A NEW APPROACH TO GENERALIZED FUNCTIONS FOR MATHEMATICAL PHYSICS

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ABSTRACT. The main aim of the present work is to arrive at a mathematical theory near to Cauchy-Dirac's original conception of generalized function, i.e. set theoretical functions defined, and with values, in a suitable ring of scalars and having several properties in common with ordinary smooth functions, like composition and nonlinear operations. This is as they are still used in some informal calculations in Physics. We introduce a category of generalized functions as smooth set-theoretical maps on (multidimensional) points of a ring of scalars containing infinitesimals and infinities. This category extends Schwartz distributions. The calculus of these generalized functions is closely related to classical analysis, with point values, composition, non linear operations and the generalization of several classical theorems of calculus. Some applications in Mathematical Physics are presented. Finally, we extend this category of generalized functions to a quasi-topos which includes diffeological spaces, in particular finite and infinite-dimensional manifolds and their function spaces.

CONTENTS

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1. Introduction: foundations of generalized functions for Mathematical Physics

The aim of the present work is to lay the foundations for a new approach to the theory of generalized functions. In developing such a theory, various objectives can be pursued, and our motivations mainly come from applications in Mathematical Physics, where the need for such a nonlinear theory is well known (see, e.g., [44, 109, 49, 18, 8, 93, 84, 11, 45] and references therein).

In particular, our aim is to arrive at a mathematical theory close to Cauchy-Dirac's original conception of generalized function, [21, 78, 64]. In essence, the idea of Cauchy and Dirac (as well as of authors such as Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside, who informally worked with "numbers" which also comprise infinitesimals and infinite scalars) was to view generalized functions as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, [64]. More generally, in this approach generalized functions are seen as set-theoretical functions defined on, and attaining values in, a suitable non-Archimedean ring of scalars containing infinitesimals and infinities, as well as sharing essential properties of ordinary smooth functions. In the present work, we will develop this point of view, and prove that it generalizes the mentioned Cauchy-Dirac approach. In our view, the main benefits of this theory lie in a clarification of a number of foundational issues in the theory of generalized functions, namely:

(i) It includes all Schwartz distributions, see Thm. 23.
(ii) It allows nonlinear operations on generalized functions, Sec. 3, and to compose them unrestrictedly, Thm. 26.
(iii) It is simpler than standard approaches as it allows us to treat generalized functions more closely to classical smooth functions. In particular, it allows us to prove a number of analogues of theorems of classical analysis: e.g., mean value theorem, intermediate value theorem, extreme value theorem, Taylor's theorem, local and global inverse function theorem, integrals via primitives, multidimensional integrals, theory of compactly supported functions, Banach like fixed point theorem, Picard-Lindelöf theorem, among others. See Sec. 6. Therefore, this approach to generalized functions results in a flexible and rich

framework which allows both the formalization of several calculations appearing in physics and the development of new applications in mathematical physics; see Sec. 11.

(iv) It allows the viewing of these new generalized functions as smooth functionals, sharing many properties with classical Schwartz distributions, Sec. 9.

(v) The closure with respect to composition lends to a solution concept of differential equations close to the classical one, Sec. 10.

(vi) It also includes physically meaningful generalized functions like an idealized mollifier $f(x) = i^{-1} \cdot \varphi(i \cdot x)$, where i is infinitesimal and $\varphi \in \mathcal{D}(\mathbb{R})$ is a compactly supported smooth function, or like $g = I \cdot \chi_{B_i(q)}$, where $\chi_{B_i(q)}$ is the characteristic function of a ball of infinitesimal radius i and I is an infinite density, or like a wave with infinite (or infinitesimal) frequency $h(t) = (\cos(I \cdot t), \sin(I \cdot t))$. Using the local structure theorem for distributions, it is possible to show that f, g and h are not distributions.

Moreover, we think that a satisfactory theory of generalized functions as used in Mathematical Physics should also provide an extension to function spaces, possibly in a Cartesian closed category or, better, in a quasi-topos. The use of a Cartesian closed category as a useful framework for Mathematical Physics can be motivated in several ways:

(i) In Physics, the necessity to use infinite-dimensional spaces frequently appears. A classical example is the space $\mathbf{Man}(M, N)$ of all smooth mappings between two smooth manifolds M and N, or some of its subspaces, e.g. the space of all the diffeomorphisms of a smooth manifold. Typically, we are interested in infinite dimensional Lie groups, because they appear, e.g., in the study of both compressible and incompressible fluids, in magnetohydrodynamics, in plasma-dynamics or in electrodynamics (see e.g. [2] and references therein). It is also well established (see e.g. [29, 35, 32]) that Cartesian closedness is a desirable condition in the calculus of variations.

(ii) The convenient setting, [77, 29], is the most advanced theory of smooth spaces extending the theory of Banach manifolds. Some applications of this notion to classical field theory can be found in [1]. In addition, several other approaches to a new notion of smooth space have been motivated by problems of Physics. For example, the notion of diffeological space has been used in [106, 107, 108], starting also from a variant of [15], to study quantization of coadjoint orbits in infinite dimensional groups of diffeomorphisms. Diffeological spaces form a Cartesian closed, complete, co-complete quasi-topos, [60, 7, 68].

(iii) Another motivation towards Cartesian closedness is surely the role of topos theory in foundational issues of quantum theory, quantum gravity and the intuitionistic approach to general relativity. The literature concerning this approach is vast, see e.g. [61, 62, 63, 13, 14, 54, 94, 82, 69, 55]. One characteristic of this approach is its critique of standard implicit hypotheses of most physical theories, e.g. to view space-time as a manifold, while gaining meaningful interpretations in quantum theory or in general relativity (see e.g. [50, 47, 51, 52, 53]).

For these reasons, we close this work by introducing a generalization of diffeological space that allows the embedding of our category of generalized functions into a quasi-topos; see Sec. 12.

Finally, a theory of generalized functions for Mathematical Physics frequently appears coupled with a theory of actual infinitesimals and infinities (see e.g. [58, 68, 67, 48]). This is natural, since informal descriptions of these functions used in many calculations in Physics employ a language including infinitesimals or infinities. Historically, it has turned out that approaches requiring a substantial background knowledge in mathematical logic are only reluctantly accepted by some physicists. Therefore, even if sometimes they appear less powerful, theories that do not need such knowledge ([33, 103, 16]) are more easily accepted.

The structure of the paper is as follows to write

2. The ring of scalars and its topologies

Exactly as real numbers can be seen as equivalence classes of sequences $(q_n)_{n\in\mathbb{N}}$ of rationals[1], it is very natural to consider a non-Archimedean extension of $\mathbb{R}$ defined by a quotient ring $\mathcal{R}/\sim$, where $\mathcal{R}\subseteq\mathbb{R}^I$. Here $\mathcal{R}$ is a subalgebra of nets $(x_\varepsilon)_{\varepsilon\in I}\in\mathbb{R}^I$ defined on a directed set $(I,\le)$, and with pointwise algebraic operations. Instead of $I=\mathbb{N}$, we consider $I:=(0,1]$, corresponding to $\varepsilon\to 0^+$, $\varepsilon\in I$. In this work, we will denote ε-dependent nets simply by $(x_\varepsilon):=(x_\varepsilon)_{\varepsilon\in I}$, and the corresponding equivalence class simply by $[x_\varepsilon]:=[(x_\varepsilon)]_\sim\in\mathcal{R}/\sim$. We aim at constructing the quotient ring $\widetilde{\mathbb{R}}:=\mathcal{R}/\sim$ so that it contains infinitesimals and infinities. The following observation points to a natural way of achieving this goal. Let us assume that $[z_\varepsilon]=0\in\widetilde{\mathbb{R}}$ and $[J_\varepsilon]\in\widetilde{\mathbb{R}}$ is generated by an infinite net (J_ε), i.e. such that $\lim_{\varepsilon\to 0^+}|J_\varepsilon|=+\infty$. Then we would have

$$\begin{aligned}[z_\varepsilon]\cdot[J_\varepsilon]&=0\cdot[J_\varepsilon]=0\\&=[z_\varepsilon\cdot J_\varepsilon].\end{aligned}\tag{2.1}$$

Finally, let us assume that

$$\forall[w_\varepsilon]\in\widetilde{\mathbb{R}}:\ [w_\varepsilon]=0\ \Rightarrow\ \lim_{\varepsilon\to 0^+}w_\varepsilon=0.\tag{2.2}$$

Under these assumptions, (2.1) yields $\lim_{\varepsilon\to 0^+}z_\varepsilon\cdot J_\varepsilon=0$, and hence

$$\exists\varepsilon_0\in I\,\forall\varepsilon\in(0,\varepsilon_0]:\ |z_\varepsilon|\le\left|J_\varepsilon^{-1}\right|.\tag{2.3}$$

We can state this property by saying that *the nets* (z_ε) *representing* 0*, i.e. such that* $(z_\varepsilon)\sim 0$*, must be dominated by the reciprocals of every infinite number* $[J_\varepsilon]\in\widetilde{\mathbb{R}}$. It is not hard to prove that if every infinite net (J_ε) is in the subalgebra $\mathcal{R}$, then (2.3) implies that the equivalence relation $\sim$ must be trivial:

$$\exists\varepsilon_0\in I\,\forall\varepsilon\in(0,\varepsilon_0]:\ z_\varepsilon=0.\tag{2.4}$$

This situation corresponds to the Schmieden-Laugwitz model, [105].

