GENERALIZED HOLOMORPHIC FUNCTIONS: SKETCHES OF A NEW THEORY

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Abstract. We start from the simple idea to define a generalized holomorphic function (GHF) without already assuming the Cauchy-Riemann equations but using a natural definition of complex differentiability, i.e. of limit of the incremental ratio. The setting is that of Robinson-Colombeau ring, which generalizes the ring of Colombeau generalized numbers by considering a more general "growth condition" (ρ_ε) (called *gauge*) instead of the usual $\rho_\varepsilon = \varepsilon$. This natural definition actually uses *two* gauges in order to define two different sharp topologies on the domain and on the codomain, and hence a new notion of limit and little-oh in the definition of GHF. This conduct us to a more general theory, where several classical theorems of differential calculus can be extended from the ordinary holomorphic case to generalized holomorphic framework, overpassing several drawbacks of Colombeau theory of holomorphic functions.

1. Introduction

Since the beginning of Colombeau theory of generalized functions (CGF) it was natural to define an holomorphic generalized functions using Cauchy-Riemann equations (CRE), see e.g. [21, 2, 5, 1]. Indeed, the notion of partial derivatives of a CGF was already clear and, mainly, using pointwise evaluation of CGF one can define the complex function of a complex variable starting from its real and imaginary parts. Note that the latter step is not so easy using Schwartz distributions. Even if an important result such as the CRE is directly taken in the definition, this approach probably appeared more natural than considering the limit of the incremental ratio in a *ring* with a sharp topology usually managed through valuation and sharp norm. On the other hand, already in [2] a more general notion of differentiability in the Colombeau setting, both for the complex and the real case, is considered using a classical Newton quotient. This allows [2] to show that all CGF are also differentiable in this more classical way.

Similarly, in this article we want to define a GHF by using some kind of limit of the incremental ratio, and to prove from this the CRE. It is not immediate to understand that this apparently stylistic problem leads to a *more* general theory and is also deeply related to a non trivial drawback of the Colombeau approach. Indeed, any good theory of GHF has to be well linked with a good notion of power series. However, in the ring $\widetilde{\mathbb{R}}$ of Colombeau generalized numbers (see below Def. 1), we have that $(x_n)_{n\in\mathbb{N}} \in \widetilde{\mathbb{R}}^{\mathbb{N}}$ is a Cauchy sequence if and only if $\lim_{n\to+\infty} |x_{n+1} - x_n| =$

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0 (in the sharp topology; see [15, 19]). As a consequence, a series of CGN

$$\sum_{n=0}^{\infty} a_n \text{ converges} \iff a_n \to 0 \text{ (in the sharp topology).} \tag{1.1}$$

Once again, this is a well-known property of every ultrametric space, cf., e.g. [13]. For example, $\sum_{n=1}^{\infty} \frac{1}{n^2} \in \widetilde{\mathbb{R}}$ converges in the sharp topology if and only if $\frac{1}{n^2} \to 0$ in the same topology, for $n \to +\infty$, $n \in \mathbb{N}_{>0}$. But the sharp topology on $\widetilde{\mathbb{R}}$ necessarily has to deal with balls having infinitesimal radius $r \in \widetilde{\mathbb{R}}$ (because generalized functions can have infinite derivatives and are continuous in this topology), and thus $\frac{1}{n^2} \not\to 0$ if $n \to +\infty$, $n \in \mathbb{N}_{>0}$, because we never have $\mathbb{R}_{>0} \ni \frac{1}{n^2} < r$ if r is infinitesimal. Similarly, one can easily prove that if the exponential series $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ converges, then necessarily $x \approx 0$ must be an infinitesimal number because if $\frac{x^n}{n!} \to 0$ in the sharp topology, then $|x_\varepsilon| \le n!\varepsilon^{1/n}$ for all $n \in \mathbb{N}$ sufficiently large.

