda_\pm)}s(\lambda_\pm)\mathrm{e}^{Q_1(x,t,\lambda_\pm)}$.

4.4.1 Symmetries

The symmetries are similar to the case $A_1 < A_2$. In particular,

(1)

$$|s_{11}(\lambda_+)|^2 - |s_{12}(\lambda_+)|^2 = 1, \quad \lambda \in \dot{\Sigma}_2. \tag{4.109}$$

(2)

$$|\frac{s_{12}(\lambda_+)}{s_{11}(\lambda_+)}| \le 1, \quad \lambda \in \dot{\Sigma}_2 \tag{4.110}$$

(3)

$$s_{11}(\lambda_+) = s_{22}(\lambda_-), \quad \lambda \in \dot{\Sigma}_2, \tag{4.111a}$$
$$s_{11}(\lambda_+) = \mathrm{i}s_{12}(\lambda_-), \quad \lambda \in \dot{\Sigma}_0, \tag{4.111b}$$
$$s_{11}(\lambda_-) = -\mathrm{i}s_{12}(\lambda_+), \quad \lambda \in \dot{\Sigma}_0. \tag{4.111c}$$

(4)

$$|\frac{s_{12}(\lambda_+)}{s_{11}(\lambda_+)}| = 1, \quad \lambda \in \dot{\Sigma}_0 \tag{4.112}$$

(5)

$$(D_j^{-1}\tilde{\Phi}_j)((-\lambda)_-) = -\sigma_3(D_j^{-1}\tilde{\Phi}_j)(\lambda_+)\sigma_3, \quad \lambda_+ \in \dot{\Sigma}_j. \tag{4.113}$$

(6)

$$\overline{(D_j^{-1}\tilde{\Phi}_j^{(j)})(\overline{\lambda})} = -(D_j^{-1}\tilde{\Phi}_j^{(j)})(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_j, \tag{4.114}$$

(7)

$$(D_1^{-1}\tilde{\Phi}_1^{(1)})(-\lambda) = -\sigma_3(D_1^{-1}\tilde{\Phi}_1^{(1)})(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_1, \tag{4.115a}$$
$$(D_2^{-1}\tilde{\Phi}_2^{(2)})(-\lambda) = \sigma_3(D_2^{-1}\tilde{\Phi}_2^{(2)})(\lambda), \quad \lambda \in \mathbb{C} \setminus \Sigma_2. \tag{4.115b}$$

(8)

$$D_j^{-1}(\lambda_-)\tilde{\Phi}_j^{(j)}(\lambda_-) = (-\mathrm{i}D_j^{-1}(\lambda_+)\tilde{\Phi}_j(\lambda_+)\sigma_1)^{(j)}, \quad \lambda \in \dot{\Sigma}_2, \tag{4.116a}$$
$$D_1^{-1}(\lambda_-)\tilde{\Phi}_1^{(1)}(\lambda_-) = (-\mathrm{i}D_1^{-1}(\lambda_+)\tilde{\Phi}_1(\lambda_+)\sigma_1)^{(1)}, \quad \lambda \in \dot{\Sigma}_0, \tag{4.116b}$$
$$D_2^{-1}(\lambda_-)\tilde{\Phi}_2^{(2)}(\lambda_-) = D_2^{-1}(\lambda_+)\tilde{\Phi}_2^{(2)}(\lambda_+), \quad \lambda \in \dot{\Sigma}_0. \tag{4.116c}$$

4.4.2 Discrete spectrum

It can be shown in a similar way as for the case $A_1 < A_2$ that discrete spectrum is located on $(-\frac{1}{A_1}, \frac{1}{A_1})$ (assuming that spectral singularities do not arise in the branch points).