If we do not want to have the trivial model (2.4), we can hence either negate the natural property (2.2) (this is the case of nonstandard analysis; see [19] for more details) or to restrict the class of all the nets (J_ε) generating infinite numbers in $\widetilde{\mathbb{R}}$. Since we want to start from a subalgebra $\mathcal{R}\subseteq\mathbb{R}^I$, a *first* natural idea is to consider the following class of infinite nets

$$\mathcal{I}:=\left\{(\varepsilon^{-a})\mid a\in\mathbb{R}_{>0}\right\}.\tag{2.5}$$

[1]In the naturals $\mathbb{N}=\{0,1,2,\dots\}$ we include zero.

and hence to consider the subalgebra $\mathcal{R} \subseteq \mathbb{R}^I$ containing nets $(b_\varepsilon) \in \mathbb{R}^I$ bounded by some $(J_\varepsilon) \in \mathcal{I}$. This idea is generalized in the following definition, where we take exactly (2.3) as the widest possible definition of $(z_\varepsilon) \sim 0$:

Definition 1. Let $\rho = (\rho_\varepsilon) \in \mathbb{R}^I$ be a net such that $(\rho_\varepsilon) \downarrow 0$ (in the following, such a net will be called a *gauge*), then

(i) $\mathcal{I}(\rho) := \{(\rho_\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\}$ is called the *asymptotic gauge* generated by ρ.

(ii) If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0 \varepsilon : \mathcal{P}(\varepsilon)$ to denote $\exists \varepsilon_0 \in I \, \forall \varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$. We can read $\forall^0 \varepsilon$ as *for ε small*.

(iii) We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ *is ρ-moderate*, and we write $(x_\varepsilon) \in \mathbb{R}_\rho$ if

$$\exists (J_\varepsilon) \in \mathcal{I}(\rho) : \; x_\varepsilon = O(J_\varepsilon) \text{ as } \varepsilon \to 0^+,$$

i.e. if

$$\exists N \in \mathbb{N} \, \forall^0 \varepsilon : \; |x_\varepsilon| \le \rho_\varepsilon^{-N}.$$

(iv) Let (x_ε), $(y_\varepsilon) \in \mathbb{R}^I$, then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if

$$\forall (J_\varepsilon) \in \mathcal{I}(\rho) : \; x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1}) \text{ as } \varepsilon \to 0^+,$$

that is if

$$\forall n \in \mathbb{N} \, \forall^0 \varepsilon : \; |x_\varepsilon - y_\varepsilon| \le \rho_\varepsilon^n.$$

This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$$^\rho\widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho,$$

which we call *Robinson-Colombeau ring of generalized numbers*, [95, 16].

In the following, ρ will always denote a net as in Def. 1, even if we will sometimes omit the dependence on the infinitesimal ρ, when this is clear. We will see below that we can choose ρ e.g. depending on the class of differential equations we need to solve for the generalized functions we are going to introduce.

We can also define an order relation on $^\rho\widetilde{\mathbb{R}}$ by saying $[x_\varepsilon] \le [y_\varepsilon]$ if there exists $(z_\varepsilon) \in \mathbb{R}^I$ such that $(z_\varepsilon) \sim_\rho 0$ (we then say that (z_ε) is *ρ-negligible*) and $x_\varepsilon \le y_\varepsilon + z_\varepsilon$ for ε small. Equivalently, we have that $x \le y$ if and only if there exist representatives $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ such that $x_\varepsilon \le y_\varepsilon$ for all ε. The following result follows directly from the previous definitions:

Theorem 2. *$^\rho\widetilde{\mathbb{R}}$ is a partially ordered ring. The real numbers $r \in \mathbb{R}$ are embedded in $^\rho\widetilde{\mathbb{R}}$ by viewing them as constant nets $[r] \in {}^\rho\widetilde{\mathbb{R}}$.*

Even if the order $\le$ is not total, we still have the possibility to define the infimum $[x_\varepsilon] \wedge [y_\varepsilon] := [\min(x_\varepsilon, y_\varepsilon)]$, and analogously the supremum function $[x_\varepsilon] \vee [y_\varepsilon] := [\max(x_\varepsilon, y_\varepsilon)]$ and the absolute value $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$. In the following, we will also use the customary notation $^\rho\widetilde{\mathbb{R}}^*$ for the set of invertible generalized numbers.

As in every non-Archimedean ring, we have the following

Definition 3. Let $x \in {}^\rho\widetilde{\mathbb{R}}$ be a generalized number, then we say that

(i) x is *infinitesimal* if $|x| < r$ for all $r \in \mathbb{R}_{>0}$. If $x = [x_\varepsilon]$, this is equivalent to $\lim_{\varepsilon \to 0^+} x_\varepsilon = 0$. We write $x \approx y$ if $x - y$ is infinitesimal.

(ii) x is *infinite* if $|x| > r$ for all $r \in \mathbb{R}_{>0}$. If $x = [x_\varepsilon]$, this is equivalent to $\lim_{\varepsilon \to 0^+} |x_\varepsilon| = +\infty$.

(iii) x is *finite* if $|x| < r$ for some $r \in \mathbb{R}_{>0}$.

For example, setting $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, we have that $\mathrm{d}\rho^n \in {}^\rho\widetilde{\mathbb{R}}$, $n \in \mathbb{N}_{>0}$, is an invertible infinitesimal, whose reciprocal is $\mathrm{d}\rho^{-n} = [\rho_\varepsilon^{-n}]$ which is necessarily a positive infinite number. Of course, in the ring ${}^\rho\widetilde{\mathbb{R}}$ we have generalized numbers which are not in any of the previous classes, like e.g. $x_\varepsilon = \frac{1}{\varepsilon}\sin\left(\frac{1}{\varepsilon}\right)$.

Our notations for intervals are: $[a,b] := \{x \in {}^\rho\widetilde{\mathbb{R}} \mid a \le x \le b\}$, $[a,b]_{\mathbb{R}} := [a,b] \cap \mathbb{R}$, and analogously for segments $[x,y] := \{x + r\cdot(y-x) \mid r \in [0,1]\} \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $[x,y]_{\mathbb{R}^n} = [x,y] \cap \mathbb{R}^n$.

2.1. Topologies on ${}^\rho\widetilde{\mathbb{R}}^n$.

On the ${}^\rho\widetilde{\mathbb{R}}$-module ${}^\rho\widetilde{\mathbb{R}}^n$ we can consider the natural extension of the Euclidean norm, i.e. $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$, where $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$. Even if this generalized norm takes values in ${}^\rho\widetilde{\mathbb{R}}$, it shares several properties with the usual norms:

$$
\begin{aligned}
&|x| = x \vee (-x)\\
&|x| \ge 0\\
&|x| = 0 \Rightarrow x = 0\\
&|y\cdot x| = |y|\cdot|x|\\
&|x+y| \le |x| + |y|\\
&||x| - |y|| \le |x-y|.
\end{aligned}
$$

It is therefore natural to consider on ${}^\rho\widetilde{\mathbb{R}}^n$ topologies generated by balls defined by this generalized norm and a set of radii:

Definition 4. We say that $\mathfrak{R}$ is a *set of radii* if

(i) $\mathfrak{R} \subseteq {}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$ is a non-empty subset of positive invertible generalized numbers.
(ii) For all r, $s \in \mathfrak{R}$ the infimum $r \wedge s \in \mathfrak{R}$.
(iii) $k \cdot r \in \mathfrak{R}$ for all $r \in \mathfrak{R}$ and all $k \in \mathbb{R}_{>0}$.

Moreover, if $\mathfrak{R}$ is a set of radii and x, $y \in {}^\rho\widetilde{\mathbb{R}}$, then:

(i) We write $x <_{\mathfrak{R}} y$ if $\exists r \in \mathfrak{R}: \ r \le y - x$.
(ii) $B_r^{\mathfrak{R}}(x) := \left\{ y \in {}^\rho\widetilde{\mathbb{R}}^n \mid |y-x| <_{\mathfrak{R}} r \right\}$ for each $r \in \mathfrak{R}$.
(iii) $B_\rho^{\mathrm{E}}(x) := \{y \in \mathbb{R}^n \mid |y-x| < \rho\}$, for each $\rho \in \mathbb{R}_{>0}$, denotes an ordinary Euclidean ball in $\mathbb{R}^n$.

For example, ${}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$ and $\mathbb{R}_{>0}$ are sets of radii.

