Recent results in generalized smooth functions theory

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Preliminary notions

Definitions

Let $\rho = (\rho_\varepsilon) : (0,1] \longrightarrow (0,1] =: I$ be a net such that $(\rho_\varepsilon) \uparrow 0$ (a *gauge*)

1. We write $\forall^0 \varepsilon : \mathcal{P}(\varepsilon)$ for $\exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$
2. ${}^\rho\widetilde{\mathbb{R}}$ is the ring of CGN with (ρ_ε) instead of (ε); $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, $\mathrm{d}\rho := [\rho_\varepsilon]$
3. $x < y$ if $x \le y$ and $y - x$ is invertible in ${}^\rho\widetilde{\mathbb{R}}$

Definition

Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. We say that $f : X \longrightarrow Y$ is a GSF if

1. $f : X \longrightarrow Y$ is a set-theoretical function
2. There exists a net $(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^I$ such that for all $[x_\varepsilon] \in X$:

   2a. $f(x) = [f_\varepsilon(x_\varepsilon)]$

   2b. $\forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon))$ is $\rho -$ moderate

The language of subpoints

Definitions

1. If $L, K \subseteq I$, then $K \subseteq_0 L \ :\iff \ 0 \in \overline{K}$ and $K \subseteq L$ (K *is co-final in* L); Using only nets $(x_\varepsilon)_{\varepsilon \in L}$ we obtain a ring ${}^{\rho}\widetilde{\mathbb{R}}^n|_L$
2. If $x = [(x_\varepsilon)_{\varepsilon \in L}] \in {}^{\rho}\widetilde{\mathbb{R}}^n|_L$ and $x' = [(x'_\varepsilon)_{\varepsilon \in K}] \in {}^{\rho}\widetilde{\mathbb{R}}^n|_K$, we say $x' = x|_K \ :\iff \ x'_\varepsilon = x_\varepsilon \ \forall \varepsilon \in K$
3. $x <_L y \ :\iff \ x|_L < y|_L$ (x *is less than* y *on* L)
4. $x <_{\mathrm{s}} y \ :\iff \ \exists L \subseteq_0 I : \ x <_L y$ (x *is less than* y *on subpoints*)

Lemma

Let $x, y \in {}^{\rho}\widetilde{\mathbb{R}}$, *then*

1. $x \not\leq y \iff x >_{\mathrm{s}} y$
2. $x \not< y \iff x \geq_{\mathrm{s}} y$
3. $x \neq y \iff x >_{\mathrm{s}} y$ *or* $x <_{\mathrm{s}} y$

Tri- and quadricotomy

Lemma

Let $x, y \in {}^{\rho}\widetilde{\mathbb{R}}$, *then*

1. $x \le y$ *or* $x >_{\mathrm{s}} y$
2. $\neg(x >_{\mathrm{s}} y \text{ and } x \le y)$
3. $x = y$ *or* $x <_{\mathrm{s}} y$ *or* $x >_{\mathrm{s}} y$
4. $x \le y \ \Rightarrow\ x <_{\mathrm{s}} y$ *or* $x = y$
5. $x \le_{\mathrm{s}} y \iff x <_{\mathrm{s}} y$ *or* $x =_{\mathrm{s}} y$
6. $x \le y$ *or* $x \ge y$ *or* $\exists L \subseteq_0 I : \ L^c \subseteq_0 I, \ x \ge_L y$ *and* $x \le_{L^c} y$
7. *If for all* $L \subseteq_0 I$ *the following implication holds*

$$x \le_L y, \text{ or } x \ge_L y \ \Rightarrow\ \forall^0 \varepsilon \in L : \ \mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\},$$

or the following implication holds

$$x <_L y, \text{ or } x >_L y \text{ or } x =_L y \ \Rightarrow\ \forall^0 \varepsilon \in L : \ \mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\},$$

then $\forall^0 \varepsilon : \ \mathcal{P}\left\{x_\varepsilon, y_\varepsilon\right\}$

Supremum and completeness from a side [with Mukhammadiev, Tiwari, Apaaboah]