Intuitively, the only way to have $\frac{1}{n^2} < r \approx 0$ is to take for $n \in \widetilde{\mathbb{R}}$ an infinite number, in this case an infinite natural number. However, we intuitively would like to have $\frac{1}{\log n} \to 0$, but we have $\frac{1}{\log n_\varepsilon} < \varepsilon^q$ if and only if $n_\varepsilon > e^{\varepsilon^{-q}}$ and the net $\left(e^{\varepsilon^{-q}}\right)_\varepsilon$ is not moderate. In order to settle this problem, it is hence important to generalize the role of the net (ε) as used in Colombeau theory, into a more general $\rho = (\rho_\varepsilon) \to 0$ (which is called a *gauge*), and hence to generalize $\widetilde{\mathbb{R}}$ into some ${}^\rho\widetilde{\mathbb{R}}$ (see Def. 1). The aforementioned set of infinite natural numbers (called *hypernatural numbers*) would be ${}^\rho\widetilde{\mathbb{N}} := \left\{[n_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N}\ \forall\varepsilon\right\}$, and we have $\frac{1}{\log n} \to 0$ in ${}^\rho\widetilde{\mathbb{R}}$ as $n \to +\infty$ for $n \in {}^\sigma\widetilde{\mathbb{N}}$, but only for a suitable gauge σ (depending on ρ), whereas this limit does not exist if $\sigma = \rho$ (cf. [15, Example 33]). This is also related to the former problem of series of generalized numbers $a_n \in {}^\rho\widetilde{\mathbb{R}}$ because instead of ordinary series, we have better results summing over all $n \in {}^\sigma\widetilde{\mathbb{N}}$: e.g. we have ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$ for all $x \in {}^\rho\widetilde{\mathbb{R}}$ where the exponential is moderate, i.e. if $|x_\varepsilon| \le \log\left(\rho_\varepsilon^{-R}\right)$ for some $R \in \mathbb{N}$. Intuitively, the smaller is the second gauge σ, the greater are the infinite numbers we can consider with it (i.e. represented by σ-moderate nets), and hence the greater is the number of summands in summations of the form ${}^\rho\sum_{n\in{}^\sigma\widetilde{\mathbb{N}}} a_n \in {}^\rho\widetilde{\mathbb{R}}$. This kind of summations are called *hyperseries*, and their theory has been developed in [19, 20].

Based on these (a posteriori!) motivations, it is now natural that we generalize the notions of *hyperlimit* of the incremental ratio, of little-oh in ${}^\rho\widetilde{\mathbb{C}}$, and finally the definition of GHF, by considering *two* gauges, one on the domain and one on the codomain.

1.1. The Ring of Robinson Colombeau Numbers.

In this section, we introduce our non-Archimedean ring of scalars. For more details and proofs about the basic notions introduced here, the reader can refer e.g. to [10, 4, 9]. As we mentioned above, in order to accomplish the theory of hyperlimits, it is important to generalize Colombeau generalized numbers by taking an arbitrary asymptotic scale instead of the usual net (ε) (see also [14] for a more general notion of scale, and [7] and references therein for a comparison).

Definition 1. Let $\rho = (\rho_\varepsilon) : (0,1] \to (0,1] =: I$ be a net such that $(\rho_\varepsilon) \to 0$ as $\varepsilon \to 0^+$ (in the following, such a net will be called a *gauge* and all the asymptotic relation will be for $\varepsilon \to 0^+$), then

(i) We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ is *ρ-moderate*, and we write $(x_\varepsilon) \in \mathbb{R}_\rho$ if $\exists N \in \mathbb{N} : x_\varepsilon = \mathcal{O}(\rho_\varepsilon^{-N})$.

(ii) Let (x_ε), $(y_\varepsilon) \in \mathbb{R}^I$, then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if $\forall N \in \mathbb{N} : |x_\varepsilon - y_\varepsilon| = \mathcal{O}(\rho_\varepsilon^N)$. This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$$ {}^\rho\widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho, $$

which we call *Robinson-Colombeau ring of generalized numbers*. The corresponding equivalence classes are simply denoted by $x = [x_\varepsilon]$ or $x = [x_\varepsilon]_\rho$ in case we have to underscore the dependence from the gauge ρ. In particular, $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$.

(iii) If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0\varepsilon : \mathcal{P}(\varepsilon)$ to denote $\exists \varepsilon_0 \in I\, \forall \varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$. We can read $\forall^0\varepsilon$ as for ε small.

(iv) Let x, $y \in {}^\rho\widetilde{\mathbb{R}}$. We write $x \le y$ if for all representative $[x_\varepsilon] = x$, there exists $[y_\varepsilon] = y$ such that $\forall^0\varepsilon : \ x_\varepsilon \le y_\varepsilon$.

(v) We denote by ${}^\rho\widetilde{\mathbb{R}}_{>0}$ the set of positive invertible generalized numbers. In general, we write $x < y$ to say that $x \le y$ and $x - y$ is invertible.

