

# THE CLASSICAL THEORY OF CALCULUS OF VARIATIONS FOR GENERALIZED FUNCTIONS

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ABSTRACT. We present an extension of the classical theory of calculus of variations to generalized functions. The framework is the category of generalized smooth functions, which includes Schwartz distributions while sharing many nonlinear properties with ordinary smooth functions. We prove full connections between extremals and Euler-Lagrange equations, classical necessary and sufficient conditions to have a minimizer, the necessary Legendre condition, Jacobi's theorem on conjugate points and Noether's theorem. We close with an application to low regularity Riemannian geometry.

## 1. INTRODUCTION AND MOTIVATIONS

Singular problems in the calculus of variations have longly been studied both in mathematics and in relevant applications (see, e.g., [18, 7, 43, 26] and references therein). In this paper, we introduce an approach to variational problems involving singularities that allows the extension of the classical theory with very natural statements and proofs. We are interested in extremizing functionals which are either distributional themselves or whose set of extremals includes generalized functions. Clearly, distribution theory, being a linear theory, has certain difficulties when nonlinear problems are in play.

To overcome this type of problems, we are going to use the category of generalized smooth functions, see [11, 12, 14, 15]. This theory seems to be a good candidate, since it is an extension of classical distribution theory which allows the modeling of nonlinear singular problems, while at the same time sharing many nonlinear properties with ordinary smooth functions, like the closure with respect to composition and several non trivial classical theorems of the calculus. One could describe generalized smooth functions as a methodological restoration of Cauchy-Dirac's original conception of generalized function, see [8, 29, 25]. In essence, the idea of Cauchy and Dirac (but also of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view generalized functions as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, cf. [25].

In the present work, the foundation of the calculus of variations is set for functionals defined by arbitrary generalized functions. This in particular applies to any Schwartz distribution and any Colombeau generalized function (see e.g. [5, 6]), and hence justifies the title of the present paper.

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For example, during the last years, the study of low regular Riemannian and Lorentzian geometry was intensified and made a huge amount of progress (cf. [28, 27, 41, 34, 30, 39]). It was shown that the exponential map is a bi-Lipschitz homeomorphism when metrics  $g \in \mathcal{C}^{1,1}$  are considered, [35, 28], or that Hawking's singularity theorem still holds when  $g \in \mathcal{C}^{1,1}$ , see [27]. However, calculus of variations in the classical sense may cease to hold when metrics with  $\mathcal{C}^{1,1}$  regularity, or below, are considered [20, 31]. This motivates the search for an alternative. In fact, if  $p, q \in \mathbb{R}^d$  and  $\Omega(p, q)$  denotes the set of all Lipschitz continuous curves connecting  $p$  and  $q$ , the natural question about what curves  $\gamma \in \Omega(p, q)$  realize the minimal  $g$ -length leads to the corresponding geodesic equation, but the Jacobi equation is not rigorously defined. To be more precise: The Riemannian curvature tensor exists only as an  $L_{\text{loc}}^\infty$  function on  $\mathbb{R}^d$  and is evaluated along  $\gamma$ . However, the image  $\text{Im}(\gamma)$  of  $\gamma$  has Lebesgue-measure zero, if  $d > 1$ . Thus we cannot state the Jacobi equations properly.

In order to present a possible way out of the aforementioned problems, the singular metric  $g$  is embedded as a generalized smooth function. In this way, the embedding  $\iota(g)$  has derivatives of all orders, valued in a suitable non-Archimedean ring<sup>1</sup>  ${}^{\rho}\widetilde{\mathbb{R}} \supseteq \mathbb{R}$ . Despite the total disconnectedness of the ground ring, the final class of smooth functions on  ${}^{\rho}\widetilde{\mathbb{R}}$  behaves very closely to that of standard smooth functions; this is a typical step one can recognize in other topics such as analytic space theory, [4, 37], and non-Archimedean analysis, see e.g. [17] and references therein. We then apply our extended calculus of variations to the generalized Riemannian space  $({}^{\rho}\widetilde{\mathbb{R}}^d, \iota(g))$ , and sketch a way to translate the given problem into the language of generalized smooth functions, solve it there, and translate it back to the standard Riemannian space  $(\mathbb{R}^d, g)$ . Clearly, the process of embedding the singular metric  $g$  using  $\iota(g)$  introduces infinitesimal differences. This is typical in a non-Archimedean setting, but the notion of *standard part* comes to help: if  $x \in {}^{\rho}\widetilde{\mathbb{R}}$  is infinitely close to a standard real number  $s$ , i.e.  $|x - s| \leq r$  for all  $r \in \mathbb{R}_{>0}$ , then the standard part of  $x$  is exactly  $s$ . We then show that (assuming that  $(\mathbb{R}^d, g)$  is geodesically complete) the standard part of the minimal length in the sense of generalized smooth functions is the minimal length in the classical sense, and give a simple way to check if a given (classical) geodesic is a minimizer of the length functional or not. In this way, the framework of generalized smooth functions is presented as a method to solve standard problems rather than a proposal to switch into a new setting.

The structure of the present paper is as follows. We start with an introduction into the setting of generalized smooth functions and give basic notions concerning generalized smooth functions and their calculus that are needed for the calculus of variations (Section 2). The paper is self-contained in the sense that it contains all the statements required for the proofs of calculus of variations we are going to present. If proofs of preliminaries are omitted, we clearly give references to where they can be found. Therefore, to understand this paper, only a basic knowledge of distribution theory is needed.

In Section 3, we obtain some preliminary lemmas regarding the calculus of variations with generalized smooth functions. The first variation and the notion of critical point will be defined and studied in section 4. We prove the fundamental lemma of calculus of variations and the full connection between critical points of a given functional and solutions of the corresponding Euler-Lagrange equation. In section 5, we study the second variation and define the notion of local minimizer. We also extend to generalized functions classical necessary and sufficient conditions to have a minimizer, and we give a proof of the Legendre condition. In Section

<sup>1</sup>I.e. a ring that contains infinitesimal and infinite numbers.

6, we introduce the notion of Jacobi field and extend to generalized functions the definition of conjugate points, so as to prove the corresponding Jacobi theorem. In Section 7, we extend the classical Noether's theorem. We close with an application to  $\mathcal{C}^{1,1}$  Riemannian geometry in Section 8.

Note that the work [26] already established the calculus of variations in the setting of Colombeau generalized functions by using a comparable methodological approach. Indeed, generalized smooth functions are related to Colombeau generalized functions, and one could say that the former is a minimal extension of the latter so as to get more general domains for generalized functions and hence the closure with respect to composition and a better behaviour on unbounded sets. However, there are some conceptual advantages in our approach.

- (i) Whereas generalized smooth functions are closed with respect to composition, Colombeau generalized functions are not. This forces [26] to consider only functionals defined using compactly supported Colombeau generalized functions, i.e. functions assuming only finite values, or tempered generalized function.
- (ii) The authors of [26] are forced to consider the so called compactly supported points  $c(\Omega)$  (i.e. finite points in  $\Omega \subseteq \mathbb{R}^n$ ), where the setting of generalized smooth functions gives the possibility to consider more natural domains like the interval  $[a, b] \subseteq {}^\rho\tilde{\mathbb{R}}$ . This leads us to extend in a natural way the statements of classical results of calculus of variations. Moreover, all our results still hold when we take as  $a, b \in {}^\rho\tilde{\mathbb{R}}$  two infinite numbers such that  $a < b$ , or as boundary points two unbounded points  $p, q \in {}^\rho\tilde{\mathbb{R}}^d$ .
- (iii) Furthermore, the theory of generalized smooth functions was developed to be very user friendly, in the sense that one can avoid cumbersome “ $\varepsilon$ -wise” proofs quite often, whereas the proofs in [26] frequently use this technique. Thus, one could say that some of the proofs based on generalized smooth functions are more “intrinsic” and close to the classical proofs in a standard smooth setting. This allows a smoother approach to this new framework.
- (iv) The setting of generalized smooth functions depends on a fixed infinitesimal net  $(\rho_\varepsilon)_{\varepsilon \in (0,1]} \downarrow 0$ , whereas the Colombeau setting considers only  $\rho_\varepsilon = \varepsilon$ . This added degree of freedom allows to solve singular differential equations that are unsolvable in the classical Colombeau setting and to prove a more general Jacobi theorem on conjugate points.
- (v) In [26] only the notion of global minimizer is defined, whereas we define the notion of local minimizer as in [10] using a natural topology in space of generalized smooth curves.
- (vi) We obtain more classical results like the Legendre condition, and the classical results about Jacobi fields and conjugate points.
- (vii) In addition, note that the Colombeau generalized functions can be embedded into generalized smooth functions. Thus our approach is a natural extension of [26].

## 2. BASIC NOTIONS

*The new ring of scalars.* In this work,  $I$  denotes the interval  $(0, 1] \subseteq \mathbb{R}$  and we will always use the variable  $\varepsilon$  for elements of  $I$ ; we also denote  $\varepsilon$ -dependent nets  $x \in \mathbb{R}^I$  simply by  $(x_\varepsilon)$ . By  $\mathbb{N}$  we denote the set of natural numbers, including zero.

We start by defining the new simple non-Archimedean ring of scalars that extends the real field  $\mathbb{R}$ . The entire theory is constructive to a high degree, e.g. no ultrafilter or non-standard method is used. For all the proofs of results in this section, see [15, 11, 14].

**Definition 1.** Let  $\rho = (\rho_\varepsilon) \in \mathbb{R}^I$  be a net such that  $\lim_{\varepsilon \rightarrow 0} \rho_\varepsilon = 0^+$ , then

- (i)  $\mathcal{I}(\rho) := \{(\rho_\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\}$  is called the *asymptotic gauge* generated by  $\rho$ . The net  $\rho$  is called a *gauge*.
- (ii) If  $\mathcal{P}(\varepsilon)$  is a property of  $\varepsilon \in I$ , we use the notation  $\forall^0 \varepsilon : \mathcal{P}(\varepsilon)$  to denote  $\exists \varepsilon_0 \in I \forall \varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$ . We can read  $\forall^0 \varepsilon$  as *for  $\varepsilon$  small*.
- (iii) We say that a net  $(x_\varepsilon) \in \mathbb{R}^I$  is  $\rho$ -moderate, and we write  $(x_\varepsilon) \in \mathbb{R}_\rho$  if  $\exists (J_\varepsilon) \in \mathcal{I}(\rho) : x_\varepsilon = O(J_\varepsilon)$  as  $\varepsilon \rightarrow 0^+$ .
- (iv) Let  $(x_\varepsilon), (y_\varepsilon) \in \mathbb{R}^I$ , then we say that  $(x_\varepsilon) \sim_\rho (y_\varepsilon)$  if  $\forall (J_\varepsilon) \in \mathcal{I}(\rho) : x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1})$  as  $\varepsilon \rightarrow 0^+$ . This is a congruence relation on the ring  $\mathbb{R}_\rho$  of moderate nets with respect to pointwise operations, and we can hence define

$${}^\rho \widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho,$$

which we call *Robinson-Colombeau ring of generalized numbers*, [38, 5, 6]. We denote the equivalence class  $\tilde{x} \in {}^\rho \widetilde{\mathbb{R}}$  simply by  $x := [x_\varepsilon] := [(x_\varepsilon)]_\sim \in {}^\rho \widetilde{\mathbb{R}}$ .

In the following,  $\rho$  will always denote a net as in Def. 1. The infinitesimal  $\rho$  can be chosen depending on the class of differential equations we need to solve for the generalized functions we are going to introduce, see [16]. For motivations concerning the naturality of  ${}^\rho \widetilde{\mathbb{R}}$ , see [14].

We can also define an order relation on  ${}^\rho \widetilde{\mathbb{R}}$  by saying that  $[x_\varepsilon] \leq [y_\varepsilon]$  if there exists  $(z_\varepsilon) \in \mathbb{R}^I$  such that  $(z_\varepsilon) \sim_\rho 0$  (we then say that  $(z_\varepsilon)$  is  $\rho$ -negligible) and  $x_\varepsilon \leq y_\varepsilon + z_\varepsilon$  for  $\varepsilon$  small. Equivalently, we have that  $x \leq y$  if and only if there exist representatives  $[x_\varepsilon] = x$  and  $[y_\varepsilon] = y$  such that  $x_\varepsilon \leq y_\varepsilon$  for all  $\varepsilon$ . Clearly,  ${}^\rho \widetilde{\mathbb{R}}$  is a partially ordered ring. The usual real numbers  $r \in \mathbb{R}$  are embedded in  ${}^\rho \widetilde{\mathbb{R}}$  considering constant nets  $[r] \in {}^\rho \widetilde{\mathbb{R}}$ .

Even if the order  $\leq$  is not total, we still have the possibility to define the infimum  $\min([x_\varepsilon], [y_\varepsilon]) := [\min(x_\varepsilon, y_\varepsilon)]$ , and analogously the supremum function  $\max([x_\varepsilon], [y_\varepsilon]) := [\max(x_\varepsilon, y_\varepsilon)]$  and the absolute value  $||[x_\varepsilon]|| := |[x_\varepsilon]| \in {}^\rho \widetilde{\mathbb{R}}$ . Note, e.g., that  $x \leq z$  and  $-x \leq z$  imply  $|x| \leq z$ . In the following, we will also use the customary notation  ${}^\rho \widetilde{\mathbb{R}}^*$  for the set of invertible generalized numbers. Our notations for intervals are:  $[a, b] := \{x \in {}^\rho \widetilde{\mathbb{R}} \mid a \leq x \leq b\}$ ,  $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$ , and analogously for segments  $[x, y] := \{x + r \cdot (y - x) \mid r \in [0, 1]\} \subseteq {}^\rho \widetilde{\mathbb{R}}^n$  and  $[x, y]_{\mathbb{R}^n} = [x, y] \cap \mathbb{R}^n$ . Finally, we write  $x \approx y$  to denote that  $|x - y|$  is an infinitesimal number, i.e.  $|x - y| \leq r$  for all  $r \in \mathbb{R}_{>0}$ . This is equivalent to  $\lim_{\varepsilon \rightarrow 0^+} |x_\varepsilon - y_\varepsilon| = 0$  for all representatives  $x = [x_\varepsilon]$  and  $y = [y_\varepsilon]$ .

*Topologies on  ${}^\rho \widetilde{\mathbb{R}}^n$ .* On the  ${}^\rho \widetilde{\mathbb{R}}$ -module  ${}^\rho \widetilde{\mathbb{R}}^n$ , we can consider the natural extension of the Euclidean norm, i.e.  $||[x_\varepsilon]|| := |[x_\varepsilon]| \in {}^\rho \widetilde{\mathbb{R}}$ , where  $[x_\varepsilon] \in {}^\rho \widetilde{\mathbb{R}}^n$ . Even if this generalized norm takes values in  ${}^\rho \widetilde{\mathbb{R}}$ , it shares several properties with usual norms, like the triangular inequality or the property  $|y \cdot x| = |y| \cdot |x|$ . It is therefore natural to consider on  ${}^\rho \widetilde{\mathbb{R}}^n$  topologies generated by balls defined by this generalized norm and a set of radii  $\mathfrak{R}$ :

**Definition 2.** Let  $\mathfrak{R} \in \left\{ {}^\rho \widetilde{\mathbb{R}}_{\geq 0}^*, \mathbb{R}_{>0} \right\}$ ,  $c \in {}^\rho \widetilde{\mathbb{R}}^n$  and  $x, y \in {}^\rho \widetilde{\mathbb{R}}$ , then:

- (i) We write  $x <_{\mathfrak{R}} y$  if  $\exists r \in \mathfrak{R} : r \leq y - x$ .
- (ii)  $B_r^{\mathfrak{R}}(c) := \{x \in {}^\rho \widetilde{\mathbb{R}}^n \mid |x - c| <_{\mathfrak{R}} r\}$  for each  $r \in \mathfrak{R}$ .
- (iii)  $B_r^E(c) := \{x \in \mathbb{R}^n \mid |x - c| < r\}$ , for each  $r \in \mathbb{R}_{>0}$ , denotes an ordinary Euclidean ball in  $\mathbb{R}^n$ .

The relation  $<_{\mathfrak{R}}$  has better topological properties as compared to the usual strict order relation  $a \leq b$  and  $a \neq b$  (that we will *never* use) because for  $\mathfrak{R} \in \left\{ {}^\rho \widetilde{\mathbb{R}}_{\geq 0}^*, \mathbb{R}_{>0} \right\}$

the set of balls  $\{B_r^{\mathfrak{A}}(c) \mid r \in \mathfrak{A}, c \in {}^\rho\widetilde{\mathbb{R}}^n\}$  is a base for a topology on  ${}^\rho\widetilde{\mathbb{R}}^n$ . The topology generated in the case  $\mathfrak{A} = {}^\rho\widetilde{\mathbb{R}}_{>0}^*$  is called *sharp topology*, whereas the one with the set of radii  $\mathfrak{A} = \mathbb{R}_{>0}$  is called *Fermat topology*. We will call *sharply open set* any open set in the sharp topology, and *large open set* any open set in the Fermat topology; clearly, the latter is coarser than the former. The existence of infinitesimal neighborhoods implies that the sharp topology induces the discrete topology on  $\mathbb{R}$ . This is a necessary result when one has to deal with continuous generalized functions which have infinite derivatives. In fact, if  $f'(x_0)$  is infinite, we have  $f(x) \approx f(x_0)$  only for  $x \approx x_0$ , see [13, 12]. With an innocuous abuse of language, we write  $x < y$  instead of  $x <_{{}^\rho\widetilde{\mathbb{R}}_{\geq 0}^*} y$  and  $x <_{\mathbb{R}} y$  instead of  $x <_{\mathbb{R}_{>0}} y$ . For example,  ${}^\rho\widetilde{\mathbb{R}}_{\geq 0}^* = {}^\rho\widetilde{\mathbb{R}}_{>0}$ . We will simply write  $B_r(c)$  to denote an open ball in the sharp topology and  $B_r^f(c)$  for an open ball in the Fermat topology. Also open intervals are defined using the relation  $<$ , i.e.  $(a, b) := \{x \in {}^\rho\widetilde{\mathbb{R}} \mid a < x < b\}$ .

The following result is useful to deal with positive and invertible generalized numbers (cf. [19]).

**Lemma 3.** *Let  $x \in {}^\rho\widetilde{\mathbb{R}}$ . Then the following are equivalent:*

- (i)  $x$  is invertible and  $x \geq 0$ , i.e.  $x > 0$ .
- (ii) For each representative  $(x_\varepsilon) \in \mathbb{R}_\rho$  of  $x$  we have  $\forall^0 \varepsilon : x_\varepsilon > 0$ .
- (iii) For each representative  $(x_\varepsilon) \in \mathbb{R}_\rho$  of  $x$  we have  $\exists m \in \mathbb{N} \forall^0 \varepsilon : x_\varepsilon > \rho_\varepsilon^m$ .

We will also need the following result.