- Sup/Inf of the set of all the infinitesimals do not exist in ${}^{\rho}\widetilde{\mathbb{R}}$
- σ is the *closed supremum* (or *supremum in the sharp topology*) of $S \subseteq {}^{\rho}\widetilde{\mathbb{R}} \ :\iff \ \forall s \in S : \ s \le \sigma$ and $\forall q \in \mathbb{N}\, \exists \bar{s} \in S : \ \sigma - \mathrm{d}\rho^{q} \le \bar{s}$ [Garetto, Vernaeve 2011]
- A lot of properties of Sup/Inf hold (IF they exist!)
- Let S, $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}$, then we say that *$S$ is complete from above for U* if for all $q \in \mathbb{N}$, $u \in U$ and $s \in S$

$$\exists u' \in U\, \exists s' \in S : \ s \le s' \le u' \le u, \ u' - s' < \mathrm{d}\rho^{q}$$

- If S is complete from above for the set of all the upper bounds, then:
  - supremum coincides with least upper bound
  - supremum exists if and only if there exists an upper bound of S
  - supremum can be approximated with sequences of S

- $\frac{1}{n} \not\to 0$ in the sharp topology as $n \to +\infty$, $n \in \mathbb{N}_{>0}$ because $\frac{1}{n} \not< \mathrm{d}\rho$
- *Hypernatural numbers*: ${}^\rho\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall\varepsilon\right\}$; but $1 = \left[1 - \rho_\varepsilon^{1/\varepsilon}\right] = \left[1 + \rho_\varepsilon^{1/\varepsilon}\right]$ so the integer part is not well-def.
- *Nearest integer function*: If $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{N}}$ and $\mathrm{ni}\left([x_\varepsilon]\right) = (n_\varepsilon)$ then $\forall^0\varepsilon: \; n_\varepsilon = \lfloor x_\varepsilon + \frac{1}{2} \rfloor$ and $[n_\varepsilon] = [x_\varepsilon]$
- *Hyperlimit*: If $x : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ (a *hypersequence*), then $\forall q \in N \, \exists M \in {}^\sigma\widetilde{\mathbb{N}} \, \forall n \in {}^\sigma\widetilde{\mathbb{N}}_{\geq M}: \; |x_n - l| < \mathrm{d}\rho^q$
- Set $\sigma_\varepsilon := \exp\left(-\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right)$, then ${}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} \frac{1}{\log n} = 0 \in {}^\rho\widetilde{\mathbb{R}}$ whereas $\nexists {}^\rho\lim_{n \in {}^\rho\widetilde{\mathbb{N}}} \frac{1}{\log n}$
- Properties: algebraic, squeeze, Cauchy criterion, monotonic hypesequences and Sup/Inf, limit superior/inferior, L'Hôpital rule...

Hyperseries [with Tiwari]

- $\sum_{n\in\mathbb{N}} a_n$ converges $\iff$ $a_n \to 0$ (in the sharp topology)
- Problem: extend $\left(N \in \mathbb{N} \mapsto \sum_{n=0}^{N} a_n \in {}^{\rho}\widetilde{\mathbb{R}}\right) : \mathbb{N} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$ to ${}^{\sigma}\widetilde{\mathbb{N}}$: there are representatives $[a_{n\varepsilon}] = 0$ such that $\left(\sum_{n=0}^{N_\varepsilon} a_{n\varepsilon}\right)$ is not moderate or it depends on $(a_{n\varepsilon})$. Only certain sequences $(a_n)_{n\in\mathbb{N}}$ of ${}^{\rho}\widetilde{\mathbb{R}}$ can be considered in hyperseries
- $\forall q \in \mathbb{N}\, \exists M \in {}^{\sigma}\widetilde{\mathbb{N}}\, \forall N \in {}^{\sigma}\widetilde{\mathbb{N}}_{\ge M} : \left|\sum_{n=0}^{N} a_n - s\right| < \mathrm{d}\rho^q$
- Particularity: a series of positive terms can be convergent, divergent, or the supremum of its summands does not exist
- Divergent series for ω infinite: $\sum_{n=1}^{\omega} 1 = \omega \in {}^{\rho}\widetilde{\mathbb{R}} \setminus \mathbb{R}$, $\sum_{n=0}^{\omega}(-1)^n = \frac{1}{2}(-1)^{\omega+1} + \frac{1}{2}$, $\sum_{n=0}^{\omega} 2^{n-1} = 2^{\omega} - 1 \in {}^{\sigma}\widetilde{\mathbb{R}}$ if ω is ρ-moderate, etc.
- Classical examples, convergence tests, Cauchy product, integral test using GSF (!)

