Applications of generalized smooth functions

Applicant and PI: Paolo Giordano

Co-authors: M. Kunzinger

PhD student: A. Bryzgalov

Contents

The main aim of the proposed project[2] is to develop new theoretical developments and some fundamental applications of the theory of generalized smooth functions (GSF[3]). The nonlinear theory of generalized smooth functions has recently emerged as a minimal extension of Colombeau's theory that allows for more general domains for generalized

[1]This proposal is a resubmission, and a part of this section has been improved.

[2]This proposal is a resubmission. Some parts are new and elaborated in accordance with reviewers' indications. The *main* changes are underscored using footnotes.

[3]A complete list of acronyms can be found at the end of this document; To help the reader, Adobe Acrobat produces small windows (tooltips) near acronyms, equation references, figures and citations (near the end of the citation).

functions (GF), resulting in the closure with respect to composition, a better behaviour on unbounded sets and new general existence results. The central topics we propose to develop are:

1. Newton method and the Pontryagin principle for singular problems;
2. Singular Hamiltonian mechanics;
3. Numerical and graphical tools to visualize GSF.

The proposal thus aims at showing the flexibility of GSF theory in these classical topics, and at widely extending well-known results in different fields of mathematical analysis.

1 Aims and research objectives

The main objective of the present research project is to develop new theoretical developments and some fundamental applications of the theory of GSF. This type of GF are an extension of classical distribution theory which makes it possible to model nonlinear singular problems, while at the same time sharing several fundamental properties with ordinary smooth functions, such as the closure with respect to composition and several non trivial theorems of the calculus, see [Giordano-Kunzinger-Ver15, Giordano-Kunzinger18a, Giordano-Kun-Ver18, Giordano-Kunzinger16, LL-Giordano16]. One could describe GSF as a methodological restoration of Cauchy-Dirac's original conception of generalized function, see [Dir26, Laug89, KatTal12]. In essence, the idea of Cauchy and Dirac (but also of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view GF as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. GSF are a minimal extension of Colombeau's theory of generalized functions (CGF), see [Col92, NePiSc98, Obe92, Pil94]. In fact, when the domain is the set $\widetilde{\Omega}_c$ of compactly supported generalized points in the open set $\Omega \subseteq \mathbb{R}^n$, then the two spaces of GF coincide, cf. [Giordano-Kunzinger-Ver15]. Therefore, we expect that the directions envisaged in the present project will also exert a considerable impact on Colombeau's theory. For these reasons, the department of Mathematics of the University of Vienna, and in particular the research group of Prof. M. Kunzinger (see `http://www.mat.univie.ac.at/~mike/` and the included CV), constitute the ideal place where to implement the present research project, because of the group's specific competencies on GF, partial differential equations (PDE) and functional analysis.

A concise presentation of the project's main aims is as follows (WP = work package):

WP 1: Newton method and the Pontryagin principle for singular problems

Problems and motivations: To our knowledge, the only general method to solve equations with GSF is the Banach fixed point method developed in [L-Giordano16a]. The well-known Newton method is another root-finding algorithm that seems approachable in this framework. A first analysis of its proofs shows that the use of ${}^{\rho}\widetilde{\mathbb{R}}$-valued norms and the related completeness could be the basic methods needed for this extension. The availability of Newton method would open the possibility to rigorously prove and extend the Pontryagin's maximum principle, by following the proof e.g. of [Eva10] and [DmOs14].

The idea and the plan: The simple method of generalizing the steps of the classical proofs by using the ring of scalars ${}^{\rho}\widetilde{\mathbb{R}}$ instead of the field $\mathbb{R}$ proved to be surprisingly effective e.g. in the development of both the Banach fixed point theorem and several results of calculus of variations, see

[L-Giordano16a, LL-Giordano16].

Innovative features and deliverables: The main delivery of a proof of Newton method would be a complete proof of Pontryagin principle, and hence an important extension of calculus of variations and control theory to singular problems.

WP 2: Singular Hamiltonian mechanics

Problems and motivations: Since many important physical systems are described by singular Hamiltonians (see below), a well-established thread of research aims at developing classical mechanics using some kind of GF. Since the modern formulation of Hamiltonian mechanics is formulated in the language of symplectic manifold, this generalization would also be an occasion to develop a number of important notions of differential geometry for GSF.

The idea and the plan: The framework of GSF presents flexible properties that are not available for Sobolev-Schwartz distributions. Among them it is important to underscore: existence and uniqueness theorems for ODE (and PDE); a Grothendieck topos that permits to extend in a very general way the notion of GSF from arbitrary subsets of generalized points of ${}^{\rho}\widetilde{\mathbb{R}}^n$ to extensions of manifolds; a language of nilpotent infinitesimals that allow us to develop several topics of differential geometry in an intrinsic way, along the lines of previous works of the PI, [Giordano10a, Giordano10b, Giordano11a, Giordano-Wu16].

Innovative features and deliverables: Even a partial development of this part of the project would represent a milestone of the theory of GSF, because of its applicability to singular classical mechanics and of the corresponding theoretical results in differential geometry.

WP 3: Numerical and graphical tools to visualize GSF

Problems and motivations: Even if the theories of GSF and CGF can nowadays be considered well developed from the theoretical point of view, there are no general and easy-to-use tools to visualize generalized solutions of nonlinear singular differential equations (DE). In particular, tools showing the "dynamics" of the solutions as $\varepsilon \to 0$ can be useful to develop a strong intuition on this type of GF. We expect this to lead to a substantial strengthening of the intuition underlying the theory, in particular with respect to infinitesimal properties.

The idea: The idea is to use Matlab object oriented programming to easily implement particular interesting cases of GSF: Cauchy-Dirac GSF, i.e. of the form $f(x) = [\varphi(x_\varepsilon, p_\varepsilon)]$ (see below), where φ is an ordinary smooth function and $p = [p_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n$ are fixed parameters; GSF defined by different cases in ε, i.e. $f_\varepsilon := f_{k\varepsilon}$ if $\varepsilon \in I_k \subseteq I$ and $I_k \cap I_j = \emptyset$ if $k \neq j$; embedding of Sobolev-Schwartz distributions, etc. Given a DE, among others, we plan to implement a visualization tool to display "ε-moving solutions", i.e. as $\varepsilon \to 0^+$, so as to investigate infinitesimal and infinite properties of solutions.

The plan: The aim of this part of the proposal is ***not*** to develop ***new*** numerical methods, but to use well-known Matlab-solvers and oriented programming to visualize GSF. Another approach is to use Picard-Lindelöf iterates (see below for the Picard-Lindelöf theorem for ODE and PDE with GSF).

Innovative features and deliverables: We plan to prepare a new Matlab toolbox and to (freely) deliver it to the mathematical community. This could help the dissemination and applications of the theory. Suitable visualization tools can also greatly help in developing the intuition related to this type of GF. A comparison with the well-known numerical results of [Col07b] is planned.

The present research project is designed for five co-workers: the applicant P. Giordano, co-authors M. Kunzinger, A. Bryzgalov (PhD candidate), a new post-doc and a new PhD candidates. See in the proposal the CV of the three cited co-workers. See also Sec. 5 for the organization of the research work.

2 State of the art

2.1 State of the art in the research field

Generalized functions:

J.F. Colombeau's theory of GF allows one to perform non linear operations (of polynomial growth) between embedded distributions, avoiding the difficulty of Schwartz impossibility theorem. See e.g. [Ned-Pil06, GrKuObSt01, NePiSc98, Pil94, Obe92, Col92] for an introduction with applications. This theory makes it possible to find generalized solutions of some well-known PDE which do not have solutions in the classical space of distributions, see [Obe92], and has manifold applications, e.g. to the theory of elasticity, fluid mechanics and in the theory of shock waves (see e.g. [Col92, Obe92]), to differential geometry and relativity theory [GrKuObSt01, Kun04, SteVic06] and to quantum field theory [CoGs08].

A new and fundamental step in the theory of GF based on CGN, which presents several analogies with our present proposal, has first been achieved in [Ar-Fe-Ju05, Ar-Fe-Ju12]. In this work, the basic idea is to generalize the derivative as a limit of an incremental ratio taken with respect to the e-norm, [Ar-Fe-Ju05, Ar-Fe-Ju09] and with increments which are asymptotic to invertible infinitesimals of the form $[\varepsilon^r] \in \widetilde{\mathbb{R}}$, for $r \in \mathbb{R}_{\geq 0}$. This theory extends the usual classical notion of derivative and smoothness to set-theoretical functions on CGN, e.g. of the form $f : \widetilde{\mathbb{R}}^n \longrightarrow \widetilde{\mathbb{R}}^d$, and enables one to prove that every CGF is infinitely differentiable in this new sense. Several important applications have already been achieved (see [Ar-Fe-Ju12]) and hence the theory promises to be very relevant. As explained in greater detail in [Giordano-Kunzinger-Ver15, Giordano-Kunzinger16, Giordano-Kun-Ver18], the notion of smoothness developed in [Ar-Fe-Ju05] includes functions like $i(x) = 1$ if x is infinitesimal and $i(x) = 0$ otherwise. This makes it impossible to prove classical theorems like the intermediate value one, whereas for GSF this theorem holds. The theory of GSF, on the other hand, while fully compatible with the approach in [Ar-Fe-Ju05], singles out a subclass of smooth functions with more favourable compatibility properties with respect to classical calculus and hence may be viewed as a refinement of that theory.

Newton method and the Pontryagin principle

Newton method, also called Newton-Raphson method, is a well-known method for computing a zero of a nonlinear equation. If X and Y are Banach spaces and $F : X \to Y$ is a Fréchet differentiable function, the Newton method can be written as

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k) \quad k \in \mathbb{N} \tag{1}$$

where F' is the Fréchet derivative of F [Ar08].

Newton method helps in finding approximate solutions to nonlinear problems. Convergence properties, error estimates, numerical stability and computational complexity of the method has been proven, see e.g. [Oz04, Ga00, Hi10]. Concerning its convergence properties, Kantorovich theorem assumes conditions that ensure existence and uniqueness of solution of nonlinear equations yielding the convergence of Newton method, [Or68]. Whereas Newton method may be considered a first order method, Halley method is a root-finding algorithm which extends to higher order using Taylor expansion. It yields iteratively a sequence of approximations to the root and has a cubic rate of convergence. We also mention that Householder method is an iterative root-finding algorithm which has a convergence rate of order $d+1$ and generalizes both Newton and Halley methods [SeGo01]. [Oz04] presents a variant of the Newton method based on harmonic mean and midpoint integration. The method has been applied to computation of eigenvalues and eigenvectors of symmetric matrices [FuZe86], nonlinear wave equations, nonlinear partial differential equations and nonsmooth analysis, see e.g. [Hi10], [KoJeMi03] and references therein.

In control theory, the state of a physical system is governed by an ODE, and the possibility to influence system's behaviour is achieved by adding suitable control variables. Given $A \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$, the set of admissible controls can be represented as: $\mathscr{A} = \{a : [0,\infty) \to A \mid a \text{ is a measurable}\}$. If $a \in \mathscr{A}$, then given an ODE system, a solution is found for the evolution of the system: $\dot{x}(t) = f(x(t), a(t))$, $t \geq 0$, $x(0) = x^0$. A pay-off functional is introduced as: $P[a] = \int_0^T r(x(t), a(t))\,\mathrm{d}t + g(x(T))$, where $T > 0$, $r : \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$. The Pontryagin maximum principle states that if $a \in \mathcal{A}$ is optimal for the given ODE, for P and x^*, then there exists a function $p^* : [0,T] \to \mathbb{R}^n$ such that

$$\begin{array}{lr} ODE & \dot{x}^*(t) = \nabla_p H(x^*(t), p^*(t), a^*(t)) \\ ADJ & \dot{p}^*(t) = -\nabla_x H(x^*(t), p^*(t), a^*(t)) \\ M & H(x^*(t), p^*(t), a^*(t)) = \max_{a \in \mathcal{A}} H(x^*(t), p^*(t), a). \end{array} \tag{2}$$

Moreover, the map $t \to H(x^*(t), p^*(t), a^*(t))$ is constant and T satisfies $\dot{p}^*(T) = \nabla_x H(x^*(T))$. The original proof of Pontryagin principle did not fit into classical methods of the calculus of variations. There have been several attempts to reprove the method [DmOs14, Eva10], and efforts were also made to find rigorous proof for the classical problem of Pontryagin type. As far as we know, as of yet there is no extension of the Pontryagin principle to GF.