Lemma 5. *Let $\mathfrak{R}$ be a set of radii and x, y, $z \in {}^\rho\widetilde{\mathbb{R}}$, then*

(i) $\neg\,(x <_{\mathfrak{R}} x)$.
(ii) $x <_{\mathfrak{R}} y$ *and* $y <_{\mathfrak{R}} z$ *imply* $x <_{\mathfrak{R}} z$.
(iii) $\forall r \in \mathfrak{R}: \ 0 <_{\mathfrak{R}} r$.

Proof. (i): $x <_{\mathfrak{R}} x$ implies $r \le 0$ for some $r \in {}^\rho\widetilde{\mathbb{R}}^*$. Therefore $r^{-1}r = 1 \le 0$.
(ii): If $r \le y - x$ and $s \le z - y$ for r, $s \in \mathfrak{R}$, then $2(r \wedge s) \le r + s \le z - x$.
(iii): In fact, we have $0 <_{\mathfrak{R}} r$ if and only if $s \le r$ for some $s \in \mathfrak{R}$. □

The relation $<_{\mathfrak{R}}$ has better topological properties as compared to the usual strict order relation $x \le y$ and $x \ne y$ (that we will never use) because of the following result

Theorem 6. *The set of balls* $\left\{B_r^{\mathfrak{R}}(x) \mid r \in \mathfrak{R},\ x \in {}^\rho\widetilde{\mathbb{R}}^n\right\}$ *generated by a set of radii* $\mathfrak{R}$ *is a base for a topology on* ${}^\rho\widetilde{\mathbb{R}}^n$.

Proof. It suffices to consider $z \in B_r^{\mathfrak{R}}(x) \cap B_s^{\mathfrak{R}}(y)$ and to prove that $B_\rho^{\mathfrak{R}}(z) \subseteq B_r^{\mathfrak{R}}(x) \cap B_s^{\mathfrak{R}}(y)$ for some $\rho \in \mathfrak{R}$. The proof is essentially a reformulation of the classical proof in metric spaces. In fact, we have $\bar{r} \le r - |x - z|$ and $\bar{s} \le s - |y - z|$ for some $\bar{r}$, $\bar{s} \in \mathfrak{R}$. Set $\rho := \bar{r} \wedge \bar{s} \in \mathfrak{R}$. The inequality $|w - z| <_{\mathfrak{R}} \rho$ implies $\sigma \le \rho - |w-z|$ for some $\sigma \in \mathfrak{R}$. Therefore, $|w-x| \le |w-z| + |z-x| \le \rho - \sigma + r - \bar{r}$ and whereby $\sigma \le \bar{r} + \sigma - \rho \le r - |w - x|$, i.e. $|w - x| <_{\mathfrak{R}} r$. This proves that $B_\rho^{\mathfrak{R}}(z) \subseteq B_r^{\mathfrak{R}}(x)$, and the other inclusion follows analogously. □

Henceforth, we will only consider the sets of radii ${}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$ and $\mathbb{R}_{>0}$. The topology generated in the former case is called *sharp topology*, whereas the latter is called *Fermat topology*. We will call *sharply open set* any open set in the sharp topology, and *large open set* any open set in the Fermat topology; clearly, the latter is coarser than the former. Let us note explicitly that taking an infinitesimal radius $r \in {}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$ we can consider infinitesimal neighborhoods of $x \in {}^\rho\widetilde{\mathbb{R}}^n$ in the sharp topology. Of course, this is not possible in the Fermat topology. The existence of infinitesimal neighborhoods implies that the sharp topology induces the discrete topology on $\mathbb{R}$. This phenomenon occurs necessarily in any theory containing continuous generalized functions which have infinite derivatives; In fact, from the mean value theorem Thm. 48.(i) below, we have $f(x) - f(x_0) = f'(c) \cdot (x - x_0)$ for some $c \in [x, x_0]$. Therefore, we have $f(x) \in B_r(f(x_0))$, for a given $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, if and only if $|x - x_0| \cdot |f'(c)| < r$, which yields an infinitesimal neighborhood of x_0 in case $f'(c)$ is infinite; see [38, 39] for precise statements and proofs corresponding to this intuition. By an innocuous abuse of language, we write $x < y$ instead of $x <_{{}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}} y$ and $x <_{\mathbb{R}} y$ instead of $x <_{\mathbb{R}_{>0}} y$. For example, ${}^\rho\widetilde{\mathbb{R}}^*_{\ge 0} = {}^\rho\widetilde{\mathbb{R}}_{>0}$. We will simply write $B_r(x)$ to denote an open ball in the sharp topology and $B_r^{\mathrm{F}}(x)$ for an open ball in the Fermat topology.

The following result is useful to deal with positive and invertible generalized numbers.

Lemma 7. *Let* $x \in {}^\rho\widetilde{\mathbb{R}}$. *Then the following are equivalent:*

(i) *x is invertible and $x \ge 0$, i.e. $x > 0$.*
(ii) *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x we have $\forall^0\varepsilon : \ x_\varepsilon > 0$.*
(iii) *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x we have $\exists m \in \mathbb{N}\, \forall^0\varepsilon : \ x_\varepsilon > \rho_\varepsilon^m$.*

Proof. (i) ⇒ (ii): Since x is positive, we can find a representative $[x_\varepsilon] = x$ such that $x_\varepsilon \ge 0$ for all ε. But x is also invertible, so for all ε we can also write $x_\varepsilon y_\varepsilon = 1 + z_\varepsilon$, where $(z_\varepsilon) \sim_\rho 0$ is a negligible net. By contradiction, assume that $x_{\varepsilon_k} \le 0$ for each $k \in \mathbb{N}$, where $(\varepsilon_k)_{k\in\mathbb{N}} \to 0^+$. Then $x_{\varepsilon_k} = 0$ and hence $x_{\varepsilon_k} y_{\varepsilon_k} = 0 = 1 + z_{\varepsilon_k} \to 1$ for $k \to +\infty$, which is a contradiction.
(ii) ⇒ (iii): Assume that there exists a representative $[x_\varepsilon] = x$ such that $x_{\varepsilon_k} \le \rho_{\varepsilon_k}^k$ for each $k \in \mathbb{N}$, where $(\varepsilon_k)_{k\in\mathbb{N}} \to 0^+$ monotonically. We then define an ρ-moderate net by $\hat{x}_\varepsilon := 0$ if $\varepsilon = \varepsilon_k$ and $\hat{x}_\varepsilon := x_\varepsilon$ otherwise. For each $n \in \mathbb{N}$, if k is sufficiently big, we have $|x_{\varepsilon_k} - \hat{x}_{\varepsilon_k}| \le \rho_{\varepsilon_k}^k \le \rho_{\varepsilon_k}^n$. This implies that $(x_\varepsilon) \sim_\rho (\hat{x}_\varepsilon)$. Therefore $(\hat{x}_\varepsilon)$ is another representative of x which contradicts (ii) by construction.
(iii) ⇒ (i): By assumption, $\lim_{\varepsilon\to 0^+} \rho_\varepsilon = 0^+$. This and (iii) yield that $x_\varepsilon > \rho_\varepsilon^m > 0$ for ε small, say for $\varepsilon \le \varepsilon_0$. Therefore, $0 < y_\varepsilon := x_\varepsilon^{-1} \le \rho_\varepsilon^{-m}$ for $\varepsilon \le \varepsilon_0$ (and y_ε

arbitrarily defined elsewhere) is ρ-moderate and hence it is a representative of the inverse of x. $\square$

2.2. **Open, closed and bounded sets generated by nets.** A natural way to obtain sharply open, closed and bounded sets in ${}^{\rho}\widetilde{\mathbb{R}}^n$ is by using a net (A_ε) of subsets $A_\varepsilon \subseteq \mathbb{R}^n$. We have two ways of extending the membership relation $x_\varepsilon \in A_\varepsilon$ to generalized points $[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n$:

Definition 8. Let (A_ε) be a net of subsets of $\mathbb{R}^n$, then

(i) $[A_\varepsilon] := \left\{[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid \forall^0\varepsilon : x_\varepsilon \in A_\varepsilon\right\}$ is called the *internal set* generated by the net (A_ε).

(ii) Let (x_ε) be a net of points of $\mathbb{R}^n$, then we say that $x_\varepsilon \in_\varepsilon A_\varepsilon$, and we read it as (x_ε) *strongly belongs to* (A_ε), if
   (i) $\forall^0\varepsilon : x_\varepsilon \in A_\varepsilon$.
   (ii) If $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$, then also $x'_\varepsilon \in A_\varepsilon$ for ε small.

   Moreover, we set $\langle A_\varepsilon\rangle := \left\{[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\varepsilon A_\varepsilon\right\}$, and we call it the *strongly internal set* generated by the net (A_ε).

(iii) We say that the internal set $K = [A_\varepsilon]$ is *sharply bounded* if there exists $M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $K \subseteq B_M(0)$.