**Lemma 4.** *Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  such that  $a < b$ , then the interior  $\text{int}([a, b])$  in the sharp topology is dense in  $[a, b]$ .*

*Internal and strongly internal sets.* A natural way to obtain sharply open, closed and bounded sets in  ${}^\rho\widetilde{\mathbb{R}}^n$  is by using a net  $(A_\varepsilon)$  of subsets  $A_\varepsilon \subseteq \mathbb{R}^n$ . We have two ways of extending the membership relation  $x_\varepsilon \in A_\varepsilon$  to generalized points  $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ :

**Definition 5.** Let  $(A_\varepsilon)$  be a net of subsets of  $\mathbb{R}^n$ , then

- (i)  $[A_\varepsilon] := \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid \forall^0 \varepsilon : x_\varepsilon \in A_\varepsilon\}$  is called the *internal set* generated by the net  $(A_\varepsilon)$ . See [36] for the introduction and an in-depth study of this notion in the case  $\rho_\varepsilon = \varepsilon$ .
- (ii) Let  $(x_\varepsilon)$  be a net of points of  $\mathbb{R}^n$ , then we say that  $x_\varepsilon \in_\varepsilon A_\varepsilon$ , and we read it as  $(x_\varepsilon)$  *strongly belongs to*  $(A_\varepsilon)$ , if  $\forall^0 \varepsilon : x_\varepsilon \in A_\varepsilon$  and if  $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$ , then also  $x'_\varepsilon \in A_\varepsilon$  for  $\varepsilon$  small. Moreover, we set  $\langle A_\varepsilon \rangle := \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\varepsilon A_\varepsilon\}$ , and we call it the *strongly internal set* generated by the net  $(A_\varepsilon)$ .
- (iii) Finally, we say that the internal set  $K = [A_\varepsilon]$  is *sharply bounded* if there exists  $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$  such that  $K \subseteq B_r(0)$ . Analogously, a net  $(A_\varepsilon)$  is *sharply bounded* the internal set  $[A_\varepsilon]$  is sharply bounded.

Therefore,  $x \in [A_\varepsilon]$  if there exists a representative  $[x_\varepsilon] = x$  such that  $x_\varepsilon \in A_\varepsilon$  for  $\varepsilon$  small, whereas this membership is independent from the chosen representative in the case of strongly internal sets. Note explicitly that an internal set generated by a constant net  $A_\varepsilon = A \subseteq \mathbb{R}^n$  is simply denoted by  $[A]$ .

The following theorem shows that internal and strongly internal sets have dual topological properties:

**Theorem 6.** *For  $\varepsilon \in I$ , let  $A_\varepsilon \subseteq \mathbb{R}^n$  and let  $x_\varepsilon \in \mathbb{R}^n$ . Then we have*

- (i)  $[x_\varepsilon] \in [A_\varepsilon]$  if and only if  $\forall q \in \mathbb{R}_{>0} \forall^0 \varepsilon : d(x_\varepsilon, A_\varepsilon) \leq \rho_\varepsilon^q$ . Therefore  $[x_\varepsilon] \in [A_\varepsilon]$  if and only if  $[d(x_\varepsilon, A_\varepsilon)] = 0 \in {}^\rho\widetilde{\mathbb{R}}$ .

- (ii)  $[x_\varepsilon] \in \langle A_\varepsilon \rangle$  if and only if  $\exists q \in \mathbb{R}_{>0} \forall^0 \varepsilon : d(x_\varepsilon, A_\varepsilon^c) > \rho_\varepsilon^q$ , where  $A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon$ . Therefore, if  $(d(x_\varepsilon, A_\varepsilon^c)) \in \mathbb{R}_\rho$ , then  $[x_\varepsilon] \in \langle A_\varepsilon \rangle$  if and only if  $[d(x_\varepsilon, A_\varepsilon^c)] > 0$ .
- (iii)  $[A_\varepsilon]$  is sharply closed and  $\langle A_\varepsilon \rangle$  is sharply open.
- (iv)  $[A_\varepsilon] = [\text{cl}(A_\varepsilon)]$ , where  $\text{cl}(S)$  is the closure of  $S \subseteq \mathbb{R}^n$ . On the other hand  $\langle A_\varepsilon \rangle = [\text{int}(A_\varepsilon)]$ , where  $\text{int}(S)$  is the interior of  $S \subseteq \mathbb{R}^n$ .

*Generalized smooth functions and their calculus.* Using the ring  ${}^\rho\widetilde{\mathbb{R}}$ , it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by  $f(x, \sigma)$ , and if we set  $\sigma = [\sigma_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ , where  $\sigma \approx 0$ , we obtain the net of smooth functions  $(f(-, \sigma_\varepsilon))_{\varepsilon \in I}$ . This is the basic idea we are going to develop in the following

**Definition 7.** Let  $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$  and  $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$  be arbitrary subsets of generalized points. Then we say that

$f : X \longrightarrow Y$  is a *generalized smooth function*

if there exists a net  $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$  defining  $f$  in the sense that  $X \subseteq \langle \Omega_\varepsilon \rangle$ ,  $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \in Y$  and  $(\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}_\rho^d$  for all  $x = [x_\varepsilon] \in X$  and all  $\alpha \in \mathbb{N}^n$ . The space of generalized smooth functions (GSF) from  $X$  to  $Y$  is denoted by  ${}^\rho\mathcal{GC}^\infty(X, Y)$ .

Let us note explicitly that this definition states minimal logical conditions to obtain a set-theoretical map from  $X$  into  $Y$  and defined by a net of smooth functions. In particular, the following Thm. 8 states that the equality  $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)]$  is meaningful, i.e. that we have independence from the representatives for all derivatives  $[x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$ ,  $\alpha \in \mathbb{N}^n$ .

**Theorem 8.** Let  $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$  and  $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$  be arbitrary subsets of generalized points. Let  $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$  be a net of smooth functions that defines a generalized smooth map of the type  $X \longrightarrow Y$ , then

- (i)  $\forall \alpha \in \mathbb{N}^n \forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}_\rho^n : [x_\varepsilon] = [x'_\varepsilon] \in X \Rightarrow (\partial^\alpha u_\varepsilon(x_\varepsilon)) \sim_\rho (\partial^\alpha u_\varepsilon(x'_\varepsilon))$ .
- (ii)  $\forall [x_\varepsilon] \in X \forall \alpha \in \mathbb{N}^n \exists q \in \mathbb{R}_{>0} \forall^0 \varepsilon : \sup_{y \in B_{\varepsilon^q}^{\mathbb{R}^n}(x_\varepsilon)} |\partial^\alpha u_\varepsilon(y)| \leq \varepsilon^{-q}$ .
- (iii) For all  $\alpha \in \mathbb{N}^n$ , the GSF  $g : [x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in \widetilde{\mathbb{R}}^d$  is locally Lipschitz in the sharp topology, i.e. each  $x \in X$  possesses a sharp neighborhood  $U$  such that  $|g(x) - g(y)| \leq L|x - y|$  for all  $x, y \in U$  and some  $L \in {}^\rho\widetilde{\mathbb{R}}$ .
- (iv) Each  $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$  is continuous with respect to the sharp topologies induced on  $X, Y$ .
- (v) Assume that the GSF  $f$  is locally Lipschitz in the Fermat topology and that its Lipschitz constants are always finite:  $L \in \mathbb{R}$ . Then  $f$  is continuous in the Fermat topology.
- (vi)  $f : X \longrightarrow Y$  is a GSF if and only if there exists a net  $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$  defining a generalized smooth map of type  $X \longrightarrow Y$  such that  $f = [v_\varepsilon(-)]|_X$ .
- (vii) Subsets  $S \subseteq {}^\rho\widetilde{\mathbb{R}}^s$  with the trace of the sharp topology, and generalized smooth maps as arrows form a subcategory of the category of topological spaces. We will call this category  ${}^\rho\mathcal{GC}^\infty$ , the category of GSF.

The differential calculus for GSF can be introduced by showing existence and uniqueness of another GSF serving as incremental ratio.

**Theorem 9** (Fermat-Reyes theorem for GSF). Let  $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$  be a sharply open set, let  $v = [v_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$ , and let  $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$  be a generalized smooth map generated by the net of smooth functions  $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$ . Then

- (i) There exists a sharp neighborhood  $T$  of  $U \times \{0\}$  and a generalized smooth map  $r \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$ , called the generalized incremental ratio of  $f$  along  $v$ , such that

$$\forall (x, h) \in T : f(x + hv) = f(x) + h \cdot r(x, h).$$

- (ii) Any two generalized incremental ratios coincide on a sharp neighborhood of  $U \times \{0\}$ .
- (iii) We have  $r(x, 0) = \left[ \frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon) \right]$  for every  $x \in U$  and we can thus define  $Df(x) \cdot v := \frac{\partial f}{\partial v}(x) := r(x, 0)$ , so that  $\frac{\partial f}{\partial v} \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ .

If  $U$  is a large open set, then an analogous statement holds replacing sharp neighborhoods by large neighborhoods.

Note that this result permits the consideration of the partial derivative of  $f$  with respect to an arbitrary generalized vector  $v \in {}^\rho\widetilde{\mathbb{R}}^n$  which can be, e.g., infinitesimal or infinite. Using this result, we can also define subsequent differentials  $D^j f(x)$  as  $j$ -multilinear maps, and we set  $D^j f(x) \cdot h^j := D^j f(x)(h, \dots, h)$ . The set of all the  $j$ -multilinear maps  $({}^\rho\widetilde{\mathbb{R}}^n)^j \rightarrow {}^\rho\widetilde{\mathbb{R}}^d$  over the ring  ${}^\rho\widetilde{\mathbb{R}}$  will be denoted by  $L^j({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^d)$ . For  $A = [A_\varepsilon(-)] \in L^j({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^d)$ , we set  $|A| := [|A_\varepsilon|]$ , the generalized number defined by the operator norms of the multilinear maps  $A_\varepsilon \in L^j(\mathbb{R}^n, \mathbb{R}^d)$ .

The following result follows from the analogous properties for the nets of smooth functions defining  $f$  and  $g$ .

**Theorem 10.** *Let  $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$  be an open subset in the sharp topology, let  $v \in {}^\rho\widetilde{\mathbb{R}}^n$  and  $f, g : U \rightarrow {}^\rho\widetilde{\mathbb{R}}$  be generalized smooth maps. Then*

- (i)  $\frac{\partial(f+g)}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v}$
- (ii)  $\frac{\partial(r \cdot f)}{\partial v} = r \cdot \frac{\partial f}{\partial v} \quad \forall r \in {}^\rho\widetilde{\mathbb{R}}$
- (iii)  $\frac{\partial(f \cdot g)}{\partial v} = \frac{\partial f}{\partial v} \cdot g + f \cdot \frac{\partial g}{\partial v}$
- (iv) For each  $x \in U$ , the map  $df(x) \cdot v := \frac{\partial f}{\partial v}(x) \in {}^\rho\widetilde{\mathbb{R}}$  is  ${}^\rho\widetilde{\mathbb{R}}$ -linear in  $v \in {}^\rho\widetilde{\mathbb{R}}^n$
- (v) Let  $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$  and  $V \subseteq {}^\rho\widetilde{\mathbb{R}}^d$  be open subsets in the sharp topology and  $g \in {}^\rho\mathcal{GC}^\infty(V, U)$ ,  $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$  be generalized smooth maps. Then for all  $x \in V$  and all  $v \in {}^\rho\widetilde{\mathbb{R}}^d$ , we have  $\frac{\partial(f \circ g)}{\partial v}(x) = df(g(x)) \cdot \frac{\partial g}{\partial v}(x)$ .

We also have a generalization of the Taylor formula:

**Theorem 11.** *Let  $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$  be a generalized smooth function defined in the sharply open set  $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ . Let  $a, x \in {}^\rho\widetilde{\mathbb{R}}^n$  such that the line segment  $[a, x] \subseteq U$ . Then, for all  $n \in \mathbb{N}$  we have*

$$\exists \xi \in [a, x] : f(x) = \sum_{j=0}^n \frac{D^j f(a)}{j!} \cdot (x-a)^j + \frac{D^{n+1} f(\xi)}{(n+1)!} \cdot (x-a)^{n+1}. \quad (2.1)$$

If we further assume that all the  $n$  components  $(x-a)_k \in {}^\rho\widetilde{\mathbb{R}}$  of  $x-a \in {}^\rho\widetilde{\mathbb{R}}^n$  are invertible, then there exists  $\rho \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ ,  $\rho \leq |x-a|$ , such that

$$\forall k \in B_\rho(0) \exists \xi \in [a-k, a+k] : f(a+k) = \sum_{j=0}^n \frac{D^j f(a)}{j!} \cdot k^j + \frac{D^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} \quad (2.2)$$

$$\frac{D^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} \approx 0. \quad (2.3)$$

Formula (2.1) corresponds to a direct generalization of Taylor formulas for ordinary smooth functions with Lagrange remainder. On the other hand, in (2.2) and (2.3), the possibility that the differential  $D^{n+1} f$  may be infinite at some point is considered, and the Taylor formulas are stated so as to have infinitesimal remainder.

The following local inverse function theorem will be used in the proof of Jacobi's theorem (see [11] for a proof).

**Theorem 12.** *Let  $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ , let  $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^n)$  and suppose that for some  $x_0$  in the sharp interior of  $X$ ,  $Df(x_0)$  is invertible in  $L({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^n)$ . Then there exists a sharp neighborhood  $U \subseteq X$  of  $x_0$  and a sharp neighborhood  $V$  of  $f(x_0)$  such that  $f : U \rightarrow V$  is invertible and  $f^{-1} \in {}^\rho\mathcal{GC}^\infty(V, U)$ .*

We can define right and left derivatives as e.g.  $f'(a) := f'_+(a) := \lim_{a < t} f'(t)$ , which always exist if  $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ . One dimensional integral calculus of GSF is based on the following

**Theorem 13.** *Let  $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$  be a generalized smooth function defined in the interval  $[a, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$ , where  $a < b$ . Let  $c \in [a, b]$ . Then, there exists one and only one generalized smooth function  $F \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$  such that  $F(c) = 0$  and  $F'(x) = f(x)$  for all  $x \in [a, b]$ . Moreover, if  $f$  is defined by the net  $f_\varepsilon \in C^\infty(\mathbb{R}, \mathbb{R})$  and  $c = [c_\varepsilon]$ , then  $F(x) = \left[ \int_{c_\varepsilon}^{x_\varepsilon} f_\varepsilon(s) ds \right]$  for all  $x = [x_\varepsilon] \in [a, b]$ .*

We can thus define

**Definition 14.** Under the assumptions of Theorem 13, we denote by  $\int_c^{(-)} f := \int_c^{(-)} f(s) ds \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$  the unique generalized smooth function such that:

- (i)  $\int_c^c f = 0$
- (ii)  $\left( \int_u^{(-)} f \right)'(x) = \frac{d}{dx} \int_u^x f(s) ds = f(x)$  for all  $x \in [a, b]$ .

All the classical rules of integral calculus hold in this setting:

**Theorem 15.** *Let  $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$  and  $g \in {}^\rho\mathcal{GC}^\infty(V, {}^\rho\widetilde{\mathbb{R}})$  be generalized smooth functions defined on sharply open domains in  ${}^\rho\widetilde{\mathbb{R}}$ . Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  with  $a < b$  and  $c, d \in [a, b] \subseteq U \cap V$ , then*

- (i)  $\int_c^d (f + g) = \int_c^d f + \int_c^d g$
- (ii)  $\int_c^d \lambda f = \lambda \int_c^d f \quad \forall \lambda \in {}^\rho\widetilde{\mathbb{R}}$
- (iii)  $\int_c^d f = \int_c^e f + \int_e^d f$  for all  $e \in [a, b]$
- (iv)  $\int_c^d f = - \int_d^c f$
- (v)  $\int_c^d f' = f(d) - f(c)$
- (vi)  $\int_c^d f' \cdot g = [f \cdot g]_c^d - \int_c^d f \cdot g'$

**Theorem 16.** *Let  $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$  and  $\varphi \in {}^\rho\mathcal{GC}^\infty(V, U)$  be generalized smooth functions defined on sharply open domains in  ${}^\rho\widetilde{\mathbb{R}}$ . Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$ , with  $a < b$ , such that  $[a, b] \subseteq V$ ,  $\varphi(a) < \varphi(b)$ ,  $[\varphi(a), \varphi(b)] \subseteq U$ . Finally, assume that  $\varphi([a, b]) \subseteq [\varphi(a), \varphi(b)]$ . Then*

$$\int_{\varphi(a)}^{\varphi(b)} f(t) dt = \int_a^b f[\varphi(s)] \cdot \varphi'(s) ds.$$

*Embedding of Schwartz distributions and Colombeau functions.* We finally recall two results that give a certain flexibility in constructing embeddings of Schwartz distributions. Note that both the infinitesimal  $\rho$  and the embedding of Schwartz distributions have to be chosen depending on the problem we aim to solve. A trivial example in this direction is the ODE  $y' = y/\varepsilon$ , which cannot be solved for  $\rho = (\varepsilon)$ , but it has a solution for  $\rho = (e^{-1/\varepsilon})$ . As another simple example, if we need the property  $H(0) = 1/2$ , where  $H$  is the Heaviside function, then we have to choose the embedding of distributions accordingly. See also [16, 33] for further details.

If  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ ,  $r \in \mathbb{R}_{>0}$  and  $x \in \mathbb{R}^n$ , we use the notations  $r \odot \varphi$  for the function  $x \in \mathbb{R}^n \mapsto \frac{1}{r^n} \cdot \varphi\left(\frac{x}{r}\right) \in \mathbb{R}$  and  $x \oplus \varphi$  for the function  $y \in \mathbb{R}^n \mapsto \varphi(y - x) \in \mathbb{R}$ .



These notations permit to highlight that  $\odot$  is a free action of the multiplicative group  $(\mathbb{R}_{>0}, \cdot, 1)$  on  $\mathcal{D}(\mathbb{R}^n)$  and  $\oplus$  is a free action of the additive group  $(\mathbb{R}_{>0}, +, 0)$  on  $\mathcal{D}(\mathbb{R}^n)$ . We also have the distributive property  $r \odot (x \oplus \varphi) = rx \oplus r \odot \varphi$ .

**Lemma 17.** *Let  $b \in {}^\rho\widetilde{\mathbb{R}}$  be a net such that  $\lim_{\varepsilon \rightarrow 0^+} b_\varepsilon = +\infty$ . Let  $d \in (0, 1)$ . There exists a net  $(\psi_\varepsilon)_{\varepsilon \in I}$  of  $\mathcal{D}(\mathbb{R}^n)$  with the properties:*

- (i)  $\text{supp}(\psi_\varepsilon) \subseteq B_1(0)$  for all  $\varepsilon \in I$ .
- (ii)  $\int \psi_\varepsilon = 1$  for all  $\varepsilon \in I$ .
- (iii)  $\forall \alpha \in \mathbb{N}^n \exists p \in \mathbb{N} : \sup_{x \in \mathbb{R}^n} |\partial^\alpha \psi_\varepsilon(x)| = O(b_\varepsilon^p)$  as  $\varepsilon \rightarrow 0^+$ .
- (iv)  $\forall j \in \mathbb{N} \forall \varepsilon : 1 \leq |\alpha| \leq j \Rightarrow \int x^\alpha \cdot \psi_\varepsilon(x) dx = 0$ .
- (v)  $\forall \eta \in \mathbb{R}_{>0} \forall \varepsilon : \int |\psi_\varepsilon| \leq 1 + \eta$ .
- (vi) If  $n = 1$ , then the net  $(\psi_\varepsilon)_{\varepsilon \in I}$  can be chosen so that  $\int_{-\infty}^0 \psi_\varepsilon = d$ .