Extension of Hamiltonian mechanics using generalized functions

One of the main reasons to study Hamiltonian mechanics from a mathematical point of view is due to the fact that Hamiltonian mechanics can be formulated on an arbitrary symplectic manifold. A recent review of this symplectic description can be found in [deL14] (note, however, that this work is mainly oriented towards the study of classical field theories on the basis of results for Hamiltonian and Lagrangian Mechanics).

The motivation to introduce a suitable kind of GF formalism in classical mechanics is clear: is it possible to generalize Hamiltonian and Langrangian mechanics so that the Hamiltonian can be presented as a generalized function? This would undoubtedly be of an applicable advantage, since many relevant systems are described by singular Hamiltonians. In fact, this type of problems is widely studied (e.g. [Kunzle96, LeMoSj91, Tan93, LiHuZh14, Lim89]), but the presented solutions are not general and hold only for special conditions and effective potentials.

In this sense, the fact that since J.D. Marsden's works [Mar68, Mar69] (who uses Sobolev-Schwartz distributions) there has not been any further attempt to use GF for the description of Hamiltonian mechanics can be considered as a clue that the classical Sobolev-Schwartz distributional framework is not well suited to face this problem in general terms. In [Mar68, Mar69], J.D. Marsden introduces distributions on manifolds based on flows. In fact, since the traditional system of Hamilton's equations breaks down, Marsden considers the flow as a limit of smooth ones. On the other hand, Kunzinger et al. in [KuObStVi] (using CGF) critically analysed the regularization approach put forward by Marsden and constructed a counter example for the main flow theorems of [Mar68]. Among recent works we single out [Col07, Giun03].

In this field a number of problems remains open, see [Mar68]. For example, in the non-smooth case the variational theorems fail, and the study of the virial equation for hard spheres in a box is still an open question.

Numerical solutions of (singular) nonlinear differential equations

As far as we know, the numerical solution of singular nonlinear DE that possess generalized solutions has not been studied by many authors. Here, we can cite e.g. [Col92, Col07b]. In [Col07b], the numerical solution of 2D systems using dimension splitting, which was introduced by Godunov [Go59], and the solution of the Riemann problem for 1D systems by the method of splitting of equations have been considered. Results were achieved for problems which included scalar conservation laws using Heaviside GF [CoRo88] and Godunov's scheme and the Lax-Friedrichs method were used for the numerical computations. Further results included charged black holes moving at light speed, predator-prey models, and collisions of solids with strong deformation. [CaCo89] investigated discontinuous generalized solutions of nonlinear nonconservative hyperbolic equations that arise in elastodynamics. For this kind of problems, it is not possible to define a weak solution depending on distribution theory. Existence of global solutions of the Cauchy problem for a system of two equations was proved by a compactness argument from a convergent numerical scheme using generalized solutions obtained by the theory of CGF.

In the present project proposal, we do ***not*** aim at introducing ***new*** numerical schemes for solving DE, but to use Matlab implementation of well-established numerical methods to visualize ε-wise solutions of nonlinear singular DE in the context of GSF. For these reasons, we review only these Matlab methods. Matlab has proven to have good solvers both for ODE and PDE. Matlab ODE solvers can be grouped into *stiff* and *nonstiff* problem solvers. For stiff problems, the solvers include: `ode23s` (based on order 2 of a modified Rosenbrock formula), `ode23t` (based on implementation of the trapezoidal rule using a "free" interpolant), `ode23tb` (based on implementation of an implicit Runge-Kutta formula with a first stage that is a trapezoidal rule step and a second stage that is a backward differentiation formula of order two) and `ode15s` (a variable order solver based on the numerical differentiation formulas). The solvers for nonstiff problems include: `ode45` (based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair), `ode23` (based on an explicit Runge-Kutta (2,3) pair of Bogacki and Shampine), `ode113` (a variable order Adams-Bashforth-Moulton PECE solver) and `ode15i` (variable order method). The Matlab Toolbox has two PDE solvers. The first solver is `pdepe` which solves initial-boundary value problems for parabolic-elliptic PDE in $1-D$. The ODE that result from discretization in space are integrated to obtain approximate solutions at specified times. The `pdepe` function returns values of the solution on a input provided mesh. The second solver is `pdeval`, which evaluates numerical solution of PDE using output of `pdepe`. The solver is executed using $[uout, duoutdx] = pdeval(m, x, ui, xout)$, where m is a symmetry of the problem, ui is the same as $xmesh$ in $pdepe$, $xout$ is a vector of points of the interval. See [Mat18].

2.2 State of the art in applicant's research

In this section, we briefly introduce some of the key notions of the present research proposal: nets in the variable $\varepsilon \in I := (0,1]$ are written as (x_ε); if (x_ε) is a net of real numbers, $x = [x_\varepsilon]$ denotes the corresponding equivalence class with respect to the equivalence relation $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ iff $|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^m)$ for every $m \in \mathbb{N} = \{0,1,2,\ldots\}$, where $\rho = (\rho_\varepsilon) \downarrow 0$ is an increasing net called *gauge* that generalizes the classical $\rho_\varepsilon = \varepsilon$.

The ring of Colombeau ${}^\rho\widetilde{\mathbb{R}}$

The ring ${}^\rho\widetilde{\mathbb{R}}$ is the quotient of the ring of *ρ-moderate nets* ($\exists N \in \mathbb{N} :\ x_\varepsilon = O(\rho_\varepsilon^{-N})$) modulo *$\rho$-negligible nets* ($\forall n \in \mathbb{N} :\ x_\varepsilon = O(\rho_\varepsilon^{n})$). The point of view of GSF is frequently that of a theory where ${}^\rho\widetilde{\mathbb{R}}$ acts as the ring of scalars for all the subsequent constructions. For example, sharp topology is preferably defined using the absolute value $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$ and the balls $B_r(x) := \{y \in {}^\rho\widetilde{\mathbb{R}}^d \mid |y-x| < r\}$, where $r > 0$ is a *strictly positive* generalized number, i.e. $r \in {}^\rho\widetilde{\mathbb{R}}_{\ge 0}$ and r is invertible. In this proposal, we use the notation $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$.

Generalized smooth functions as a category of smooth set-theoretical maps

If $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ are arbitrary subsets of generalized numbers, a GSF $f \in {}^\rho\mathcal{GC}^\infty(X,Y)$ can be simply defined as a set-theoretical map $f : X \longrightarrow Y$ such that

$$\exists (f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^I \,\forall [x_\varepsilon] \in X \,\forall \alpha \in \mathbb{N}^n :\ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \text{ is } \rho-\text{moderate and } f(x) = [f_\varepsilon(x_\varepsilon)], \tag{3}$$

see [Giordano-Kunzinger-Ver15, Giordano-Kun-Ver18]. If (3) holds, we say that the net (f_ε) *defines* f. If $X = \widetilde{\Omega}_c$, the set of compactly supported points in the open set $\Omega \subseteq \mathbb{R}^n$, then ${}^\rho\mathcal{GC}^\infty(\widetilde{\Omega}_c, {}^\rho\widetilde{\mathbb{R}})$ coincides exactly with the set-theoretical maps induced by all the CGF of the algebra $\mathcal{G}^{\mathrm{s}}(\Omega)$. The greater flexibility in the choice of the domains X leads e.g. to the closure of GSF with respect to composition, to the extreme value theorem on closed intervals bounded by infinite numbers, to purely infinitesimal solutions of ODE or also to inverses of given GSF, see [Giordano-Kunzinger-Ver15, Giordano-Kun-Ver18, L-Giordano16a, Giordano-Kunzinger16].
Classical theorems like the chain rule, existence and uniqueness of primitives, integration by change of variables, the intermediate value theorem, mean value theorems, the extreme value theorem, Taylor's theorem in several forms for the remainder, suitable sheaf properties, the local inverse and implicit function theorems, some global inverse function theorems, the Banach fixed point theorem, the Picard-Lindelöf theorem and several results in the classical theory of calculus of variations, hold for these GSF, see [Giordano-Kunzinger-Ver15, Giordano-Kun-Ver18, Giordano-Kunzinger16, L-Giordano16a, LL-Giordano16]. One of the peculiar properties of GSF is that these extensions of classical theorems for smooth functions have very natural statements, formally similar to the classical ones. All this underscores the different philosophical approach as compared to [Ar-Fe-Ju05] (and to the more classical Colombeau theory), which constitutes a more general approach, but where some of these classical theorems do not hold.

Particularly interesting for the present research proposal are the structure of ${}^\rho\widetilde{\mathbb{R}}$-Fréchet space on a solid compactly supported set and the notion of hyperseries. Firstly, a functionally compact set is a sharply bounded internal sets $K = [K_\varepsilon] = \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in K_\varepsilon \text{ for } \varepsilon \text{ small}\} \subseteq B_r(0)$, for some $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, generated by a net $K_\varepsilon \Subset \mathbb{R}^n$ of compact sets. Secondly, a *solid set* is a set $S \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ whose interior in the sharp topology is dense in S; the latter allow us to deal with partial derivatives at boundary points. For example, every closed interval $[a,b]\subseteq{}^{\rho}\widetilde{\mathbb{R}}$ is functionally compact and solid. On functionally compact sets, GSF satisfy the extreme value theorem and hence on every closed interval they can be integrated $\int_a^b f\in{}^{\rho}\widetilde{\mathbb{R}}$. The space ${}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^d)$ share many properties with the classical Fréchet spaces of ordinary smooth functions defined on a compact set. In particular, these spaces are sharply Cauchy complete and their sharp topology can be defined using a countable family $\|f\|_i:=\left[\max_{\substack{|\alpha|\le i\\1\le k\le d}}\sup_{x\in\mathbb{R}^n}|\partial^\alpha f_\varepsilon^k(x))|\right]\in{}^{\rho}\widetilde{\mathbb{R}}$, $i\in\mathbb{N}$, of ${}^{\rho}\widetilde{\mathbb{R}}$-valued norms, see [Giordano-Kunzinger18a, L-Giordano16a]. We also proved a generalization of the Banach fixed point theorem and of the corresponding Picard-Lindelöf theorem that are applicable to any Cauchy problem with a normal generalized PDE, see [Giordano-L-Kunzinger18]. The basic idea is the notion of loss of derivatives: if $K\subseteq{}^{\rho}\widetilde{\mathbb{R}}^n$ is a solid functionally compact set, and $y_0\in X\subseteq{}^{\rho}\mathcal{GC}^{\infty}(K,{}^{\rho}\widetilde{\mathbb{R}}^d)$, then we say that $P:X\longrightarrow X$ is a *contraction on X with loss of derivatives $L\in\mathbb{N}$ starting from y_0* if

$$\forall i\in\mathbb{N}\,\exists\alpha_i\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}:\ \|P(u)-P(v)\|_i\le\alpha_i\cdot\|u-v\|_{i+L}\ \forall u,v\in X$$

and

$$\lim_{\substack{n,m\to+\infty\\n\le m}}\alpha_{i+mL}^n\cdot\|P(y_0)-y_0\|_{i+mL}=0,$$

where the limit is taken with respect to the sharp topology and with n, $m\in\mathbb{N}$. We proved that if $\alpha_i\le\alpha_{i+1}$ and X is sharply Cauchy complete, then P is sharply continuous, $\exists\lim_{n\to+\infty}P^n(y_0)=:y$ and $P(y)=y$. Note explicitly that, in general, we don't have the uniqueness of the fixed point y, exactly because we can have a loss of $L>0$ derivatives. If $T\subseteq{}^{\rho}\widetilde{\mathbb{R}}$, $S\subseteq{}^{\rho}\widetilde{\mathbb{R}}^n$ are solid functionally compact sets, $Y\subseteq{}^{\rho}\mathcal{GC}^{\infty}(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ and the set-theoretical map $F:T\times S\times Y\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}^d$ satisfies $F(-,-,y)\in{}^{\rho}\mathcal{GC}^{\infty}(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ for all $y\in Y$, then we say that F is *uniformly Lipschitz on Y with constants $(\Lambda_i)_{i\in\mathbb{N}}\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}^{\mathbb{N}}$ and loss of derivatives $L\in\mathbb{N}$ if*

$$\forall i\in\mathbb{N}\,\forall u,v\in Y:\ \|F(-,-,u)-F(-,-,v)\|_i\le\Lambda_i\cdot\|u-v\|_{i+L}.$$