(iv) Finally, we say that the (A_ε) is a *sharply bounded net* if there exists $N \in \mathbb{R}_{>0}$ such that $\forall^0\varepsilon\, \forall x \in A_\varepsilon : |x| \le \rho_\varepsilon^{-N}$.

Therefore, $x \in [A_\varepsilon]$ if there exists a representative $[x_\varepsilon] = x$ such that $x_\varepsilon \in A_\varepsilon$ for ε small, whereas this membership is independent from the chosen representative in case of strongly internal sets. An internal set generated by a constant net $A_\varepsilon = A \subseteq \mathbb{R}^n$ will simply be denoted by $[A]$.

The following theorem shows that internal and strongly internal sets have dual topological properties:

Theorem 9. *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then we have*

(i) $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $\forall q \in \mathbb{R}_{>0}\, \forall^0\varepsilon : d(x_\varepsilon, A_\varepsilon) \le \rho_\varepsilon^q$. *Therefore* $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $[d(x_\varepsilon, A_\varepsilon)] = 0 \in {}^{\rho}\widetilde{\mathbb{R}}$.

(ii) $[x_\varepsilon] \in \langle A_\varepsilon\rangle$ *if and only if* $\exists q \in \mathbb{R}_{>0}\, \forall^0\varepsilon : d(x_\varepsilon, A_\varepsilon^c) > \rho_\varepsilon^q$, *where* $A_\varepsilon^c := \mathbb{R}^n \setminus A_\varepsilon$. *Therefore, if* $(d(x_\varepsilon, A_\varepsilon^c)) \in \mathbb{R}_\rho$, *then* $[x_\varepsilon] \in \langle A_\varepsilon\rangle$ *if and only if* $[d(x_\varepsilon, A_\varepsilon^c)] > 0$.

(iii) $[A_\varepsilon]$ *is sharply closed.*

(iv) $\langle A_\varepsilon\rangle$ *is sharply open.*

(v) $[A_\varepsilon] = [\mathrm{cl}\,(A_\varepsilon)]$, *where* $\mathrm{cl}\,(S)$ *is the closure of* $S \subseteq \mathbb{R}^n$.

(vi) $\langle A_\varepsilon\rangle = \langle \mathrm{int}(A_\varepsilon)\rangle$, *where* $\mathrm{int}\,(S)$ *is the interior of* $S \subseteq \mathbb{R}^n$.

Proof. (i) $\Rightarrow$: We have $x'_\varepsilon \in A_\varepsilon$ for some representative $[x'_\varepsilon] = [x_\varepsilon] \in [A_\varepsilon]$. But $d(x_\varepsilon, A_\varepsilon) \le |x_\varepsilon - x'_\varepsilon| + d(x'_\varepsilon, A_\varepsilon)$, from which the conclusion follows.

(i) $\Leftarrow$: Since the net $(\inf_{a\in A_\varepsilon} |x_\varepsilon - a|)$ is ρ-negligible, we can find a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}} \downarrow 0$ such that $\inf_{a\in A_\varepsilon} |x_\varepsilon - a| < \rho_\varepsilon^n$ for $\varepsilon \le \varepsilon_n$. Hence, for each $\varepsilon \in (\varepsilon_{n+1}, \varepsilon_n]_{\mathbb{R}}$ we can find $x'_\varepsilon \in A_\varepsilon$ such that $|x_\varepsilon - x'_\varepsilon| \le \rho_\varepsilon^n$. Therefore, (x'_ε) is another representative of $[x_\varepsilon]$ and $x'_\varepsilon \in A_\varepsilon$ for $\varepsilon \le \varepsilon_0$.

(ii): Let $[x_\varepsilon] \in \langle A_\varepsilon\rangle$ and suppose to the contrary that there exists a sequence $\varepsilon_k \searrow 0$ such that $d(x_{\varepsilon_k}, A_{\varepsilon_k}^c) \le \rho_{\varepsilon_k}^k$ for all $k \in \mathbb{N}$. For each k, pick some $x'_k \in A_{\varepsilon_k}^c$ with $|x'_k - x_{\varepsilon_k}| < 2\rho_{\varepsilon_k}^k$ and choose $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$ such that $x'_{\varepsilon_k} = x'_k$ for all k. Then $x'_{\varepsilon_k} \notin A_{\varepsilon_k}$ for all k, contradicting $x_\varepsilon \in_\varepsilon A_\varepsilon$. Conversely, let $d(x_\varepsilon, A_\varepsilon^c) > \rho_\varepsilon^q$ for ε

small. Then in particular, $x_\varepsilon \in A_\varepsilon$. Also, if $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$ then $d(x'_\varepsilon, A^c_\varepsilon) > (1/2)\rho^q_\varepsilon$ for ε small, so $x'_\varepsilon \in A_\varepsilon$. Thus, $[x_\varepsilon] \in \langle A_\varepsilon \rangle$.
(iii): Let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \setminus [A_\varepsilon]$. Then (i) yields that $d(x_{\varepsilon_k}, A_{\varepsilon_k}) > \rho^q_{\varepsilon_k}$ for some $q \in \mathbb{R}_{>0}$ and some sequence $(\varepsilon_k)_{k\in\mathbb{N}} \downarrow 0$. Set $\rho := \frac{1}{2}\mathrm{d}\rho^q$, then $y \in B_\rho(x)$ implies that for some representative $[y_\varepsilon] = y$ we have $d(y_{\varepsilon_k}, A_{\varepsilon_k}) \ge d(x_{\varepsilon_k}, A_{\varepsilon_k}) - |x_{\varepsilon_k} - y_{\varepsilon_k}| > \rho^q_{\varepsilon_k} - \frac{1}{2}\rho^q_{\varepsilon_k}$. Thereby (i) gives $y \notin [A_\varepsilon]$. This proves that ${}^\rho\widetilde{\mathbb{R}}^n \setminus [A_\varepsilon]$ is sharply open.
(iv): (ii) yields that $[x_\varepsilon] \in \langle A_\varepsilon \rangle$ if and only if $[x_\varepsilon]$ is in the interior of $\langle A_\varepsilon \rangle$ with respect to the sharp topology.
(v), (vi): Directly from (i) and (ii). □

For example, it is not hard to show that the closure in the sharp topology of a ball of center $c = [c_\varepsilon]$ and radius $r = [r_\varepsilon] > 0$ is

$$\overline{B_r(c)} = \left\{x \in {}^\rho\widetilde{\mathbb{R}}^d \mid |x - c| \le r\right\} = \left[\overline{B^{\mathrm{E}}_{r_\varepsilon}(c_\varepsilon)}\right]. \tag{2.6}$$

In fact, it suffices to prove these equalities for $c = 0$, because the translation $x \mapsto x - c$ is sharply continuous. If (x_n) is a sequence of $\{x \mid |x| \le r\}$ that converges to x_0, then $|x_0| \le |x_0 - x_n| + |x_n| \le |x_0 - x_n| + r$. For $n \to +\infty$, this shows that $\{x \mid |x| \le r\}$ is closed. Vice versa, if $|x| \le r$, to prove that x is an adherent point of $B_r(0)$, we need to show that

$$\forall s \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \exists \bar{x} \in B_r(0) \cap B_s(x).$$

Take $k \in \mathbb{N}$ such that $\frac{3}{2}\mathrm{d}\rho^k < \min(r, s)$ and a representative $[x_\varepsilon] = x$ such that $|x_\varepsilon| \le r_\varepsilon$ for small ε. The point $\bar{x}_\varepsilon := x_\varepsilon$ if $|x_\varepsilon| < r_\varepsilon - \rho^k_\varepsilon$ and $\bar{x}_\varepsilon := x_\varepsilon - \frac{x_\varepsilon}{2|x_\varepsilon|}\rho^k_\varepsilon$ otherwise satisfies the desired conditions. This proves the first equality in (2.6). The proof that $\overline{B_r(0)} \supseteq \left[\overline{B^{\mathrm{E}}_{r_\varepsilon}(0)}\right]$ is easy. Vice versa, if $|\bar{x}_\varepsilon| \le r_\varepsilon + z_\varepsilon$ for some representatives $[\bar{x}_\varepsilon] = x$ and $[z_\varepsilon] = 0$, then setting $x_\varepsilon := \bar{x}_\varepsilon$ if $|\bar{x}_\varepsilon| \le r_\varepsilon$ and $x_\varepsilon := \frac{\bar{x}_\varepsilon}{|\bar{x}_\varepsilon|}r_\varepsilon$ otherwise gives another representative of x that shows that $x \in \left[\overline{B^{\mathrm{E}}_{r_\varepsilon}(0)}\right]$. In a similar way, it can be shown that for every x, $y \in {}^\rho\widetilde{\mathbb{R}}$

$$y \ge x \Leftrightarrow y \in \overline{\left\{z \in {}^\rho\widetilde{\mathbb{R}} \mid z > x\right\}}. \tag{2.7}$$