In particular  $\psi_\varepsilon^b := b_\varepsilon^{-1} \odot \psi_\varepsilon$  satisfies (ii) - (v).

It is worth noting that the condition (iv) of null moments is well known in the study of convergence of numerical solutions of singular differential equations, see e.g. [42, 9, 22] and references therein.

Concerning embeddings of Schwartz distributions, we have the following result, where  $c(\Omega) := \{[x_\varepsilon] \in [\Omega] \mid \exists K \Subset \Omega \forall \varepsilon : x_\varepsilon \in K\}$  is called the set of *compactly supported points* in  $\Omega \subseteq \mathbb{R}^n$ .

**Theorem 18.** *Under the assumptions of Lemma 17, let  $\Omega \subseteq \mathbb{R}^n$  be an open set and let  $(\psi_\varepsilon^b)$  be the net defined in 17. Then the mapping*

$$\iota_\Omega^b : T \in \mathcal{E}'(\Omega) \mapsto [(T * \psi_\varepsilon^b)(-)] \in {}^\rho\mathcal{GC}^\infty(c(\Omega), {}^\rho\widetilde{\mathbb{R}})$$

uniquely extends to a sheaf morphism of real vector spaces

$$\iota^b : \mathcal{D}' \longrightarrow {}^\rho\mathcal{GC}^\infty(c((-)), {}^\rho\widetilde{\mathbb{R}}),$$

and satisfies the following properties:

- (i) If  $b \geq d\rho^{-a}$  for some  $a \in \mathbb{R}_{>0}$ , then  $\iota^b|_{\mathcal{C}^\infty(-)} : \mathcal{C}^\infty(-) \longrightarrow {}^\rho\mathcal{GC}^\infty(c((-)), {}^\rho\widetilde{\mathbb{R}})$  is a sheaf morphism of algebras;
- (ii) If  $T \in \mathcal{E}'(\Omega)$  then  $\text{supp}(T) = \text{supp}(\iota_\Omega^b(T))$ ;
- (iii)  $\lim_{\varepsilon \rightarrow 0^+} \int_\Omega \iota_\Omega^b(T)_\varepsilon \cdot \varphi = \langle T, \varphi \rangle$  for all  $\varphi \in \mathcal{D}(\Omega)$  and all  $T \in \mathcal{D}'(\Omega)$ ;
- (iv)  $\iota^b$  commutes with partial derivatives, i.e.  $\partial^\alpha (\iota_\Omega^b(T)) = \iota_\Omega^b(\partial^\alpha T)$  for each  $T \in \mathcal{D}'(\Omega)$  and  $\alpha \in \mathbb{N}^n$ .

Concerning the embedding of Colombeau generalized functions, we recall that the special Colombeau algebra on  $\Omega$  is defined as the quotient  $\mathcal{G}^s(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}^s(\Omega)$  of *moderate nets* over *negligible nets*, where the former is

$$\mathcal{E}_M(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\}$$

and the latter is

$$\mathcal{N}^s(\Omega) := \{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega \forall \alpha \in \mathbb{N}^n \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^m)\}.$$

Using  $\rho = (\varepsilon)$ , we have the following compatibility result:

**Theorem 19.** *A Colombeau generalized function  $u = (u_\varepsilon) + \mathcal{N}^s(\Omega)^d \in \mathcal{G}^s(\Omega)^d$  defines a generalized smooth map  $u : [x_\varepsilon] \in c(\Omega) \longrightarrow [u_\varepsilon(x_\varepsilon)] \in \widetilde{\mathbb{R}}^d$  which is locally Lipschitz on the same neighborhood of the Fermat topology for all derivatives. This assignment provides a bijection of  $\mathcal{G}^s(\Omega)^d$  onto  ${}^\rho\mathcal{GC}^\infty(c(\Omega), {}^\rho\widetilde{\mathbb{R}}^d)$  for every open set  $\Omega \subseteq \mathbb{R}^n$ .*

**2.1. Extreme value theorem and functionally compact sets.** For GSF, suitable generalizations of many classical theorems of differential and integral calculus hold: intermediate value theorem, mean value theorems, a sheaf property for the Fermat topology, local and global inverse function theorems, Banach fixed point theorem and a corresponding Picard-Lindelöf theorem, see [15, 14, 32, 11].

Even though the intervals  $[a, b] \subseteq \widetilde{\mathbb{R}}$ ,  $a, b \in \mathbb{R}$ , are neither compact in the sharp nor in the Fermat topology (see [15, Thm. 25]), analogously to the case of smooth functions, a GSF satisfies an extreme value theorem on such sets. In fact, we have:

**Theorem 20.** *Let  $f \in \mathcal{GC}^\infty(X, \widetilde{\mathbb{R}})$  be a generalized smooth function defined on the subset  $X$  of  $\widetilde{\mathbb{R}}^n$ . Let  $\emptyset \neq K = [K_\varepsilon] \subseteq X$  be an internal set generated by a sharply bounded net  $(K_\varepsilon)$  of compact sets  $K_\varepsilon \in \mathbb{R}^n$ , then*

$$\exists m, M \in K \forall x \in K : f(m) \leq f(x) \leq f(M). \quad (2.4)$$

We shall use the assumptions on  $K$  and  $(K_\varepsilon)$  given in this theorem to introduce a notion of “compact subset” which behaves better than the usual classical notion of compactness in the sharp topology.

**Definition 21.** A subset  $K$  of  $\widetilde{\mathbb{R}}^n$  is called *functionally compact*, denoted by  $K \in_f \widetilde{\mathbb{R}}^n$ , if there exists a net  $(K_\varepsilon)$  such that

- (i)  $K = [K_\varepsilon] \subseteq \widetilde{\mathbb{R}}^n$
- (ii)  $(K_\varepsilon)$  is sharply bounded
- (iii)  $\forall \varepsilon \in I : K_\varepsilon \in \mathbb{R}^n$

If, in addition,  $K \subseteq U \subseteq \widetilde{\mathbb{R}}^n$  then we write  $K \in_f U$ . Finally, we write  $[K_\varepsilon] \in_f U$  if (ii), (iii) and  $[K_\varepsilon] \subseteq U$  hold.

We motivate the name *functionally compact subset* by noting that on this type of subsets, GSF have properties very similar to those that ordinary smooth functions have on standard compact sets.

*Remark 22.*

- (i) By [36, Prop. 2.3], any internal set  $K = [K_\varepsilon]$  is closed in the sharp topology. In particular, the open interval  $(0, 1) \subseteq \mathbb{R}$  is not functionally compact since it is not closed.
- (ii) If  $H \in \mathbb{R}^n$  is a non-empty ordinary compact set, then  $[H]$  is functionally compact. In particular,  $[0, 1] = [[0, 1]_{\mathbb{R}}]$  is functionally compact.
- (iii) The empty set  $\emptyset = \emptyset \in_f \widetilde{\mathbb{R}}$ .
- (iv)  $\widetilde{\mathbb{R}}^n$  is not functionally compact since it is not sharply bounded.
- (v) The set of compactly supported points  $\widetilde{\mathbb{R}}_c$  is not functionally compact because the GSF  $f(x) = x$  does not satisfy the conclusion (2.4) of Prop. 20.

In the present paper, we need the following properties of functionally compact sets.

**Theorem 23.** *Let  $K \subseteq X \subseteq \widetilde{\mathbb{R}}^n$ ,  $f \in \mathcal{GC}^\infty(X, \widetilde{\mathbb{R}}^d)$ . Then  $K \in_f \widetilde{\mathbb{R}}^n$  implies  $f(K) \in_f \widetilde{\mathbb{R}}^d$ .*

As a corollary of this theorem and Rem. (22).(ii) we get

**Corollary 24.** *If  $a, b \in \widetilde{\mathbb{R}}$  and  $a \leq b$ , then  $[a, b] \in_f \widetilde{\mathbb{R}}$ .*

Let us note that  $a, b \in \widetilde{\mathbb{R}}$  can also be infinite, e.g.  $a = [-\varepsilon^{-N}]$ ,  $b = [\varepsilon^{-M}]$  or  $a = [\varepsilon^{-N}]$ ,  $b = [\varepsilon^{-M}]$  with  $M > N$ . Finally, in the following result we consider the product of functionally compact sets:

**Theorem 25.** *Let  $K \in_f \widetilde{\mathbb{R}}^n$  and  $H \in_f \widetilde{\mathbb{R}}^d$ , then  $K \times H \in_f \widetilde{\mathbb{R}}^{n+d}$ . In particular, if  $a_i \leq b_i$  for  $i = 1, \dots, n$ , then  $\prod_{i=1}^n [a_i, b_i] \in_f \widetilde{\mathbb{R}}^n$ .*

A theory of compactly supported GSF has been developed in [11], and it closely resembles the classical theory of LF-spaces of compactly supported smooth functions. It establishes that for suitable functionally compact subsets, the corresponding space of compactly supported GSF contains extensions of all Colombeau generalized functions, and hence also of all Schwartz distributions.

### 3. PRELIMINARY RESULTS FOR CALCULUS OF VARIATIONS WITH GSF

In this section, we study extremal values of generalized functions at sharply interior points of intervals  $[a, b] \subseteq {}^o\mathbb{R}$ . As in the classical calculus of variations, this will provide the basis for proving necessary and sufficient conditions for general variational problems. Since the new ring of scalars  ${}^o\mathbb{R}$  has zero divisors and is not totally ordered, the following extension requires a more refined analysis than in the classical case.

The following lemma shows that we can interchange integration and differentiation while working with generalized functions.

**Lemma 26.** *Let  $a, b, c, d \in {}^o\mathbb{R}$ , with  $a < b$  and  $c < d$ . Let  $f \in {}^o\mathcal{GC}^\infty(X, Y)$  and assume that  $[a, b] \times [c, d] \subseteq X \subseteq {}^o\mathbb{R}^2$  and  $Y \subseteq {}^o\mathbb{R}^d$ . Then for all  $s \in [c, d]$ , we have*

$$\frac{d}{ds} \int_a^b f(\tau, s) d\tau = \int_a^b \frac{\partial}{\partial s} f(\tau, s) d\tau. \quad (3.1)$$

*Proof.* We first note that  $f(\cdot, s) \in {}^o\mathcal{GC}^\infty([a, b], Y)$  by the closure of GSF with respect to composition. Therefore,  $\frac{\partial}{\partial s} f(\cdot, s) \in {}^o\mathcal{GC}^\infty([a, b], {}^o\mathbb{R}^d)$ , and the right hand side of (3.1) is well defined as an integral of a GSF. In order to show that also the left hand side of (3.1) is well-defined, we need to prove that also  $\sigma \in [c, d] \mapsto \int_a^b f(\tau, \sigma) d\tau \in {}^o\mathbb{R}^d$  is a GSF. Let  $f$  be defined by the net  $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ , with  $X \subseteq \langle \Omega_\varepsilon \rangle$ . Let  $[\sigma_\varepsilon] \in [c, d]$ , then  $[a, b] \times \{[\sigma_\varepsilon]\} \in {}^o\mathbb{R}^2$  and the extreme value theorem 20 applied to  $\frac{\partial^n f}{\partial \sigma^n}$  yields the existence of  $N \in \mathbb{R}_{>0}$  such that

$$\left| \frac{d^n}{d\sigma^n} \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon(\tau, \sigma_\varepsilon) d\tau \right| \leq \int_{a_\varepsilon}^{b_\varepsilon} \left| \frac{\partial^n}{\partial \sigma^n} f_\varepsilon(\tau, \sigma_\varepsilon) \right| d\tau \leq \rho_\varepsilon^{-N} \cdot (b_\varepsilon - a_\varepsilon).$$

This proves that also the left hand side of (3.1) is well-defined as a derivative of a GSF. From the classical derivation under the integral sign, the Fermat-Reyes theorem 9, and Thm. 13 about definite integrals of GSF, we obtain

$$\begin{aligned} \frac{d}{ds} \int_a^b f(\tau, s) d\tau &= \frac{d}{ds} \int_a^b [f_\varepsilon(\tau, s)] d\tau \\ &= \frac{d}{ds} \left[ \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon(\tau, s) d\tau \right] \\ &= \left[ \frac{d}{ds} \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon(\tau, s) d\tau \right] \\ &= \int_a^b \left[ \frac{\partial}{\partial s} f_\varepsilon(\tau, s) \right] d\tau \\ &= \int_a^b \frac{\partial}{\partial s} f(\tau, s) d\tau. \end{aligned}$$

□

The next result will frequently be used in the following

**Lemma 27.** *Let  $(D, \geq)$  be a directed set and let  $f : D \rightarrow {}^\rho\widetilde{\mathbb{R}}$  be a set-theoretical map such that  $f(d) \geq 0$  for all  $d \in D$ , and  $\exists \lim_{d \in D} f(d) \in {}^\rho\widetilde{\mathbb{R}}$  in the sharp topology. Then  $\lim_{d \in D} f(d) \geq 0$ .*

*Proof.* Note that the internal set  $[0, +\infty) = [[0, +\infty)_{\mathbb{R}}]$  is sharply closed by (iii) of Thm. 6.  $\square$

*Remark 28.*

- (i) If  $x \in {}^\rho\widetilde{\mathbb{R}}$ , then  $x \geq 0$  if and only if  $\exists A \in \mathbb{R}_{>0} \forall a \in \mathbb{R}_{>A} : x \geq -d\rho^a$ . Indeed, it suffices to let  $a \rightarrow +\infty$  in  $f(a) = x + d\rho^a$ .
- (ii) Assume that  $x, y \in {}^\rho\widetilde{\mathbb{R}}^n$  and

$$\exists s_0 \in {}^\rho\widetilde{\mathbb{R}}_{>0} \forall s \in {}^\rho\widetilde{\mathbb{R}}_{>0} : s \leq s_0 \Rightarrow |x| \leq s|y|.$$

Then taking  $s \rightarrow 0$  in  $f(s) = s|y| - |x|$  we get  $x = 0$ .

**Definition 29.** We call  $x = (x_1, \dots, x_d) \in {}^\rho\widetilde{\mathbb{R}}^d$  *componentwise invertible* if and only if for all  $k \in \{1, \dots, d\}$  we have that  $x_k \in {}^\rho\widetilde{\mathbb{R}}$  is invertible.

**Lemma 30.** *Let  $f \in {}^\rho\mathcal{GC}^\infty(U, Y)$  where  $Y \subseteq {}^\rho\widetilde{\mathbb{R}}$  and  $U \subseteq {}^\rho\widetilde{\mathbb{R}}^d$  is a sharply open subset. Then  $f \geq 0$  if and only if  $f(x) \geq 0$  for all componentwise invertible  $x \in U$ .*

*Proof.* By Lem. 3, it follows that for  $V \subseteq {}^\rho\widetilde{\mathbb{R}}$ , the set of invertible points in  $V$ , i.e.  $V \cap {}^\rho\widetilde{\mathbb{R}}^* \subseteq V$  is dense in  $V$  (with respect to the sharp topology). This implies that  $U \cap ({}^\rho\widetilde{\mathbb{R}}^d)^* \subseteq U$  is dense. By Thm. 8.(iv),  $f$  is sharply continuous, so Lem. 27 yields that  $f(x) \geq 0$ . The other direction is obvious.  $\square$

Analogously to the classical case, we say that  $x_0 \in X$  is a local minimum of  $f \in {}^\rho\mathcal{GC}^\infty(X)$  if there exists a sharply open neighbourhood (in the trace topology)  $Y \subseteq X$  of  $x_0$  such that  $f(x_0) \leq f(y)$  for all  $y \in Y$ . A local maximum is defined accordingly. We will write  $f(x_0) = \min!$ , which is a short hand notation to denote that  $x_0$  is a (local) minimum of  $f$ .

**Lemma 31.** *Let  $X \subseteq {}^\rho\widetilde{\mathbb{R}}$  and let  $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ . If  $x_0 \in X$  is a sharply interior local minimum of  $f$  then  $f'(x_0) = 0$ .*

*Proof.* Without loss of generality, we can assume  $x_0 = 0$ , because of the closure of GSF with respect to composition. Let  $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$  be such that  $B_{2r}(0) =: U \subseteq X$  and  $f(0) = \min!$  over  $U$ . Take any  $x \in {}^\rho\widetilde{\mathbb{R}}$  such that  $0 < |x| < r$ , so that  $[-|x|, |x|] \subseteq U$ . Thus, if  $x > 0$ , by Taylor's theorem 11 there exists  $\xi \in [0, x]$  such that

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(\xi)}{2} \cdot x^2.$$

Set  $K := [B_{r_\varepsilon}(0)] \Subset_f B_{2r}(0) \subseteq U$  and  $M := \max_{x \in K} |f''(x)| \in {}^\rho\widetilde{\mathbb{R}}_{\geq 0}$ . Due to the fact that  $f(0)$  is minimal, we have

$$f'(0) \cdot x + \frac{f''(\xi)}{2} \cdot x^2 = f(x) - f(0) \geq 0. \quad (3.2)$$

Thus  $-f'(0) \cdot x \leq \frac{M}{2}x^2$  and  $-f'(0) \leq \frac{M}{2}|x|$  since  $x > 0$ . Analogously, if we take  $x < 0$ , we get  $f'(0) \leq -\frac{M}{2}x = \frac{M}{2}|x|$ . Therefore,  $|f'(0)| \leq \frac{M}{2}|x|$  and the conclusion follows by Rem. 28.(ii).  $\square$

As a corollary of Lem. 4 and Thm. 8.(iv), we have

**Lemma 32.** *Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  with  $a < b$  and let  $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$  such that  $f(x) = 0$  for all sharply interior points  $x \in [a, b]$ . Then  $f = 0$  on  $[a, b]$ .*

Now, we are able to prove the “second - derivative - test” for GSF.

**Lemma 33.** *Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  with  $a < b$  and let  $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$  such that  $f(x_0) = \min!$  for some sharply interior  $x_0 \in [a, b]$ . Then  $f''(x_0) \geq 0$ . Vice versa, if  $f'(x_0) = 0$  and  $f''(x_0) > 0$ , then  $f(x_0) = \min!$ .*

*Proof.* As above, we can assume  $x_0 = 0$ . Let  $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$  be such that  $B_{2r}(0) =: U \subseteq X$  and  $f(0) = \min!$  over  $U$ . Take any  $x \in {}^\rho\widetilde{\mathbb{R}}$  such that  $0 < x < r$ , so that  $[0, x] \subseteq U$ , and set  $K := [B_{r_\varepsilon}(0)] \Subset_f B_{2r}(0) \subseteq U$  and  $M := \max_{x \in K} |f'''(x)| \in {}^\rho\widetilde{\mathbb{R}}_{\geq 0}$ . By Taylor's theorem 11, we obtain that for some  $\xi \in [0, x]$

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(\xi)x^3.$$

By assumption, we have that for all  $a \in \mathbb{R}_{>0}$

$$0 \leq f(x) - f(0) + d\rho^a.$$

By Lemma 31, we know that  $f'(0) = 0$ . Thus, we obtain for all  $a \in \mathbb{R}_{>0}$

$$f(x) - f(0) = \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(\xi)x^3 \geq -d\rho^a.$$

Therefore, also  $\frac{1}{2}f''(0)x^2 + \frac{1}{6}Mx^3 \geq -d\rho^a$ . In this inequality we can set  $x = d\rho^{a/3}$ , assuming that  $a > A$  and  $d\rho^A < r$ . We get  $f''(0) \geq -(2 + \frac{M}{3})d\rho^{a/3}$ , and the conclusion follows from Lem. 27 as  $a \rightarrow +\infty$ .