(vi) A generalized complex number $z := x + iy \in {}^\rho\widetilde{\mathbb{C}}$, where x, $y \in {}^\rho\widetilde{\mathbb{R}}$ and i is the imaginary unit.

On ${}^\rho\widetilde{\mathbb{R}}$, we consider the natural extension of the Euclidean norm, i.e. $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$. Even if this generalized norm takes value in ${}^\rho\widetilde{\mathbb{R}}$, it shares essential properties with classical norms. It is therefore natural to consider on ${}^\rho\widetilde{\mathbb{R}}$ the topology generated by balls $B_r(x) := \left\{ y \in {}^\rho\widetilde{\mathbb{R}} : |x-y| < r \right\}$, $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, which is called *sharp topology*.

A natural way to obtain some particular sets in ${}^\rho\widetilde{\mathbb{R}}$ is by using a net (A_ε) of subset $A_\varepsilon \subseteq \mathbb{R}$. We have two ways of extending the membership relation $x_\varepsilon \in A_\varepsilon$ to generalized points $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$.

Definition 2. Let (A_ε) be a net of subsets of $\mathbb{R}$, then

(i) A set of the type

$$ [A_\varepsilon] := \left\{ [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid \forall^0\varepsilon : \ x_\varepsilon \in A_\varepsilon \right\} $$

is called *internal set*, and it is sharply closed.

(ii) A set of the type

$$ \langle A_\varepsilon \rangle := \left\{ x \in {}^\rho\widetilde{\mathbb{R}} \mid \forall [x_\varepsilon] = x\, \forall^0\varepsilon : \ x_\varepsilon \in A_\varepsilon \right\} $$

is called *strongly internal set*, and it is sharply open.

A thorough investigation of internal sets can be found in [17, 1]; see [9] for strongly internal sets.

2. Generalized Holomorphic Functions

2.1. **Hyperlimit and Little-oh.** Starting from the theory of GSF and CGF, we still want to have generalized function of the form $f(z) = [f_\varepsilon(z_\varepsilon)]$ but without considering the differentiability of f_ε, we therefore define the notion of basic function below:

Definition 3. Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$. We say that $f : U \to {}^\rho\widetilde{\mathbb{C}}$ is a *basic function*, if

(i) $f : U \to {}^\rho\widetilde{\mathbb{C}}$ is a set-theoretical function;

(ii) There exists a net (f_ε) such that $f_\varepsilon : U_\varepsilon \to \mathbb{C}$, $U \subseteq \langle U_\varepsilon \rangle$, and for all $z = [z_\varepsilon] \in U$, $f(z) = [f_\varepsilon (z_\varepsilon)]$.

Note that a similar concept was first introduced in [6] to define basic linear maps. Since we aim to define a generalized holomorphic function using a natural definition of complex differentiability, we also need to define the limit of a basic function. This can be accomplished using two gauges to generate two topologies in the domain and codomain respectively.

Definition 4. Let σ, ρ be two gauges, then we say $\sigma \le \rho$ if $\forall^0\varepsilon : \ \sigma_\varepsilon \le \rho_\varepsilon$.

The relation $(-) \le (-)$ is reflexive, transitive, and antisymmetric in the sense that $\sigma \le \rho$ and $\rho \le \sigma$ imply $\sigma_\varepsilon = \rho_\varepsilon$ for ε small. Clearly, $\sigma \le \rho$ implies the inclusion of ρ-moderate nets $\mathbb{R}_\rho \subseteq \mathbb{R}_\sigma$. This means that we have larger infinities and smaller infinitesimals.