Multidimensional integration [with Vernaeve, Kunzinger]

Problems with integration of CGF: they are defined only on finite points and $N_\varepsilon := \text{int}\left(\rho_\varepsilon^{-1/\varepsilon}\right)$, $K_\varepsilon := \{\rho_\varepsilon^{1/\varepsilon}, 2\rho_\varepsilon^{1/\varepsilon}, \ldots, N_\varepsilon \rho_\varepsilon^{1/\varepsilon}\}$, then $[0,1] = [K_\varepsilon]$ and $\left[\int_{[0,1]_{\mathbb{R}}} f_\varepsilon \,\mathrm{d}\lambda\right] \neq \left[\int_{K_\varepsilon} f_\varepsilon \,\mathrm{d}\lambda\right] = 0$

Definition

Let μ be a measure on $\mathbb{R}^n$ and let K be a functionally compact subset of ${}^{\rho}\widetilde{\mathbb{R}}^n$. Then

- We call K μ-measurable if the limit $\mu(K) := \lim_{m\to\infty}[\mu(\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon))]$ exists for some $[K_\varepsilon] = K$
- If $f \in {}^{\rho}\mathcal{GC}^\infty(K, {}^{\rho}\widetilde{\mathbb{R}})$, then always

  $\exists \int_K f \,\mathrm{d}\mu := \lim_{m\to\infty}\left[\int_{\overline{B^{\mathrm{E}}}_{\rho_\varepsilon^m}(K_\varepsilon)} f_\varepsilon \,\mathrm{d}\mu\right]$ and does not depend on (K_ε)

Multidimensional integration

Theorem

- *There exists* $[K_\varepsilon] = K$ *such that* $\int_K f \, \mathrm{d}\mu = \left[\int_{K_\varepsilon} f_\varepsilon \, \mathrm{d}\mu \right]$
- *change of variables, integration via primitives, properties about 1-dim integration and order*
- *If* $[\mu(K_\varepsilon \cap L_\varepsilon)] = 0$ *for all* $[K_\varepsilon] = K$, $[L_\varepsilon] = L$, *then* $\int_{K \vee L} f \, \mathrm{d}\mu = \int_K f \, \mathrm{d}\mu + \int_L f \, \mathrm{d}\mu$, *where* $K \vee L$ *is the interleaving*
- *If* $\exists \, {}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} f_n(x)$ *for each* $x \in K$, *then* ${}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} \int_K f_n \, \mathrm{d}\mu = \int_K {}^\rho\lim_{n \in {}^\sigma\widetilde{\mathbb{N}}} f_n \, \mathrm{d}\mu$

Future persectives

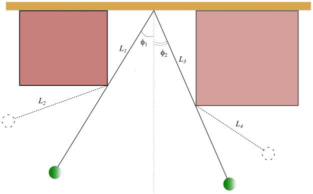
- Sigma additivity with "countable interleaving"?
- ${}^\rho\widetilde{\mathbb{R}}$-valued probability theory of events $A \subseteq {}^\rho\widetilde{\mathbb{R}}^n$?
- Generality: we generalize 1-dim integration of "Ye, Liu, *The distributional Henstock–Kurzweil integral and applications*", 2016

Calculus of variations [with Lecke, Luperi Baglini, Frederico, Bryzgalov, Lazo]

- 1-dim. integration: Euler-Lagrange equations with arbitrary GSF in action (closure w.r.t. composition), classical prop. of 2nd variation, Legendre condition, conjugate points and Jacobi's theorem
- n-dim. integration: higher order Euler-Lagrange equations, higher order Noether's theorem, starting of optimal control and Pontryagin Maximum Principle