We can prove that any PDE of the form $\partial_t y(t,x)=G\left[t,x,\partial_x y(t,x)\right]$, where G is a GSF, defines a uniformly Lipschitz map on the space

$$Y=\left\{y\in{}^{\rho}\mathcal{GC}^{\infty}(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)\mid\|y-y_0\|_i\le r_i\ \forall i\in\mathbb{N}\right\}.\tag{4}$$

We finally proved the following generalization of the Picard-Lindelöf theorem: let $t_0\in{}^{\rho}\widetilde{\mathbb{R}}$, α, $r_i\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $T_\alpha:=[t_0-\alpha,t_0+\alpha]$. Let $y_0\in{}^{\rho}\mathcal{GC}^{\infty}(S,H)$, where $H\subseteq{}^{\rho}\widetilde{\mathbb{R}}^d$ is a sharply closed set such that $\overline{B_r(y_0(x))}\subseteq H$ for all $x\in S$. Define Y_α as in (4), but using T_α instead of T, and assume that F is uniformly Lipschitz on Y_α with constants $(\Lambda_i)_{i\in\mathbb{N}}$ and loss of derivatives L. Finally, assume that

$$\begin{aligned}&\Lambda_i\le\Lambda_{i+1}\quad\forall i\in\mathbb{N}\\&\|F(-,-,y)\|_i\le M_i(y)\quad\forall y\in Y_\alpha\\&\alpha\cdot M_i(y)\le r_i\quad\forall i\in\mathbb{N}\\&\lim_{\substack{n,m\to+\infty\\n\le m}}\alpha^{n+1}\cdot\Lambda_{i+mL}^n\cdot\|F(-,-,y_0)\|_{i+mL}=0\end{aligned}$$

Then there exists a solution $y \in {}^{\rho}\mathcal{GC}^{\infty}(T_{\alpha} \times S, {}^{\rho}\widetilde{\mathbb{R}}^{d})$ of the Cauchy problem

$$\begin{cases} \partial_t y(t,x) = F(t,x,y) & \forall (t,x) \in T_{\alpha} \times S \\ y(0,x) = y_0(x) & \forall x \in S \end{cases}$$

Note explicitly that this is only an existence result and nothing is stated about the uniqueness of the solution. A generalization of these results to k-th order PDE can be easily proved because, due to the closure of GSF with respect to composition, every higher order PDE can be reduced to a system of first order PDE, see [Giordano-L-Kunzinger18]. For example, by taking $S = [-\mathrm{d}\rho^{-1}, \mathrm{d}\rho^{-1}]^n \supseteq \mathbb{R}^n$ it is possible to prove the existence, local in the normal variable t, but global in the "space" variable $x \in S \supseteq \mathbb{R}^n$ of every polynomial PDE with real coefficients, i.e. where $G \in \mathbb{R}[t, x, d]$. This includes an infinite class of PDE that cannot even be formulated e.g. within the theory of Sobolev-Schwartz distributions. Note also that this kind of results is not possible for CGF either due to the missing of closure with respect to arbitrary compositions and because generally speaking the domain $T_{\alpha} \times S$ also includes non compactly supported points[4].

To introduce the notion of hyperlimit and hyperseries, we first consider the set of *hypernatural numbers* in ${}^{\rho}\widetilde{\mathbb{R}}$, i.e. ${}^{\rho}\widetilde{\mathbb{N}} := \left\{ [n_{\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}} \mid n_{\varepsilon} \in \mathbb{N} \quad \forall \varepsilon \right\}$. If σ, ρ are *two* gauges and $a : {}^{\sigma}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ is a σ-generalized sequence of ρ-generalized numbers, then the hyperlimit $l = {}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n$ is simply the limit of this sequence in the sharp topology, i.e.

$$\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall n \in \mathbb{N}_{\sigma} : \ n \geq M \ \Rightarrow \ |a_n - l| < \mathrm{d}\rho^q.$$

The importance to consider two gauges lies in the fact that if $\sigma_{\varepsilon} := \exp\left(-\rho_{\varepsilon}^{-\frac{1}{\rho_{\varepsilon}}}\right)$, then ${}^{\rho}\lim_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} \frac{1}{\log n} = 0 \in {}^{\rho}\widetilde{\mathbb{R}}$, whereas $\nexists\, {}^{\rho}\lim_{n \in {}^{\rho}\widetilde{\mathbb{N}}} \frac{1}{\log n}$, see [MTA-Giordano]. The notion of hyperseries is a particular case: let $a : \mathbb{N} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ be a sequence of ${}^{\rho}\widetilde{\mathbb{R}}$ and let $s \in {}^{\rho}\widetilde{\mathbb{R}}$. Assume that the partial sums with summands $a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ can be extended to ${}^{\sigma}\widetilde{\mathbb{N}}$ as $N \in {}^{\sigma}\widetilde{\mathbb{N}} \mapsto \sum_{n=0}^{N} a_n := \left[\sum_{n=0}^{\mathrm{nint}(N)_{\varepsilon}} a_{n\varepsilon}\right] \in {}^{\rho}\widetilde{\mathbb{R}}$ (here $\mathrm{nint}(k)$ is the nearest integer function), then we set ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} a_n := {}^{\rho}\lim_{N \in {}^{\sigma}\widetilde{\mathbb{N}}} \sum_{n=0}^{N} a_n$ whenever this hyperlimit exists. For example, one can easily prove that ${}^{\rho}\sum_{n \in {}^{\sigma}\widetilde{\mathbb{N}}} k^n = \frac{1}{1-k}$ for all $k \in {}^{\rho}\widetilde{\mathbb{R}}_{<1}$ and ${}^{\rho}\sum_{n \in {}^{\rho}\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$ for all $x \in {}^{\rho}\widetilde{\mathbb{R}}$ finite, see [T-Giordano20, T-Giordano].

We finally mention that the sheaf property for GSF, see [Giordano-Kun-Ver18], can be used to prove that the functor ${}^{\rho}\mathcal{GC}^{\infty}(-, T)$ belongs to a suitable Grothendieck topos ${}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}$ of sheaves. This topos can be considered a Cartesian closed universe of generalized sets and functions which is closed with respect to set theoretical operations such as $X \cup Y$, $X \cap Y$, Y^X, $X \times Y$, $\mathcal{P}(X)$, subsets, etc. This allows us to consider a framework of infinite dimensional spaces of GF like ${}^{\rho}\mathcal{GC}^{\infty}(C, D)^{{}^{\rho}\mathcal{GC}^{\infty}(A,B)} = {}^{\rho}\mathrm{T}\mathcal{GC}^{\infty}({}^{\rho}\mathcal{GC}^{\infty}(A, B), {}^{\rho}\mathcal{GC}^{\infty}(C, D))$.

3 Work program

In this section, we describe the methods we plan to employ in carrying out the research program sketched above. For each one of the three parts of the research project, we will also give a (subjective but justified) judgement of its feasibility. This qualitative judgement of feasibility will also be used to quantify and support the project's time planning.

[4] This proposal is a resubmission, and a part of this section has been improved.

3.1 WP 1: Newton method and the Pontryagin principle for singular problems[5]

A proof of Newton method can be accomplished along different lines. For our aims of extension to the GSF case, a very promising approach is presented in [Ben66], where the problem is reduced to the study of a contraction, and hence to the Banach fixed point theorem (which was already proved for GSF). Different approaches seems anyway repeatable in our context, e.g. using the Fermat-Reyes theorem for GSF (i.e. the possibility to see derivatives of GSF as generalized incremental ratios, see [Giordano-Kun-Ver18]) or Taylor's formulae. In any case, it is essential to consider the structure of ${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet space (see Sec. 2.2) on a functionally compact solid set, and hence its Cauchy completeness with respect to the sharp topology. We also take into account the possibility to consider an infinite number $N \in {}^{\rho}\widetilde{\mathbb{N}} \setminus \mathbb{N}$ of iterations in Newton's algorithm.

We also plan to consider suitable generalization of Newton method, such as Householder's method, Halley's method ([SeGo01]) and possibly situations where the function F is defined between infinite-dimensional ${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet spaces of GSF and not only between sets of generalized points. This would allows an applications of Newton methods to the solution of DE.

As we mentioned above, this extension of Newton method to the GSF framework, would permit us to consider a possible proof of Pontryagin's maximum principle. As it is clearly stated in [DmOs14, Eva10], one of the problem considered by the present research on this topic is to find a proof satisfying the present requirements of rigour used in modern calculus of variations. Clearly, we add to this the goal to find a wide extension to the GSF setting. Our work plan will be in accordance with the following methodological lines:

1. The general setting for variational problems and control theory have to be the same used in our generalization of classical results of calculus of variations to the GSF context, see [LL-Giordano16]. From here we already know what function spaces we need to consider and what kind of topology we have to take on these spaces.

2. The setting for unique solution of ODE will clearly be the one presented in [L-Giordano16a], where we proved existence and uniqueness of local and global solutions of ODE (using Picard-Lindelöf (possibly hyperfinite $N \in {}^{\rho}\widetilde{\mathbb{N}}$) iterates), continuous dependence from initial conditions, relations with classical distributional solutions and several other examples.

3. We hence plan to prepare a series of seminars about the rigorous proof presented in [DmOs14] but using the notations of [Eva10]. These seminars have the main aim to check where the level of rigour is considered as sufficient by our group and, mainly, where a possible generalization to the GSF setting would be more problematic. We also plan here an intuitively very clear presentation of the ideas used in the proofs.

4. For example, in the proof presented in [DmOs14], the more problematic step, from the point of view of its extensibility to GSF, is the *finite intersection property*. This is mainly where [Eva10] differs from [DmOs14] because the former uses a suitable form of Newton method.

5. We plan to extend the Pontryagin principle so as to allow for GSF as admissible control parameters. This includes e.g. the possibility to also consider infinitesimal or infinite control parameters.

6. Particular problems to which we plan to apply these general theoretical results are:

[5] This proposal is a resubmission, and a part of this section has been improved.

a) Minimization of the energy functional for:

   i. A classical particle (or high-frequency waves) moving through discontinuous media containing barriers or interfaces where the Hamiltonian is discontinuous: see e.g. [SiJe13, JiWuHu08] and references therein; see also [CrLi83]. We can view the physical and geometrical parameters characterizing the different media as the set of admissible controls.

   ii. Discontinuous Lagrangian in geometrical optics: see e.g. [LaGhTh11]. As above, the admissible controls can be the physical and geometrical parameters of the media traversed by light rays.

   iii. A harmonic oscillator forced by a Dirac comb or a more general periodic singular force (formalized using a GSF). As above, the movement in a discontinuous medium can also be considered.

   iv. Departure from the classical Hook's law for steel deformation (the stress-strain relation is not differentiable): see e.g. [UgFe11, ChCh11]. We can consider the stress-strain relation as depending on a suitable number of admissible control parameters. The final aim is hence to find the best values of these parameters so as to get the desired nonlinear dynamics in case of deformation.

b) Applications of variational principles in quantum mechanics (see e.g. [Gri95, SaNa13]). For example, solution of the stationary Schrödinger equation for an infinite rectangular potential well (a case that cannot be formalized using Sobolev-Schwartz distributions, see e.g. [GaPa90]) or a rectangular potential well with periodic barriers changing at infinite frequency, and application to high frequency laser pulses acting on quantum objects (see e.g. [VKRKPW]);

c) Design of potentials in Alfa-decay modelling, see e.g. [Zak96]: by describing the potential well using suitable admissible control parameters, the problem is to find solutions of a stationary Schrödinger equation that allow for an increasing or decreasing of the probability of nucleus decay. Note that for solving this problem, the easiness in defining a GSF (as compared e.g. to Sobolev-Schwartz distributions) is of fundamental importance.

d) Deduction of the Schrödinger equation from variational principles using a Lagrangian GSF. This derivation can be easily obtained by generalizing the well-known calculations presented e.g. in [And06] to the calculus of variations we developed in [LL-Giordano16]. We also plan to apply the same idea to the Dirac, and hence in particular to the Pauli, equation. To solve this problem, we also collaborate with researchers of the recently approved FWF project *Functional analysis of infinite bounded operators* (PI P. Giordano), which also includes a section about the foundation of quantum mechanics using GSF and non-Archimedean analysis.