Even assuming that we need two gauges σ and ρ to define the notion of hyperlimit of a basic function, at the end we aim at defining a GHF of the type $f : U \longrightarrow {}^\rho\widetilde{\mathbb{C}}$, where $U \subseteq {}^\rho\widetilde{\mathbb{C}}$. On the other hand, we want to consider $f(z_0 + h) \in {}^\rho\widetilde{\mathbb{C}}$ and take $h \to 0$ in ${}^\sigma\widetilde{\mathbb{C}}$. We therefore need to link "small" numbers in ${}^\sigma\widetilde{\mathbb{C}}$ with those in ${}^\rho\widetilde{\mathbb{C}}$: If $\sigma \le \rho$, we define ${}^\rho_\sigma\widetilde{\mathbb{C}} := \left\{ [x_\varepsilon] \in {}^\sigma\widetilde{\mathbb{C}} \mid (x_\varepsilon) \in \mathbb{C}_\rho \right\}$ as the set of ${}^\sigma\widetilde{\mathbb{C}}$ numbers which are ρ-moderate. We can also well-define a natural map $\iota : [x_\varepsilon]_\sigma \in {}^\rho_\sigma\widetilde{\mathbb{C}} \mapsto [x_\varepsilon]_\rho \in {}^\rho\widetilde{\mathbb{C}}$ because $\sigma \le \rho$, and we simply write $x_\iota := \iota(x)$ where no risk of confusion exists. This map is surjective but generally not injective, even if $\iota(x) = \iota(y)$ implies $|x - y| \le [\rho_\varepsilon]^q_\sigma$ for all $q \in \mathbb{R}_{\ge 0}$. Similarly, we can define ${}^\rho_\sigma\widetilde{\mathbb{R}}$.

In the next definition, we introduce in our framework hyperlimit of basic functions; see [15] for a deeper study of hyperlimit of hypersequences, i.e. of functions of the form $s : {}^\sigma\widetilde{\mathbb{N}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$.

Definition 5. Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharp neighborhood of 0. Let σ be a gauge such that $\sigma \le \rho$, and $R : U \to {}^\rho\widetilde{\mathbb{C}}$ be a basic function. We say that the (ρ-)hyperlimit of f, as h tends to 0 in the (σ-)sharp topology is 0, and we write ${}^\rho\lim_{h \xrightarrow{\sigma} 0} R(h) = 0$, if the following property holds:

$$\forall q \in \mathbb{N}\, \exists H \in {}^\rho_\sigma\widetilde{\mathbb{R}}_{>0}\, \forall h \in {}^\rho_\sigma\widetilde{\mathbb{C}} : \ 0 < |h| < H \Rightarrow |R(h_\iota)| < \mathrm{d}\rho^q. \tag{2.1}$$

Note that if $B_r(0) \subseteq U$, $r = [r_\varepsilon]_\rho \in (0, 1] \subseteq {}^\rho\widetilde{\mathbb{R}}$, then for $\bar{r} := [r_\varepsilon]_\sigma \in {}^\sigma\widetilde{\mathbb{R}}_{>0}$, we have $B_{\bar{r}}(0) \subseteq \left\{ h \in {}^\rho_\sigma\widetilde{\mathbb{C}} \mid h_\iota \in B_r(0) \right\}$. Therefore, the map $h \mapsto R(h_\iota)$ is defined in a σ-neighborhood of 0, and the hyperlimit (2.1) is at most only one.

The following Thm. 6 equivalently states ε-wise convergence of hyperlimit.

Theorem 6. *Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharply neighborhood of 0 and $R : U \to {}^\rho\widetilde{\mathbb{C}}$ be a basic function such that $R(h) := [R_\varepsilon(h_\varepsilon)]$ for all $h = [h_\varepsilon] \in U$. There exists σ such that $\sigma \le \rho$ and ${}^\rho\lim_{h \xrightarrow{\sigma} 0} R(h) = 0$ if and only if for ε small $\exists \lim_{h \to 0} R_\varepsilon(h) =: \ell_\varepsilon$, and $(\ell_\varepsilon) \sim_\rho 0$, i.e. it is ρ-negligible.*