Now assume that  $f'(0) = 0$  and  $f''(0) > 0$ , so that  $f''(0) > d\rho^a$  for some  $a \in \mathbb{R}_{>0}$  by Lem. 3. Since  $f'(0) = 0$ , for all  $x \in B_r(0)$ , Taylor's formula gives

$$f(x) - f(0) = \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(\xi_x)x^3,$$

where  $\xi_x \in [0, x]$ . Therefore,  $f(x) - f(0) > x^2 (\frac{1}{2}d\rho^a + \frac{1}{6}f'''(\xi_x)x)$ . Now

$$\left| \frac{1}{6}f'''(\xi_x)x \right| \leq \frac{1}{6}M|x| \rightarrow 0 \text{ as } x \rightarrow 0.$$

Thus

$$\exists s \in {}^\rho\widetilde{\mathbb{R}}_{>0} : s < r, \forall x \in B_s(0) : -\frac{1}{4}d\rho^a < \frac{1}{6}f'''(\xi_x)x < \frac{1}{4}d\rho^a.$$

We can hence write  $f(x) - f(0) > x^2 (\frac{1}{2}d\rho^a - \frac{1}{4}d\rho^a) = x^2 \frac{1}{4}d\rho^a \geq 0$  for all  $x \in B_s(0)$ , which proves that  $x = 0$  is a local minimum.  $\square$

For the generalization of Lem. 31 and Lem. 33 to the multivariate case, one can proceed as above, using the ideas of [26]. Note, however, that we do not need this generalization in the present work.

#### 4. FIRST VARIATION AND CRITICAL POINTS

In this section, we define the first variation of a functional and prove that some classical results have their counterparts in this generalized setting, for example the fundamental lemma (Lem. 37) or the connection between critical points and the Euler-Lagrange equations (Thm. 38).

**Definition 34.** If  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  and  $a < b$ , we define

$${}^\rho\mathcal{GC}_0^\infty(a, b) := \left\{ \eta \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}^d) : \eta(a) = 0 = \eta(b) \right\}.$$

When the use of the points  $a, b$  is clear from the context, we adopt the simplified notation  ${}^\rho\mathcal{GC}_0^\infty$ . We also note here that  ${}^\rho\mathcal{GC}_0^\infty(a, b)$  is an  ${}^\rho\widetilde{\mathbb{R}}$ -module.

One of the positive features of the use of GSF for the calculus of variations is their closure with respect to composition. For this reason, the next definition of functional is formally equal to the classical one, though it can be applied to arbitrary generalized functions  $F$  and  $u$ .

**Theorem 35.** Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  with  $a < b$ . Let  $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$  and let  $F \in {}^\rho\mathcal{GC}^\infty([a, b] \times {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$  and define

$$I(u) := \int_a^b F(t, u, \dot{u}) dt. \quad (4.1)$$

Let  $\eta \in {}^\rho\mathcal{GC}_0^\infty$ , then

$$\delta I(u; \eta) := \left. \frac{d}{ds} I(u + s\eta) \right|_{s=0} = \int_a^b \eta \left( F_u(t, u, \dot{u}) - \frac{d}{dt} F_{\dot{u}}(t, u, \dot{u}) \right) dt.$$

*Proof.* We have (we use Thm. 10, Thm. 15 and Lemma 26)

$$\begin{aligned} \left. \frac{d}{ds} I(u + s\eta) \right|_{s=0} &= \left. \frac{d}{ds} \int_a^b F(t, u + s\eta, \dot{u} + s\dot{\eta}) dt \right|_{s=0} \\ &= \int_a^b \left. \frac{\partial}{\partial s} F(t, u + s\eta, \dot{u} + s\dot{\eta}) \right|_{s=0} dt \\ &= \int_a^b \eta F_u(t, u, \dot{u}) + \dot{\eta} F_{\dot{u}}(t, u, \dot{u}) dt \\ &= [\eta F_{\dot{u}}(t, u, \dot{u})]_a^b + \int_a^b \eta \left( F_u(t, u, \dot{u}) - \frac{d}{dt} F_{\dot{u}}(t, u, \dot{u}) \right) dt \\ &= \int_a^b \eta \left( F_u(t, u, \dot{u}) - \frac{d}{dt} F_{\dot{u}}(t, u, \dot{u}) \right) dt. \end{aligned}$$

□

We call  $\delta I(u; \eta)$  the *first variation* of  $I$ . In addition we call  $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$  a *critical point* of  $I$  if  $\delta I(u; \eta) = 0$  for all  $\eta \in {}^\rho\mathcal{GC}_0^\infty$ .

To prove the fundamental lemma of the calculus of variations, Lem. 37, we first show that every GSF can be approximated using generalized strict delta nets.

**Lemma 36.** Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  be such that  $a < b$  and let  $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ . Let  $x \in [a, b]$  and  $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$  be such that  $B_R(x) \subseteq [a, b]$ . Assume that  $G_t \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$  satisfy

- (i)  $\int_{-R}^R G_t = 1$  for  $t \in {}^\rho\widetilde{\mathbb{R}}_{>0}$  small.
- (ii) For  $t$  small,  $(G_t)_{t \in {}^\rho\widetilde{\mathbb{R}}_{>0}}$  is zero outside every ball  $B_\delta(0)$ ,  $0 < \delta < R$ , i.e.

$$\forall \delta \in {}^\rho\widetilde{\mathbb{R}}_{>0} \exists \rho \in {}^\rho\widetilde{\mathbb{R}}_{>0} \forall t \in B_\rho(0) \cap {}^\rho\widetilde{\mathbb{R}}_{>0} \forall y \in [-R, -\delta] \cup [\delta, R] : G_t(y) = 0. \quad (4.2)$$

- (iii)  $\exists M \in {}^\rho\widetilde{\mathbb{R}}_{>0} \exists \rho \in {}^\rho\widetilde{\mathbb{R}} \forall t \in B_\rho(0) : \int_{-R}^R |G_t(y)| dy \leq M$ .

Then

$$\lim_{t \rightarrow 0^+} \int_{-R}^R f(x-y) G_t(y) dy = f(x).$$

Moreover  $\int_{-R}^R f(x-y) G_t(y) dy = \int_{x-R}^{x+R} f(y) G_t(x-y) dy$ .

*Proof.* We only have to generalize the classical proof concerning limits of convolutions with strict delta nets. We first note that

$$\int_{-R}^R f(x-y) G_t(y) dy = \int_{x-R}^{x+R} f(y) G_t(x-y) dy$$

so that these integrals exist because  $(x - R, x + R) = B_R(x) \subseteq [a, b]$ . Using (i), for  $t$  small, let's say for  $0 < t < S \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ , we get

$$\begin{aligned} \left| \int_{-R}^R f(x-y)G_t(y) dy - f(x) \right| &= \left| \int_{-R}^R [f(x-y) - f(x)] G_t(y) dy \right| \\ &\leq \int_{-R}^R |f(x-y) - f(x)| \cdot |G_t(y)| dy. \end{aligned}$$

For each  $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ , sharp continuity of  $f$  at  $x$  yields  $|f(x-y) - f(x)| < r$  for all  $y$  such that  $|y| < \delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ , and we can take  $\delta < R$ . By (ii), for  $0 < |t| < \min(\rho, S)$ , we have

$$\left| \int_{-R}^R f(x-y)G_t(y) dy - f(x) \right| \leq r \int_{-\delta}^{+\delta} |G_t(y)| dy. \quad (4.3)$$

The right hand side of (4.3) can be taken arbitrarily small in  ${}^\rho\widetilde{\mathbb{R}}_{>0}$  because  $[-\delta, \delta] \in {}^\rho\mathbb{R}$ , (iii) and because of the extreme value theorem 20 applied to the GSF  $G_t$ .  $\square$

**Lemma 37** (Fundamental Lemma of the Calculus of Variations). *Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  such that  $a < b$ , and let  $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ . If*

$$\int_a^b f(t)\eta(t) dt = 0 \text{ for all } \eta \in {}^\rho\mathcal{GC}_0^\infty, \quad (4.4)$$

then  $f = 0$ .

*Proof.* Let  $x \in [a, b]$ . Because of Thm. 8.(iv) and Lem. 4, without loss of generality we can assume that  $x$  is a sharply interior point, so that  $B_R(x) \subseteq [a, b]$  for some  $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ . Let  $\varphi \in \mathcal{D}_{[-1,1]}(\mathbb{R})$  be such that  $\int \varphi = 1$ . Set  $G_{t,\varepsilon}(x) := \frac{1}{t_\varepsilon} \varphi\left(\frac{x}{t_\varepsilon}\right)$ , where  $x \in \mathbb{R}$  and  $t \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ , and  $G_t(x) := [G_{t,\varepsilon}(x_\varepsilon)]$  for all  $x \in {}^\rho\widetilde{\mathbb{R}}$ . Then, for  $t$  sufficiently small, we have  $G_t(x - \cdot) \in {}^\rho\mathcal{GC}_0^\infty$  and (4.4) yields  $\int_a^b f(y)G_t(x-y) dy = 0$ . For  $t$  small, we both have that  $G_t(x - \cdot) = 0$  on  $[a, x-R] \cup [x+R, b]$  and the assumptions of Lem. 36 hold. Therefore

$$\begin{aligned} 0 &= \int_a^b f(y)G_t(x-y) dy = \int_{x-R}^{x+R} f(y)G_t(x-y) dy = \\ &= \int_{-R}^R f(x-y)G_t(y) dy, \end{aligned}$$

and Lem. 36 hence yields  $f(x) = 0$ .  $\square$

Thus we obtain the following

**Theorem 38.** *Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  such that  $a < b$ , and let  $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ . Then  $u$  solves the Euler-Lagrange equations*

$$F_u - \frac{d}{dt}F_{\dot{u}} = 0 \quad (4.5)$$

for  $I$  given by (4.1), if and only if  $\delta I(u; \eta) = 0$  for all  $\eta \in {}^\rho\mathcal{GC}_0^\infty$ , i.e. if and only if  $u$  is a critical point of  $I$ .

## 5. SECOND VARIATION AND MINIMIZERS

As in the classical case (see e.g. [10]), thanks to the extreme value theorem 20 and the property of the interval  $[a, b]$  of being functionally compact, we can naturally define a topology on the space  ${}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ :

**Definition 39.** Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$ , with  $a < b$ . Let  $m \in \mathbb{N}$  and  $v \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ . Then

$$\|v\|_m := \max_{\substack{n \leq m \\ 1 \leq i \leq d}} \max \left( \left| \frac{d^n}{dt^n} v^i(M_{ni}) \right|, \left| \frac{d^n}{dt^n} v^i(m_{ni}) \right| \right) \in {}^\rho\widetilde{\mathbb{R}},$$

where  $M_{ni}, m_{ni} \in [a, b]$  satisfy

$$\forall t \in [a, b] : \frac{d^n}{dt^n} v^i(m_{ni}) \leq \frac{d^n}{dt^n} v^i(t) \leq \frac{d^n}{dt^n} v^i(M_{ni}).$$

The following result permits the calculation of the (generalized) norm  $\|v\|_m$  using any net  $(v_\varepsilon)$  that defines  $v$ .

**Lemma 40.** Under the assumptions of Def. 39, let  $a = [a_\varepsilon]$  and  $b = [b_\varepsilon]$  be such that  $a_\varepsilon < b_\varepsilon$  for all  $\varepsilon$ . Then we have:

- (i) If the net  $(v_\varepsilon)$  defines  $v$ , then  $\|v\|_m = \left[ \max_{\substack{n \leq m \\ 1 \leq i \leq d}} \max_{t \in [a_\varepsilon, b_\varepsilon]} \left| \frac{d^n}{dt^n} v_\varepsilon^i(t) \right| \right]$ ;
- (ii)  $\|v\|_m \geq 0$ ;
- (iii)  $\|v\|_m = 0$  if and only if  $v = 0$ ;
- (iv)  $\forall c \in {}^\rho\widetilde{\mathbb{R}} : \|c \cdot v\|_m = |c| \cdot \|v\|_m$ ;
- (v) For all  $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ , we have  $\|u+v\|_m \leq \|u\|_m + \|v\|_m$  and  $\|u \cdot v\|_m \leq c_m \cdot \|u\|_m \cdot \|v\|_m$  for some  $c_m \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ .

*Proof.* By the standard extreme value theorem applied  $\varepsilon$ -wise, we get the existence of  $\bar{m}_{ni\varepsilon}, \bar{M}_{ni\varepsilon} \in [a_\varepsilon, b_\varepsilon]$  such that

$$\forall t \in [a_\varepsilon, b_\varepsilon] : \frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon}) \leq \frac{d^n}{dt^n} v_\varepsilon^i(t) \leq \frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon}).$$

Hence  $\left| \frac{d^n}{dt^n} v_\varepsilon^i(t) \right| \leq \max \left( \left| \frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon}) \right|, \left| \frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon}) \right| \right)$ . Thus

$$\max_{\substack{n \leq m \\ 1 \leq i \leq d}} \max_{t \in [a_\varepsilon, b_\varepsilon]} \left| \frac{d^n}{dt^n} v_\varepsilon^i(t) \right| \leq \max_{\substack{n \leq m \\ 1 \leq i \leq d}} \max \left( \left| \frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon}) \right|, \left| \frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon}) \right| \right).$$

But  $\bar{m}_{ni\varepsilon}, \bar{M}_{ni\varepsilon} \in [a_\varepsilon, b_\varepsilon]$ , so

$$\begin{aligned} \left[ \max_{\substack{n \leq m \\ 1 \leq i \leq d}} \max_{t \in [a_\varepsilon, b_\varepsilon]} \left| \frac{d^n}{dt^n} v_\varepsilon^i(t) \right| \right] &= \left[ \max_{\substack{n \leq m \\ 1 \leq i \leq d}} \max \left( \left| \frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon}) \right|, \left| \frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon}) \right| \right) \right] = \\ &= \max_{\substack{n \leq m \\ 1 \leq i \leq d}} \max \left( \left| \frac{d^n}{dt^n} v^i(\bar{m}_{ni}) \right|, \left| \frac{d^n}{dt^n} v^i(\bar{M}_{ni}) \right| \right). \end{aligned}$$

This proves both that  $\|v\|_m$  is well-defined, i.e. it does not depend on the particular choice of points  $m_{ni}, M_{ni}$  as in Def. 39, and the claim (i). The remaining properties (ii) - (v) follows directly from (i) and the usual properties of standard  $\mathcal{C}^m$ -norms.  $\square$

Using these  ${}^\rho\widetilde{\mathbb{R}}$ -valued norms, we can naturally define a topology on the space  ${}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ .

**Definition 41.** Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$ , with  $a < b$ . Let  $m \in \mathbb{N}$ ,  $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ ,  $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ , then

- (i)  $B_r^m(u) := \left\{ v \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d) \mid \|v - u\|_m < r \right\}$
- (ii) If  $U \subseteq {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ , then we say that  $U$  is a *sharply open set* if

$$\forall u \in U \exists m \in \mathbb{N} \exists r \in {}^\rho\widetilde{\mathbb{R}}_{>0} : B_r^m(u) \subseteq U.$$



As in [15, Thm. 2], one can easily prove that sharply open sets form a topology on  ${}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}}^d)$ . Using this topology, we can define when a curve is a minimizer of the functional  $I$ . Note explicitly that there are no restrictions on the generalized numbers  $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$ ,  $a < b$ . E.g. they can also both be infinite.

**Definition 42.** Let  $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$ , with  $a < b$  and let  $u \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}}^d)$ , then

(i) For all  $p, q \in {}^{\rho}\widetilde{\mathbb{R}}^d$ , we set

$${}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q) := \left\{ v \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}}^d) \mid v(a) = p, v(b) = q \right\}.$$

Note that  ${}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(0, 0) = {}^{\rho}\mathcal{GC}_0^{\infty}$ . The subscript “bd” stands here for “boundary values”.

(ii) We say that  $u$  is a *local minimizer of  $I$  in  ${}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q)$*  if  $u \in {}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q)$  and

$$\exists r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} \exists m \in \mathbb{N} \forall v \in B_r^m(u) \cap {}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q) : I(v) \geq I(u) \quad (5.1)$$

(iii) We define the *second variation of  $I$  in direction  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$*  as

$$\delta^2 I(u; \eta) := \left. \frac{d^2}{ds^2} \right|_0 I(u + s\eta).$$

Note also explicitly that the points  $p, q \in {}^{\rho}\widetilde{\mathbb{R}}^d$  can have infinite norm, e.g.  $|p_{\varepsilon}| \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ . We calculate, by using the standard Einstein’s summation conventions

$$\begin{aligned} \delta^2 I(u; \eta) &= \left. \frac{d^2}{ds^2} \right|_0 \int_a^b F(t, u + s\eta, \dot{u} + s\dot{\eta}) dt \\ &= \int_a^b \left. \frac{\partial^2}{\partial s^2} \right|_0 F(t, u + s\eta, \dot{u} + s\dot{\eta}) dt \\ &= \int_a^b F_{u^i u^j}(t, u, \dot{u}) \eta^i \eta^j + 2F_{u^i \dot{u}^j}(t, u, \dot{u}) \eta^i \dot{\eta}^j + F_{\dot{u}^i \dot{u}^j}(t, u, \dot{u}) \dot{\eta}^i \dot{\eta}^j dt, \end{aligned}$$

which we abbreviate as

$$\delta^2 I(u; \eta) = \int_a^b F_{uu}(t, u, \dot{u}) \eta \eta + 2F_{u\dot{u}}(t, u, \dot{u}) \eta \dot{\eta} + F_{\dot{u}\dot{u}}(t, u, \dot{u}) \dot{\eta} \dot{\eta} dt.$$

The following results establish classical necessary and sufficient conditions to decide if a function  $u$  is a minimizer for the given functional (4.1).

**Theorem 43.** Let  $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$  with  $a < b$ , let  $F \in {}^{\rho}\mathcal{GC}^{\infty}([a, b] \times {}^{\rho}\widetilde{\mathbb{R}}^d \times {}^{\rho}\widetilde{\mathbb{R}}^d, {}^{\rho}\widetilde{\mathbb{R}})$ , let  $p, q \in {}^{\rho}\widetilde{\mathbb{R}}^d$  and let  $u$  be a local minimizer of  $I$  in  ${}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q)$ . Then

- (i)  $\delta I(u; \eta) = 0$  for all  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ ;
- (ii)  $\delta^2 I(u; \eta) \geq 0$  for all  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ .