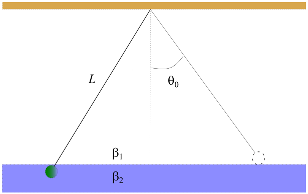
Applications:

- Rigorous deduction of Snell's laws (discontinuous Lagrangians in optics)
- Finite and infinite potential wells in QM
- Singular variable length pendulum

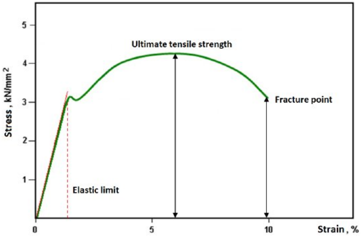


Applications

- Oscillatory motion of the pendulum in the interface of two media



- Non linear strain-stress model



Equations with Banach and Newton [with Luperi Baglini, Apaaboah]

- For $K=[K_\varepsilon]\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ functionally compact such that $K=\overline{\mathring{K}}$, we can set $\|f\|_m=\left[\max_{\substack{|\alpha|\le m\\1\le i\le d}}\sup_{x\in K_\varepsilon}\left|\partial^\alpha f_\varepsilon^i(x)\right|\right]\in{}^\rho\widetilde{\mathbb{R}}$ to obtain an ${}^\rho\widetilde{\mathbb{R}}$-Fréchet space ${}^\rho\mathcal{GC}^\infty(K,{}^\rho\widetilde{\mathbb{R}}^d)\supseteq X$
- Tame contraction starting from y_0: is a set-theoretical map $P:X\longrightarrow X$
  1. $\forall i\in\mathbb{N}\,\exists\alpha_i\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall u,v\in X:\ \|P(u)-P(v)\|_i\le\alpha_i\cdot\|u-v\|_{i+L}$, $\alpha_i\le\alpha_{i+1}$
  2. For all $i\in\mathbb{N}$, we have $\lim_{\substack{n,m\to+\infty\\n\le m}}\alpha_{i+mL}^n\cdot\|P(y_0)-y_0\|_{i+mL}=0$ (sharp top.)
- Banach fixed point Thm: $\lim_{n\to+\infty}P^n(y_0)=y$, $P(y)=y$ (uniqueness if $L=0$)
- Newton: reduced to Banach
- If $n\in{}^\sigma\widetilde{\mathbb{N}}$ is ρ-moderate and we have $n^n\in{}^\sigma\widetilde{\mathbb{N}}$, then we can also consider hyperfinite iterations: $[P_\varepsilon^{n_\varepsilon}(y_{0\varepsilon})(-)]$

Picard-Lindelöf for ODE and PDE [with Luperi Baglini]

- "only" normal PDE, and we set $\partial_t^D y = G\left(t, x, \left(\partial_t^j \partial_x^\alpha y\right)_{j,\alpha}\right) =: F(t, x, y)$
- using the closure w.r.t. composition we can always reduce higher order PDE to 1st order
- In $Y_\alpha := \{y \in {}^\rho\mathcal{GC}^\infty([t_0 - \alpha, t_0 + \alpha] \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N}\}$, $S = [S_\varepsilon]$ arbitrary functionally compt., we ask $\forall i \in \mathbb{N}, y, u, v \in Y_\alpha$:

$$F(-,-,y) \in {}^\rho\mathcal{GC}^\infty\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$$

$$\|F(-,-,u) - F(-,-,v)\|_i \le \Lambda_i \cdot \|u - v\|_{i+L}$$

- PLT for PDE: If

$$\|F(-,-,y)\|_i \le M_i(y), \quad \alpha \cdot M_i(y) \le r_i$$

$$\lim_{n,m\to\infty, n\le m} \alpha^{n+1} \cdot \Lambda_{i+mL}^n \cdot \|F(-,-,y_0)\|_{i+mL} = 0$$

then there exists a solution $y \in Y_\alpha$ of

$$\begin{cases} \partial_t y(t,x) = G\left(t, x, \left(\partial_x^a y\right)_a\right) = F(t,x,y) \\ y(t_0, x) = y_0(x) \end{cases}$$