4.4.3 RH problem parametrized by (x, t)

Notations. In this case it is convenient to introduce $\check{\rho}$ as

$$\check{\rho}(\lambda) = \frac{s_{12}(\lambda_+)}{s_{11}(\lambda_+)}, \quad \lambda \in \dot{\Sigma}_2 \cup \dot{\Sigma}_0. \tag{4.117}$$

Observe that (4.110) and (4.112) imply that

$$|\check{\rho}(\lambda)| \leq 1, \quad \lambda \in \dot{\Sigma}_2, \tag{4.118a}$$

$$|\check{\rho}(\lambda)| = 1, \quad \lambda \in \dot{\Sigma}_0. \tag{4.118b}$$

Recalling the analytic properties of eigenfunctions and scattering coefficients, we introduce the matrix-valued function

$$N(x,t,\lambda) = \left((D_1^{-1}\tilde{\Phi}_1^{(1)})(x,t,\lambda), \frac{(D_2^{-1}\tilde{\Phi}_2^{(2)})(x,t,\lambda)}{s_{11}(\lambda)\mathrm{e}^{p_1(x,t,\lambda)-p_2(x,t,\lambda)}} \right), \quad \lambda \in \mathbb{C} \setminus \Sigma_2, \tag{4.119}$$

meromorphic in $\mathbb{C} \setminus \Sigma_2$, where p_j, $j = 1, 2$, are defined in (4.14b). Since $D_j^{-1}(\lambda)\tilde{\Phi}_j(x,t,\lambda) = \Phi_j(x,t,\lambda)\mathrm{e}^{Q_j(x,t,\lambda)}$, $N(x,t,\lambda)$ can be written as

$$N(x,t,\lambda) = \left(\Phi_1^{(1)}(x,t,\lambda), \frac{\Phi_2^{(2)}(x,t,\lambda)}{s_{11}(\lambda)} \right) \mathrm{e}^{p_1(x,t,\lambda)\sigma_3}.$$

Proceeding as in case $A_1 < A_2$, we conclude that $N(x,t,\lambda)$ can be characterized as the solution of the following Riemann–Hilbert problem:

Find a 2×2 meromorphic matrix $N(x,t,\lambda)$ that satisfies the following conditions:

- The *jump* condition

$$N^+(x,t,\lambda) = N^-(x,t,\lambda)G(x,t,\lambda), \quad \lambda \in \dot{\Sigma}_2 \cup \dot{\Sigma}_0, \tag{4.120a}$$

where

$$G(x,t,\lambda) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-p_1(\lambda_+)} & 0 \\ 0 & \mathrm{e}^{p_1(\lambda_+)} \end{pmatrix} G_0(\lambda) \begin{pmatrix} \mathrm{e}^{p_1(\lambda_+)} & 0 \\ 0 & \mathrm{e}^{-p_1(\lambda_+)} \end{pmatrix} \tag{4.120b}$$

with

$$G_0(\lambda) = \begin{cases} \begin{pmatrix} 1 & -\check{\rho}(\lambda) \\ \overline{\check{\rho}(\lambda)} & 1-|\check{\rho}(\lambda)|^2 \end{pmatrix}, & \lambda \in \dot{\Sigma}_2, \\ \begin{pmatrix} 1 & -\check{\rho}(\lambda) \\ \frac{1}{\check{\rho}(\lambda)} & 0 \end{pmatrix}, & \lambda \in \dot{\Sigma}_0. \end{cases} \tag{4.120c}$$

- The *normalization* condition:

$$N(x,t,\lambda) = \begin{cases} \sqrt{\frac{1}{2}} \begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty, \quad \lambda \in \mathbb{C}^+, \\ \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty, \quad \lambda \in \mathbb{C}^-, \end{cases} \tag{4.121}$$

- *Singularity* conditions: the singularities of $N(x,t,\lambda)$ at $\pm\frac{1}{A_j}$ are of order not bigger than $\frac{1}{4}$.

- *Residue* conditions (if any): given $\{\check{\lambda}_k, \check{\kappa}_k\}_1^{\check{N}}$ with $\check{\lambda}_k \in (0, \frac{1}{A_1})$ and $\check{\kappa}_k \in \mathbb{R} \setminus \{0\}$, $N^{(2)}(x,t,\lambda)$ has simple poles at $\{\check{\lambda}_k, -\check{\lambda}_k\}_1^{\check{N}}$, with the residues satisfying the equations

$$\operatorname{Res}_{\pm\check{\lambda}_k} N^{(2)}(x,t,\lambda) = \check{\kappa}_k \mathrm{e}^{-2p_1(\check{\lambda}_k)} N^{(2)}(x,t,\pm\check{\lambda}_k). \tag{4.122}$$

Remark 4.4.1. The solution of the RH problem above, if exists, satisfies the following properties:

1. $\det N \equiv 1$.