*Proof.* Let  $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$  be such that (5.1) holds. Since  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ , the map  $s \in {}^{\rho}\widetilde{\mathbb{R}} \mapsto u + s\eta \in {}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q)$  is well defined and continuous with respect to the sharp topology in its codomain. Therefore, we can find  $\bar{r} \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$  such that  $u + s\eta \in B_r^m(u) \cap {}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q)$  for all  $s \in B_{\bar{r}}(0)$ . We hence have  $I(u + s\eta) \geq I(u)$ . This shows that the GSF  $s \in B_{\bar{r}}(0) \mapsto I(u + s\eta) \in {}^{\rho}\widetilde{\mathbb{R}}$  has a local minimum at  $s = 0$ . Now, we employ Lem. 31 and Lem. 33 and thus the claims are proven.  $\square$

**Theorem 44.** Let  $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$  with  $a < b$  and  $p, q \in {}^{\rho}\widetilde{\mathbb{R}}^d$ . Let  $u \in {}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q)$  be such that

- (i)  $\delta I(u; \eta) = 0$  for all  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ .
- (ii)  $\delta^2 I(u; \eta) \geq 0$  for all  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$  and for all  $v \in B_r^m(u) \cap {}^{\rho}\mathcal{GC}_{\text{bd}}^{\infty}(p, q)$ , where  $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$  and  $m \in \mathbb{N}$ .

Then  $u$  is a local minimizer of the functional  $I$  in  ${}^\rho\mathcal{GC}_{\text{bd}}^\infty(p, q)$ .

Moreover, if  $\delta^2 I(v; \eta) > 0$  for all  $\eta \in {}^\rho\mathcal{GC}_0^\infty$  such that  $\|\eta\|_m > 0$  and for all  $v \in B_{2r}^m(u) \cap \mathcal{GC}_{\text{bd}}^\infty(p, q)$ , then  $I(v) > I(u)$  for all  $v \in B_r^m(u) \cap \mathcal{GC}_{\text{bd}}^\infty(p, q)$  such that  $\|v - u\|_m > 0$ .

*Proof.* For any  $v \in B_r^m(u) \cap \mathcal{GC}_{\text{bd}}^\infty(p, q)$ , we set  $\psi(s) := I(u + s(v - u)) \in {}^\rho\widetilde{\mathbb{R}}$  for all  $s \in B_1(0)$  so that  $u + s(v - u) \in B_r^m(u)$ . Since  $(v - u)(a) = 0 = (v - u)(b)$ , we have  $v - u \in {}^\rho\mathcal{GC}_0^\infty$ , and properties (i), (ii) yield  $\psi'(0) = \delta I(u; v - u) = 0$  and  $\psi''(s) = \delta^2 I(u + s(v - u); v - u) \geq 0$  for all  $s \in B_1(0)$ . We claim that  $s = 0$  is a minimum of  $\psi$ . In fact, for all  $s \in B_1(0)$  by Taylor's theorem 11

$$\psi(s) = \psi(0) + s\psi'(0) + \frac{s^2}{2}\psi''(\xi)$$

for some  $\xi \in [0, s]$ . But  $\psi'(0) = 0$  and hence  $\psi(s) - \psi(0) = \frac{s^2}{2}\psi''(\xi) \geq 0$ . Finally, Lem. 27 yields

$$\lim_{s \rightarrow 1^-} \psi(s) = I(v) \geq \psi(0) = I(u),$$

which is our conclusion. Note explicitly that if  $\delta^2 I(v; \eta) = 0$  for all  $\eta \in {}^\rho\mathcal{GC}_0^\infty$  and for all  $v \in B_r^m(u) \cap \mathcal{GC}_{\text{bd}}^\infty(p, q)$ , then  $\psi''(\xi) = 0$  and hence  $I(v) = I(u)$ .

Now, assume that  $\delta^2 I(v; \eta) > 0$  for all  $\eta \in {}^\rho\mathcal{GC}_0^\infty$  such that  $\|\eta\|_m > 0$  and for all  $v \in B_{2r}^m(u) \cap \mathcal{GC}_{\text{bd}}^\infty(p, q)$ , and take  $v \in B_r^m(u) \cap \mathcal{GC}_{\text{bd}}^\infty(p, q)$  such that  $\|v - u\|_m > 0$ . As above set  $\psi(s) := I(u + s(v - u)) \in {}^\rho\widetilde{\mathbb{R}}$  for all  $s \in B_{3/2}(0)$  so that  $u + s(v - u) \in B_{2r}^m(u)$ . We have  $\psi'(0) = 0$  and  $\psi''(s) = \delta^2 I(u + s(v - u); v - u) > 0$  for all  $s \in B_{3/2}(0)$  because  $\|v - u\|_m > 0$ . Using Taylor's theorem, we get  $\psi(1) = \psi(0) + \frac{1}{2}\psi''(\xi)$  for some  $\xi \in [0, 1]$ . Therefore  $\psi(1) - \psi(0) = I(v) - I(u) = \frac{1}{2}\psi''(\xi) > 0$ .  $\square$

**Lemma 45.** Let  $(a_k)_{k \in \mathbb{N}}$ ,  $(b_k)_{k \in \mathbb{N}}$  and  $(c_k)_{k \in \mathbb{N}}$  be sequences in  ${}^\rho\widetilde{\mathbb{R}}_{>0}$ . Assume that both  $(a_k)_k$ ,  $(b_k)_k \rightarrow 0$  and  $\frac{c_k}{a_k + b_k} \rightarrow 1$  in the sharp topology as  $k \rightarrow +\infty$ . Let  $f \in {}^\rho\mathcal{GC}^\infty([a_1, b_1], {}^\rho\widetilde{\mathbb{R}})$ . Finally, let  $a_k < t < b_k$  for all  $k \in \mathbb{N}$ , then it holds that

$$f(t) = \lim_{k \rightarrow \infty} \frac{1}{c_k} \int_{t-a_k}^{t+b_k} f(s) ds.$$

*Proof.* We can apply the integral mean value theorem for each  $\varepsilon$  and each defining net  $(f_\varepsilon)$  of  $f$  to get the existence of  $\tau_k \in [t - a_k, t + b_k]$  such that

$$\begin{aligned} f(\tau_k) &= \frac{1}{b_k + a_k} \int_{t-a_k}^{t+b_k} f(s) ds \\ &= \frac{c_k}{b_k + a_k} \frac{1}{c_k} \int_{t-a_k}^{t+b_k} f(s) ds. \end{aligned}$$

Now, we take the limit for  $k \rightarrow \infty$ , and the claim follows by assumption and by Thm. 8.(iv), i.e. by sharp continuity of  $f$ .  $\square$

We now derive the so-called necessary Legendre condition:

**Theorem 46.** Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}$  with  $a < b$  and let  $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$  be a minimizer of the functional  $I$ . Then

$$F_{\dot{u}\dot{u}}(t, u(t), \dot{u}(t))$$

is positive semi definite for all  $t \in [a, b]$ , i.e.

$$F_{\dot{u}\dot{u}}(t, u(t), \dot{u}(t))\lambda^i \lambda^j \geq 0, \quad \forall \lambda = (\lambda^1, \dots, \lambda^d) \in {}^\rho\widetilde{\mathbb{R}}^d. \quad (5.2)$$

*Proof.* Let  $\lambda = [\lambda_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^d$  and  $k, h \in \mathbb{N}$  be arbitrary. Let  $t = [t_\varepsilon] \in [a, b]$ . We can assume that  $t$  is a sharply interior point, because otherwise we can use sharp continuity of the left hand side of (5.2) and Lem. 27. We can also assume that  $\lambda$  is componentwise invertible because of Lem. 30. We want to mimic the classical

proof of [24, Thm. 1.3.2], but considering a “regularized” version of the triangular function used there (see Fig. 5.1). In particular: (1) the smoothed triangle must have an infinitesimal height which is proportional to  $\lambda$ , and we will take  $\rho_\varepsilon^k$  as this infinitesimal; (2) in the proof we need that the derivative at  $t$  is equal to  $\lambda$ , and this justifies the drawing of the peak in Fig. 5.1; (3) to regularize the singular points of the triangular function, we need a smaller infinitesimal, and we can take e.g.  $\rho_\varepsilon^{2k}$ . So, consider a net of smooth functions  $\vartheta_\varepsilon$  on  $[a_\varepsilon, b_\varepsilon]$ , such that the following properties hold:

- (i)  $\vartheta_\varepsilon(x) = 0$ , for  $x \leq t_\varepsilon - \rho_\varepsilon^k - \rho_\varepsilon^{2k}$ .
- (ii)  $\vartheta_\varepsilon(x) = 0$ , for  $x \geq t_\varepsilon + \rho_\varepsilon^k + \rho_\varepsilon^{2k}$ .
- (iii)  $\vartheta_\varepsilon(x) = \lambda(x - t_\varepsilon) + \rho_\varepsilon^k \lambda$  for  $x \in [t_\varepsilon - \rho_\varepsilon^k + \rho_\varepsilon^{2k}, t_\varepsilon]$ .
- (iv)  $\vartheta_\varepsilon(x) = -\lambda(x - t_\varepsilon) + \rho_\varepsilon^k \lambda$  for  $x \in [t_\varepsilon + \rho_\varepsilon^{2k}, t_\varepsilon + \rho_\varepsilon^k - \rho_\varepsilon^{2k}]$ .
- (v)  $|\vartheta_\varepsilon(x)| \leq \rho_\varepsilon^k \cdot |\lambda| + 2\rho_\varepsilon^{2k}|\lambda|$ .
- (vi)  $|\dot{\vartheta}_\varepsilon(x)| \leq 2|\lambda|$  for all  $x$

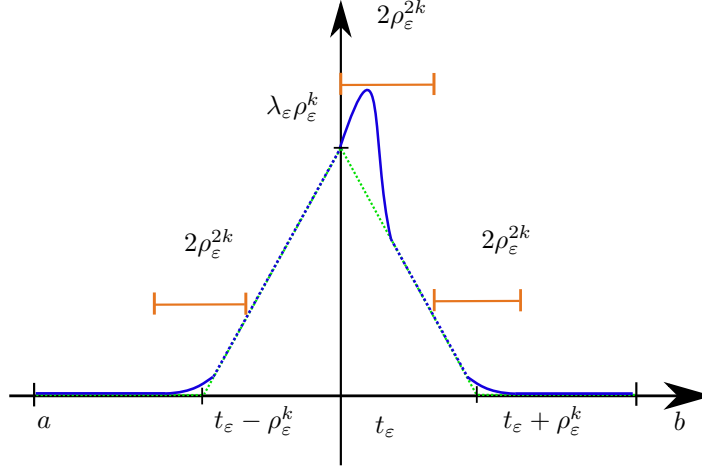


FIGURE 5.1. This figure illustrates the function  $\vartheta_\varepsilon$  we are considering (blue). The dotted green triangle symbolizes the function which is used in the classical proofs of the Legendre necessary condition (cf. [24, Thm. 1.3.2]).

The net  $(\vartheta_\varepsilon)$  defines a GSF  $\vartheta := [\vartheta_\varepsilon(-)] \in {}^\rho\mathcal{GC}_0^\infty$  because  $t$  is a sharply interior point. Setting for simplicity  $a_k := d\rho^k + d\rho^{2k}$ , by assumption we have

$$0 \leq \delta^2 I(u, \vartheta) = \int_{t-a_k}^{t+a_k} F_{uu}(t, u, \dot{u}) \vartheta \dot{\vartheta} + 2F_{\dot{u}u}(t, u, \dot{u}) \dot{\vartheta} \vartheta + F_{\dot{u}\dot{u}}(t, u, \dot{u}) \dot{\vartheta} \dot{\vartheta} dt. \quad (5.3)$$

Now, setting  $M := \max_{[a,b]} |F_{uu}(t, u, \dot{u})|$  and  $N := \max_{[a,b]} |F_{\dot{u}\dot{u}}(t, u, \dot{u})|$ , by (v) we

$$\left| \int_{t-a_k}^{t+a_k} F_{uu}(t, u, \dot{u}) \vartheta \dot{\vartheta} dt \right| \leq M \cdot |\vartheta(t)|^2 \cdot 2a_k = O(d\rho^{3k}),$$

where we used the evident notation  $G_k = O(d\rho^k)$  to denote that there exists some  $A \in {}^\rho\widetilde{\mathbb{R}}_{>0}$  such that  $G_k \leq A \cdot d\rho^k$  for all  $k \in \mathbb{N}$ . Using (v) and (vi), we analogously have

$$\left| \int_{t-a_k}^{t+a_k} F_{\dot{u}\dot{u}}(t, u, \dot{u}) \dot{\vartheta} \dot{\vartheta} dt \right| \leq 4N \cdot |\vartheta(t)| \cdot a_k \cdot |\lambda| = O(d\rho^{2k}).$$

Note that there always exists  $C \in {}^\rho\widetilde{\mathbb{R}}$  such that  $|\lambda| \leq Cd\rho^k$ . Therefore

$$\lim_{k \rightarrow +\infty} \frac{1}{2d\rho^k} \int_{t-a_k}^{t+a_k} F_{uu}(t, u, \dot{u})\vartheta\vartheta + 2F_{i\dot{u}}(t, u, \dot{u})\dot{\vartheta}\vartheta dt = 0. \quad (5.4)$$

Using Lemma 45, (5.4), (5.3) and Lem. 27, we obtain that

$$F_{i\dot{u}}(t, u(t)\dot{u}(t))\dot{\vartheta}(t)\dot{\vartheta}(t) = \lim_{k \rightarrow +\infty} \frac{1}{2d\rho^k} \int_{t-a_k}^{t+a_k} F_{i\dot{u}}(t, u, \dot{u})\dot{\vartheta}\dot{\vartheta} dt \geq 0.$$

But (iii) yields  $\dot{\vartheta}(t) = \lambda$ , and this concludes the proof.  $\square$

## 6. JACOBI FIELDS

As in the classical case, Thm. 43.(ii) motivates us to define the *accessory* integral

$$Q(\eta) := \int_a^b \psi(t, \eta, \dot{\eta}) dt \quad \forall \eta \in {}^\rho\mathcal{GC}_0^\infty, \quad (6.1)$$

where

$$\psi(t, l, v) := F_{uu}(t, u, \dot{u})ll + 2F_{u\dot{u}}(t, u, \dot{u})lv + F_{i\dot{u}}(t, u, \dot{u})vv \quad (6.2)$$

for all  $t \in [a, b]$  and  $(l, v) \in {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d$ . Note that if  $u$  minimizes  $I$ , then

$$Q(\eta) \geq 0 \quad \forall \eta \in {}^\rho\mathcal{GC}_0^\infty.$$

As usual, we note that  $\eta = 0$  is a minimizer of the functional  $Q$  and we are interested whether there are others. In order to solve this problem, we consider the *Euler-Lagrange* equations for  $Q$ , which are given by

$$\frac{d}{dt} \psi_{\dot{\eta}}(t, \eta, \dot{\eta}) = \psi_\eta(t, \eta, \dot{\eta}), \quad (6.3)$$

in other words

$$\frac{d}{dt} \{F_{i\dot{u}}(t, u, \dot{u})\dot{\eta} + F_{u\dot{u}}(t, u, \dot{u})\eta\} = F_{u\dot{u}}(t, u, \dot{u})\dot{\eta} + F_{uu}(t, u, \dot{u})\eta. \quad (6.4)$$

Since  $u$  is given, (6.4) is an  ${}^\rho\widetilde{\mathbb{R}}$ -linear system of second order equations in the unknown GSF  $\eta$  and with time dependent coefficients in  ${}^\rho\widetilde{\mathbb{R}}$ . We call (6.4) the *Jacobi equations of  $I$  with respect to  $u$* . As in the classical setting, we define

**Definition 47.** A solution  $\eta \in {}^\rho\mathcal{GC}_0^\infty$  of the Jacobi equations (6.4) is called a *Jacobi field along  $u$* .

The following result confirms that the intuitive interpretation of a Jacobi field as the tangent space of a smooth family of solutions of the Euler-Lagrange equation still holds in this generalized setting.

**Lemma 48.** *Let  $u \in {}^\rho\mathcal{GC}^\infty([-\delta, \delta] \times [a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ , where  $\delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ . We write  $u_s := u(s, -)$  for all  $s \in [-\delta, \delta]$ . Assume that each  $u_s$  satisfies the Euler-Lagrange equations (4.5):*

$$\frac{d}{dt} F_{\dot{u}}(t, u_s, \dot{u}_s) = F_u(t, u_s, \dot{u}_s) \quad \forall s \in [-\delta, \delta].$$

Then

$$\eta(t) := \left. \frac{d}{ds} \right|_0 u_s(t) \quad \forall t \in [a, b]$$

is a Jacobi field along  $u$ .

*Proof.* A straight forward calculation gives:

$$\begin{aligned} 0 &= \frac{d}{ds} \Big|_0 \left( \frac{d}{dt} F_{\dot{u}}(t, u_s, \dot{u}_s) - F_u(t, u_s, \dot{u}_s) \right) \\ &= \frac{d}{dt} (F_{\dot{u}\dot{u}}(t, u, \dot{u})\dot{\eta} + F_{u\dot{u}}(t, u, \dot{u})\eta) - F_{u\dot{u}}(t, u, \dot{u})\dot{\eta} - F_{uu}(t, u, \dot{u})\eta. \end{aligned}$$

□

**6.1. Conjugate points and Jacobi's theorem.** The classical key result concerning Jacobi fields relates conjugate points and minimizers. The main aim of the present section is to derive this theorem in our generalized framework by extending the ideas of the proof of [24, Thm. 1.3.4].

A crucial notion is hence that of piecewise GSF:

**Definition 49.** We call *piecewise GSF* an  $n$ -tuple  $(f_1, \dots, f_n)$  such that

- (i) For all  $i = 1, \dots, n$  there exist  $a_i, a_{i+1} \in {}^\rho\widetilde{\mathbb{R}}$  such that  $a_i < a_{i+1}$  and  $f_i \in {}^\rho\mathcal{GC}^\infty([a_i, a_{i+1}], {}^\rho\widetilde{\mathbb{R}}^d)$ . Note that  $[a, b] = [a', b']$  implies  $a = a'$  and  $b = b'$  because the relation  $\leq$  is antisymmetric. Therefore, the points  $a_i, a_{i+1}$  are uniquely determined by the set-theoretical function  $f_i$ .
- (ii) For all  $i = 1, \dots, n$ , we have  $f_i(a_{i+1}) = f_{i+1}(a_{i+1})$ .

Every pointwise GSF  $(f_1, \dots, f_n)$  defines a set-theoretical function:

- (iii) For all  $t \in \bigcup_{i=1}^n [a_i, a_{i+1}]$ , we set  $(f_1, \dots, f_n)(t) := f_i(t)$  if  $t \in [a_i, a_{i+1}]$ .

We also use the arrow notation  $(f_1, \dots, f_n) : \bigcup_{i=1}^n [a_i, a_{i+1}] \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$  to say that both (i) and (ii) hold.