Here the expertise of A. Bryzgalov in physical modelling will be of great help.

e) Determine $\min_{u\in\mathfrak{M}_p}\int_K\left(|\nabla u|^2+W(u)\right)\mathrm{d}x$, where $K\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ is a functionally compact subset, $\mathfrak{M}_p=\left\{u\in{}^\rho\mathcal{GC}^\infty(K,{}^\rho\widetilde{\mathbb{R}})\mid\int_K|u|^p\,\mathrm{d}x=1\right\}$ and W is a GSF (see [BrSq18, BeLuSq20]).

f) Existence and uniqueness of solutions for nonlinear integral equations of Fredholm type $\varphi(x)=f(x)+\lambda\int_a^b K(x,t)\varphi(t)^p\,\mathrm{d}t$, where $x\in[a,b]\subseteq{}^\rho\widetilde{\mathbb{R}}$ (where $a<b$ can also be infinite

numbers), $\lambda \in {}^\rho\widetilde{\mathbb{R}}$, $p \in {}^\rho\widetilde{\mathbb{R}}_{\ge 2}$, f and K are GSF; see [GuHeSa04] for a classical approach in the particular case of continuous f, K.

Note that all these problems cannot be solved or even formulated either for Sobolev-Schwartz distribution or for CGF if K or $[a, b]$ contain infinite numbers.[6]

Risks: The generalization of Newton method does not seem to present particular difficulties. The possibility to reduce it to the Banach fixed point theorem increases the feasibility of this part. An extension of the Pontryagin principle could be considered as more problematic because unexpected topological properties may be necessary in the proof.
Solutions: On the other hand, our work on calculus of variations, [LL-Giordano16], clearly shows that the sharp topology on ${}^\rho\widetilde{\mathbb{R}}$-Fréchet spaces of GSF is the right one for this kind of variational problems.
Subjective assessment of feasibility: For these reasons, in our opinion this part of the project has a *medium-high* assessment of feasibility.

3.2 WP 2: Singular Hamiltonian mechanics

The main problem in extending modern versions of Hamiltonian mechanics mainly corresponds to a generalization of results of differential geometry and of an interrelated theory of ODE on these geometrical spaces. In our case, the main problem is to give a flexible definition of GSF on an extended manifolds and hence to prove local existence and uniqueness of solutions of ODE on these new spaces. The solution presented in [KuObStVi], in the framework of CGF, presents constraints such as c-boundedness conditions for the composition of CGF and only extension of sharply bounded regions of the manifold. Clearly, a generalization to infinite dimensional spaces such as $\mathbf{Man}(M, N)$, the space of all the smooth mappings between two manifolds, or to manifolds with singular points is not immediately possible using the idea of [KuObStVi] because of the lacking of Cartesian closedness of the category of smooth manifolds.

The Grothendieck topos ${}^\rho\mathrm{T}\mathcal{GC}^\infty$ of GSF, would permits a more general and flexible approach. To briefly illustrate the main idea in the particular case of a smooth manifold M, let $\varphi : U \longrightarrow \varphi(U) \subseteq M$ and $\psi : V \longrightarrow \psi(V) \subseteq M$ be charts of an atlas on M; we introduce an equivalent relation $(x, \varphi) \sim (y, \psi)$, if $\varphi(x) = \psi(y) \in M$. The isomorphism $M \simeq \mathrm{Charts}/\sim$, where Charts is the set of all the pairs of the form (x, φ), can be written in more general categorical terms as

$$M \simeq \operatorname*{colim}_{U \in \mathbf{Man}/M} U \tag{5}$$

i.e. as a colimit (quotient set) of all the charts $U \in \mathbf{Man}/M$ over the manifold M modulo the aforementioned equivalence relation.

We therefore plan to consider the following steps:

1. Take the extension $\langle U \rangle$ of each open set $U \subseteq \mathbb{R}^n$ as the corresponding strongly open set generated by the constant net $\varepsilon \mapsto U$. We have that $\langle U \rangle \in {}^\rho\mathrm{T}\mathcal{GC}^\infty$.

2. Introduce the negligibility relation on the set of *all* the nets M^I: $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if there exists a net of charts $\varphi_\varepsilon : U_\varepsilon \longrightarrow \varphi_\varepsilon(U_\varepsilon) \subseteq M$ (of the same atlas) such that $(\varphi_\varepsilon^{-1}(x_\varepsilon)) \sim_\rho (\varphi_\varepsilon^{-1}(y_\varepsilon))$ for ε small.

[6]This proposal is a resubmission, and a part of this section has been improved.

3. Extends each net of charts $\varphi_\varepsilon : U_\varepsilon \longrightarrow \varphi_\varepsilon(U_\varepsilon)$ as $\langle\varphi_\varepsilon\rangle([x_\varepsilon]) := [\varphi_\varepsilon(x_\varepsilon)]$ and define the equivalence relation $([x_\varepsilon], \langle\varphi_\varepsilon\rangle) \sim ([y_\varepsilon], \langle\psi_\varepsilon\rangle)$ if $\langle\varphi_\varepsilon\rangle([x_\varepsilon]) = \langle\psi_\varepsilon\rangle([y_\varepsilon])$.

4. Define the extension of the manifold as $\langle M\rangle := \mathrm{colim}_{U\in\mathbf{Man}/M}\langle U\rangle = \langle\mathrm{Chart}\rangle/\sim$, where the index set is the same as in (5), but where the colimit (quotient) is now taken in the category ${}^\rho\mathrm{T}\mathcal{GC}^\infty$. In this definition, $\langle\mathrm{Chart}\rangle$ is the set of all the pairs $([x_\varepsilon], \langle\varphi_\varepsilon\rangle)$. A GSF in this setting is simply an arrow $f \in {}^\rho\mathrm{T}\mathcal{GC}^\infty(\langle M\rangle, \langle N\rangle)$ in the topos.

5. Prove that in the case where M is an open set of $\mathbb{R}^n$, then this new construction and the classical one always give the same space $\langle M\rangle \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ (up to a natural isomorphism). Prove the preservation properties of the functor $\langle-\rangle : \mathbf{Man} \longrightarrow {}^\rho\mathrm{T}\mathcal{GC}^\infty$ by following analogous ideas used in [Giordano-Wu16]. We call $\langle-\rangle$ the *Colombeau functor*.

6. Generalize the same construction from manifold to arbitrary diffeological spaces (for a self-contained introduction to diffeological spaces, see [Giordano-Kun-Ver18, Giordano-Wu16, Giordano11a]) so as to include also infinite dimensional spaces such as ${}^\rho\mathrm{T}\mathcal{GC}^\infty(\langle M\rangle, \langle N\rangle)$, i.e. the infinite dimensional space of all the GSF between the extended manifolds $\langle M\rangle$ and $\langle N\rangle$.

7. Use the language of nilpotent infinitesimals introduced for infinitesimal Taylor formula of GSF (see [Giordano-Kun-Ver18, Thm. 53]) to define tangent vectors, vector fields, flows, Lie brackets and differential forms along the lines of [Giordano10b]. Use the results of [L-Giordano16a] about local existence of solutions of ODE to prove the existence of generalized flows.

8. Definition of symplectic structure and generalized Poisson bracket; Hamiltonians as GSF; generalized Hamiltonian vector fields; Legendre transform and relations with Lagrangian approach; Hamiltonian systems as fiber bundles over ${}^\rho\widetilde{\mathbb{R}}$; generalized smooth canonical transformations, symplectomorphisms and symmetries.

9. Using the preservation properties of the functor $\langle-\rangle$, and the well-known relations between GSF and CGF, compare this approach with that of [KuObStVi, Mar68, Mar69].

10. Compare the preservation properties of the functor $\langle-\rangle$ with a more classical ultrapower extension used in nonstandard analysis; see e.g. [Tod15, Tod13, Tod11] for an approach to CGF in the framework of nonstandard analysis.

At this point, we have all the geometrical instruments we need to consider Lagrangian mechanics on manifolds (clearly using [LL-Giordano16]), symplectic manifolds and the canonical formalism for Hamiltonian dynamics. Note that the possibility to extend the aforementioned geometrical tools to the case of arbitrary diffeological space would allow an extension to infinite dimensional spaces, such as in fluidodynamics. Anyway, this part of the project will *probably not* be considered due to a lack to time. On the other hand, we will definitely consider the physical examples listed in WP 2 in 6a) and 6b).

Risks and solutions: This part of the project is well grounded on papers such as [LL-Giordano16] for the calculus of variations; [Giordano-Kun-Ver18] for the Grothendieck topos of GSF; for ODE theory with GSF see [L-Giordano16a]; finally, for the study of the Colombeau functor and the language of nilpotent infinitesimals in differential geometry, see [Giordano10b, Giordano-Wu16] for a very similar approach. For these reasons, and for its very high and important goals, we assess as successful even a partial development of this part. We also take into account the possibility that the construction outlined above will really work only for bounded subsets of the manifolds. Even if our ideas try to avoid this constraint, similar conditions appears both in the independent works [Mar68, Mar69] and [KuObStVi]. Anyway, the expertises of P. Giordano on the aforementioned categorical construction and on the use of nilpotent infinitesimals, and those of M. Kunzinger in singular differential geometry applied to mathematical general relativity (see [GrKuSa19, KuSa18, GrGrKu18]) will surely help in finding the correct solutions.

Subjective assessment of feasibility: For these reasons, in our opinion this part of the project has a *medium-high* assessment of feasibility.

3.3 WP 3: Numerical and graphical tools to visualize GSF

In this part of the project, our aim is twofold: on the one hand, we want to develop new tools to visualize the behaviour of a meaningful subclass of GSF. On the other hand, using *standard* Matlab solvers for DE, we want to implement a new toolbox for the numerical solutions of nonlinear singular DE.

By using Matlab object oriented programming, we can create an easy-to-use numerical framework to input and study GSF. We recall that, using overloading of operators in object oriented programming, we can insert a GSF by using suitable *creation methods*, and then to define new GSF by using the common notation for sum, product, composition, powers, etc. of previously defined GSF. The creation methods we want to implement are:

1. Input of a gauge $\rho = [\rho_\varepsilon]$ and the possibility to use in Matlab the notation $\mathrm{d}\rho$.

2. Input of a Cauchy-Dirac GSF, i.e. of a GSF of the form $f(x) = [\varphi(x_\varepsilon, p_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$, where $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^P, \mathbb{R}^d)$ is an ordinary smooth function, and $p = [p_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^P$ is a vector of generalized parameters. For example, $f(x) = \mathrm{d}\rho \cdot \sin\left(\frac{x}{\mathrm{d}\rho}\right)$ is an example of Cauchy-Dirac GSF (and of use of overloading of the product symbol $\cdot$, and of the symbol sin).

3. GSF defined by cases in ε. For example, $f(x) = [f_{1\varepsilon}(x_\varepsilon)]$ if $\varepsilon = \frac{1}{n}$ for some $n \in \mathbb{N}$ and $f(x) = [f_{2\varepsilon}(x_\varepsilon)]$ otherwise.

4. GSF defined by smoothly interpolating two (or more) given smooth functions. For example, we can interpolate $f_{1\varepsilon} \in \mathcal{C}^\infty([-1, -\varepsilon], \mathbb{R})$ and $f_{2\varepsilon} \in \mathcal{C}^\infty([\varepsilon, 1], \mathbb{R})$ using a smooth function $i_\varepsilon \in \mathcal{C}^\infty([-\varepsilon, \varepsilon], \mathbb{R})$ which satisfies the conditions $i_\varepsilon(0) = 1/2$, $i'_\varepsilon(0) = 1/\varepsilon$. Different smooth interpolations will be considered.

5. GSF defined using the classical Colombeau embedding, i.e. the inverse Fourier transform of a given function of $\mathcal{S}(\mathbb{R}^n)$ which is identically equal to 1 in a neighbourhood of the origin.

6. GSF defined by hyperseries, i.e. hyper-analytic GSF, see [T-Giordano20, T-Giordano].

7. GSF defined by (hyperfinite) Fourier transform, see [M-Giordano20].

8. GSF defined as ε-wise solutions of a boundary value problem for an ODE or a PDE. In this part of the project we will *not* develop *new* numerical methods for the solution of nonlinear singular DE. Therefore, in case of non-convergent solutions, our Matlab implementation will simply output a warning to indicate that more refined methods are necessary. Even under these constraints, the competencies in implementing numerical solutions of both linear and non-linear problems of co-author A. Bryzgalov (see CV) and of Prof. H. Schichl of the Computational Mathematics

Group of the University of Vienna (see Sec. 4 and the enclosed collaboration agreement letter) will be of great help.