2. *Symmetries*:

$$N(-\lambda) = -\sigma_3 N(\lambda)\sigma_3, \qquad \overline{N(\overline{\lambda})} = -N(\lambda), \qquad \lambda \in \mathbb{C} \setminus \Sigma_1, \tag{4.123a}$$

$$N((-\lambda)_-) = -\sigma_3 N(\lambda_+)\sigma_3, \qquad \overline{N(\lambda_-)} = -N(\lambda_+), \qquad \lambda \in \dot{\Sigma}_2. \tag{4.123b}$$

where $N(\lambda) \equiv N(x,t,\lambda)$ (follows from the respective symmetries of the jump matrix and the residue conditions, assuming the uniqueness of the solution).

4.4.4 Eigenfunctions near $\lambda = 0$

Introducing $\tilde{\Phi}_{0,j}$ as in (4.84) and proceeding as in case $A_1 < A_2$, the following development of $N(x,t,\lambda)$ near $\lambda = 0$ holds:

$$N(x,t,\lambda) = \mathrm{i}\begin{pmatrix} 0 & b_1(x,t) \\ b_1^{-1}(x,t) & 0 \end{pmatrix} + \mathrm{i}\lambda \begin{pmatrix} b_2(x,t) & 0 \\ 0 & b_3(x,t) \end{pmatrix} + \mathrm{O}(\lambda^2), \quad (4.124)$$

where

$$b_1(x,t) = \mathrm{e}^{\frac{1}{2A_1}\int_{-\infty}^{x}(m(\xi,t)-A_1)\mathrm{d}\xi}, \quad (4.125a)$$

$$b_2(x,t) = \Big(\int_{-\infty}^{x} e^{-(x-\xi)}\frac{m-A_1}{2}\mathrm{d}\xi + \frac{A_1}{2}\Big)\mathrm{e}^{-\frac{1}{2A_1}\int_{-\infty}^{x}(m(\xi,t)-A_1)\mathrm{d}\xi}, \quad (4.125b)$$

$$b_3(x,t) = \Big(\int_{x}^{\infty} e^{(x-\xi)}\frac{m-A_2}{2}\mathrm{d}\xi + \frac{A_2}{2}\Big)\mathrm{e}^{\frac{1}{2A_1}\int_{-\infty}^{x}(m(\xi,t)-A_1)\mathrm{d}\xi}. \quad (4.125c)$$

Proposition 4.4.2. *$u(x,t)$ and $u_x(x,t)$ can be algebraically expressed in terms of the coefficients $b_j(x,t)$, $j = 1,3$ in the development* (4.124) *of $N(x,t,\lambda)$ as follows:*

$$u(x,t) = b_1(x,t)b_2(x,t) + b_1^{-1}(x,t)b_3(x,t), \quad (4.126a)$$

$$u_x(x,t) = -b_1(x,t)b_2(x,t) + b_1^{-1}(x,t)b_3(x,t). \quad (4.126b)$$

4.4.5 RH problem in the (y,t) scale

Introducing the new space variable $\check{y}(x,t)$ by

$$\check{y}(x,t) = x + \frac{1}{A_1}\int_{-\infty}^{x}(m(\xi,t) - A_1)\mathrm{d}\xi - A_1^2 t \quad (4.127)$$

and introducing $\hat{N}(\check{y},t,\lambda)$ so that $N(x,t,\lambda) = \hat{N}(\check{y}(x,t),t,\lambda)$, the jump condition (4.120a) becomes

$$\hat{N}^+(\check{y},t,\lambda) = \hat{N}^-(\check{y},t,\lambda)\hat{G}(\check{y},t,\lambda), \quad \lambda \in \dot{\Sigma}_2 \cup \dot{\Sigma}_0, \quad (4.128a)$$

where

$$\hat{G}(\check{y},t,\lambda) := \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\hat{p}_1(\check{y},t,\lambda_+)} & 0 \\ 0 & \mathrm{e}^{\hat{p}_1(\check{y},t,\lambda_+)} \end{pmatrix} G_0(\lambda) \begin{pmatrix} \mathrm{e}^{\hat{p}_1(\check{y},t,\lambda_+)} & 0 \\ 0 & \mathrm{e}^{-\hat{p}_1(\check{y},t,\lambda_+)} \end{pmatrix}, \tag{4.128b}$$