To obtain large open sets starting from a net of subsets $A_\varepsilon \subseteq \mathbb{R}^n$, we can consider the analogue of $\langle A_\varepsilon \rangle$ but using the radii of the Fermat topology:

Definition 10. Let (A_ε) be a net of subsets of $\mathbb{R}^n$ and let (x_ε), (x'_ε) be nets of points of $\mathbb{R}^n$. Then

(i) We write $(x_\varepsilon) \sim_{\mathrm{F}} (x'_\varepsilon)$ to denote the property $|x_\varepsilon - x'_\varepsilon| < r$ for all $r \in \mathbb{R}_{>0}$, i.e., $\lim_{\varepsilon\to 0^+} |x_\varepsilon - x'_\varepsilon| = 0$.
(ii) We say that $x_\varepsilon \in_{\mathrm{F}} A_\varepsilon$, and we read it as (x_ε) *strongly belongs to* (A_ε) *in the Fermat topology*, if
  (i) $\forall^0\varepsilon:\ x_\varepsilon \in A_\varepsilon$.
  (ii) If $(x'_\varepsilon) \sim_{\mathrm{F}} (x_\varepsilon)$, then also $x'_\varepsilon \in A_\varepsilon$ for ε small.
  Moreover, we set $\langle A_\varepsilon \rangle_{\mathrm{F}} := \left\{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_{\mathrm{F}} A_\varepsilon\right\}$, and we call it the *strongly internal set* generated by the net (A_ε) *in the Fermat topology.*

The following result can be proved simply by generalizing the proof of Thm. 9.

Theorem 11. *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then we have*
(i) $[x_\varepsilon] \in \langle A_\varepsilon \rangle_{\mathrm{F}}$ *if and only if* $\exists r \in \mathbb{R}_{>0}\, \forall^0\varepsilon:\ d(x_\varepsilon, A^c_\varepsilon) > r$.

(ii) $\langle A_\varepsilon \rangle_{\mathrm{F}}$ *is a Fermat open set.*

Sharply bounded internal sets (which are always sharply closed by Thm. 9 (iii)) serve as compact sets for our generalized functions. We will show this by proving for them an extreme value theorem (see Thm. 50); for a deeper study of this type of sets in the case $\rho = (\varepsilon)$ see [39]; in the same particular setting, the notion of sharp topology has been introduced in [10, 99]; see also [83, 48] for an analogue of Lem. 7; see [89] for the study of internal sets, and see [41] for strongly internal sets.

3. Generalized functions as smooth set-theoretical maps

3.1. Definition and sharp continuity.

Using the ring ${}^\rho\widetilde{\mathbb{R}}$, it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by $f(x, \sigma)$, and if we set $\sigma = [\sigma_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, where $\sigma \approx 0$, we obtain the net of smooth functions $(f(-, \sigma_\varepsilon))_{\varepsilon \in I}$. This is the basic idea we are going to develop in the following definitions. We will first introduce the notion of a net (f_ε) defining a generalized smooth map of the type $X \longrightarrow Y$, where $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. This is a net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ which induces well defined maps of the form $[\partial^\alpha f_\varepsilon(-)] : \langle \Omega_\varepsilon \rangle \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$, for every multi-index $\alpha \in \mathbb{N}^n$.

Definition 12. Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of smooth functions, with $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$. Then we say that

$$(f_\varepsilon) \textit{ defines a generalized smooth map} : X \longrightarrow Y$$

if:

(i) $X \subseteq \langle \Omega_\varepsilon \rangle$ and $[f_\varepsilon(x_\varepsilon)] \in Y$ for all $[x_\varepsilon] \in X$.
(ii) $\forall [x_\varepsilon] \in X \, \forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_\rho$.

The notation

$$\forall [x_\varepsilon] \in X : \ \mathcal{P}\{(x_\varepsilon)\}$$

means

$$\forall (x_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] \in X \ \Rightarrow \ \mathcal{P}\{(x_\varepsilon)\},$$

i.e. for all representatives (x_ε) generating a point $[x_\varepsilon] \in X$, the property $\mathcal{P}\{(x_\varepsilon)\}$ holds.

A generalized smooth map is simply a function of the form $f = [f_\varepsilon(-)]|_X$:

Definition 13. Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points, then we say that

$$f : X \longrightarrow Y \text{ is a } \textit{generalized smooth function}$$

if there exists a net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defining a generalized smooth map of type $X \longrightarrow Y$, in the sense of Def. 12, such that f is the map.

$$f = [f_\varepsilon(-)]|_X. \tag{3.1}$$

We will also say that f *is defined by the net of smooth functions* (f_ε). The set of all these generalized smooth functions (GSF) will be denoted by ${}^\rho\mathcal{GC}^\infty(X, Y)$ or simply by $\mathcal{GC}^\infty(X, Y)$.

Let us note explicitly that definitions 12 and 13 state minimal logical conditions to obtain a set-theoretical map from X into Y and defined by a net of smooth functions. In particular, the following Thm. 14 states that the equality (3.1) is meaningful, i.e. that we have independence from the representatives for all derivatives $[x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$, $\alpha \in \mathbb{N}^n$.

Theorem 14. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of smooth functions, with $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$. Assume that (f_ε) defines a generalized smooth map of the type $X \longrightarrow Y$, then*

$$\forall \alpha \in \mathbb{N}^n \, \forall (x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho : \ [x_\varepsilon] = [x'_\varepsilon] \in X \ \Rightarrow \ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \sim_\rho (\partial^\alpha f_\varepsilon(x'_\varepsilon))$$

Proof. Let $\alpha \in \mathbb{N}^n$ and (x_ε), (x'_ε) be two representatives of the same point $x = [x_\varepsilon] = [x'_\varepsilon] \in X \subseteq \langle \Omega_\varepsilon \rangle$. Thm. 9 (ii) yields

$$d(x_\varepsilon, \Omega^c_\varepsilon) > \rho^q_\varepsilon \tag{3.2}$$

for some $q \in \mathbb{R}_{>0}$ and ε small. Thus $B^{\mathrm{E}}_{\rho^q_\varepsilon}(x_\varepsilon) \subseteq \Omega_\varepsilon$ for these values of ε. Choose $r \in \mathbb{R}_{>0}$ sufficiently big so that

$$2\rho^r_\varepsilon < \rho^q_\varepsilon \tag{3.3}$$

for ε small. Since $(x_\varepsilon) \sim_\rho (x'_\varepsilon)$ we have that

$$x'_\varepsilon \in B^{\mathrm{E}}_{\rho^r_\varepsilon}(x_\varepsilon) \tag{3.4}$$

for ε small, and the entire segment $\overline{x_\varepsilon x'_\varepsilon}$ connecting x_ε and x'_ε lies in $B^{\mathrm{E}}_{\rho^r_\varepsilon}(x_\varepsilon)$. Suppose that (3.2), (3.3) and (3.4) hold for $\varepsilon \in (0, \varepsilon_0]$. Now set $\mu_\varepsilon(t) := \partial^\alpha f_\varepsilon(x_\varepsilon + t(x'_\varepsilon - x_\varepsilon))$ for $t \in [0,1]$ and $\varepsilon \in (0, \varepsilon_0]$. By the classical mean value theorem $\partial^\alpha f_\varepsilon(x'_\varepsilon) - \partial^\alpha f_\varepsilon(x_\varepsilon) = \mu_\varepsilon(1) - \mu_\varepsilon(0) = \mu'_\varepsilon(\theta_\varepsilon)$ for some $\theta_\varepsilon \in (0,1)$, and hence for all $\varepsilon \in (0, \varepsilon_0]$ we get

$$\partial^\alpha f_\varepsilon(x'_\varepsilon) - \partial^\alpha f_\varepsilon(x_\varepsilon) = \sum_{k=1}^n \partial^{\alpha+e_k} f_\varepsilon(\xi_\varepsilon) \cdot (x'^k_\varepsilon - x^k_\varepsilon), \tag{3.5}$$

where $\xi_\varepsilon := x_\varepsilon + \theta_\varepsilon(x'_\varepsilon - x_\varepsilon)$ and $e_k := (0, \overset{k-1}{\ldots}, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n$. The generalized point $[\xi_\varepsilon] = [x_\varepsilon] \in X$ since $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$. Therefore by Def. 12 (ii) we get that every derivative $(\partial^{\alpha+e_k} f_\varepsilon(\xi_\varepsilon))$ is ρ-moderate. From this and the equivalence $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$, equation (3.5) yields the conclusion $(\partial^\alpha f_\varepsilon(x'_\varepsilon)) \sim_\rho (\partial^\alpha f_\varepsilon(x_\varepsilon))$. □

Note that taking arbitrary subsets $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ in Def. 12, we can also consider GSF defined on closed sets, like the set of all infinitesimals, or like a closed interval $[a, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$. We can also consider GSF defined at infinite generalized points. A simple case is the exponential map

$$e^{(-)} : [x_\varepsilon] \in \left\{ x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists z \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ x \le \log z \right\} \mapsto [e^{x_\varepsilon}] \in {}^\rho\widetilde{\mathbb{R}}. \tag{3.6}$$

The domain of this map depends on the infinitesimal net ρ. For instance, if $\rho = (\varepsilon)$ then all its points are bounded by generalized numbers of the form $[-N \log \varepsilon]$, $N \in \mathbb{N}$; whereas if $\rho = \left(e^{-\frac{1}{\varepsilon}}\right)$, all points are bounded by $[N\varepsilon^{-1}]$, $N \in \mathbb{N}$.