9. A comparison of this toolbox with the well-known numerical results of [Col07b], obtained in the setting of CGF, is planned.

Even if, of course, this would permit us only the implementation of a small subclass of GSF, we think it could completely solve the goals of this part of the project. For example, it would easily permit us to consider GSF such as $f(x) = \delta(\delta(\delta(x))) + \cos(H(\delta(x))) + \mathrm{d}\rho \cdot \delta(H(x))^3 - N(\mathrm{d}\rho^{-1}, x)$, where $N(\sigma, x)$ is a Gaussian density with standard deviation σ. Note that the generalized function $f(x)$ is neither a Sobolev-Schwartz distribution nor a CGF.

An important tool we also plan to develop is the creation of a video that represents a given GSF (e.g. originating as solution of a nonlinear PDE) as $\varepsilon \to 0^+$. Such a visualization can be really helpful in understanding blowing up of solutions or infinitesimal/infinite properties of the solution or a meaningful discontinuous behaviour with respect to ε. In our experience, this kind of "experimental mathematics" could accomplish strong intuition culminating in interesting theoretical results (e.g. visualization of wave front sets).

Clearly, Matlab is not the only possibility to implement all these numerical tools. Even if we prefer to focus our attention on Matlab because of its diffusion in industries and because of our specific competencies, we will also study, if the time would allow it, the possibility to have analogous implementation in Wolfram Mathematica, Maple, Python or Fortran.

Risks: The plan is feasible, without particular risks. The results, even if it has clear differences with respect to the full theory, can be surely considered as an effective tool to develop a strong intuition on GSF.

Subjective assessment of feasibility: For these reasons, and based on the experience of the co-author A. Bryzgalov in numerical algorithms, in our opinion this part of the project has a *very high* assessment of feasibility.

4 Scientific relevance, originality and expected benefits for potential users

The present research proposal takes place in the following international research frameworks:

- It fits well in current threads of Austrian research, in particular those of the DIANA group of Prof. M. Kunzinger at the University of Vienna, who is also one of the main developers of the theory of GSF.
- It also fits well into the research interests of the international community of CGF, where the interest for numerical implementation, applications of Colombeau's theory and its dissemination was clearly voiced in several conferences.
- The WP 3 about numerical and graphical representation of GSF will certainly profit from our collaboration with the Computational Mathematics Group of Prof. Hermann Schichl, also situated at the Faculty of Mathematics in Vienna. We will establish close interactions and thereby strengthen the already existing collaboration with this highly active group (see the enclosed collaboration agreement letter).[7]

[7]This proposal is a resubmission, and a part of this section has been improved.

Originality, innovations and benefits of the present proposal:

- A broad theoretical development of control theory with applications of the Pontryagin principle to singular problems.
- A general solution of the extension of classical mechanics to singular models which is well grounded on differential geometry and topos theory.
- A flexible and easy-to-use toolbox to visualize and enhance intuition of students and researchers in studying, developing and applying GSF.

Potential users can hence be foreseen both in pure and applied mathematics, in physics and engineering applications.

4.1 Importance for human resources

A very important peculiarity of the present research proposal is that its topics have been adapted to the mathematical competencies of the co-author PhD. Aleksandr Bryzgalov (WP3: Extension of Hamiltonian mechanics using GSF; He already has a Russian analogue of a PhD degree, called *Candidate of Physical and Mathematical Sciences*. However, he is highly motivated to get a PhD in pure mathematics at the University of Vienna). A new PhD candidate will also work on WP1 and WP2: Newton method and Pontryagin principle and visualization tools for GSF. The participation to all the important conferences in this sector will introduce these new researchers into this community.

The new results achieved in the present proposal would allow to write an important monograph about GSF, with both very original new theoretical and applied results. This is an important step to consolidate P. Giordano toward an important ERC grant. For the CV of all the members of the research group, see below in this proposal.

4.2 Ethical Issues

There are no ethical, security-related or regulatory aspects of the proposed research project.

4.3 Sex-specific and gender-related aspects

There are no Sex-specific and gender-related aspects of the proposed research project.

5 Dissemination strategy and time planning

Work organization[8]

We recall that the present research project is designed for five co-workers: the applicant P. Giordano, co-author M. Kunzinger, a new post-doc, A. Bryzgalov (PhD candidate), and a new PhD candidate. Almost all the ideas of this project have been originally developed by P. Giordano. For this reason, he and the new post-doc collaborator will supervise all the other members of the group in all the WP; M. Kunzinger will also supervise, more specifically, on WP 2 because of his interest and expertise in differential geometry. The PhD work of A. Bryzgalov will focus on WP 2 and 3, mainly for his competencies in physical modelling and numerical calculus; the new PhD candidate will focus on WP

[8]This proposal is a resubmission, and a part of this section has been improved.

1 and 3. The post-doc researcher will be employed full time, whereas, as stated by FWF, the two PhD candidates will be employed at 75% (30 hours per week) so as to have time for their PhD studies.

Moreover, we plan to organize several joint meetings in order to initiate and improve the collaboration, both in the solutions of problems and to get new ideas:

- Weekly meetings of P. Giordano with the new post-doc and the two PhD students (three separate meetings per week).
- Weekly meeting of the new post-doc with both PhD students (at least two separate meetings per week).
- Monthly seminar with all the member of the research group (and possibly interested external people), with presentation of present state, open problems and new results.
- In developing WP 3, we plan to organize two mini-workshops per year with Prof. H. Schichl's and all the interested people of the Computational Mathematics Group of the University of Vienna.

Dissemination and collaboration strategy

Our strategy for the dissemination of the results of this proposal is addressed both to an internal and an international audience:

- More typical seminars of the DIANA group of the University of Vienna addressed to all the interested colleagues are planned and with presentation of present state, open problems and new results.
- Prof. Yu. A. Korovin, head of General and Special Physics Department of the National Research Nuclear University IATE MEPhI (https://eng.mephi.ru/), Moscow, showed an interest to plan a bidirectional transfer of knowledge concerning the results of the project. This collaboration is particularly important for the applications in physics planned in WP 1. We plan one travel per year, lasting one working week, since the main part of the collaboration will be performed on-line. Travel expenses will be covered by this project (see *Annex 1: financial aspects*).
- Contributions for two international conferences per year per person for the presentation of relevant results are planned. In particular, we are thinking of conferences such as the International Conference on Generalized Functions in 2024, the 15th Viennese Conference on Optimal Control and Dynamic Games in 2021, the International Congress of Mathematical Physics 2021 and the International Congress of Theoretical and Applied Mechanics 2021.
- Several articles for peer reviewed green open access journals and the related dissemination by means of preprint-servers is planned. We aim at journals such as: *Archive for Rational Mechanics and Analysis*, *AIP Journal of Mathematical Physics*, *Acta Applicandae Mathematicae*, *Advances in Nonlinear Analysis, Calculus of Variations and Partial Differential Equations*, *SIAM Journal on Numerical Analysis.*

Time planning

To estimate the total amount of work to be dedicated into each one of the three parts of this project, we plan about six months to fully understand the background on GSF and hence 28 months for the

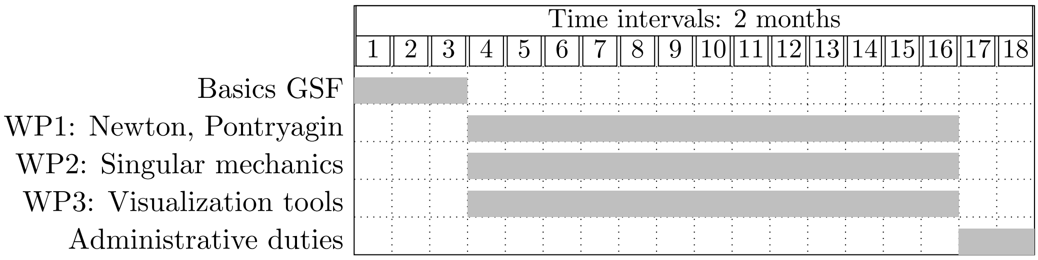

Figure 1: Basic time organization

development of each WP. A period of two months is planned to accomplish administrative duties such as theses writing and correction. The entire research project is hence planned to be concluded in 36 months and the time planning is represented in Fig. 1.

Nomenclature

CGF	Colombeau generalized function(s)
DE	Differential equation(s)
GF	Generalized function(s)
GSF	Generalized smooth function(s)
ODE	Ordinary differential equation(s)
PDE	Partial differential equation(s)
WP	work package(s)

Annex 1: financial aspects

Available personnel and infrastructure

The University of Vienna (AT) is the planned research institution to host the present research project. Available personnel is: Dr. P. Giordano, who is already self-employed as PI of the FWF project P33538-N, and Prof. M. Kunzinger, who is already employed at the University of Vienna. Almost all the ideas of the present project have been originally developed by P. Giordano. For this reason, he will actively supervise the entire project and his contribution is essential to the success of this application.

Personnel costs

In our view, work on the project goals can be pursued by funding the entire PhD candidate position of A. Bryzgalov (we recall that he is also planning to take a PhD in mathematics, so that he will be employed at 75% as stated by FWF), a new 100% post-doc position for 30 months (NN1) and a new PhD candidate (NN2) at 75% for 2 years. As it is quite common at the University of Vienna, the remaining funding for the completion of the post-doc position to 36 months and for the PhD studies of the last candidate will be requested in separate specific project proposals. Therefore, on the basis of the 2020 FWF salary rates, we have the following

- 1 post-doc position for 30 months (NN1): 69'040 €/y * 2.5 y = 172'600 €.
- 1 Ph.D. candidate positions for 3 years (A. Bryzgalov) = 39'210 €/y * 3 y = 117'630 €.
- 1 Ph.D. candidate positions for 2 years (NN2) = 39'210 €/y * 2 y = 78'420 €.

This amounts to a total of 368'650 €.

Equipment and material costs

These costs concern three notebooks, one for each PhD candidate and one of the post-doc researcher, suitably adapted for the non trivial numerical computations of work package 3 of the proposal, and three graphic tablets (which should not be confused with a tablet computer; see e.g. https://www.youtube.com/watch?v=eEJtMFdkzzU) for exchange of mathematical handwritten notes and PDF annotations. A classical graphic tablet could be the Wacom Pth-860-S, size L (490 € Amazon.it price as of 11 June 2020). A good notebook for project's aims could be the MSI GS75 or similar (1'999.99 € Amazon.it price as of 11 June 2020) because its numerical and graphical performances are well suited for the development of work package 3. Note that this kind of IT tools are not available as standard equipment of the University of Vienna. Moreover, even respecting the total amount of working hours, the creative work in mathematics must be organized with a certain flexibility: this justify why notebooks are preferable with respect to non-portable workstations. This is particularly true for programming in numerical calculus, as planned in the work package 3 of the proposal. Matlab licenses will be provided by the University of Vienna.

- 1 notebook * 3 researchers = 2'000 € * 3 = 6'000 €.
- 1 graphic tablet * 3 researchers = 490 € * 3 = 1'470 €.

This amounts to a total of 7'470 €.

Travel costs

We plan to pay travel expenses for A. Bryzgalov, NN1, NN2 (one travel per year, lasting one working week; the main part of the collaboration will be performed online) at the *General and Special Physics Department* of the *National Research Nuclear University IATE MEPhI*, Moscow, which showed an interest to plan a bidirectional transfer of knowledge concerning the results of the project. This collaboration is particularly important for the applications in physics planned in work package 1. We plan 450 € per travel per 3 person.

- 3 travels * 450 €/researcher * 3 researchers = 4'050 €.

Therefore, the total amount requested for the present proposal (considering 5% of general costs) is 399'178.50 €.

Annex 2: List of references

[And06] A.V. Andreev, *Atomic Spectoscopy: Introdution to the Theory of Hyperfine Structure*. Springer Science+Business Media Inc., 2006.

[Ar-Fe-Ju05] J. Aragona, R. Fernandez, S.O. Juriaans, A Discontinuous Colombeau Differential Calculus. Monatsh. Math. 144 (2005), 13-29.

[Ar-Fe-Ju09] J. Aragona, R. Fernandez, S.O. Juriaans, Natural topologies on Colombeau algebras, Topol. Methods Nonlinear Anal. 34 (2009), no. 1, 161-180.