*Remark 50.*

- (i) Clearly,  $t \in [a_i, a_{i+1}] \cap [a_{i+1}, a_{i+2}]$  implies  $t = a_{i+1}$ , so that condition (ii) yields that the evaluation (iii) is well defined.
- (ii) Since the order relation  $\leq$  is not a total one, we do not have that  $[a_i, a_{i+1}] \cup [a_{i+1}, a_{i+2}] = [a_i, a_{i+2}]$ .
- (iii) If  $\nu : [a_1, a_2] \cup [a_2, a_3] \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$  is a set-theoretical function originating from a piecewise GSF  $(f_1, f_2)$ , then neither the GSF  $f_i$  nor the points  $a_i$  are uniquely determined by  $\nu$ . For this reason, we prefer to stress our notations with symbols like  $(f_1, f_2)(t) \in {}^\rho\widetilde{\mathbb{R}}^d$ .
- (iv) Every GSF  $f \in {}^\rho\mathcal{GC}^\infty([a_1, a_2], {}^\rho\widetilde{\mathbb{R}}^d)$  can be seen as a particular case of a piecewise GSF.
- (v) If  $(g_1, \dots, g_n), (f_1, \dots, f_n) : \bigcup_{i=1}^n [a_i, a_{i+1}] \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$  and  $r \in {}^\rho\widetilde{\mathbb{R}}$ , then also  $(g_1, \dots, g_n) + (f_1, \dots, f_n) := (g_1 + f_1, \dots, g_n + f_n)$  and  $r \cdot (f_1, \dots, f_n) := (r \cdot f_1, \dots, r \cdot f_n)$  are piecewise GSF, and we hence have a structure of  ${}^\rho\widetilde{\mathbb{R}}$ -module.
- (vi) If  $(f_1, \dots, f_n) : \bigcup_{i=1}^n [a_i, a_{i+1}] \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$  and  $F \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}}^n)$ , then we can define the composition

$$F \circ (f_1, \dots, f_n) := (F \circ f_1, \dots, F \circ f_n) : \bigcup_{i=1}^n [a_i, a_{i+1}] \longrightarrow {}^\rho\widetilde{\mathbb{R}}^n.$$

Piecewise GSF inherit from their defining components a well-behaved differential and integral calculus. The former is even more general and taken from [1].

**Definition 51.** Let  $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ , then

- (i)  $\nu(x) := \sup \{b \in \mathbb{R} \mid |x_\varepsilon| = O(\rho_\varepsilon^b)\} \in \mathbb{R} \cup \{+\infty\}$ .
- (ii)  $|x|_e := e^{-\nu(x)} \in \mathbb{R}_{\geq 0}$ .
- (iii)  $d\rho(x) := d\rho^{-\log |x|_e} \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ .

It is worth noting that  $|\cdot|_e : {}^\rho\widetilde{\mathbb{R}} \rightarrow \mathbb{R}_{\geq 0}$  induces an ultrametric on  ${}^\rho\widetilde{\mathbb{R}}$  that generates exactly the sharp topology, see e.g. [2, 13] and references therein. However, we will not use this ultrametric structure in the present paper, and we only introduced it to get an invertible infinitesimal  $d\rho(x)$  that goes to zero with  $x$ : it is in fact easy to show that

$$\lim_{x \rightarrow 0} \frac{x}{d\rho(x)} = 1$$

in the sharp topology.

**Definition 52.** Let  $T \subseteq {}^\rho\widetilde{\mathbb{R}}$  and let  $f : T \rightarrow {}^\rho\widetilde{\mathbb{R}}^d$  be an arbitrary set-theoretical function. Let  $t_0 \in T$  be a sharply interior point of  $T$ . Then we say that  $f$  is *differentiable at  $t_0$*  if<sup>2</sup>

$$\exists m \in {}^\rho\widetilde{\mathbb{R}}^d : \lim_{h \rightarrow 0} \frac{f(t_0 + h) - f(t_0) - m \cdot h}{d\rho(h)} = 0. \quad (6.5)$$

In this case, using Landau little-oh notation, we can hence write

$$f(t_0 + h) = f(t_0) + m \cdot h + o(d\rho(h)) \quad \text{as } h \rightarrow 0. \quad (6.6)$$

As in the classical case, (6.6) implies the uniqueness of  $m \in {}^\rho\widetilde{\mathbb{R}}^d$ , so that we can define  $f'(t_0) := \dot{f}(t_0) := m$ , and the usual elementary rules of differential calculus. By the Fermat-Reyes theorem, this definition of derivative generalizes that given for GSF.

In particular, this notion of derivative applies to the set-theoretical function induced by a piecewise GSF  $(f_1, \dots, f_n)$ . We therefore have that  $(f_1, \dots, f_n)(-)$  is differentiable at each  $a_i < t < a_{i+1}$ , and  $(f_1, \dots, f_n)'(t) = f'_i(t)$ , but clearly there is no guarantee that  $(f_1, \dots, f_n)(-)$  is also differentiable at each point  $a_i$ .

The notion of definite integral is naturally introduced in the following

**Definition 53.** Let  $(f_1, \dots, f_n) : \bigcup_{i=1}^n [a_i, a_{i+1}] \rightarrow {}^\rho\widetilde{\mathbb{R}}^d$  be a piecewise GSF, then

$$\int_{a_1}^{a_{n+1}} (f_1, \dots, f_n)(t) dt := \sum_{i=1}^n \int_{a_i}^{a_{i+1}} f_i(t) dt.$$

Since our main aim in using piecewise GSF is to prove Jacobi's theorem, we do not need to prove that the usual elementary rules of integration hold, since we will always reduce to integrals of GSF.

Having a notion of derivative and of definite integral, also for piecewise GSF we can consider functionals

$$\nu := (f_1, \dots, f_n), \quad a_1 = a, \quad a_n = b \quad \implies \quad I(\nu) := \int_a^b F(t, \nu(t), \dot{\nu}(t)) dt \in {}^\rho\widetilde{\mathbb{R}}, \quad (6.7)$$

and the concept of *piecewise GSF (global) minimizer*:  $I(\nu) \leq I(\tilde{\nu})$  for all  $\tilde{\nu} \in {}^\rho\mathcal{GC}_0^\infty$ . For the proof of Jacobi's theorem, we will only need this particular notion of global minimizer. Note explicitly that in (6.7) we only need existence of right and left derivatives of GSF, because of Def. 53 and of Def. 14 of definite integral of GSF.

Classically, several proofs of Jacobi's theorem use both some form of implicit function theorem and of uniqueness of solution for linear ODE.

**Theorem 54** (Implicit function theorem). *Let  $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ ,  $V \subseteq {}^\rho\widetilde{\mathbb{R}}^d$  be sharply open sets. Let  $F \in {}^\rho\mathcal{GC}^\infty(U \times V, {}^\rho\widetilde{\mathbb{R}}^d)$  and  $(x_0, y_0) \in U \times V$ . If  $\partial_2 F(x_0, y_0)$  is invertible in  $L({}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}}^d)$ , then there exists a sharply open neighbourhood  $U_1 \times V_1 \subseteq U \times V$  of  $(x_0, y_0)$  such that*

$$\forall x \in U_1 \exists! y_x \in V_1 : F(x, y_x) = F(x_0, y_0). \quad (6.8)$$

<sup>2</sup>This definition is based on [1, Def. 2.2].

Moreover, the function  $f(x) := y_x$  for all  $x \in U_1$  is a GSF  $f \in {}^\rho\mathcal{GC}^\infty(U_1, V_1)$  and satisfies

$$Df(x) = -(\partial_2 F(x, f(x)))^{-1} \circ \partial_1 F(x, f(x)). \quad (6.9)$$

*Proof.* The usual deduction of the implicit function theorem from the inverse function theorem in Banach spaces can be easily adapted using Thm. 12 and noting that  $\det[\partial_2 F(-, -)]$  is a GSF such that  $|\det[\partial_2 F(x_0, y_0)]| \in {}^\rho\mathbb{R}_{>0}$ .  $\square$

In the next theorem, the dependence of the entire theory on the initial infinitesimal net  $\rho = (\rho_\varepsilon) \downarrow 0$  plays an essential role. Indirectly, the same important role will reverberate in the final Jacobi's theorem.

**Theorem 55** (Solution of first order linear ODE). *Let  $A \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^{d \times d})$ , where  $a, b \in {}^\rho\widetilde{\mathbb{R}}$ ,  $a < b$ , and  $t_0 \in [a, b]$ ,  $y_0 \in {}^\rho\widetilde{\mathbb{R}}^d$ . Assume that*

$$\left| \int_{t_0}^t A(s) \, ds \right| \leq -C \cdot \log d\rho \quad \forall t \in [a, b], \quad (6.10)$$

where  $C \in \mathbb{R}_{>0}$ . Then there exists one and only one  $y \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$  such that

$$\begin{cases} y'(t) = A(t) \cdot y(t) & \text{if } t \in [a, b] \\ y(t_0) = y_0 \end{cases} \quad (6.11)$$

Moreover, this  $y$  is given by  $y(t) = \exp\left(\int_{t_0}^t A(s) \, ds\right) \cdot y_0$  for all  $t \in [a, b]$ .

*Proof.* We first note that  $\exp\left(\int_{t_0}^t A(s) \, ds\right) = \left[\exp\left(\int_{t_0\varepsilon}^{t\varepsilon} A_\varepsilon(s) \, ds\right)\right]$ , where  $t = [t_\varepsilon]$ ,  $t_0 = [t_{0\varepsilon}]$  and  $A(s) = [A_\varepsilon(s_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^{d \times d}$ . This exponential matrix in  ${}^\rho\widetilde{\mathbb{R}}^{d \times d}$  is a GSF because for all  $t \in [a, b]$ , we have

$$\exp\left(\int_{t_0}^t A(s) \, ds\right) \leq e^{-C \log d\rho} \leq d\rho^{-C}.$$

Therefore, all values of  $y(t) = \exp\left(\int_{t_0}^t A(s) \, ds\right) \cdot y_0$  are  $\rho$ -moderate. Analogously, one can prove that also  $y^{(k)}(t)$  are moderate for all  $k \in \mathbb{N}$  and  $t \in [a, b]$ . Considering that derivatives can be calculated  $\varepsilon$ -wise, we have that this GSF  $y$  satisfies (6.11), and this proves the existence part.

To show uniqueness, we can proceed as in the smooth case. Assume that  $z \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$  satisfies (6.11), and set  $h(t) := \exp\left(-\int_{t_0}^t A(s) \, ds\right)$  for all  $t \in [a, b]$ . Since  $h' = -A \cdot h$ , we have  $(hz)' = h'z + hz' = -Ahz + hAz = -Ahz + Ahz = 0$ . From uniqueness of primitives of GSF, Thm. 13, we have that  $h \cdot z = h(t_0) \cdot z(t_0) = y_0$ . Therefore  $z = h^{-1} \cdot y_0$ .  $\square$

If  $\alpha, \beta \in {}^\rho\widetilde{\mathbb{R}}$ , we write  $\alpha = O_{\mathbb{R}}(\beta)$  to denote that there exists  $C \in \mathbb{R}_{>0}$  such that  $|\alpha| \leq C \cdot |\beta|$ . Therefore, assumption (6.10) can be written as  $\int_{t_0}^t A(s) \, ds = O_{\mathbb{R}}(\log d\rho)$ . Note that this assumption is weaker, in general, than

$$(b-a) \cdot \max_{t \in [a, b]} |A(t)| = O_{\mathbb{R}}(\log d\rho).$$

The following result is the key regularity property we need to prove Jacobi's theorem.

**Lemma 56.** *Let  $a, a', b \in {}^\rho\widetilde{\mathbb{R}}$ , with  $a < a' < b$ , and let  $K \in {}^\rho\mathcal{GC}^\infty([a, b] \times {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$ . Let  $\nu = (\eta, \beta) : [a, a'] \cup [a', b] \rightarrow {}^\rho\widetilde{\mathbb{R}}^d$  be a piecewise GSF which satisfies the Euler-Lagrange equation*

$$K_u(t, \nu(t), \dot{\nu}(t)) - \frac{d}{dt} K_{\dot{u}}(t, \nu(t), \dot{\nu}(t)) = 0 \quad \forall t \in [a, a'] \cup (a', b]. \quad (6.12)$$

Finally, assume that  $\det (K_{\dot{u}_i \dot{u}_j}(a', \eta(a'), \dot{\eta}(a'))_{i,j=i,\dots,d}) \in {}^{\rho}\widetilde{\mathbb{R}}$  is invertible, then

$$\lim_{\substack{t \rightarrow a' \\ t < a'}} \dot{\nu}(t) = \lim_{\substack{t \rightarrow a' \\ a' < t}} \dot{\nu}(t) = \dot{\eta}(a'). \quad (6.13)$$

In particular, if  $\beta \equiv 0|_{[a',b]}$ , then  $\dot{\eta}(a') = 0$ .

*Proof.* Set  $\Phi(t, l, v, q) := K_{\dot{u}}(t, l, v) - q$  for all  $t \in [a, b]$  and all  $l, v, q \in {}^{\rho}\widetilde{\mathbb{R}}^d$ . For simplicity, set  $(t_0, l_0, v_0, q_0) := (a', \eta(a'), \dot{\eta}(a'), K_{\dot{u}}(a', \eta(a'), \dot{\eta}(a'))$ . Our assumption on the invertibility of  $K_{\dot{u}\dot{u}}(a', \eta(a'), \dot{\eta}(a')) = \partial_v \Phi(t_0, l_0, v_0, q_0)$  makes it possible to apply the implicit function Thm. 54 to conclude that there exists a neighbourhood  $T \times L \times V \times Q$  of  $(t_0, l_0, v_0, q_0)$  such that

$$\forall (t, l, q) \in T \times L \times Q \exists ! v \in V : \Phi(t, l, v, q) = \Phi(t_0, l_0, v_0, q_0). \quad (6.14)$$

But  $\Phi(t_0, l_0, v_0, q_0) = K_{\dot{u}}(a', \eta(a'), \dot{\eta}(a')) - q_0 = 0$ . Moreover, the unique function  $\varphi$  defined by  $\Phi(t, l, \varphi(t, l, q), q) = 0$  for all  $(t, l, q) \in T \times L \times Q$  is a GSF  $\varphi \in {}^{\rho}\mathcal{GC}^{\infty}(T \times L \times Q, V)$ . Now, for all  $t \in [a, a'] \cup (a', b]$ , we have

$$\Phi(t, \nu(t), \dot{\nu}(t), K_{\dot{u}}(t, \nu(t), \dot{\nu}(t))) = K_{\dot{u}}(t, \nu(t), \dot{\nu}(t)) - K_{\dot{u}}(t, \nu(t), \dot{\nu}(t)) = 0.$$

Therefore, uniqueness in (6.14) yields

$$\dot{\nu}(t) = \varphi(t, \nu(t), K_{\dot{u}}(t, \nu(t), \dot{\nu}(t))) \quad \forall t \in [a, a'] \cup (a', b]. \quad (6.15)$$

We now integrate the Euler-Lagrange equation (6.12) on  $[a, t]$ , obtaining

$$K_{\dot{u}}(t, \nu(t), \dot{\nu}(t)) = \int_a^t K_u(s, \nu(s), \dot{\nu}(s)) ds + K_{\dot{u}}(a, \eta(a), \dot{\eta}(a)) \quad \forall t \in [a, a'] \cup (a', b].$$

This entails that we can write

$$\dot{\nu}(t) = \varphi \left( t, \nu(t), \int_a^t K_u(s, \nu(s), \dot{\nu}(s)) ds + K_{\dot{u}}(a, \eta(a), \dot{\eta}(a)) \right) \quad \forall t \in [a, a'] \cup (a', b]. \quad (6.16)$$

But the function  $t \in [a, a'] \cup (a', b] \mapsto \int_a^t K_u(s, \nu(s), \dot{\nu}(s)) ds \in {}^{\rho}\widetilde{\mathbb{R}}^d$  has equal limits on the left and on the right of  $a'$  because on  $[a, a']$  and on  $(a', b]$  it is a GSF; in fact for  $t < a'$  we have

$$\left| \int_a^t K_u(s, \nu(s), \dot{\nu}(s)) ds - \int_a^{a'} K_u(s, \nu(s), \dot{\nu}(s)) ds \right| \leq \max_{t \in [a, a']} |K_u(s, \eta(s), \dot{\eta}(s))| \cdot |t - a'|,$$

and this goes to 0 as  $t \rightarrow a'$ ,  $t < a'$ . Analogously we can proceed for  $t > a'$  using  $\beta$ . Therefore

$$\lim_{\substack{t \rightarrow a' \\ t < a'}} \int_a^t K_u(s, \nu(s), \dot{\nu}(s)) ds = \lim_{\substack{t \rightarrow a' \\ t > a'}} \int_a^t K_u(s, \nu(s), \dot{\nu}(s)) ds.$$

Applying this equality in (6.16), we get  $\lim_{\substack{t \rightarrow a' \\ t < a'}} \dot{\nu}(t) = \dot{\eta}(a') = \lim_{\substack{t \rightarrow a' \\ a' < t}} \dot{\nu}(t)$  as claimed. Finally, if  $\beta \equiv 0|_{[a',b]}$ , then  $\lim_{\substack{t \rightarrow a' \\ a' < t}} \dot{\nu}(t) = 0$ .  $\square$

In the following definition and below, we use the complete notation  ${}^{\rho}\mathcal{GC}_0^{\infty}(a, a')$  (see Def. 34).

**Definition 57.** Let  $a, a', b \in {}^{\rho}\widetilde{\mathbb{R}}$ , where  $a < a' < b$ . We call  $a'$  conjugate to  $a$  with respect to the variational problem (4.1) if there exists a non identically vanishing Jacobi field  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}(a, a')$  along  $u|_{[a, a']}$  such that  $\eta(a) = 0 = \eta(a')$ , where  $\psi$  is given by (6.2).



Jacobi's theorem shows that we cannot have minimizers if there are interior points which are conjugate to  $a$ . In order to prove it in the present generalized context, we finally need the following

**Lemma 58.** *Let  $u \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}}^d)$  and let  $a' \in (a, b)$ . Let  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}(a, a')$  be a Jacobi field along  $u|_{[a, a']}$ , with  $\eta(a) = 0 = \eta(a')$ . Then*

$$\int_a^{a'} \psi(t, \eta, \dot{\eta}) dt = 0.$$

*Proof.* Since  $\psi$  is  ${}^{\rho}\widetilde{\mathbb{R}}$ -homogeneous of second order in  $(\eta, \dot{\eta})$ , we have

$$2\psi(t, \eta, \dot{\eta}) = \psi_{\eta}(t, \eta, \dot{\eta})\eta + \psi_{\dot{\eta}}(t, \eta, \dot{\eta})\dot{\eta}.$$

Thus we calculate:

$$\begin{aligned} 2 \int_a^{a'} \psi(t, \eta, \dot{\eta}) dt &= \int_a^{a'} \eta \psi_{\eta}(t, \eta, \dot{\eta}) + \dot{\eta} \psi_{\dot{\eta}}(t, \eta, \dot{\eta}) dt \\ &= \int_a^{a'} \eta \left( \psi_{\eta}(t, \eta, \dot{\eta}) - \frac{d}{dt} \psi_{\dot{\eta}}(t, \eta, \dot{\eta}) \right) dt \text{ by integration by parts} \\ &= 0 \text{ since } \eta \text{ is a Jacobi field.} \end{aligned}$$

□

After these preparations, we can finally prove

**Theorem 59 (Jacobi).** *Let  $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$  be such that  $a < b$ . Suppose that  $F \in {}^{\rho}\mathcal{GC}^{\infty}([a, b] \times {}^{\rho}\widetilde{\mathbb{R}}^d \times {}^{\rho}\widetilde{\mathbb{R}}^d, {}^{\rho}\widetilde{\mathbb{R}})$  and  $u \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}})$  are such that*

- (i)  $a' \in (a, b)$  is conjugate to  $a$
- (ii)  $\det F_{\dot{u}\dot{u}}(t, u(t), \dot{u}(t)) \in {}^{\rho}\widetilde{\mathbb{R}}$  is invertible for all  $t \in [a, b]$ .
- (iii) For all  $t \in [a, a']$

$$\begin{aligned} \int_{a'}^t F_{\dot{u}\dot{u}}^{-1}(s, u(s), \dot{u}(s)) \cdot \left[ \frac{d}{ds} F_{u\dot{u}}(s, u(s), \dot{u}(s)) - F_{uu}(s, u(s), \dot{u}(s)) \right] ds = \\ = O_{\mathbb{R}}(\log d\rho) \end{aligned}$$

$$\int_{a'}^t F_{\dot{u}\dot{u}}^{-1}(s, u(s), \dot{u}(s)) \cdot \frac{d}{ds} F_{\dot{u}\dot{u}}(s, u(s), \dot{u}(s)) ds = O_{\mathbb{R}}(\log d\rho).$$

Then  $u$  cannot be a local minimizer of  $I$ . Therefore, for any  $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$  there exists  $v \in {}^{\rho}\mathcal{GC}_{ba}^{\infty}(u(a), u(b))$  and  $m \in \mathbb{N}$  such that  $\|v - u\|_m < r$  but  $I(u) \not\leq I(v)$ .