A first regularity property of GSF is the continuity with respect to the sharp topology, as proved in the following

Theorem 15. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ and $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ be a net of smooth functions that defines a GSF of the type $X \longrightarrow Y$. Then* *(i)* $\forall[x_\varepsilon]\in X\,\forall\alpha\in\mathbb{N}^n\,\exists q\in\mathbb{R}_{>0}\,\forall^0\varepsilon:\ \sup_{y\in B^{\mathrm{E}}_{\rho_\varepsilon^q}(x_\varepsilon)}|\partial^\alpha f_\varepsilon(y)|\le\rho_\varepsilon^{-q}$.

(ii) *For all $\alpha\in\mathbb{N}^n$, the GSF $g:[x_\varepsilon]\in X\mapsto[\partial^\alpha f_\varepsilon(x_\varepsilon)]\in\widetilde{\mathbb{R}}^d$ is locally Lipschitz in the sharp topology, i.e. each $x\in X$ possesses a sharp neighborhood U such that $|g(x)-g(y)|\le L|x-y|$ for all x, $y\in U$ and some $L\in{}^\rho\widetilde{\mathbb{R}}$.*

(iii) *Each $f\in{}^\rho\mathcal{GC}^\infty(X,Y)$ is continuous with respect to the sharp topologies induced on X, Y.*

(iv) *Assume that the GSF f is locally Lipschitz in the Fermat topology and that its Lipschitz constants are always finite: $L\in\mathbb{R}$. Then f is continuous in the Fermat topology.*

Proof. We first prove (i) by contradiction, assuming that for some $[x_\varepsilon]\in X$ and some α there exists $(\varepsilon_k)_k\downarrow 0$ and a sequence $(y_k)_k$ of points in $\mathbb{R}^n$ such that $|x_{\varepsilon_k}-y_k|<\rho_{\varepsilon_k}^k$ but $|\partial^\alpha f_\varepsilon(y_k)|>\rho_{\varepsilon_k}^{-k}$. Define $x'_\varepsilon:=y_k$ for $\varepsilon=\varepsilon_k$ and $x'_\varepsilon:=x_\varepsilon$ otherwise. Then $(x'_\varepsilon)\sim_\rho(x_\varepsilon)$ but $(\partial^\alpha f_\varepsilon(x'_\varepsilon))$ is not ρ-moderate, which contradicts Def. 12 (ii).

To prove (ii), we apply (i) to each derivative $\partial^{\alpha+e_k}f_\varepsilon$ to obtain

$$\forall k=1,\dots,n\,\exists q_k\in\mathbb{R}_{>0}\exists\varepsilon_k\in I\,\forall\varepsilon\in(0,\varepsilon_k]:\ \sup_{y\in B^{\mathrm{E}}_{\rho_\varepsilon^{q_k}}(x_\varepsilon)}|\partial^{\alpha+e_k}f_\varepsilon(y)|\le\rho_\varepsilon^{-q_k}. \tag{3.7}$$

Set $q:=\max_{k=1,\dots,n}q_k$, so that for y, $z\in B_{\mathrm{d}\rho^q}(x)$ we get

$$\exists\varepsilon_0\,\forall\varepsilon\in(0,\varepsilon_0]:\ \overline{y_\varepsilon z_\varepsilon}\subseteq B^{\mathrm{E}}_{\rho_\varepsilon^q}(x_\varepsilon). \tag{3.8}$$

This and (3.7) yield that for ε small we can write

$$|\partial^\alpha f_\varepsilon(y_\varepsilon)-\partial^\alpha f_\varepsilon(z_\varepsilon)|=\left|\sum_{k=1}^n\partial^{\alpha+e_k}f_\varepsilon(\zeta_\varepsilon)\cdot(y_\varepsilon^k-z_\varepsilon^k)\right|$$

where $\zeta_\varepsilon:=y_\varepsilon+\sigma_\varepsilon(z_\varepsilon-y_\varepsilon)$ for some $\sigma_\varepsilon\in(0,1)$. Therefore $\zeta_\varepsilon\in B^{\mathrm{E}}_{\rho_\varepsilon^q}(x_\varepsilon)\subseteq B^{\mathrm{E}}_{\rho_\varepsilon^{q_k}}(x_\varepsilon)$ and

$$|\partial^\alpha f_\varepsilon(y_\varepsilon)-\partial^\alpha f_\varepsilon(z_\varepsilon)|\le\sqrt{n}\cdot\rho_\varepsilon^{-q}|y_\varepsilon-z_\varepsilon|.$$

Property (iii) follows setting $\alpha=0$ in (ii). Property (iv) follows directly from the definition of locally Lipschitz function in the Fermat topology. In fact, we have that $L|x-y|<r\in\mathbb{R}$ if $y\in B^{\mathrm{F}}_{r/L}(x)$, which is an open ball in the Fermat topology because $L\in\mathbb{R}$. ☐

In the following result, we show that the dependence of the domains Ω_ε on ε can be avoided since it does not lead to a larger class of generalized functions. In spite of this possibility, we preferred to formulate Def. 12 using ε-dependent domains because the extension of $f_\varepsilon\in\mathcal{C}^\infty(\Omega_\varepsilon,\mathbb{R}^d)$ to the whole of $\mathbb{R}^n$ is not unique and hence introduces extrinsic elements.

Lemma 16. *Let $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ and $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points, then $f:X\longrightarrow Y$ is a GSF if and only if there exists a net $v_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)$ defining a generalized smooth map of type $X\longrightarrow Y$ such that $f=[v_\varepsilon(-)]|_X$.*

Proof. The stated condition is clearly sufficient. Conversely, assume that $f:X\longrightarrow Y$ is defined by the net $f_\varepsilon\in\mathcal{C}^\infty(\Omega_\varepsilon,\mathbb{R}^d)$. For every $\varepsilon\in I$ let $\Omega'_\varepsilon:=\left\{x\in\Omega_\varepsilon\mid d(x,\Omega_\varepsilon^c)>\rho_\varepsilon^{-\frac{1}{\varepsilon}}\right\}$ and choose $\chi_\varepsilon\in\mathcal{C}^\infty(\Omega_\varepsilon)$ with $\mathrm{supp}(\chi_\varepsilon)\subseteq\Omega'_{\varepsilon/2}$ and $\chi_\varepsilon=1$ in a neighborhood of Ω'_ε. Set $f_\varepsilon:=0$ on $\mathbb{R}^n\setminus\Omega_\varepsilon$ and $v_\varepsilon:=\chi_\varepsilon\cdot f_\varepsilon$, so that $v_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)$. If $x=[x_\varepsilon]\in X\subseteq\langle\Omega_\varepsilon\rangle$, then $x_\varepsilon\in\Omega'_\varepsilon\subseteq\Omega_\varepsilon$ for ε small by

Thm. 9, so for all $\alpha\in\mathbb{N}^n$ we get $\partial^\alpha v_\varepsilon(x_\varepsilon)=\partial^\alpha f_\varepsilon(x_\varepsilon)$. Therefore, $(v_\varepsilon)_\varepsilon$ defines a GSF of the type $X\longrightarrow Y$ and clearly $f=[f_\varepsilon(-)]|_X=[v_\varepsilon(-)]|_X$. $\square$

Consider a GSF $f:X\longrightarrow Y$. We want to show that for a large class of domains X, the function f is uniquely determined by its values on particularly well behaving points $x\in X$. These domains and these points are introduced in the following

Definition 17.

(i) Let $x\in{}^\rho\widetilde{\mathbb{R}}^n$, then we say that the point x is *near-standard* if there exists a representative (x_ε) of x such that $\exists\lim_{\varepsilon\to0^+}x_\varepsilon=:x^\circ\in\mathbb{R}^n$ (x° is called the standard part of x). Clearly, this limit does not depend on the representative of x.

(ii) If $\Omega\subseteq\mathbb{R}^n$, then $\Omega^\bullet:=\left\{x\in{}^\rho\widetilde{\mathbb{R}}^n\mid\exists x^\circ\in\Omega\right\}$.