[Ar08] I.K. Argyros, Convergence and applications of Newton-type iterations. Springer Science and Business Media, 2008.

[Ar-Fe-Ju12] J. Aragona, R. Fernandez, S.O. Juriaans and M. Oberguggenberger, Differential calculus and integration of generalized functions over membranes, Monatsh Math (2012) 166: 1. https://doi.org/10.1007/s00605-010-0275-z

[BeLuSq20] V. Benci, L. Luperi Baglini, M. Squassina, Generalized solutions of variational problems and applications, Adv. Nonlinear Anal. 9, pp. 124-147, 2020.

[Ben66] A. Ben-Israel, A Newton-Raphson Method for the Solution Systems of Equations, Journal of mathematical analysis and applications, 15, 243-252 (1966)

[BrSq18] L. Brasco, M. Squassina, Optimal solvability for a nonlocal problem at critical growth, J. Differential Equations 264, no. 3, 2242-2269, 2018.

[CaCo89] J.J. Cauret, J.F. Colombeau, A.Y. Le Roux., Discontinuous generalized solutions of nonlinear nonconservative hyperbolic equations. Journal of mathematical analysis and applications 139.2 (1989): 552-573.

[ChCh11] R. Champion, W.L. Champion, Departure from linear mechanical behaviour of a helical spring, Mathematical and Computer Modelling 53, 915-926, 2011.

[Col92] J.F. Colombeau, *Multiplication of Distributions; A Tool in Mathematics, Numerical Engineering and Theoretical Physics*, Lecture Notes in Mathematics, 1532, Springer-Verlag, Berlin, 1992.

[Col07] J.F. Colombeau, Mathematical problems on generalized functions and the canonical Hamiltonian formalism, 2007 see https://arxiv.org/abs/0708.3425

[Col07b] J.F. Colombeau, Nonlinear generalized functions and nonlinear numerical simulations in fluid and solid continuum mechanics, 2007. See arXiv:math-ph/0702014

[CoRo88] Colombeau, J. F., and A. Y. Le Roux., Multiplications of distributions in elasticity and hydrodynamics. Journal of Mathematical Physics 29.2 (1988): 315-319.

[CoGs08] J.F. Colombeau, A. Gsponer, The Heisenberg-Pauli Canonical Formalism of Quantum Field Theory in the Rigorous Setting of Nonlinear Generalized Functions. arXiv 0807-0289v2, 2008.

[CrLi83] Crandall, M.G., Lions, P.L. Viscosity solutions of Hamilton-Jacobi equations. Trans. Am. Math. Soc., 277, 1-42, 1983.

[deL14] M. de Leon, M. Salgado and S. Vilarino *Methods of Differential Geometry in Classical Field Theories: k-symplectic and k-cosymplectic approaches*, World Scientific Publishing Company, 2015

[Dir26] P.A.M. Dirac, The physical interpretation of the quantum dynamics, Proc. R. Soc. Lond. A, **113**, 1926-27, 621-641.

[DmOs14] A. Dmitruk, N. Osmolovskii, On the proof of Pontryagin's maximum principle by means of needle variations, arXiv preprint arXiv:1412.2363 (2014).

[Eva10] Evans, L.C., *Partial Differential Equations*, Second Edition, Graduate Studies in Mathematics, American Mathematical Society, 2010.

[FuZe86] Fusco, G., and V. Zecca, Applications of Newton method to some numerical problems in matrix theory. Calcolo 23.4 (1986): 285-303.

[Ga00] Galántai, A., The theory of Newton method. Journal of Computational and Applied Mathematics 124.1-2 (2000): 25-44.

[GaPa90] Galindo, A., Pascual, P., *Quantum Mechanics I*, Springer-Verlag Berlin Heidelberg, 1990.

[Giordano10a] Giordano, P., The ring of fermat reals, Advances in Mathematics 225 (2010), pp. 2050-2075.

[Giordano10b] Giordano, P., Infinitesimals without logic, Russian Journal of Mathematical Physics, 17(2), pp.159-191, 2010.

[Giordano11a] Giordano, P., Fermat-Reyes method in the ring of Fermat reals. Advances in Mathematics 228, pp. 862-893, 2011.

[Giordano-Kunzinger16] Giordano, P., Kunzinger, M., Inverse Function Theorems for Generalized Smooth Functions. Invited paper for the Special issue ISAAC - Dedicated to Prof. Stevan Pilipovic for his 65 birthday. Eds. M. Oberguggenberger, J. Toft, J. Vindas and P. Wahlberg, Springer series Operator Theory: Advances and Applications, Birkhaeuser Basel, 2016.

[Giordano-Kunzinger18a] Giordano, P., Kunzinger, M., A convenient notion of compact sets for generalized functions. Proceedings of the Edinburgh Mathematical Society, Volume 61, Issue 1, February 2018, pp. 57-92.

[Giordano-Kun-Ver18] Giordano P., Kunzinger M., Vernaeve H., The Grothendieck topos of generalized smooth functions I: basic theory. See http://www.mat.univie.ac.at/~giordap7/ToposI.pdf.

[Giordano-Kunzinger-Ver15] Giordano, P., Kunzinger, M., Vernaeve, H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56-71.

[Giordano-L-Kunzinger18] Giordano, P., Luperi Baglini, L., Kunzinger, M., The Grothendieck topos of generalized smooth functions III: normal PDE. Preprint, see http://www.mat.univie.ac.at/~giordap7/ToposIIIa.pdf and http://www.mat.univie.ac.at/~giordap7/ToposIIIb.pdf.

[Giordano-Wu16] Giordano, P., Wu, E., Calculus in the ring of Fermat reals. Part I: Integral calculus. Advances in Mathematics 289 (2016) 888-927.

[Giun03] Z. Giunashvili, Bott Connection and Generalized Functions on Poisson Manifold, 2003 see https://arxiv.org/abs/math/0301364

[Go59] S.K. Godunov, A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics, Matematicheskii Sbornik 89.3 (1959): 271-306.

[GrGrKu18] M. Graf, J.D.E. Grant, M. Kunzinger, R. Steinbauer, The Hawking-Penrose singularity theorem for $C^{1,1}$-Lorentzian metrics, Comm. Math. Phys., 360 (2018), no. 3, 1009-1042.

[GrKuSa19] J.D.E. Grant, M. Kunzinger, C. Sämann, Inextendibility of spacetimes and Lorentzian length spaces, Ann. Global Anal. Geom. 55:133-147 (2019).

[Gri95] Griffiths, D. J., *Introduction to Quantum Mechanics*. Upper Saddle River, New Jersey: Prentice Hall, 1995.

[GrKuObSt01] M. Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer, *Geometric theory of generalized functions with applications to general relativity*, Mathematics And Its Applications, Kluwer Academic Publishers, 2001.

[GuHeSa04] J.M. Gutiérrez, M.A. Hernández, M.A. Salanova, On the approximate solution of some Fredholm integral equations by Newton's method, Southwest Journal of Pure and Applied Mathematics, Issue 1 July 2004, pp. 1-9.

[Hi10] Hintermüller, M., Semismooth Newton methods and applications. Department of Mathematics, Humboldt-University of Berlin (2010).

[Horm63] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[Hor99] Hörmann, G., Integration and Microlocal Analysis in Colombeau Algebras of Generalized Functions, Journal of Mathematical Analysis and Applications 239, 332-348, 1999.

[JiWuHu08] S. Jin, H. Wu, Z. Huang, A Hybrid Phase-Flow Method for Hamiltonian Systems with Discontinuous Hamiltonians, SIAM J. Sci. Comput., 31(2), 1303-1321, 2008.

[KatTal12] Katz, M.G., Tall, D., A Cauchy-Dirac delta function. *Foundations of Science*, 2012. See http://dx.doi.org/10.1007/s10699-012-9289-4 and http://arxiv.org/abs/1206.0119.

[KoJeMi03] P. Kornerup, .J.M. Muller. Choosing starting values for Newton-Raphson computation of reciprocals, square-roots and square-root reciprocals. Diss. INRIA, 2003.

[Kun04] M. Kunzinger, Nonsmooth Differential Geometry and algebras of generalized functions, *J. Math. Anal. Appl.* 297, pp. 456-471, 2004.

[KuObStVi] M. Kunzinger, M. Oberguggenberger, R. Steinbauer, J. A. Vickers, Generalized Flows and Singular ODEs on Differentiable Manifolds, Acta Applicandae Mathematicae, 80(2), pp 221-241, 2004.

[KuSa18] M. Kunzinger, C. Sämann, Lorentzian length spaces, Ann. Global Anal. Geom. 54, no. 3, 399-447 (2018).

[Kunzle96] A.F. Künzle, Singular Hamiltonian systems and symplectic capacities, Singularities and Differential Equations Banach Center Publications, 33, pp.171-187, 1996. http://pldml.icm.edu.pl/pldml/element/bwmeta1.element.bwnjournal-article-bcpv33z1p171bwm

[LaGhTh11] V. Lakshminarayanan, A.K. Ghatak, K. Thyagarajan, *Lagrangian Optics*, Springer Netherlands, 2011.

[Laug89] D. Laugwitz, Definite values of infinite sums: aspects of the foundations of infinitesimal analysis around 1820, Arch. Hist. Exact Sci. 39 (3): 195-245, 1989.

[LL-Giordano16] Lecke, A., Luperi Baglini, L., Giordano, P., The classical theory of calculus of variations for generalized functions. Advances in Nonlinear Analysis 2017, DOI: 10.1515/anona-2017-0150

[LeMoSj91] E. Lerman, R. Montgomery, R. Sjamaar, Examples of singular reduction. Symplectic geometry, London Math. Soc. Lecture Note Ser., 192, Cambridge Univ. Press, Cambridge, 1993.

[Lew57] H. Lewy, An example of a smooth linear partial differential equation without a solution, Ann. of Math. 66, pp. 155-158, 1957.

[LiHuZh14] F. Li, Q. Hua, S. Zhang, New periodic solutions of singular Hamiltonian systems with fixed energies, *J Inequal Appl* 2014: 400. https://doi.org/10.1186/1029-242X-2014-400

[LigRob75] A.H. Lightstone, A. Robinson, *Nonarchimedean Fields and Asymptotic Expansions*, North-Holland, Amsterdam, 1975.

[Lim89] C.C. Lim, On singular Hamiltonians: the existence of quasi-periodic solutions and nonlinear stability, Bull. Amer. Math. Soc. (N.S.) 20(1) 35-40, 1989.

[L-Giordano16a] Luperi Baglini, L., Giordano, P., The Grothendieck topos of generalized smooth functions II: ODE, Preprint see http://www.mat.univie.ac.at/~giordap7/ToposII.pdf.

[Mar68] Marsden, J.E., Generalized Hamiltonian mechanics, Arch. Rat. Mech. Anal., 28(4), pp. 323-361, 1968.

[Mar69] Marsden, J.E., Non-smooth geodesic flows and classical mechanics, Canad. Math. Bull., 12, pp. 209-212, 1969.

[Mat18] https://fr.mathworks.com/help/matlab/numerical-integration-and-differential-equations.html

[MTA-Giordano] Mukhammadiev, A., Tiwari, D., Apaaboah, G., Giordano, P., Supremum, Infimum and and hyperlimits of Colombeau generalized numbers. Article in preparation, 2020. See http://www.mat.univie.ac.at/~giordap7/Hyperlim.pdf.

[M-Giordano20] Mukhammadiev, A., Giordano, P., A Fourier transform for all generalized functions. Article in preparation, 2020.

[Ned-Pil06] M. Nedeljkov and S. Pilipović, Generalized function algebras and PDEs with singularities. A survey. *Zb. Rad. (Beogr.)* 11(19), pp. 61-120, 2006.

[NePiSc98] M. Nedeljkov, S. Pilipović and D. Scarpalezos, *Linear Theory of Colombeau's Generalized Functions*, Addison Wesley, Longman, 1998.

[Obe92] M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*. Pitman Research Notes in Mathematics Series, 259. Longman, Harlow, 1992.

[Or68] Ortega, J.M, The Newton-Kantorovich theorem, The American Mathematical Monthly 75.6 (1968): 658-660.

[Oz04] Özban, A.Y., Some new variants of Newton method, Applied Mathematics Letters 17.6 (2004): 677-682.

[Pil94] S. Pilipović, *Colombeau's generalized functions and the pseudo-differential calculus*, Lecture Notes in Mathematics, Sci., Univ. Tokyo, 1994.