*Proof.* By contradiction, assume that  $u$  is a local minimizer, and let  $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}(a, a')$  be a Jacobi field along  $u|_{[a, a']}$  such that the conditions from Def. 57 hold for  $\eta$ . We want to prove that  $\eta \equiv 0$ . Define  $\nu := (\eta, 0|_{[a', b]})$ , which is a piecewise GSF since  $\eta(a') = 0$ . Since also  $\eta(a) = 0$ , Lem. 58 and homogeneity of  $\psi$  yield

$$Q(\nu) = \int_a^b \psi(t, \nu(t), \dot{\nu}(t)) dt = \int_a^{a'} \psi(t, \eta(t), \dot{\eta}(t)) dt + \int_{a'}^b \psi(t, 0, 0) dt = 0.$$

Therefore, Thm. 43 (necessary condition for  $u$  being a minimizer) gives  $Q(\tilde{\nu}) \geq 0 = Q(\nu)$  for all  $\tilde{\nu} \in {}^{\rho}\mathcal{GC}_0^{\infty}(a, b)$ . Thus,  $\nu$  is a minimizer of the functional  $Q$ . Since  $\nu$  is only a piecewise GSF, we cannot directly apply Thm. 38 (Euler-Lagrange equations). But for all  $\varphi \in {}^{\rho}\mathcal{GC}_0^{\infty}(a, b)$  and all  $s \in {}^{\rho}\widetilde{\mathbb{R}}$ , we have

$$\begin{aligned} Q(\nu + s\varphi) &= \int_a^b \psi(t, \nu + s\varphi, \dot{\nu} + s\dot{\varphi}) dt \\ &= \int_a^{a'} \psi(t, \eta + s\varphi, \dot{\eta} + s\dot{\varphi}) dt + \int_{a'}^b \psi(t, s\varphi, s\dot{\varphi}) dt. \end{aligned} \quad (6.17)$$

This shows that  $s \in {}^\rho\widetilde{\mathbb{R}} \mapsto Q(\nu + s\varphi) \in {}^\rho\widetilde{\mathbb{R}}$  is a GSF, and hence  $s = 0$  is a minimum for this function. By Lem. 31 and (6.17), we get

$$\begin{aligned} \delta Q(\nu, \varphi) &= 0 = \left. \frac{d}{ds} Q(\nu + s\varphi) \right|_0 \\ &= \int_a^{a'} \left( \psi_\eta(t, \eta, \dot{\eta}) - \frac{d}{dt} \psi_{\dot{\eta}}(t, \eta, \dot{\eta}) \right) \varphi dt + \\ &\quad \int_{a'}^b (\varphi \psi_\eta(t, 0, 0) + \dot{\varphi} \psi_{\dot{\eta}}(t, 0, 0)) dt \\ &= \int_a^{a'} \left( \psi_\eta(t, \eta, \dot{\eta}) - \frac{d}{dt} \psi_{\dot{\eta}}(t, \eta, \dot{\eta}) \right) \varphi dt. \end{aligned}$$

By the fundamental Lem. 37, this implies that  $\eta$  satisfies the Euler-Lagrange equations for  $\psi$  in the interval  $[a, a']$ . Therefore,  $\nu$  satisfies the same equations in  $[a, a'] \cup (a', b]$ . Moreover,  $\psi_{\dot{\eta}\dot{\eta}}(a', \eta(a'), \dot{\eta}(a')) = F_{\dot{u}\dot{u}}(a', u(a'), \dot{u}(a'))$  is invertible by assumption (ii). Thus, all the hypotheses of the regularity Lem. 56 hold, and we derive that  $\dot{\eta}(a') = 0$ .

For all  $t \in [a, b]$ , we define

$$\begin{aligned} \xi(t) &:= -F_{\dot{u}\dot{u}}^{-1} \cdot \left[ \frac{d}{dt} F_{u\dot{u}}(t, u, \dot{u}) - F_{uu}(t, u, \dot{u}) \right], \text{ and} \\ \vartheta(t) &:= -F_{\dot{u}\dot{u}}^{-1} \cdot \frac{d}{dt} F_{u\dot{u}}(t, u, \dot{u}), \end{aligned}$$

so that we can re-write the Jacobi equations (6.4) for  $\eta$  on  $[a, a']$  as a system of first order ODE

$$\begin{cases} \dot{y} := \begin{pmatrix} \dot{\eta} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \xi & \vartheta \end{pmatrix} \cdot \begin{pmatrix} \eta \\ z \end{pmatrix} =: A \cdot y & \forall t \in [a, a'] \\ y(a') = \begin{pmatrix} \eta(a') \\ \dot{\eta}(a') \end{pmatrix} = 0 \end{cases}$$

By assumptions (iii), we obtain  $\int_a^t A(t) = O_{\mathbb{R}}(\log d\rho)$  for all  $t \in [a, a']$ , and we can hence apply Thm. 55 obtaining  $y \equiv 0$  and thus  $\eta \equiv 0$ .  $\square$

Note that if one of the quantities in (iii) depends even only polynomially on  $\varepsilon$ , then we are forced to take e.g.  $\rho_\varepsilon = \varepsilon^{1/\varepsilon}$  to fulfill this assumption. This underlines the importance of the parameter  $\rho$  making the entire theory dependent on the parameter  $\rho$ , in order to avoid unnecessary constraints on the scope of the functionals we look upon.

## 7. NOETHER'S THEOREM

In this section, we state and prove Noether's theorem following the lines of [3]. We first note that any  $X \in {}^\rho\mathcal{GC}^\infty(J \times X, Y)$ , where  $J \subseteq {}^\rho\widetilde{\mathbb{R}}$ , can also be considered as a family in GSF which smoothly depends on the parameter  $s \in J$ . In this case, we hence say that  $(X_s)_{s \in J}$  is a *generalized smooth family in  ${}^\rho\mathcal{GC}^\infty(X, Y)$* . In particular, we can reformulate in the language of GSF the classical definition of *one-parameter group of generalized diffeomorphisms of  $X$*  as follows:

- (i)  $(X_s)_{s \in {}^\rho\widetilde{\mathbb{R}}}$  is a generalized smooth family in  ${}^\rho\mathcal{GC}^\infty(X, X)$ ,
- (ii) For all  $s \in {}^\rho\widetilde{\mathbb{R}}$ , the map  $X_s : X \rightarrow X$  is invertible, and  $X_s^{-1} \in {}^\rho\mathcal{GC}^\infty(X, X)$ ,
- (iii)  $X_0(x) = x$  for all  $x \in X$ ,
- (iv)  $X_s \circ X_t = X_{s+t}$  for all  $s, t \in {}^\rho\widetilde{\mathbb{R}}$ .

In our proofs, we will in fact only use properties (i) and (iii).

The proof of Noether's theorem is classically anticipated by the following time-independent version, which the general case is subsequently reduced to.

**Theorem 60.** *Let  $K \in {}^\rho\mathcal{GC}^\infty(L \times V, {}^\rho\widetilde{\mathbb{R}})$ , where  $L, V \subseteq {}^\rho\widetilde{\mathbb{R}}^n$  are sharply open sets. Let  $w \in {}^\rho\mathcal{GC}^\infty((a, b), L)$  be a solution of the Euler-Lagrange equation corresponding to  $K$ , i.e. for all  $t \in (a, b)$*

$$\dot{w}(t) \in V, \quad K_u(w(t), \dot{w}(t)) = \frac{d}{dt} K_{\dot{u}}(w(t), \dot{w}(t)). \quad (7.1)$$

Suppose that  $0$  is a sharply interior point of  $J \subseteq {}^\rho\widetilde{\mathbb{R}}$  and  $(X_s)_{s \in J}$  is a generalized smooth family in  ${}^\rho\mathcal{GC}^\infty(L, L)$ , such that for all  $t \in (a, b)$

- (i)  $\frac{\partial}{\partial t} X_s(w(t)) \in V$ ,
- (ii)  $X_0(w(t)) = w(t)$ ,
- (iii)  $K$  is invariant under  $(X_s)_{s \in J}$  along  $w$ , i.e.

$$K(w(t), \dot{w}(t)) = K\left(X_s(w(t)), \frac{\partial}{\partial t} X_s(w(t))\right) \quad \forall s \in J. \quad (7.2)$$

Then, the quantity

$$K_{\dot{u}^j}(w(t), \dot{w}(t)) \frac{\partial}{\partial s} \Big|_{s=0} X_s^j(w(t)) \in {}^\rho\widetilde{\mathbb{R}}$$

is constant in  $t \in (a, b)$ .

*Proof.* We first note that both sides of (7.2) are in  ${}^\rho\mathcal{GC}^\infty((a, b), {}^\rho\widetilde{\mathbb{R}})$ . Let  $\tau \in (a, b)$  be arbitrary but fixed. Since  $s = 0 \in J$  is a sharply interior point, we can consider  $\frac{d}{ds} \Big|_{s=0}$ . We obtain

$$\begin{aligned} 0 &\stackrel{(7.2)}{=} \frac{\partial}{\partial s} \Big|_{s=0} K\left(X_s(w), \frac{\partial}{\partial t} X_s(w)\right) \\ &\stackrel{(ii)}{=} \int_a^\tau K_u(w, \dot{w}) \frac{\partial}{\partial s} \Big|_{s=0} X_s(w) + K_{\dot{u}}(w, \dot{w}) \frac{\partial}{\partial t} \frac{\partial}{\partial s} \Big|_{s=0} X_s(w) dt. \end{aligned}$$

Since the Euler-Lagrange equations (7.1) for  $K$  are given by  $K_u(w, \dot{w}) = \frac{d}{dt} K_{\dot{u}}(w, \dot{w})$  we have

$$\begin{aligned} 0 &= \frac{d}{dt} (K_{\dot{u}}(w, \dot{w})) \frac{\partial}{\partial s} \Big|_{s=0} X_s(w) + K_{\dot{u}}(w, \dot{w}) \frac{\partial}{\partial t} \frac{\partial}{\partial s} \Big|_{s=0} X_s(w) \\ &= \frac{d}{dt} \left( K_{\dot{u}}(w, \dot{w}) \frac{\partial}{\partial s} \Big|_{s=0} X_s(w) \right). \end{aligned}$$

Which is our conclusion by the uniqueness - part of Thm. 13.  $\square$

We are now able to prove Noether's theorem. For the convenience of the reader, in its statement and proof we use the variables  $t, T, l, L, v, V$  so as to recall *tempus, locus, velocitas* resp.

**Theorem 61** (E. Noether).

Let  $a, b \in {}^\rho\widetilde{\mathbb{R}}^d$ , with  $a < b$ , and  $F \in {}^\rho\mathcal{GC}^\infty([a, b] \times {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$ . Let  $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$  be a solution of the Euler-Lagrange equation (4.5) corresponding to  $F$ . Suppose that  $0$  is a sharply interior point of  $J \subseteq {}^\rho\widetilde{\mathbb{R}}$  and  $(X_s)_{s \in J}$  is a generalized smooth family in  ${}^\rho\mathcal{GC}^\infty((a, b) \times {}^\rho\widetilde{\mathbb{R}}^d, (a, b) \times {}^\rho\widetilde{\mathbb{R}}^d)$ . We denote by  $T_s(t, l) := X_s^1(t, l) \in (a, b)$  and  $L_s(t, l) := X_s^2(t, l) \in {}^\rho\widetilde{\mathbb{R}}^d$  for all  $(t, l) \in (a, b) \times {}^\rho\widetilde{\mathbb{R}}^d$ , the two projections of  $X_s$  on  $(a, b)$  and  ${}^\rho\widetilde{\mathbb{R}}^d$  resp. We assume that for all  $t \in (a, b)$

- (i)  $\frac{\partial}{\partial t} T_s(t, u(t)) \in {}^\rho\widetilde{\mathbb{R}}$  is invertible,

- (ii)  $T_0(t, u(t)) = t$  and  $L_0(t, u(t)) = u(t)$ ,  
 (iii)  $F(t, u(t), \dot{u}(t)) = F \left[ T_s(t, u), L_s(t, u), \frac{\partial}{\partial t} L_s(t, u), \frac{\partial}{\partial t} T_s(t, u) \right] \cdot \frac{\partial}{\partial t} T_s(t, u)$  for all  $s \in J$ .

Then, the quantity

$$F_{\dot{u}^j}(t, u(t), \dot{u}(t)) \frac{\partial}{\partial s} \Big|_{s=0} L_s^j(t, u(t)) + [F(t, u(t), \dot{u}(t)) - F_{\dot{u}^k}(t, u(t), \dot{u}(t)) \dot{u}^k(t)] \frac{\partial}{\partial s} \Big|_{s=0} T_s(t, u(t)) \quad (7.3)$$

is constant in  $t \in [a, b]$ .

*Proof.* Since (7.3) is a GSF in  $t \in [a, b]$ , by sharp continuity it suffices to prove the claim for all  $t \in (a, b)$ . Set  $L := (a, b) \times {}^\rho\widetilde{\mathbb{R}}^d$ ,  $V := {}^\rho\widetilde{\mathbb{R}}^* \times {}^\rho\widetilde{\mathbb{R}}^d$  (we recall that  ${}^\rho\widetilde{\mathbb{R}}^*$  denotes the set of all invertible generalized numbers in  ${}^\rho\widetilde{\mathbb{R}}$ ). Define  $K \in {}^\rho\mathcal{GC}^\infty(L \times V, {}^\rho\widetilde{\mathbb{R}})$  by

$$K(t, l; p, v) := F \left( t, l, \frac{v}{p} \right) \cdot p \quad \forall (t, l) \in L \forall (p, v) \in V, \quad (7.4)$$

and  $w \in {}^\rho\mathcal{GC}^\infty((a, b), L)$  by  $w(t) := (t, u(t))$  for all  $t \in (a, b)$ . We note that  $L, V \subseteq {}^\rho\widetilde{\mathbb{R}}^{d+1}$  are sharply open subsets and that  $\dot{w}(t) = (1, \dot{u}(t)) \in V$ . The notations for partial derivatives used in the present work result from the symbolic writing  $K(u^1, \dots, u^{d+1}; \dot{u}^1, \dots, \dot{u}^{d+1})$ , so that the variables used in (7.4) yield

$$K_{u^j}(t, l; p, v) = \begin{cases} K_t(t, l; p, v) = F_t \left( t, l, \frac{v}{p} \right) \cdot p & \text{if } j = 1 \\ K_{l^j}(t, l; p, v) = F_{u^{j-1}} \left( t, l, \frac{v}{p} \right) \cdot p & \text{if } j = 2, \dots, d+1, \end{cases} \quad (7.5)$$

and

$$K_{\dot{u}^j}(t, l; p, v) = \begin{cases} K_p(t, l; p, v) = F \left( t, l, \frac{v}{p} \right) - F_{\dot{u}^k} \left( t, l, \frac{v}{p} \right) \frac{v^k}{p} & \text{if } j = 1 \\ K_{v^j}(t, l; p, v) = F_{\dot{u}^{j-1}} \left( t, l, \frac{v}{p} \right) & \text{if } j = 2, \dots, d+1. \end{cases} \quad (7.6)$$

From these, for all  $t \in (a, b)$  and all  $j = 2, \dots, d+1$ , it follows that

$$K_{u^1}(w, \dot{w}) - \frac{d}{dt} K_{\dot{u}^1}(w, \dot{w}) = \left[ \frac{d}{dt} F_{\dot{u}^k}(t, u, \dot{u}) - F_{u^k}(t, u, \dot{u}) \right] \cdot \dot{u}^k \\ K_{u^j}(w, \dot{w}) - \frac{d}{dt} K_{\dot{u}^j}(w, \dot{w}) = F_{u^{j-1}}(t, u, \dot{u}) - \frac{d}{dt} F_{\dot{u}^{j-1}}(t, u, \dot{u}).$$

Therefore, since  $u$  satisfies the Euler-Lagrange equations for  $F$ , this entails that  $w$  is a solution of the analogous equations for  $K$  in  $(a, b)$ . Now, (i) gives

$$\frac{\partial}{\partial t} X_s(w(t)) = \left( \frac{\partial}{\partial t} T_s(t, u(t)), \frac{\partial}{\partial t} L_s(t, u(t)) \right) \in {}^\rho\widetilde{\mathbb{R}}^* \times {}^\rho\widetilde{\mathbb{R}}^d = V.$$

Moreover, (ii) gives  $X_0(w(t)) = (T_0(t, u(t)), L_0(t, u(t))) = w(t)$ . Finally

$$K(w, \dot{w}) = F(t, u, \dot{u})$$

$$K \left( X_s(w), \frac{\partial}{\partial t} X_s(w) \right) = F \left[ T_s(t, u), L_s(t, u), \frac{\partial}{\partial t} L_s(t, u), \frac{\partial}{\partial t} T_s(t, u) \right] \cdot \frac{\partial}{\partial t} T_s(t, u).$$

We can hence apply Thm. 60, and from (7.5), (7.6) we get that

$$K_{\dot{u}^j}(w, \dot{w}) \frac{\partial}{\partial s} \Big|_0 X_s^j(w) = F_{\dot{u}^j}(t, u(t), \dot{u}(t)) \frac{\partial}{\partial s} \Big|_{s=0} L_s^j(t, u(t)) + [F(t, u(t), \dot{u}(t)) - F_{\dot{u}^k}(t, u(t), \dot{u}(t)) \dot{u}^k(t)] \frac{\partial}{\partial s} \Big|_{s=0} T_s(t, u(t))$$

is constant in  $t \in (a, b)$ .  $\square$

## 8. APPLICATION TO $\mathcal{C}^{1,1}$ RIEMANNIAN METRIC

In the following, we apply what we did so far to the problem of length-minimizers in  $(\mathbb{R}^d, g)$ , where  $g \in \mathcal{C}^{1,1}$  is a Riemannian metric. Furthermore, we assume that  $(\mathbb{R}^d, g)$  is geodesically complete. Note that the seeming restriction of considering only  $\mathbb{R}^d$  as our manifold weighs not so heavy. Indeed, the question of length minimizers can be considered to be a local one, since it is not guaranteed that global minimizers exist at all, whereas local minimizers always exist. Additionally, note that it was shown that it suffices to consider smooth manifolds (cf. [21, Thm. 2.9]) instead of  $\mathcal{C}^k$  manifolds with  $1 \leq k < +\infty$ . Therefore, there is no need to consider non-smooth charts.