- Levy-Mizohata only in infinitesimal neighborhood

Infinitesimal uniqueness and continuous dependence

- $\begin{cases} \partial_t y_i(t,x) = G\left(t, x, \left(\partial_x^a y_i\right)_a\right) \\ y_i(t_0, x) = y_0(x) \end{cases}$ where $y_i : [-\alpha, \alpha] \times S \longrightarrow H$, $i = 1, 2$; $S \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $H \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ are funct. cmpt.
- Infinitesimal uniqueness: If

$$\exists M \in {}^\rho\widetilde{\mathbb{R}}\, \forall p, z, d, e\, \forall \xi \in [-\alpha, \alpha]\, \forall x \in S : \left|\partial_z^p G(\xi, x, d_a)\right|, \left|\partial_t^d \partial_x^e y_i(\xi, x)\right| \le M$$

  (e.g. G, y_i are smooth), then $y_1(h, x) = y_2(h, x)$ for all $x \in S$ and all $h \in \left\{ h \in {}^\rho\widetilde{\mathbb{R}} \mid \exists r \in \mathbb{R}_{>0} : \ |h| \le \mathrm{d}\rho^r \right\} =: D_\infty^{\mathrm{p}}$
- Continuous dependence: Let $y(-,-,y_0) : D_\infty^{\mathrm{p}} \times S \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$ be the unique solution starting from y_0, then

$$\lim_{y \to \bar{y}_0} y(-,-,y_0) = y(-,-,\bar{y}_0)$$

  in the ${}^\rho\widetilde{\mathbb{R}}$-Fréchet space ${}^\rho\mathcal{GC}^\infty([-\alpha, \alpha] \times S, {}^\rho\widetilde{\mathbb{R}}^d)$
- This is a possible solution to a zoo of counter-examples: Colombini, De Giorgi, Jannelli, Kinoshita, Spagnolo...

Global infinitesimal solutions of Navier-Stokes

- Let's consider e.g.

$$\begin{cases} \partial_t y(t,x) = G\left(t,x,y(t,x),\partial_{x_i}y(t,x),\partial^2_{x_i}y(t,x)\right) \\ y(0,x) = y_0 \end{cases}$$

where $G \in \mathbb{R}[t, x_1, \ldots, x_n, d_0, d_1, d_2]$ is a real polynomial. We set γ infinite, $S := [-\gamma, \gamma]^n \supseteq \mathbb{R}^n$, $0 < r \le r_i$ arbitrary, $K := \prod_{i=0}^{2}[\|y_0\|_i - r_i, \|y_0\|_i + r_i]^n$. Since G is a real polynomial, we have

$$\exists M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \forall i \in \mathbb{N} : \ M_i := \|G|_{T\times S\times K}\|_i \le M$$

and we get that there exists a solution in
$Y_\alpha = \{y \in {}^{\rho}\mathcal{GC}^\infty([t_0-\alpha, t_0+\alpha]\times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N}\}$

- As a particular case we get the existence of local (infinitesimal) in t and global in $x \in \mathbb{R}^n$ solutions of Navier-Stokes

Future developments

- Functional analysis of infinite bounded operators (present FWF project with M. Kunzinger and H. Vernaeve), e.g.:

$$\exists C \in {}^{\sigma}\widetilde{\mathbb{R}}_{>0}\, \forall f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}})\, \forall m \in \mathbb{N} : \ \|f\|_0 > 0 \ \Rightarrow \ \|\partial^{\alpha} f\|_m \leq C \cdot \|f\|_0$$

- Universal properties of ${}^{\rho}\widetilde{\mathbb{R}}$, CGF and GSF
- Pontryagin's maximum principle (future FWF project (?) with M. Kunzinger)
- Numerical calculus and visualization of GSF
- Start analytical mechanics with singular Hamiltonians

Contacts and references

References:

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Thank you for your attention!