(iii) Let x, $x'\in{}^\rho\widetilde{\mathbb{R}}^n$, then we say that x' is a subpoint of x if there exist representatives $x=[x_\varepsilon]$ and $x'=[x'_\varepsilon]$ such that

$$\forall\varepsilon\,\exists\bar\varepsilon\le\varepsilon:\ x'_\varepsilon=x_{\bar\varepsilon}. \tag{3.9}$$

We say that x' is a sample of x if the condition $\bar\varepsilon\le\varepsilon$ is dropped, i.e. if $\forall\varepsilon\,\exists\bar\varepsilon:\ x'_\varepsilon=x_{\bar\varepsilon}$.

(iv) We say that $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ *contains all its subpoints* if for all $x\in X$, $x'\in{}^\rho\widetilde{\mathbb{R}}^n$, if x' is a subpoint of x, then also $x'\in X$.

Note that, as a consequence of the condition $\bar\varepsilon\le\varepsilon$ in 3.9, we can define $\bar\varepsilon(\varepsilon):=\sup\{0<\hat\varepsilon\le\varepsilon\mid x'_\varepsilon=x_{\hat\varepsilon}\}$ so that we get

$$\begin{aligned}&\forall\varepsilon\in I:\ 0<\bar\varepsilon(\varepsilon)\le\varepsilon\\&\bar\varepsilon:I\longrightarrow I\\&\bar\varepsilon\to0^+\\&x'_\varepsilon=x_{\bar\varepsilon(\varepsilon)}\quad\forall\varepsilon\in I,\end{aligned}$$

therefore, we can roughly state that "being a subpoint is a property for $\varepsilon\to0^+$". An intrinsic way to state the property of being a subpoint is: for all representatives $x'=[x'_\varepsilon]$, $x=[x_\varepsilon]$ there exists negligible (z_ε), $(n_\varepsilon)\sim_\rho0$ such that

$$\forall\varepsilon\,\exists\bar\varepsilon\le\varepsilon:\ x'_\varepsilon+z_\varepsilon=x_{\bar\varepsilon}+n_{\bar\varepsilon}.$$

From this characterization it follows that if x'' is a subpoint of x' and x' is a subpoint of x, then also x'' is a subpoint of x.

Theorem 18. *Let $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$, $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d$, and let $f:X\longrightarrow Y$ be a GSF. If X contains all its subpoints and if $f(x)=0$ for all near-standard and for all infinite points $x\in X$, then $f=0$.*

Proof. In fact, suppose that f vanishes on every near-standard and every infinite point belonging to X, but that $f(x)\ne0$ for some $x\in X$. Let (x_ε) be a representative of x. Then there exist $m\in\mathbb{N}$ and $(\varepsilon_k)_k\downarrow0$ such that $|f_{\varepsilon_k}(x_{\varepsilon_k})|>\rho_{\varepsilon_k}^m$, where (f_ε) is a net that defines f. If $(x_{\varepsilon_k})_k$ is a bounded sequence, we can extract from it a convergent subsequence $\left(x_{\varepsilon_{k_l}}\right)_l$. Set $x'_\varepsilon:=x_{\varepsilon_{k_l}}$ if $\varepsilon\in[\varepsilon_{k_l},\varepsilon_{k_{l-1}})$, then $x':=[x'_\varepsilon]$ is a near-standard subpoint of x since $\varepsilon_{k_l}\le\varepsilon$ and $\exists\lim_{\varepsilon\to0^+}x'_\varepsilon=\lim_{l\to+\infty}x_{\varepsilon_{k_l}}$. But hence $x'\in X$ and $f(x')\ne0$, a contradiction. If, on the other hand, the sequence $(x_{\varepsilon_k})_k$ is unbounded, then $\lim_{k\to+\infty}|x_{\varepsilon_k}|=+\infty$, and we can proceed as above constructing an infinite point $x'\in X$ at which $f(x')\ne0$. $\square$

Analogously, we can prove the following

Theorem 19. *Let $X \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$. Let (Ω_ε) be a net of open subsets of $\mathbb{R}^n$, and (f_ε) be a net of smooth functions, with $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$. Assume that X contains all its subpoints. Then (f_ε) defines a GSF of the type $X \longrightarrow Y$ if and only if*

(i) $X \subseteq \langle \Omega_\varepsilon \rangle$ *and* $[f_\varepsilon(x_\varepsilon)] \in Y$ *for all* $[x_\varepsilon] \in X$.

(ii) $\forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_\rho$ *for all near-standard and for all infinite points* $[x_\varepsilon] \in X$.

For example, if Ω is an open subset of $\mathbb{R}^n$, and we define the set of compactly supported generalized points by

$$\mathrm{c}(\Omega) := \{[x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n \mid \exists K \Subset \Omega \ \forall^0 \varepsilon : \ x_\varepsilon \in K\} \subseteq \langle \Omega \rangle,$$

then $\mathrm{c}(\Omega)$ contains all its subpoints. Any internal set $[A]$ generated by a constant sequence $A \subseteq \mathbb{R}^n$ gives another example of subset containing all its subpoints.

The subset $\mathrm{c}(\Omega)$ is the natural domain for embedded distributions, as shown in the following section.

Using the notion of sample of a point, from Lem. 7 we can also easily derive the following partial replace of the trichotomy law

$$\forall x \in {}^{\rho}\widetilde{\mathbb{R}} \ \exists x_1, x_2, x_3 \text{ subpoints of } x : \ x_1 = 0 \quad \text{or} \quad x_2 < 0 \quad \text{or} \quad x_3 > 0.$$

and the inequality $x > 0$ holds if and only if for all x' sample of x we have

$$x' \text{ is near-standard or } x' \text{ is infinite} \quad \Longrightarrow \quad x' > 0.$$

3.2. **Embedding of Schwartz distributions.**

Introduction. One of the re-occurring themes of this work are the choices which the solution of a given problem within our framework may depend upon. For instance, (3.6) shows that the domain of a GSF depends on the infinitesimal net ρ. It is also easy to show that the trivial Cauchy problem

$$\begin{cases} x'(t) - [\varepsilon^{-1}] \cdot x(t) = 0 \\ x(0) = 1 \end{cases}$$

has no solution in ${}^{\rho}\mathcal{GC}^\infty(\mathbb{R}, \mathbb{R})$ if $\rho_\varepsilon = \varepsilon$. Nevertheless, it has the unique solution $x(t) = \left[e^{\frac{1}{\varepsilon}t}\right] \in {}^{\rho}\mathcal{GC}^\infty(\mathbb{R}, \mathbb{R})$ if $\rho_\varepsilon = e^{-\frac{1}{\varepsilon}}$. Therefore, the choice of the infinitesimal net ρ is closely tied to the possibility of solving a given class of differential equations. This illustrates the dependence of the theory on the infinitesimal net ρ.

Further choices concern the embedding of Schwartz distributions. Since we need to associate a net of smooth functions (u_ε) to a given distribution $T \in \mathcal{D}'(\Omega)$, this embedding is naturally built upon a regularization process. In our approach, this regularization will depend on an infinite number $b \in {}^{\rho}\widetilde{\mathbb{R}}$, and the choice of b depends on what properties we need from the embedding. For example, if δ is the (embedding of the) one-dimensional Dirac delta, then we have the property

$$\delta(0) = b. \tag{3.10}$$

We can also choose the embedding so as to get the property

$$H(0) = \frac{1}{2}, \tag{3.11}$$

where H is the (embedding of the) Heaviside step function. Equalities like these are used in diverse applications (see add references here...). In fact, we are going to construct a family of structures of the type $(\mathcal{G}, \partial, \iota)$, where $(\mathcal{G}, \partial)$ is a a sheaf of differential algebra and $\iota : \mathcal{D}' \longrightarrow \mathcal{G}$ is an embedding. The particular structure we need to consider depends on the problem we have to solve. Of course, one may be more interested in having an intrinsic embedding of distributions. This can be done by following the ideas of the full Colombeau algebra (see e.g. [48, 43, 42]) and by changing the indexing set from $I = (0, 1]$ to $I = \left\{\varphi \in \mathcal{D}(\mathbb{R}^n) \mid \int \varphi = 1\right\}$. Nevertheless, this choice decreases a lot the simplicity of the present approach and is incompatible with properties like (3.10) and (3.11).

The embedding. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $r \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^n$, we use the notations $r \odot \varphi$ for the function $x \in \mathbb{R}^n \mapsto \frac{1}{r^n} \cdot \varphi\left(\frac{x}{r}\right) \in \mathbb{R}$ and $x \oplus \varphi$ for the function $y \in \mathbb{R}^n \mapsto \varphi(y-x) \in \mathbb{R}$. These notations enable to highlight that $\odot$ is a free action of the multiplicative group $(\mathbb{R}_{>0}, \cdot, 1)$ on $\mathcal{D}(\mathbb{R}^n)$ and $\oplus$ is a free action of the additive group $(\mathbb{R}_{>0}, +, 0)$ on $\mathcal{D}(\mathbb{R}^n)$. We also have the distributive property $r \odot (x \oplus \varphi) = rx \oplus r \odot \varphi$.