$G_0(\lambda)$ is defined by (4.120c),

$$\hat{p}_1(\check{y},t,\lambda) := \frac{\mathrm{i}A_1k_1(\lambda)}{2}\left(\check{y} - \frac{2t}{\lambda^2}\right). \tag{4.128c}$$

Thus $G(x,t,\lambda) = \hat{G}(\check{y}(x,t),t,\lambda)$ and $p_1(x,t,\lambda) = \hat{p}_1(\check{y}(x,t),t,\lambda)$, where the jump $G(x,t,\lambda)$ and the phase $p_1(x,t,\lambda)$ are defined in (4.120b) and (4.14b), respectively.

Accordingly, the residue conditions (4.122) become

$$\operatorname{Res}_{\pm\check{\lambda}_k} \hat{N}^{(2)}(\check{y},t,\lambda) = \check{\kappa}_k \mathrm{e}^{-2\hat{p}_1(\check{y},t,\lambda_k)} \hat{N}^{(1)}(\check{y},t,\pm\check{\lambda}_k), \tag{4.129}$$

with $\check{\kappa}_k = \frac{1}{\check{b}_k s_{11}'(\check{\lambda}_k)}$.

Noticing that the normalization condition (4.121), the symmetries (4.123), and the singularity conditions at $\lambda = \pm\frac{1}{A_j}$ hold in the new scale $(\check{y},t)$, we arrive at the basic RH problem.

Basic RH problem. Given $\check{\rho}(\lambda)$ for $\lambda \in \dot{\Sigma}_2 \cup \dot{\Sigma}_0$, and $\{\check{\lambda}_k, \check{\kappa}_k\}_1^{\check{N}}$ with $\check{\lambda}_k \in (0, \frac{1}{A_1})$ and $\check{\kappa}_k \in \mathbb{R} \setminus \{0\}$, associated with the initial data $u_0(x)$ in (4.1), find a piece-wise (w.r.t. $\dot{\Sigma}_1$) meromorphic, 2×2-matrix valued function $\hat{N}(\check{y},t,\lambda)$ satisfying the following conditions:

- The jump condition (4.128) across $\dot{\Sigma}_2 \cup \dot{\Sigma}_0$ (with $G_0(\lambda)$ defined by (4.120c)).
- The residue conditions (4.129).
- The *normalization* condition:

$$\hat{N}(\check{y},t,\lambda) = \begin{cases} \sqrt{\frac{1}{2}}\begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty,\ \lambda \in \mathbb{C}^+, \\ \sqrt{\frac{1}{2}}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} + \mathrm{O}(\frac{1}{\lambda}), & \lambda \to \infty,\ \lambda \in \mathbb{C}^-. \end{cases} \tag{4.130}$$

- *Singularity* conditions: $\hat{N}(\check{y},t,\lambda)$ may have singularities at $\pm\frac{1}{A_j}$ of order $\frac{1}{4}$.
- *Symmetries*:

$$\hat{N}(-\lambda) = -\sigma_3\hat{N}(\lambda)\sigma_3, \qquad \overline{\hat{N}(\overline{\lambda})} = -N(\lambda), \qquad \lambda \in \mathbb{C}\setminus\Sigma_2, \tag{4.131a}$$

$$\hat{N}((-\lambda)_-) = -\sigma_3\hat{N}(\lambda_+)\sigma_3, \qquad \overline{\hat{N}(\lambda_-)} = -\hat{N}(\lambda_+), \qquad \lambda \in \dot{\Sigma}_1. \tag{4.131b}$$

where $\hat{N}(\lambda) \equiv \hat{N}(\check{y},t,\lambda)$.