Proof. We only give some first details of the proof that the ε-wise condition is sufficient, because they underscore why we use two gauges in our definition of hyperlimit of a basic function: There exists $\varepsilon_1 \in I$ such that for all $\varepsilon \in (0, \varepsilon_1]$ and $\eta \in \mathbb{R}_{>0}$, we can find $H_{\varepsilon,\eta} \in \mathbb{R}_{>0}$ such that for all $h \in \mathbb{C}$ satisfying $0 < |h| < H_{\varepsilon,\eta}$ we get $|R_\varepsilon(h_\varepsilon) - \ell_\varepsilon| < \eta$. For all fixed $q \in \mathbb{N}$, set $\eta = \rho_\varepsilon^q \in \mathbb{R}_{>0}$. For ε sufficiently small, we also have that $q \le \left\lceil \frac{1}{\varepsilon} \right\rceil$. Without loss of generality, we can assume to have recursively chosen $H_{\varepsilon,\rho_\varepsilon^q} =: H_{\varepsilon,q}$ such that $0 < H_{\varepsilon,q+1} \le H_{\varepsilon,q} \le 1$. Set $\widehat{H_\varepsilon} := H_{\varepsilon,\left\lceil \frac{1}{\varepsilon} \right\rceil} \in \mathbb{R}_{>0}$. The main problem is that, for an arbitrary function R, we have no control about how quickly $\widehat{H_\varepsilon} \to 0^+$ as $\varepsilon \to 0^+$, and indeed it can happen that $\widehat{H} := \left[\widehat{H_\varepsilon}\right]_\rho = 0$. On the contrary, if we set $\sigma_\varepsilon := \min\left\{\rho_\varepsilon, \widehat{H_\varepsilon}\right\} \in (0,1]$, then we have a new gauge $\sigma := (\sigma_\varepsilon) \le \rho$ and $\widehat{H} > 0$ because our definition of σ yields $\widehat{H_\varepsilon} \ge \sigma_\varepsilon$ for all ε. Using this gauge σ, for all $h := [h_\varepsilon] \in {}^\rho_\sigma\widetilde{\mathbb{C}}$ satisfying $0 < |h| < \widehat{H}$, we can prove that $|R(h_\iota)| < 2\mathrm{d}\rho^q$. $\square$

It is also possible to prove that if there exists σ such that $\sigma \le \rho$ and ${}^\rho\lim_{h \xrightarrow{\sigma} 0} R(h) = 0$, then for all other gauges $\bar{\sigma}$ such that $\bar{\sigma} \le \sigma$, we have ${}^\rho\lim_{h \xrightarrow{\bar{\sigma}} 0} R(h) = 0$. This underscores that the existence of this gauge σ is a condition of topological nature.

Definition 7. Let $H \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharp neighborhood of 0 and P, $Q : H \to {}^\rho\widetilde{\mathbb{C}}$ be basic maps defined on H. Let σ be a gauge such that $\sigma \le \rho$. Then we say that $P(h_\iota) = o(Q(h_\iota))$ as $h \xrightarrow{\sigma} 0$, if there exists a basic function $R : H \to {}^\rho\widetilde{\mathbb{C}}$ such that

$$ {}^\rho\lim_{h \xrightarrow{\sigma} 0} R(h) = 0 \quad \text{and} \quad \forall \iota(h) \in H : \; P(h_\iota) = R(h_\iota) Q(h_\iota). \tag{2.2} $$

Note that we use expression (2.2) since ${}^\rho\widetilde{\mathbb{C}}$ is not a field but only a ring.

2.2. **Complex differentiable function.**

Having a notion of little-oh working with two gauges, it is now natural the following:

Definition 8. Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharply open set, $f : U \to {}^\rho\widetilde{\mathbb{C}}$ be a basic function and $z_0 \in U$. Then f is said to be ${}^\rho\widetilde{\mathbb{C}}$-*differentiable at* z_0 if there exist $m \in {}^\rho\widetilde{\mathbb{C}}$ and a gauge σ such that $\sigma \le \rho$ and

$$ f(z_0 + h_\iota) = f(z_0) + m \cdot h_\iota + o(h) \quad \text{as} \quad h \xrightarrow{\sigma} 0 \text{ in the sharp topology.} \tag{2.3} $$

We use the notation $o(h)$ instead of $o(h_\iota)$ for simplicity. We then define the derivative of f at a point z_0 by the unique $m =: f'(z_0)$ satisfying Def. 8. We say that f is a *generalized holomorphic function* (GHF) at z_0 if it is ${}^\rho\widetilde{\mathbb{C}}$-differentiable at every point in some neighborhood of z_0. Moreover, a function f is a *GHF on a sharply open set* $V \subseteq U$, if it is ${}^\rho\widetilde{\mathbb{C}}$-differentiable at every point of V.