[SeGo01] S. Pascal, X. Gourdon, Newton's method and high order iterations, Numbers Comput., 2001.

[SaNa13] Sakurai, J.J., Napolitano, J., *Modern Quantum Mechanics*. Pearson Education India, 2nd New edition edizione, 2013.

[SiJe13] S. Sieniutycz, J. Jezowski, *Energy Optimization in Process Systems and Fuel Cells*. Elsevier, 2013.

[SteVic06] R. Steinbauer and J.A. Vickers, The use of generalized functions and distributions in general relativity, Class. Quantum Grav.23(10), R91-R114, (2006).

[Tan93] K. Tanaka, A Prescribed Energy Problem for a Singular Hamiltonian System with a Weak Force, *Journal of Functional Analysis* 113(2), pp 351-390, 1993.

[T-Giordano20] Tiwari, D., Giordano, P., Hyperseries of Colombeau generalized numbers. Preprint, see http://www.mat.univie.ac.at/~giordap7/Hyperseries.pdf.

[T-Giordano] Tiwari, D., Giordano, P., Hyper-power series and analytic generalized smooth functions. Article in preparation.

[Tod11] Todorov, T.D., An axiomatic approach to the non-linear theory of generalized functions and consistency of Laplace transforms. Integral Transforms and Special Functions, Volume 22, Is- sue 9, September 2011, p. 695-708.

[Tod13] Todorov, T.D., Algebraic Approach to Colombeau Theory. San Paulo Journal of Mathematical Sciences, 7 (2013), no. 2, 127-142.

[Tod15] Todor D. Todorov, Steady-State Solutions in an Algebra of Generalized Functions: Lightning, Lightning Rods and Superconductivity, Novi Sad Journal of Mathematics, Volume 45, Number 1, 2015.

[UgFe11] A.C. Ugural, S.K. Fenster, *Advanced mechanics of materials and elasticity*, Prentice Hall International Series in the Physical and Chemical Engineering Sciences, Prentice Hall; 5 edition, 2011.

[VKRKPW] J.R. Vázquez de Aldana, N.J. Kylstra, L. Roso, P.L. Knight, A. Patel, R.A. Worthington, Atoms interacting with intense, high-frequency laser pulses: Effect of the magnetic-field component on atomic stabilization, Phys. Rev. A 64, 013411, 2001.

[Zak96] B.N. Zakharév, New situation in quantum mechanics (possibilities of controlling spectra, scattering and decay), Copocobic Educational Magazine, No 7, pp 81-87, 1996 (in Russian).

Curriculum vitae of P. Giordano

Name:	Giordano, Paolo
ORCID:	0000-0001-7653-1017
Nationality:	Italian
Date of birth:	27 March 1966
Web site:	www.mat.univie.ac.at/~giordap7/
Affiliation:	Wolfgang Pauli Institute, Oskar-Morgenstern-Platz 1 1090 Wien
Email:	paolo.giordano@univie.ac.at

Present research interests

- Non-Archimedean geometry and analysis
- Nonlinear theories of generalized functions
- Foundation of differential geometry
- Mathematical theories of complex systems

Education

- *Habilitation*, University of Vienna, Austria, 2019.
- *Rheinischen Friedrich-Wilhelms-Universität Bonn* (DE), Ph.D. in Mathematics, 2009.
- *Università degli Studi di Milano* (IT), M.Sc. in Mathematics, 1997.

Academic experiences

Selected research activities as principal investigator or co-director (selection)

- July 2020 - present: project leader of FWF stand alone research project *Functional analysis of infinite bounded operators*. Co-applicants and collaborators of the project are Prof. M. Kunzinger and Prof. H. Vernaeve (Dep. of Mathematics, University of Ghent, Belgium); 407'000 Euro.
- August 2017 - present: project leader of FWF stand alone research project *Hyperfinite methods for generalized smooth functions*, Wolfgang Pauli Institute, Vienna. Co-applicant and collaborator of the project is Prof. M. Kunzinger; 397'000 Euro.
- December 2012 - May 2017: project leader of FWF stand alone research project *Analysis and Geometry based on generalized numbers*, Dep. of Mathematics, University of Vienna. Co-applicant and collaborator of the project is Prof. M. Kunzinger; 321'000 Euro.

- June 2013 - May 2016: project leader of FWF stand alone research project *Non-Archimedean Geometry and Analysis*, Dep. of Mathematics, University of Vienna (AT). Co-applicants of the project are Prof. M. Kunzinger and Prof. V. Benci; 349'000 Euro.
- October 2010 - September 2012: project leader of the research project *Nilpotent Infinitesimals and Generalized Functions*, Dep. of Mathematics, University of Vienna, supported by an FWF Lise Meitner grant. Co-applicant of the project: Prof. M. Kunzinger; 115'200 Euro.

5 selected invited lectures

1. Invited talk at Institute for Scientific Interchange (ISI), "MaTryCS - A mathematical theory of complex systems", 2016.
2. Invited plenary lecture at the conference "Algebra, Geometry and Mathematical Physics", Brno, Czech Republic - "Infinitesimal without Logic", 2012.
3. Invited talk at the University of Pisa - "Generalized smooth functions", 2015.
4. Invited opening talk at the workshop "Workshop on diffeologies etc", Aix en Provence, France - "Theory of infinitely near points in smooth manifolds: the Fermat functor", 2014.
5. Invited speaker at the Interdiziplinäre Zentrum für Komplexe Systeme (IZKS, Bonn, Germany) - "Dynamics of cities: A mathematical planning tool for shopping malls", 2009.

Supervision of Ph.D. students and postdoctoral fellows

- 2018 - current: A. Mukhammadiev, D. Tiwari, Ph.D. students, University of Vienna, AT.
- 2012 - 2016: L. Luperi Baglini and E. Wu, post-docs, University of Vienna, AT.
- 2006 - 2009: G.L. Ciampaglia, M. Esmaeili, Ph.D. students, University of Italian Switzerland, CH.

Reviewing activities

I am reviewer for: Acta Mathematica, American Mathematical Monthly, Advances in Complex Systems, Environmental modelling and software, Physics Letters A, Topology proceedings, Commentationes Mathematicae Universitatis Carolinae, Arabian Journal of Mathematics.

International research partner (selection)

Vieri Benci, University of Pisa, Italy; Sergio Albeverio, University of Bonn, Germany; Hans Vernaeve, University of Ghent, Belgium; Alberto Vancheri, SUPSI, Switzerland.

Main areas of research and selected results

- *Theory of Fermat reals*: we developed a new theory of nilpotent infinitesimals and its applications to infinite-dimensional spaces.
- *Theory of Generalized Smooth Functions*: a new theory of generalized functions resulting in the closure with respect to composition, a better behavior on unbounded sets and new general existence results.

- *Theory of Interaction Spaces*: a new unifying theory of complex systems which includes several types of complex systems models.
- *Colombeau theory*: we unified several different Colombeau-like algebras into a single general abstract notion having the same simplicity of the special algebra.

Most important publications

For the links to these publications and the complete list, see:
https://www.mat.univie.ac.at/~giordap7/#publications

1. Giordano P., Kunzinger M., A convenient notion of compact set for generalized functions. Proceedings of the Edinburgh Mathematical Society, Volume 61, Issue 1, February 2018, pp. 57-92. DOI: 10.1017/S0013091516000559
2. Lecke A., Luperi Baglini L., Giordano P., The classical theory of calculus of variations for generalized functions. Advances in Nonlinear Analysis, Vol. 8, issue 1, 2017. DOI: 10.1515/anona-2017-0150
3. Giordano P., Kunzinger M., Inverse Function Theorems for Generalized Smooth Functions. Chapter in "Generalized Functions and Fourier Analysis", Volume 260 of the series Operator Theory: Advances and Applications pp 95-114. DOI: 10.1007/978-3-319-51911-1_7
4. Giordano P., Wu E., Calculus in the ring of Fermat reals. Part I: Integral calculus. Advances in Mathematics 289 (2016) 888–927. DOI: 10.1016/j.aim.2015.11.021
5. Giordano P., Kunzinger M., Vernaeve H., Strongly internal sets and generalized smooth functions. Journal of Mathematical Analysis and Applications, volume 422, issue 1, 2015, pp. 56-71. DOI: 10.1016/j.jmaa.2014.08.03
6. Vancheri A., Giordano P., Andrey D., Fuzzy logic based modeling of traffic flows induced by regional shopping malls. Advances in Complex Systems Vol. 17, N. 3 & 4, 2014, (39 pages). DOI: 10.1142/S0219525914500179
7. Giordano P., Kunzinger M., Topological and algebraic structures on the ring of Fermat reals. Israel Journal of Mathematics, January 2013, Volume 193, Issue 1, pp. 459-505. DOI: 10.1007/s11856-012-0079-z
8. Giordano P., The ring of fermat reals, Advances in Mathematics 225 (2010), pp. 2050-2075. DOI: 10.1016/j.aim.2010.04.010
9. Giordano P., Infinitesimals without logic, Russian Journal of Mathematical Physics, 17(2), pp.159-191, 2010. DOI: 10.1134/S1061920810020032
10. Giordano P., Fermat-Reyes method in the ring of Fermat reals. Advances in Mathematics 228, pp. 862-893, 2011. DOI: 10.1016/j.aim.2011.06.008

Michael Kunzinger

Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
☎ *+43 1 4277 50681*
✉ *michael.kunzinger@univie.ac.at*
http://www.mat.univie.ac.at/∼mike/, ORCID: 0000-0002-7113-0588

Curriculum Vitæ

Main Areas of Research

Functional analysis, generalized functions, semi-Riemannian geometry, mathematical physics, Lie group analysis of partial differential equations.

Education

2001 **Habilitation**, *University of Vienna*, Austria.

1996 **Ph.D**, *University of Vienna*, Austria.

1993 **Mag.rer.nat. (Mathematics)**, *University of Vienna*, Austria.

2002 **Mag.rer.nat. (Physics)**, *University of Vienna*, Austria.

Employment

Since 2019 **Full Professor**, *University of Vienna*, Austria.

2001–2019 **Associate Professor**, *University of Vienna*, Austria.

06/01–09/01 **Visiting Scientist**, *University of Southampton*, GB.

1996–2001 **Assistant Professor**, *University of Vienna*, Austria.

1995–1999 **Research Assistant**, *University of Vienna*, Austria, FWF research grant P-12023MAT, 'Distributional Methods in Einstein's Theory of Gravitation'.

1994–1995 **Assistant Professor**, *University of Vienna*, Austria.

Supervised PhD Students

2018 **Melanie Graf**, *Singularity theorems and rigidity in Lorentzian geometry*.

2016 **Alexander Lecke**, *Non-smooth Lorentzian Geometry and Causality Theory*.

2015 **Milena Stojković**, *Causality Theory for $C^{1,1}$-metrics*.

2010 **Eduard Nigsch**, *A Nonlinear Theory of Tensor Distributions on Riemannian Manifolds*.

2009 **Jasmin Sahbegović**, *Short-Time Fourier Transform and Modulation Spaces in Algebras of Generalized Functions*.

2008 **Sanja Konjik**, *Group Analysis and Variational Symmetries for Non-Smooth Problems*.

2006 **Eberhard Mayerhofer**, *The wave equation on singular space-times*.

5 Selected Invited Lectures

Jul 26, 2017 **Singularity theorems in regularity $C^{1,1}$**, *Geometry and Relativity*, Erwin Schrödinger Institue, Vienna.

Sep 9, 2016 **International Conference on Generalized Functions GF2016**, *Generalized functions as set-theoretical maps*, University of Dubrovnik.

Dez 6, 2015 **Workshop on generalized functions**, *Low-regularity semi-Riemannian geometry and the singularity theorems of General Relativity*, University of Zagreb.

Sep 6, 2012 **PDEMTA, Topics in PDE, Microlocal and Time-frequency Analysis**, *Abstract regularity theory*, University of Novi Sad.

Jan 6, 2006 **1st International Workshop on Mathematical Sciences: Mathematical Analysis and Applications.**, *Nonlinear distributional geometry*, Sogang University, Seoul.

Academic Prizes/Awards

2004 **START-prize**, *Austrian Science Fund FWF*.

2003 **ÖMG-Prize**, *Austrian Mathematical Society*.

Most important peer review activities, editorships and/or memberships in academic organisations

President, *International Association for Generalized Functions*.