4.4.6 Recovering $u(x,t)$ from the solution of the RH problem

Theorem 4.4.3. *Assume that $u(x,t)$ is the solution of the Cauchy problem* (4.1) *and let $\hat{N}(\check{y},t,x)$ be the solution of the associated RH problem* (4.128)–(4.130)*, whose data are determined by $u_0(x)$. Let*

$$\hat{N}(\check{y},t,\lambda) = \mathrm{i}\begin{pmatrix} 0 & \hat{b}_1(\check{y},t) \\ \hat{b}_1^{-1}(\check{y},t) & 0 \end{pmatrix} + \mathrm{i}\lambda\begin{pmatrix} \hat{b}_2(\check{y},t) & 0 \\ 0 & \hat{b}_3(\check{y},t) \end{pmatrix} + \mathrm{O}(\lambda^2) \tag{4.132}$$

be the development of $\hat{N}(\check{y},t,x)$ at $\lambda = 0$. Then the solution $u(x,t)$ of the Cauchy problem (4.1) *can be expressed, in a parametric form, in terms of $\hat{b}_j(\check{y},t)$, $j=1,2,3$: $u(x,t) = \hat{u}(\check{y}(x,t),t)$, where*

$$\hat{u}(\check{y},t) = \hat{b}_1(\check{y},t)\hat{b}_2(\check{y},t) + \hat{b}_1^{-1}(\check{y},t)\hat{b}_3(\check{y},t), \tag{4.133a}$$

$$x(\check{y},t) = \check{y} - 2\ln\hat{b}_1(\check{y},t) + A_2^2 t. \tag{4.133b}$$

Additionally, $\hat{u}_x(\check{y},t)$ can also be algebraically expressed in terms of $\hat{b}_j(\check{y},t)$, $j=1,2,3$: $u_x(x,t) = \hat{u}_x(\check{y}(x,t),t)$, where

$$\hat{u}_x(\check{y},t) = -\hat{b}_1(\check{y},t)\hat{b}_2(\check{y},t) + \hat{b}_1^{-1}(\check{y},t)\hat{b}_3(\check{y},t). \tag{4.133c}$$

Proposition 4.4.4. *Let $\hat{M}(y,t,\mu)$ be the solution of the RH problem* (4.128)–(4.131) *whose data are associated with the initial data $u_0(x)$. Define $\hat{\mu}_1(y,t) := \hat{M}_{11}(y,t,0) + \hat{M}_{21}(y,t,0)$ and $\hat{\mu}_2(y,t) := \hat{M}_{12}(y,t,0) + \hat{M}_{22}(y,t,0)$. The solution $u(x,t)$ of the Cauchy problem* (4.1) *has x-derivative given by the parametric representation*

$$u_x(x(y,t),t) = \frac{1}{2A_1}\partial_{ty}\ln\frac{\hat{\mu}_1(y,t)}{\hat{\mu}_2(y,t)}, \tag{4.134a}$$

$$x(y,t) = y + \ln \frac{\hat{\mu}_1(y,t)}{\hat{\mu}_2(y,t)} + A_1^2 t. \tag{4.134b}$$

Proof. In what follows we will express u_x in the variables (y,t). To express a function $f(x,t)$ in (y,t) we will use the notation $\hat{f}(y,t) := f(x(y,t),t)$, e.g.,

$$\hat{u}(y,t) := u(x(y,t),t),\ \hat{u}_x(y,t) := u_x(x(y,t),t),\ \hat{m}(y,t) := m(x(y,t),t).$$

Differentiation of the identity $x(y(x,t),t) = x$ w.r.t. t gives

$$\partial_t\left(x(y(x,t),t)\right) = x_y(y,t)y_t(x,t) + x_t(y,t) = 0. \tag{4.135}$$

From (4.127) it follows that

$$x_y(y,t) = \frac{A_1}{\hat{m}(y,t)} \tag{4.136}$$

and

$$y_t(x,t) = -\frac{1}{A_1}(u^2 - u_x^2)m.$$

Substituting this and (4.136) into (4.135) we obtain

$$x_t(y,t) = \hat{u}^2(y,t) - \hat{u}_x^2(y,t). \tag{4.137}$$

Further, differentiating (4.137) w.r.t. y we get

$$x_{ty}(y,t) = (\hat{u}^2(y,t) - \hat{u}_x^2(y,t))_x x_y(y,t) = 2A_1\hat{u}_x(y,t). \tag{4.138}$$