The first step is to prove ε-wise differentiability of ${}^\rho\widetilde{\mathbb{C}}$-differentiable function:

Theorem 9. *Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharply open set and (f_ε) be a net of functions $f_\varepsilon : U_\varepsilon \to \mathbb{C}$ with $U_\varepsilon \subseteq \mathbb{C}$. If $U \subseteq \langle U_\varepsilon \rangle$ and $f = [f_\varepsilon(-)] : U \to {}^\rho\widetilde{\mathbb{C}}$ is a basic function, then f is ${}^\rho\widetilde{\mathbb{C}}$-differentiable at $z_0 := [z_{0\varepsilon}] \in U$ if and only if f_ε is complex differentiable on some open neighborhood of $z_{0\varepsilon}$ for small ε and $f'(z_0) = [f'_\varepsilon(z_{0\varepsilon})]$.*

In the proof of this theorem, a basic idea is that we can always have a basic function R satisfying Def. 7 for the $o(h)$ of (2.3), and such that $R(h) = [\widehat{R_\varepsilon}(h_\varepsilon)]$ and $\widehat{R_\varepsilon}(0) = 0$, for all $\varepsilon \in I$. We can indeed define a net of functions $\widehat{R_\varepsilon} : H_\varepsilon \to \mathbb{C}$, where $H \subseteq \langle H_\varepsilon \rangle$, and for all $h \in H_\varepsilon$

$$\widehat{R_\varepsilon}(h) := \begin{cases} \frac{f_\varepsilon(z_{0\varepsilon}+h)-f_\varepsilon(z_{0\varepsilon})}{h} - m_\varepsilon & \text{if } h \in H_\varepsilon \setminus \{0\}\,, \\ 0 & \text{otherwise.} \end{cases}$$

From Thm. 9 it also follows that if f is a GHF, then its real and imaginary parts are GSF. This result is well-known in the case of Colombeau holomorphic function satisfying CRE (cf. [5, 16, 1]). The main difference is that we consider a larger class of functions, without considering the differentiability of the representatives.

It is natural to expect that several classical theorems of differential calculus, such as algebraic properties and chain rule can be extended from the ordinary holomorphic case to the generalized holomorphic framework.

Theorem 10. *Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharply open set, f, $g : U \to {}^\rho\widetilde{\mathbb{C}}$ be basic functions, $z_0 \in U$, and $c \in {}^\rho\widetilde{\mathbb{C}}$. If f, g are ${}^\rho\widetilde{\mathbb{C}}$-differentiable at z_0, then*

(i) $f + g$ *is* ${}^\rho\widetilde{\mathbb{C}}$*-differentiable at* z_0 *and* $(f+g)'(z_0) = f'(z_0) + g'(z_0)$;
(ii) cf *is* ${}^\rho\widetilde{\mathbb{C}}$*-differentiable at* z_0 *and* $(cf)'(z_0) = cf'(z_0)$;
(iii) fg *is* ${}^\rho\widetilde{\mathbb{C}}$*-differentiable at* z_0 *and* $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.

Theorem 11 (Chain rule)**.** *Let U, $V \subseteq {}^\rho\widetilde{\mathbb{C}}$ be sharply open sets, $f : V \to U$, $g : U \to {}^\rho\widetilde{\mathbb{C}}$, and $z_0 \in V$. If f is ${}^\rho\widetilde{\mathbb{C}}$-differentiable at z_0 and g is ${}^\rho\widetilde{\mathbb{C}}$-differentiable at $f(z_0)$, then $(g \circ f)$ is ${}^\rho\widetilde{\mathbb{C}}$-differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Using the new notion of hyperlimit and of little-oh, we also define new notions of continuity, ${}^\rho\widetilde{\mathbb{R}}$-differentiability, and ${}^\rho\widetilde{\mathbb{R}}$-partial differentiability (see also [2] for similar notions, even if in our setting we use two gauges, the notion of hyperlimit and of little-oh. We also prefer to state these notions without considering quotients, i.e. without using the density of invertible elements).

Definition 12.

(i) Let $U \subseteq {}^\rho\widetilde{\mathbb{C}}$ be a sharply open set, $f : U \to {}^\rho\widetilde{\mathbb{C}}$ be a basic function and $z_0 \in U$. Then f is said to be *Landau continuous* at z_0 if there exists a gauge σ, such that $\sigma \le \rho$ and

$$f(z_0 + h_\iota) - f(z_0) = o(1) \quad \text{as} \quad h \xrightarrow{\sigma} 0.$$

(ii) Let $V \subseteq {}^\rho\widetilde{\mathbb{R}}^2$ be a sharply open set, $f : V \to {}^\rho\widetilde{\mathbb{R}}^2$ be a basic function and $z = (x, y) \in V$. Then,

  (i) A function f is said to be ${}^\rho\widetilde{\mathbb{R}}$-*differentiable* at z if there exists a gauge σ, such that $\sigma \le \rho$, and a linear mapping $T_z : {}^\rho\widetilde{\mathbb{R}}^2 \to {}^\rho\widetilde{\mathbb{R}}^2$, satisfying the condition:

$$f(z + h_\iota) - f(z) - T_z(h_\iota) = o(h) \quad \text{as} \quad h \xrightarrow{\sigma} 0.$$

  This unique T_z is called the differential of f at z and is denoted by $f'(z)$ or $\mathrm{d}f(z)$.