Editor, *Publications de l'Institut Mathematique*, Belgrade.

2008-2016 **Member of the Austrian Academy of Sciences**.

Selected Research Projects

2017–2021 **P30233**, *Regularity Theory in Algebras of Generalized Functions*, Austrian Science Fund.

2016–2021 **P28770**, *Singularity Theorems and Comparison Geometry*, Austrian Science Fund.

2008-2011 **P20525**, *Global Analysis in Algebras of Generalized Functions*, Austrian Science Fund.

2005-2011 **START-Project Y237**, *Nonlinear Distributional Geometry*, Austrian Science Fund.

International Cooperation Partners (in the last 5 years)

James D. E. Grant, *University of Surrey*, GB.

James A. Vickers, *University of Southampton*, GB.

Hans Vernaeve, *University of Ghent*, Belgium.

Darko Mitrovic, *University of Montenegro*, Montenegro.

Clemens Sämann, *University of Toronto*, Canada.

Main areas of research and selected results

Symmetry Analysis of Differential Equations: New classification methods (conditional equivalence groups, normalized classes), study of conservation laws, characteristics and potential symmetries, formal compatibility, generalized conditional symmetries, singular reduction modules.

Algebras of Generalized Functions: Geometrization of the Theory, diffeomorphism invariant embeddings of distributions, applications to singular spacetimes in General Relativity, microlocal analysis, development of a theory of generalized smooth functions.

Mathematical General Relativity: Study of highly singular (distributional) spacetimes (pp-waves), extension of the singularity theorems of Hawking and Penrose to spacetimes of regularity $C^{1,1}$, study of singular spacetimes via new methods of metric geometry.

Differential Geometry: Nonlinear distributional geometry, low regularity pseudo-Riemannian geometry: new comparison methods, development of a new metric theory of Lorentzian geometry in metric spaces: Lorentzian length spaces, study of synthetic curvature bounds in this setting and applications to extendability of spacetimes.

Partial Differential Equations: Study of highly singular PDEs in the framework of algebras of generalized functions, kinetic theory (Vlasov-Klein-Gordon), degenerate parabolic equations on compact Riemannian manifolds.

Locally convex spaces: Study of weak barrelledness properties, applications of the theory of vector valued distributions to Cauchy-Dirichlet problems.

10 Most Important Publications

For the complete list of publications, see `https://www.mat.univie.ac.at/~mike/publications.php`

Books

1. **Barrelledness, Baire-like- and (LF)-Spaces**
   Pitman Research Notes in Mathematics Vol. 298,
   Longman, Harlow 1993.
2. with E. Farkas, M. Grosser und R. Steinbauer,
   On the Foundations of Nonlinear Generalized Functions I,II
   Mem. Amer. Math. Soc. 153, No. 729, 2001.
3. with M. Grosser, M. Oberguggenberger und R. Steinbauer
   Geometric Theory of Generalized Functions
   with Applications to General Relativity,
   Mathematics and its Applications, Kluwer 2001.

Journal Publications

4. with M. Oberguggenberger,
   Characterization of Colombeau generalized functions by their pointvalues
   Math. Nachr. 203, 147–157, 1999.
5. with M. Grosser, R. Steinbauer und J. Vickers
   A global theory of algebras of generalized functions
   Advances in Math. 166, No. 1, 50–72, 2002.
   http://arxiv.org/abs/math.FA/9912216
6. with G. Rein, R. Steinbauer, and G. Teschl
   Global weak solutions of the relativistic Vlasov-Klein-Gordon System
   Comm. Math. Phys., 238 (1-2), 367-378, 2003.
   http://arxiv.org/abs/math.AP/0209303
7. with P. Giordano
   Topological and algebraic structures on the ring of Fermat reals
   Israel J. Math., 193 no. 1, 459–505, 2013.
   http://arxiv.org/abs/1104.1492
8. with S. Dave,
   Singularity structures for noncommutative spaces
   Trans. Amer. Math. Soc., 367, no. 1, 251-273, 2015.
   http://arxiv.org/abs/1111.6570
9. with V. M. Boyko and R. Popovych
   Singular reduction modules of differential equations
   J. Math. Phys. 57, 101503 (2016)
   http://arxiv.org/abs/1201.3223
10. with J.D.E. Grant, M. Graf, and R. Steinbauer
    The Hawking-Penrose singularity theorem for $C^{1,1}$-Lorentzian metrics
    Comm. Math. Phys., 360 (2018), no. 3, 1009-1042.
    https://arxiv.org/abs/1706.08426

PERSONAL INFORMATION

Aleksandr Bryzgalov

6-225, Belkinskaya St., Obninsk, Kaluga reg., 249034, Russia

+7 9038156540

sandro185@mail.ru

https://www.researchgate.net/profile/Alexander_Bryzgalov

Sex Male | Date of birth 12/01/1985 | Nationality Russian

RESEARCH INTERESTS

▪ Non-linear differential equations; Exactly solvable models; Numerical methods; Quantum Mechanics; Wigner's function; Time-dependent Moyal's equation; Time-dependent Schrodinger equation; Fourier transformation.

ONGOING PROJECTS

- **Fast Fourier Transform in Elliptic Coordinates**
- **Wigner function in polar coordinates (co-author Bondar D.)**

FINISHED PROJECTS

- **The time dynamics of electron wave functions of a two-dimensional quantum ring (co-author Karmanov F.I.)**
- **A double-well potential model and its application (co-author Karmanov F.I.)**
- **Three-dimensional axially symmetrical exactly soluble model of quantum ring (co-author Karmanov F.I.)**

EDUCATION AND TRAINING

2008 - 2012 Candidate of Physical and Mathematical Science - Thesis title: 'The method of splitting into physical processes in the non-stationary problem of electron tunnelling through the quantum rings', Mathematical modelling

National Research Nuclear University MEPhI, Moscow, Russia

2006 - 2008 Master of Physics

National Research Nuclear University MEPhI Obninsk branch, Obninsk, Russia

2002 - 2006 Bachelor of Physics

National Research Nuclear University MEPhI Obninsk branch, Obninsk, Russia

ACADEMIC EXPERIENCE

2013 - Present moment — Associate Professor

National Research Nuclear University MEPhI Obninsk branch

(1, Studgorodok St., 249040 Obninsk, Kaluga reg., Russia)

- practice of courses concerning Numerical modelling, Numerical methods and PDE's, Numerical methods, Linear and Non-linear Equations of Mathematical Physics, Computational Physics
- lectures, practice and lab work of courses concerning Physics and Theoretical Physics, General Physics, Computational Physics, Quantum Chemistry and Quantum Mechanics,
- curriculum development; development of guidelines for laboratory work

2012 - 2013 — Physics Lecturer

"Malta Crown", Malta ("Portoscala" ,Trig II-Bahhara, Marsascala ZBR 10,Malta)

▪ lectures, practice and lab work of General Physics

▪ organization of supplies of equipment for physical and chemical laboratories

2007 - 2010 (one year in total) — Associate Researcher

ENEA Research centre FPN FIS/NUC (Via Martiri di Monte Sole,4,40129 Bologna, Italy)

PEER REVIEWED PUBLICATIONS (SINCE 2014)

1) A. A. Bryzgalov *Integral Relations for Bessel Functions and Analytical Solutions for Fourier Transform in Elliptic Coordinates* //WSEAS Transactions on Mathematics, ISSN / E-ISSN: 1109-2769 / 2224-2880, Volume 17, 2018, Art. #26, pp. 205-212
https://aip.scitation.org/doi/pdf/10.1063/1.5045413

2) A. A. .Bryzgalov *Fourier Transform in Elliptic Coordinates: Case of Axial Symmetry* // Mathematical Methods and Computational Techniques in Science and Engineering II, AIP Conference Proceedings 1982, 020007 (2018); doi: 10.1063/1.5045413
http://www.wseas.org/multimedia/journals/mathematics/2018/a465906-032.php

10 MOST IMPORTANT PUBLICATIONS

1) A. A. Bryzgalov *Integral Relations for Bessel Functions and Analytical Solutions for Fourier Transform in Elliptic Coordinates* //WSEAS Transactions on Mathematics, ISSN / E-ISSN: 1109-2769 / 2224-2880, Volume 17, 2018, Art. #26, pp. 205-212
https://aip.scitation.org/doi/pdf/10.1063/1.5045413

2) A. A.Bryzgalov *Fourier Transform in Elliptic Coordinates: Case of Axial Symmetry* // Mathematical Methods and Computational Techniques in Science and Engineering II, AIP Conference Proceedings 1982, 020007 (2018); doi: 10.1063/1.5045413
http://www.wseas.org/multimedia/journals/mathematics/2018/a465906-032.php

3) E. V. Antropova A. A. Bryzgalov F. I. Karmanov *The energy levels and eigen wave functions of electrons in quantum rings in the magnetic field*// Vestnik Samarskogo Gosudarstvennogo Universiteta. Seriya fiz.-mat. nauki.- 11/2012; 1 (30):326-333 [Russian]
http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=vsgtu&paperid=1171&option_lang=eng
4) Bryzgalov A. A. and Karmanov F. I. *Method for splitting into physical processes in the problem on the time dynamics of electron wave functions of a two-dimensional quantum ring* //Mathematical Models and Computer Simulations 2011,Volume 3, Number 1, 25-34.
https://link.springer.com/article/10.1134/S2070048211010029
5) Bryzgalov A. A., Karmanov F. I. *Time dynamics of the electron wave functions of the 3D quantum ring in the alternating magnetic field* // Vestnik Samarskogo Gosudarstvennogo Universiteta. Seriya fiz.-mat. nauki. – 2011. – #1(22). – Pp. 291-296. [Russian]
http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=vsgtu&paperid=867&option_lang=eng
6) Bryzgalov A. A., Karmanov F. I. *Control of electron tunneling in the system of two concentric quantum rings using magnetic field* // Vychislitelnye metody i programmirovanie 2011. - 12. – Pp. 262-274. [Russian] http://num-meth.srcc.msu.ru/zhurnal/tom_2011/pdf/v12r132.pdf
7) Bryzgalov A. A., Karmanov F. I. *2D quantum ring: magnetic field influence to the time dynamics of the electron wave functions* //Izvestiya Vysshikh uchebnykh zavedenii. Fizika. 2010. #3/2. 31-36. [Russian] https://elibrary.ru/item.asp?id=16260826
8) Bryzgalov A. A., Karmanov F. I*. Construction of the basis functions for the calculations of method of splitting into physical interactions in the finite domain* // Vychislitelnye metody i programmirovanie 2010. 11. 289-298. [Russian]
http://num-meth.srcc.msu.ru/zhurnal/tom_2010/pdf/v11r134.pdf
9) Bryzgalov A. A., Natalenko A. A., De Rosa F., Tirini S., Voukelatou N. *Using ASTEC code to model fission products and other elements releases through reactor cooling system during the LWR severe accident* // Izvestiya Vysshikh uchebnykh zavedenii. Yadernaya energetica. 2008. #4. Pp. 3-14. [Russian] https://nuclear-power-engineering.ru/pdf/2008/04/2008-04-full-issue.pdf
10) Bryzgalov A. A., Natalenko A. A., De Rosa F., Tirini S., Voukelatou N. *Thermohydraulics simulation for PHEBUS Containment*// Izvestiya Vysshikh uchebnykh zavedenii. Yadernaya energetica. 2008. #4. Pp. 76-85. [Russian]
https://nuclear-power-engineering.ru/pdf/2008/04/2008-04-full-issue.pdf

Annex 4: collaboration letters

The following is the collaboration letter from Prof. H. Schichl, the only one stated as essential in the project description.

universität
wien

Prof. Dr. Hermann Schichl

Fakultät für Mathematik
Oskar-Morgenstern-Platz 1
A-1090 Wien
Austria

T+43-1-4277-506 66
F+43-1-4277-850666
hermann.schichl@univie.ac.at
http://www.mat.univie.ac.at/~herman

Vienna, October 21, 2019

Letter of Support for
the project proposal
of Dr. Paolo Giordano

Dear Madams or Sirs,

this is to confirm that I am familiar with the FWF-research proposal

Applications of Generalized Smooth Functions

by Dr. Paolo Giordano. As is stated in his application, he intends to collaborate with our research group on questions of the numerical and graphical representation of generalized smooth functions. I believe that this is a highly promising direction of research and will be happy to interact with the members of his team on these questions.

Sincerely,

Hermann Schichl