Therefore, we arrive at a parametric representation of $u_x(x,t)$:

$$u_x(x(y,t),t) \equiv \hat{u}_x(y,t) = \frac{1}{2A_1}\partial_{ty}x(y,t),$$
$$x(y,t) = y + \ln \frac{\hat{\mu}_1(y,t)}{\hat{\mu}_2(y,t)} + A_1^2 t,$$

which yields (4.134). □

4.5 Remarks

We have presented the Riemann–Hilbert problem approach for the modified Camassa–Holm equation on the line with step-like boundary conditions. In the proposed formalism, we have taken the branch cut of $k_j(\lambda)$ along the half-lines Σ_j (outer cuts), which is convenient since we extract the solution of the mCH equation exploiting the development of the solution of the RH problem at a point laying in the domain of analyticity. Notice that it is possible to formulate RH problem taking the branch cut of $k_j(\lambda)$ to be the segments $(-\frac{1}{A_j}, \frac{1}{A_j})$ (inner cuts). In the case with inner cuts, the properties of Jost solutions are more conventional (two of the columns are analytic in the upper half-plane and other two in the lower half-plane), but, on the other hand, possible eigenvalues are located on the jump.

We have focused on the representation results while assuming the existence of a solution of problem (4.1) in certain functional classes. To the best of our knowledge, the question of existence is still open. One of the ways to answering it is to appeal to functional analytic PDE techniques to obtain well-posedness in appropriate functional classes. However, very little is known for the cases of nonzero boundary conditions, particularly, for backgrounds having different behavior at different infinities. Since 1980s, existence problems for integrable nonlinear PDE with step-like initial conditions have been addressed using the classical Inverse Scattering Transform method [86]. A more recent progress in this direction (in the case of the Korteweg-de Vries equation) has been reported in [64, 66, 77] (see also [65]). Another way to show existence is to infer it from the RH problem formalism (see, e.g., [71] for the case of defocusing nonlinear Schrödinger equation), where a key point consists in establishing a solution of the associated RH problem and controlling its behavior w.r.t. the spatial parameter. For Camassa–Holm-type equations, where the RH problem formalism involves the change of the spatial variable, it is natural to study the existence of solution in both (x, t) and (y, t) scales. More precisely, the solvability problem splits into two problems: (i) the solvability of the RH problem parametrized by (y, t) and (ii) the bijectivity of the change of the spatial variable. Particularly, it is possible that it is the change of variables that can be responsible of the wave breaking [32, 18]. The solvability problem for problem (4.1) in the current setting will be addressed elsewhere.

4.6 Conclusions to Chapter 4

We have presented the Riemann–Hilbert problem approach for the modified Camassa–Holm equation on the line with step-like boundary conditions. In the proposed formalism, we have taken the branch cut of $k_j(\lambda)$ along the half-lines Σ_j (outer cuts), which is convenient since we extract the solution of the mCH equation exploiting the development of the solution of the RH problem at a point laying in the domain of analyticity. Notice that it is possible to formulate RH problem taking the branch cut of $k_j(\lambda)$ to be the segments $(-\frac{1}{A_j}, \frac{1}{A_j})$ (inner cuts). In the case with inner cuts, the properties of Jost solutions are more conventional (two of the columns are analytic in the upper half-plane and other two in the lower half-plane), but, on the other hand, possible eigenvalues are located on the jump. Based on the results of the research,

- We have developed the the inverse scattering transform approach in the form of Riemann–Hilbert problem for this problem in two cases: when the right background is larger than the left and when the left background is larger than the right.
- We have introduced ppropriate transformations of the Lax pair equations that allow us to study in detail the analytic properties of the corresponding Jost solutions and spectral functions.
- We have constructed the associated Jost solutions ("eigenfunctions"), and discussed the analytic and asymptotic properties of the eigenfunctions and the corresponding spectral functions (scattering coefficients), including the behavior at the branch points.
- We have investigated the symmetries of spectral functions.
- We have obtained the parametric representation of the solution of the Cauchy problem in form in terms of the solution of an associated RH

problem.