  (ii) A function f is said to be ${}^\rho\widetilde{\mathbb{R}}$-*partial differentiable* with respect to x (resp. y) if there exists a gauge σ, such that $\sigma \le \rho$, and m_x (resp. m_y)$\in {}^\rho\widetilde{\mathbb{R}}$, satisfying the condition:

$$f(x + h_\iota, y) = f(x + y) + h_\iota m_x + o(h) \quad \text{as} \quad h \xrightarrow{\sigma} 0$$

$$\left(\text{resp.}\quad f(x,y+h_\iota)=f(x+y)+h_\iota m_y+o(h)\quad\text{as}\quad h\xrightarrow{\sigma}0\right).$$

We define the ${}^{\rho}\widetilde{\mathbb{R}}$-partial derivative of f with respect to x (resp. y) as $\frac{\partial f}{\partial x}=\partial_1 f=\partial_x f:=m_x$ (or $\frac{\partial f}{\partial y}=\partial_2 f=\partial_2 f:=m_2$).

It is possible to prove that all basic functions are Landau continuous if and only if they are defined by nets of continuous functions. In this result is important to consider a smaller gauge σ, like in Thm. 6, in order to still have balls defined by σ-invertible radii (only this type of balls is closed with respect to intersection).

Finally, it is also natural to expect that ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable function satisfies the CRE. However, from Def. 8 it is also clear that necessarily the real and imaginary parts of a ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable map f are Landau continuously differentiable, and this notion, on the contrary with respect to the CRE, depends on some gauge $\sigma\le\rho$:

Theorem 13 (Cauchy-Riemann equations)**.** *Consider $U\subseteq{}^{\rho}\widetilde{\mathbb{R}}^2$ be a sharply open set and $(x_0,y_0)\in U$. Let u, $v:U\to{}^{\rho}\widetilde{\mathbb{R}}^2$ be ${}^{\rho}\widetilde{\mathbb{R}}$-differentiable functions at (x_0,y_0). Set $\bar{U}:=\left\{z\in{}^{\rho}\widetilde{\mathbb{C}}:(Re(z),Im(z))\in U\right\}$ and for all $z=(x,y)\in\overline{U}$, $f(z):=u(x,y)+iv(x,y)$. Then, f is ${}^{\rho}\widetilde{\mathbb{C}}$-differentiable at $z_0:=x_0+iy_0$ if and only if u, v are Landau continuously differentiable at (x_0,y_0) and satisfy the CRE:*

$$\partial_1 u=\partial_2 v\quad\textit{and}\quad\partial_2 u=-\partial_1 v.\tag{2.4}$$

From this result, it immediately follows that if f is a Colombeau holomorphic function, as defined in [21], then we can take $\sigma=\rho=(\varepsilon)$ and we get that f is also a GHF. The opposite is false, in general, as proved by our natural extension of the (embedding of the) Dirac delta to a subset U such that ${}^{\rho}\widetilde{\mathbb{R}}\subseteq U\subseteq{}^{\rho}\widetilde{\mathbb{C}}$ (see [18]). Note that this implies that in our setting the identity principle does *not* hold. This already occurs for generalized real analytic function (see [20]) and is a consequence of the disconnectedness of ${}^{\rho}\widetilde{\mathbb{R}}$ due to the existence of infinitesimal numbers (e.g. the set of all the infinitesimals is a clopen set). In our opinion, this is not a negative result because, on the other hand, it allows us to include interesting examples of GHF. See also [12] for general results about sets of uniqueness in the Colombeau generalized setting.

3. Conclusions

The idea to consider a definition of GHF starting from some kind of limit of the incremental ratio actually was born from a classical Hadamard sense of beauty: one of the most beautiful result of complex analysis is indeed that a simple request of differentiability implies very strong regularities on the function. It was a deep and careful attempt to prove the ε-wise version of the hyperlimit Thm. 6, starting from a similar theorem in [15], that allows the first author to understand the importance to consider two gauges. A posteriori, this was clear because of the motivations related to the use of hyperseries we explained in the introduction.

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