In this section, we fix any embedding  $(\iota_\Omega^b)_\Omega$ , where  $b \in {}^\rho\widetilde{\mathbb{R}}$  satisfies  $b \geq d\rho^{-a}$  for some  $a \in \mathbb{R}_{>0}$ , and where  $\Omega \subseteq \mathbb{R}^d$  is an arbitrary open set, see Thm. 18. Actually, the embedding also depends on the dimension  $d \in \mathbb{N}_{>0}$ , but to avoid cumbersome notations, we denote embeddings always with the symbol  $\iota$ .

By [27, Rem. 2.6.2], it follows that we can always find a net of smooth functions  $(g_{ij}^\varepsilon)$  such that setting  $\tilde{g} := \iota(g) =: [g_{ij}^\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$ , then for all  $\varepsilon$ ,  $g_{ij}^\varepsilon$  is a Riemannian metric. By Thm. 18.(iii) it follows that  $g_{ij}^\varepsilon \rightarrow g_{ij}$  in  $\mathcal{C}^0$  norm. Let  $\Gamma_{ij}^\varepsilon$  be the Christoffel symbols of  $g^\varepsilon$ , and set  $\tilde{\Gamma}_{ij} := [\Gamma_{ij}^\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}}^d)$ . A curve  $\gamma \in {}^\rho\mathcal{GC}^\infty(J, {}^\rho\widetilde{\mathbb{R}}^d)$ ,  $J$  being a sharply open subset of  ${}^\rho\widetilde{\mathbb{R}}$ , is said to be a *geodesic* of  $({}^\rho\widetilde{\mathbb{R}}^d, \tilde{g})$  if

$$\ddot{\gamma}(t) + \tilde{\Gamma}_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0 \quad \forall t \in J. \quad (8.1)$$

**Definition 62.** We say that  $({}^\rho\widetilde{\mathbb{R}}^d, \tilde{g})$  is *geodesically complete* if every solution of the geodesic equation belongs to  ${}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}^d)$ , i.e. if for all  $p \in {}^\rho\widetilde{\mathbb{R}}^d$  and all  $v \in {}^\rho\widetilde{\mathbb{R}}^d$  there exists a geodesic  $\gamma \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}^d)$  of  $({}^\rho\widetilde{\mathbb{R}}^d, \tilde{g})$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .

This definition includes also the possibility that the point  $p$  or the vector  $v$  could be infinite. By Thm. 19, it follows that if we consider only finite  $p$  and  $v$ , then any geodesic  $\gamma \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}^d)$  induces a Colombeau generalized function  $\gamma|_{\mathfrak{c}(\mathbb{R})} \in \mathcal{G}^s(\mathbb{R}^d)$ . Therefore, the space  $(\mathfrak{c}(\mathbb{R}^d), \tilde{g}|_{\mathfrak{c}(\mathbb{R}^d) \times \mathfrak{c}(\mathbb{R}^d)})$  is geodesically complete in the sense of [40]. We recall that  $\mathfrak{c}(\Omega)$  is the set of compactly supported (i.e. finite) generalized points in  $\Omega$  (see Thm. 18).

The definition of length of a (non singular) curve needs the following

*Remark 63.* We set

$$\sqrt{-} = (-)^{1/2} : x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0} \mapsto [\sqrt{x_\varepsilon}] \in {}^\rho\widetilde{\mathbb{R}}_{>0}.$$

Lem. 3 readily implies that  $\sqrt{-} \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}_{>0}, {}^\rho\widetilde{\mathbb{R}}_{>0})$ . Therefore, the square root is defined on every strictly positive infinitesimal, but it cannot be extended to  ${}^\rho\widetilde{\mathbb{R}}_{\geq 0}$ .

**Definition 64.**

(i) Let  $\tilde{p}, \tilde{q} \in {}^\rho\widetilde{\mathbb{R}}^d$ , then

$${}^\rho\mathcal{GC}_{>0}^\infty(\tilde{p}, \tilde{q}) := \left\{ \lambda \in {}^\rho\mathcal{GC}^\infty([0, 1], {}^\rho\widetilde{\mathbb{R}}^d) \mid \lambda(0) = \tilde{p}, \lambda(1) = \tilde{q}, |\dot{\lambda}(t)| > 0 \forall t \in [0, 1] \right\}.$$

Moreover, for  $\lambda \in {}^\rho\mathcal{GC}_{>0}^\infty(\tilde{p}, \tilde{q})$ , we set

$$L_{\tilde{g}}(\lambda) := \int_0^1 (\tilde{g}_{ij}(\alpha(t))\dot{\alpha}^i(t)\dot{\alpha}^j(t))^{1/2} dt \in {}^\rho\widetilde{\mathbb{R}}.$$

(ii) Let  $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$ , then we set  $\text{st}(x) := \lim_{\varepsilon \rightarrow 0} x_\varepsilon \in \mathbb{R}^d$ , if this limit exists. Note that  $x \approx \text{st}(x)$  in this case.

Note that (8.1) are the usual geodesic equations for the generalized metric  $\tilde{g}$ , whose derivation is completely analogous to that in the smooth case. Thus they are the Euler-Lagrange equations of  $L_{\tilde{g}}$ .

We are interested only in global minimizers of the functional  $L_{\tilde{g}}$ , i.e. curves  $\lambda_0 \in X(\tilde{p}, \tilde{q})$  such that  $L_{\tilde{g}}(\lambda_0) \leq L_{\tilde{g}}(\lambda)$  for all  $\lambda \in {}^\rho\mathcal{GC}_{>0}^\infty(\tilde{p}, \tilde{q})$ .

**Lemma 65.** *Let  $p, q \in \mathbb{R}^d$  and  $\tilde{p}, \tilde{q} \in {}^\rho\mathbb{R}^d$  such that  $st(\tilde{p}) = p$  and  $st(\tilde{q}) = q$ . Let  $\lambda = [\lambda_\varepsilon(-)] \in {}^\rho\mathcal{GC}_{>0}^\infty(\tilde{p}, \tilde{q})$  be such that there exists*

$\bar{\lambda} \in \mathcal{C}_{>0}^1(p, q) := \{w \in \mathcal{C}^1([0, 1]_{\mathbb{R}}, \mathbb{R}^d) \mid w(0) = p, w(1) = q, |\dot{w}(t)| > 0 \forall t \in [0, 1]_{\mathbb{R}}\}$   
such that  $\lambda_\varepsilon \rightarrow \bar{\lambda}$  in  $\mathcal{C}^1$  as  $\varepsilon \rightarrow 0$ , then

$$st(L_{\tilde{g}}(\lambda)) = L_g(\bar{\lambda}).$$

*Proof.* We calculate:

$$\begin{aligned} \left| \int_0^1 \left( g_{ij}^\varepsilon(\lambda_\varepsilon) \dot{\lambda}_\varepsilon^i \dot{\lambda}_\varepsilon^j \right)^{1/2} - \left( g_{ij}(\bar{\lambda}) \dot{\bar{\lambda}}^i \dot{\bar{\lambda}}^j \right)^{1/2} dt \right| &= \\ &= \left| \int_0^1 \frac{g_{ij}^\varepsilon(\lambda_\varepsilon) \dot{\lambda}_\varepsilon^i \dot{\lambda}_\varepsilon^j - g_{ij}(\bar{\lambda}) \dot{\bar{\lambda}}^i \dot{\bar{\lambda}}^j}{\left( g_{ij}^\varepsilon(\lambda_\varepsilon) \dot{\lambda}_\varepsilon^i \dot{\lambda}_\varepsilon^j \right)^{1/2} + \left( g_{ij}(\bar{\lambda}) \dot{\bar{\lambda}}^i \dot{\bar{\lambda}}^j \right)^{1/2}} dt \right|. \end{aligned}$$

By assumption,  $(g_{ij}^\varepsilon(\lambda_\varepsilon) \dot{\lambda}_\varepsilon^i \dot{\lambda}_\varepsilon^j)^{1/2} \rightarrow (g_{ij}(\bar{\lambda}) \dot{\bar{\lambda}}^i \dot{\bar{\lambda}}^j)^{1/2}$ , so that there exists  $C \in \mathbb{R}_{>0}$  such that

$$\begin{aligned} \left| \int_0^1 \left( g_{ij}^\varepsilon(\lambda_\varepsilon) \dot{\lambda}_\varepsilon^i \dot{\lambda}_\varepsilon^j \right)^{1/2} - \left( g_{ij}(\bar{\lambda}) \dot{\bar{\lambda}}^i \dot{\bar{\lambda}}^j \right)^{1/2} dt \right| \\ \leq C \int_0^1 \left| \left( g_{ij}^\varepsilon(\lambda_\varepsilon) - g_{ij}(\lambda_\varepsilon) + g_{ij}(\lambda_\varepsilon) - g_{ij}(\bar{\lambda}) \right) \dot{\lambda}_\varepsilon^i \dot{\lambda}_\varepsilon^j + g_{ij}(\bar{\lambda}) (\dot{\lambda}_\varepsilon^i \dot{\lambda}_\varepsilon^j - \dot{\bar{\lambda}}^i \dot{\bar{\lambda}}^j) \right| dt. \end{aligned}$$

We hence obtain the claim by the triangle inequality and by convergence of  $\lambda_\varepsilon, \dot{\lambda}_\varepsilon$  and  $g_{ij}^\varepsilon$  to  $\bar{\lambda}, \dot{\bar{\lambda}}$  and  $g_{ij}$  respectively.  $\square$

Now, we consider  $p, q \in \mathbb{R}^d$  with  $p \neq q$ . Let

$$u \in \{u \in \mathcal{C}^{2,1}([0, 1], \mathbb{R}^d) \mid u(0) = p, u(1) = q\}$$

be a solution of the geodesic equation

$$\begin{cases} \ddot{u} = -\Gamma_{ij}(u) \dot{u}^i \dot{u}^j \\ p = u(0) \\ q = u(1). \end{cases} \quad (8.2)$$

Let  $c_0 := \dot{u}(0)$ . Obviously,  $u$  is also the unique solution of

$$\begin{cases} \ddot{u} = -\Gamma_{ij}(u) \dot{u}^i \dot{u}^j \\ p = u(0) \\ c_0 = \dot{u}(0). \end{cases} \quad (8.3)$$

Using these initial conditions, for each fixed  $\varepsilon$  we can solve the following problem

$$\begin{cases} \ddot{y} = -\Gamma_{ij}^\varepsilon(y) \dot{y}^i \dot{y}^j \\ p = y(0) \\ c_0 = \dot{y}(0). \end{cases} \quad (8.4)$$

for a unique  $y_\varepsilon \in \mathcal{C}^\infty([-d_\varepsilon, d_\varepsilon]_{\mathbb{R}}, \mathbb{R}^d)$  and some  $d_\varepsilon \in \mathbb{R}_{>0}$ .

**Lemma 66.** *Let  $u$  and  $y_\varepsilon$  be as above. Then*

(i) *For  $\varepsilon$  sufficiently small, the solution  $y_\varepsilon$  can be extended to a solution  $y_\varepsilon \in \mathcal{C}^\infty([0, 1]_{\mathbb{R}}, \mathbb{R}^d)$  of (8.4) such that  $y_\varepsilon(1) = q$ .*

- (ii)  $y_\varepsilon \rightarrow u$  in  $\mathcal{C}^2$ .  
 (iii) The net  $(y_\varepsilon)$  defines a GSF, i.e.  $y := [y_\varepsilon(-)] \in {}^\rho\mathcal{GC}_{>0}^\infty(p, q)$ .

*Proof.*

**Claim (i), (ii):** For all  $i, j$ , we have that  $\Gamma_{ij}^\varepsilon \rightarrow \Gamma_{ij}$  locally uniformly. Thus, we obtain these claims by (8.2) and by continuous dependence on parameters in ODE, see e.g. [28, Lemma 2.3].

**Claim (iii) I:**  $y := [y_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty([0, 1], {}^\rho\widetilde{\mathbb{R}}^d)$

We have to show that for all  $\varepsilon$  all derivatives of  $y_\varepsilon$  are moderate. This is obviously true for  $y_\varepsilon, \dot{y}_\varepsilon$  and  $\ddot{y}_\varepsilon$ . The claim follows now from the fact that

$$\frac{d^{n+2}}{dt^{n+2}}y_\varepsilon = -\frac{d^n}{dt^n}(\Gamma_{ij}^{\varepsilon k}(y)\dot{y}_\varepsilon^i\dot{y}_\varepsilon^j)$$

so that there exists a polynomial  $P$  such that

$$\frac{d^n}{dt^n}(\Gamma_{ij}^{\varepsilon k}(y)\dot{y}_\varepsilon^i\dot{y}_\varepsilon^j) = P\left(y_\varepsilon, \frac{d}{dt}y_\varepsilon, \dots, \frac{d^{n+1}}{dt^{n+1}}y_\varepsilon, \Gamma_{ij}^{\varepsilon k}, D\Gamma_{ij}^{\varepsilon k}, \dots, D^n\Gamma_{ij}^{\varepsilon k}\right).$$

**Claim (iii) II:**  $|\dot{y}(t)| > 0$  for all  $t \in [0, 1]$

By (ii), we have that  $y_\varepsilon \rightarrow u$  in  $\mathcal{C}^2$ . Furthermore,  $g_\varepsilon \rightarrow g$  in  $\mathcal{C}^1$  by assumption, and we know that  $g(\dot{u}, \dot{u}) = c > 0$  for some  $c \in \mathbb{R}_{>0}$  since  $u$  is a  $g$ -geodesic (cf. [23, Lemma 1.4.5]). Therefore, we obtain that  $g_\varepsilon(\dot{y}_\varepsilon, \dot{y}_\varepsilon) > c/2 > 0$  for  $\varepsilon > 0$  small enough.  $\square$

Finally, the standard part of the generalized length of  $y$  is the length of  $u$ :

**Theorem 67.** *Let  $u$  and  $y_\varepsilon$  be as above. We conclude (using Lemma 65) that  $st(L_{\bar{g}}(y)) = L_g(u)$ .*

**Proposition 68.** *Let  $y = [y_\varepsilon(-)]$  be as above. In addition, assume that each  $y_\varepsilon$  is  $L_{g_\varepsilon}$ -minimizing. Then  $L_{\bar{g}}(y)$  is minimal.*

*Proof.* Let  $\lambda = [\lambda_\varepsilon(-)] \in {}^\rho\mathcal{GC}_{>0}^\infty(p, q)$ . We have that  $L_{\bar{g}}(\lambda) = [L_{g_\varepsilon}(\lambda_\varepsilon)]$  and that  $L_{\bar{g}}(y) = [L_{g_\varepsilon}(y_\varepsilon)]$ . By assumption, for all  $\varepsilon$  we have

$$L_{g_\varepsilon}(\lambda_\varepsilon) \geq L_{g_\varepsilon}(y_\varepsilon).$$

Therefore,  $L_{\bar{g}}(\lambda) \geq L_{\bar{g}}(y)$ , as claimed.  $\square$

**Corollary 69.** *Let  $\lambda \in {}^\rho\mathcal{GC}_{>0}^\infty(p, q)$  be a minimizer of  $L_{\bar{g}}$  and assume that for  $\varepsilon$  small,  $y_\varepsilon$  is  $L_{g_\varepsilon}$ -minimizing. Then  $L_{\bar{g}}(y) = L_{\bar{g}}(\lambda)$ .*

This Corollary 69 gives us a way to answer the question if a certain classical geodesic between two given classical points  $p$  and  $q$  is a length-minimizer.

Furthermore, we are able to prove the following theorem, relating generalized minimizers to classical minimizers.

**Theorem 70.** *Let  $p, q \in \mathbb{R}^d$  and let  $\gamma \in {}^\rho\mathcal{GC}_{>0}^\infty(p, q)$  such that  $L_{\bar{g}}(\gamma)$  is minimal. Assume that  $st(L_{\bar{g}}(\gamma))$  exists and that there exists  $w \in \mathcal{C}_{>0}^1(p, q)$  such that  $L_g(w) = st(L_{\bar{g}}(\gamma))$ .*

*Then  $w$  is  $g$ -minimizing and a  $g$ -geodesic.*

*Proof.* Assume to the contrary that there exists a curve  $\sigma \in \mathcal{C}^2$  connecting  $p$  and  $q$  (w.l.o.g.  $\sigma$  is a  $g$ -geodesic) such that

$$L_g(\sigma) < L_g(w).$$

Now we construct (as done above)  $g_\varepsilon, \sigma_\varepsilon$  and set  $\tilde{\sigma} := [\sigma_\varepsilon]$ . Then:

$$st(L_{\bar{g}}(\tilde{\sigma})) = L_g(\sigma) < L_g(w) = st(L_{\bar{g}}(\gamma)).$$

But, by assumption we have that  $L_{\tilde{g}}(\gamma) \leq L_{\tilde{g}}(\tilde{\sigma})$ , which implies

$$\text{st}(L_{\tilde{g}}(\gamma)) \leq \text{st}(L_{\tilde{g}}(\tilde{\sigma})) < \text{st}(L_{\tilde{g}}(\gamma)).$$

This is a contradiction.  $\square$

## 9. CONCLUSIONS

We can summarize the present work as follows

- (i) The setting of GSF allows to treat Schwartz distributions more closely to classical smooth functions. The framework is so flexible and the extensions of classical results are so natural in many ways one may treat it like smooth functions.
- (ii) One key step of the theory is the change of the ring of scalars into a non-Archimedean one and the use of the strict order relation  $<$  to deal with topological properties. So, the use of  $<$  and of  ${}^{\rho}\widetilde{\mathbb{R}}$ -valued norms allows a natural approach to topology, even of infinite dimensional spaces (cf. Def. 41). On the other hand, the use of a ring with zero divisors and a non-total order relation requires a more refined and careful analysis. However, as proved in the present work, very frequently classical proofs can be formally repeated in this context, but paying particular attention to using the relation  $<$ , using invertibility instead of being non zero in  $\mathbb{R}$  and avoiding the total order property.
- (iii) Others crucial properties are the closure of GSF with respect to composition and the use of the gauge  $\rho$ , because they do not force to narrow the theory into particular cases.
- (iv) The present extension of the classical theory of calculus of variations shows that GSF are a powerful analytical technique. The final application shows how to use them as a method to address problems in an Archimedean setting based on the real field  $\mathbb{R}$ .

Concerning possible future developments, we can note that:

- (v) A generalization of the whole construction to piecewise GSF seems possible.
- (vi) A more elegant approach to integration of piecewise GSF could use the existence of right and left limits of  $(f_1, \dots, f_n)(-)$  and hyperfinite Riemann-like sums, i.e. sums

$$\sum_{i=1}^N f(x'_i)(x_i - x_{i-1}) := \left[ \sum_{i=1}^{N_\varepsilon} f_\varepsilon(x'_{i,\varepsilon})(x_{i,\varepsilon} - x_{i-1,\varepsilon}) \right] \in {}^{\rho}\widetilde{\mathbb{R}}^d$$

extended to  $N \in \widetilde{\mathbb{N}} := \left\{ [\text{int}(x_\varepsilon)] \mid [x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}} \right\}$ , where  $\text{int}(-)$  is the integer part function.

The present work could lay the foundations for further works concerning the possibility to extend other results of the calculus of variations in this generalized setting.

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