The developed approach can be an effective basis for the investigation of the large-time behavior of solutions of the Cauchy problems adapting the nonlinear steepest descent method.

Conclusions

The Thesis aims at the development of the inverse scattering transform approach to the initial value problems for the modified Camassa–Holm equation with various boundary conditions, in particular, (i) when the solution is assumed to approach a non-zero constant at the both infinities of the space variable, and (ii) when the solution is assumed to approach two different constants at plus and minus infinity of the space variable. The specificity of our study is that we consider this equation in the case of with non-vanishing boundary conditions at infinity. Such problems are of particular interest because they can be used as models for studying expanding, oscillatory dispersive shock waves.

The method of inverse scattering problem for the modified Camassa–Holm equation on constant non-zero and step-like backgrounds was developed for the first time. In addition, for the problem on a constant non-zero background, the large time asymptotics were obtained for the first time.

For the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach a non-zero constant at the both infinities of the space variable we have obtained the following main results:

- We have developed the the inverse scattering transform approach in the form of Riemann–Hilbert problem for this problem. In particular, we have introduced the appropriate (gauge) transformation for the Lax pair equations, which reduces the original Lax pair to a "convenient" form; we have introduced the associated Jost solutions and the corresponding scattering coefficients, and analyzed their analytic and asymptotic properties; we have introduced a new (uniformising) spectral parameter which allows us to avoid non-rational dependence of the coefficients in the Lax pair equations on the spectral parameter.

- We have observed the features that distinguish the mCH equation from other CH type equations. In particular, one does not need to use a new gauge transformation to control the Jost solutions at $\lambda = 0$, , but the required form of the Lax pair comes from regrouping the terms of that appropriate for large λ.

- We have obtained the parametric representation of the solution of the Cauchy problem in form in terms of the solution of an associated RH problem.

- We have described regular and non-regular one-soliton solutions associated with the RH problems with trivial jump condition and appropriately prescribed residue conditions. In this way, we have specified two families of non-regular soliton solutions of the mCH equation: (i) peakon-type solutions, which are continuous together with their first derivative but having unbounded derivatives of order greater than 2 at the peak points; (ii) loop-shaped, multi-valued solutions, which are conventional, signal-valued solitons in the modified variables that becomes multivalued when going back to the original variables, x and t.

- We have reduced the original RH problem associated with the mCH equation that has two singularity conditions at $\mu = \pm 1$ to a regular RH problem (i.e., to a RH problem with the jump and normalization conditions only).

- Using the nonlinear steepest descent method, we have obtained the leading asymptotic terms for the solution $u(x, t)$ of the Cauchy problem, in the two sectors of the (x, t) half-plane, $1 < \frac{x}{t} < 3$ and $\frac{3}{4} < \frac{x}{t} < 3 < 1$, where the deviation from the background value is nontrivial: this term is given by modulated (with parameters depending on $\frac{x}{t}$), decaying (as $t^{-1/2}$) trigonometric oscillations.

For the modified Camassa–Holm equation on the whole line in the case when the solution is assumed to approach two different constants at plus and minus infinity of the space variable we have obtained the following main results:

- We have developed the the inverse scattering transform approach in the form of Riemann–Hilbert problem for this problem in two cases: when the right background is larger than the left and when the left background is larger than the right.
- We have introduced ppropriate transformations of the Lax pair equations that allow us to study in detail the analytic properties of the corresponding Jost solutions and spectral functions.
- We have constructed the associated Jost solutions ("eigenfunctions"), and discussed the analytic and asymptotic properties of the eigenfunctions and the corresponding spectral functions (scattering coefficients), including the behavior at the branch points.
- We have investigated the symmetries of spectral functions.
- We have obtained the parametric representation of the solution of the Cauchy problem in form in terms of the solution of an associated RH problem.

All results of the dissertation are presented with full proofs. They are of a theoretical nature and can be used in further research initial boundary value problems for equations of the Camassi-Holm type, which are promising models of physical processes of different nature.

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