Lemma 20. *There exists $\mu \in \mathcal{S}(\mathbb{R})$ such that*
(i) $\int \mu = 1$.
(ii) $\int x^\alpha \mu(x) \, \mathrm{d}x = 0$ *for all* $\alpha \in \mathbb{N}_{>0}$.
(iii) $\mu(0) = 1$.
(iv) μ *is even.*
(v) $\mu(n) = 0$ *for all* $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Consider the Fréchet space

$$F := \{\mu \in \mathcal{S}(\mathbb{R}) \mid \mu \text{ even}, \ \forall n \in \mathbb{Z} \setminus \{0\} : \mu(n) = 0\}$$

and define, for $n \in \mathbb{N}$, the continuous linear functionals $f_0(\mu) := \mu(0)$, $f_n : F \to \mathbb{R}$, $f_n(\mu) := \int x^{n-1} \mu(x) \, dx$ $(n \geq 1)$. Our objective then is to implement conditions (ii)–(iv) by showing the solvability of the system

$$f_0(\mu) = 1, \ f_1(\mu) = 1, \ f_n(\mu) = 0 \ (n \geq 2) \tag{3.12}$$

in F. Using the seminorms $p_k(\mu) = \sup_{l+m \leq k} \sup_{x \in \mathbb{R}} (1 + |x|)^l |\mu^{(m)}(x)|$, we have $|f_0(\mu)| \leq p_0(\mu)$, and $|f_n(\mu)| \leq c_n p_{n+2}(\mu)$ for constants $c_n > 0$ $(n \geq 1)$. Also, the n-value on the right hand side of these estimates is minimal. It therefore follows from [23, Bem. 3, p. 144] that indeed (3.12) has a solution in F. □

Remark 21. In addition to conditions (i)–(v) from Lemma 20 we may require that μ satisfy finitely many additional properties expressible by linearly independent functionals as in the above proof (cf. [23, Satz 2]). In particular, we may prescribe the values for μ or its derivatives at finitely many further points.

Finally, we note that any element of $\mathcal{S}(\mathbb{R})$ satisfying condition (ii) from Lemma 20 must change sign infinitely often.

We call *Colombeau mollifier* any function μ which satisfies the properties of the previous lemma. Concerning embeddings of Schwartz distributions, the idea is classically to regularize distributions using a mollifier. The use of a Colombeau mollifier allows us, on the one hand, to identify the distribution $\varphi \in \mathcal{D}(\Omega) \mapsto \int f\varphi$ with the GSF $f \in \mathcal{C}^\infty(\Omega) \subseteq {}^\rho\mathcal{GC}^\infty(\Omega, \mathbb{R})$ (thanks to property (ii)); on the other hand, it allows to completely calculate compositions as $\delta \circ \delta$, $H \circ \delta$, $\delta \circ H$ (see below).

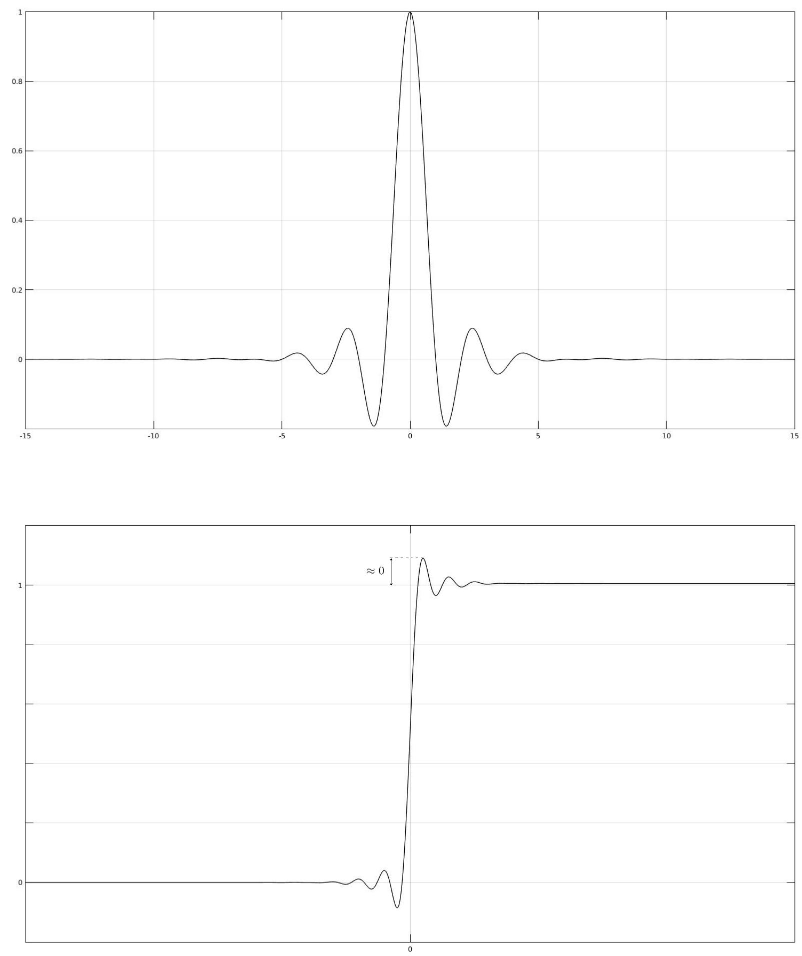

FIGURE 3.1. A Colombeau mollifier and a representation of the Heaviside function

It is worth noting that the condition (ii) of null moments is well known in the study of convergence of numerical solutions of singular differential equations, see e.g. [59, 25, 114] and references therein.

We show that the assignment $U \mapsto {}^\rho\mathcal{GC}^\infty(\mathrm{c}(U), {}^\rho\widetilde{\mathbb{R}})$ ($U \subseteq \Omega$ open) is a fine sheaf on Ω. In fact, for $V \subseteq U$, the natural restriction map ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(U), {}^\rho\widetilde{\mathbb{R}}) \to \mathcal{GC}^\infty(\mathrm{c}(V), {}^\rho\widetilde{\mathbb{R}})$ can also be written, in terms of defining nets, as $f = [f_\varepsilon] \mapsto [f_\varepsilon|_V]$. Also, $\mathrm{c}(U) \cap \mathrm{c}(V) = \mathrm{c}(U \cap V)$.

Next, suppose that Ω_j ($j \in J$) is an open covering of Ω and that for each $j \in J$ we are given $f^j = [f^j_\varepsilon] \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega_j), {}^\rho\widetilde{\mathbb{R}})$ such that $f^j|_{\mathrm{c}(\Omega_j \cap \Omega_k)} = f^k|_{\mathrm{c}(\Omega_j \cap \Omega_k)}$ for all $j,\ k \in J$. Then letting χ_j ($j \in J$) be a partition of unity subordinate to Ω_j ($j \in J$), the GSF defined by the net

$$f_\varepsilon := \sum_{j\in J} \chi_j \cdot f^j_\varepsilon \in \mathcal{C}^\infty(\Omega)$$

is the unique element of ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ with $f|_{\mathrm{c}(\Omega_j)} = f^j$ for all $j \in J$. In particular, we may define a corresponding notion of support for each $f \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ by

$$\mathrm{supp}(f) := \left(\bigcup \{\Omega' \subseteq \Omega \mid \Omega' \text{ open},\ f|_{\Omega'} = 0\}\right)^c$$

As a final technical preparation for embedding the space of distributions $\mathcal{D}'(\Omega)$ into ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ we need the following result:

Lemma 22. *Let $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi = 1$ on $\overline{B^{\mathrm{E}}_1(0)}$, and $\chi = 0$ on $\mathbb{R}^n \setminus B^{\mathrm{E}}_2(0)$. Also, let $b = [b_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ be an infinite positive number, i.e. $\lim_{\varepsilon\to 0^+} b_\varepsilon = +\infty$. Given a Colombeau mollifier μ, set*

$$\mu^b_\varepsilon(x) := (b^{-1}_\varepsilon \odot \mu)(|x|)\chi(x|\log(b^{-1}_\varepsilon)|) = b^n_\varepsilon \mu(b_\varepsilon|x|)\chi(x|\log(b^{-1}_\varepsilon)|). \tag{3.13}$$

Then

- *(i) For each ε, $\mu^b_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$ with supp$(\mu^b_\varepsilon) \subseteq B^{\mathrm{E}}_{2|\log(b^{-1}_\varepsilon)|^{-1}}(0)$.*
- *(ii) $\forall \alpha \in \mathbb{N}^n\ \exists N \in \mathbb{N}$ such that $\sup_{x\in\mathbb{R}^n} |\partial^\alpha \mu_\varepsilon(x)| = O(b^N_\varepsilon)$ $(\varepsilon \to 0)$.*
- *(iii) $\forall \alpha \in \mathbb{N}^n\ \forall q \in \mathbb{N}$: $\sup_{x\in\mathbb{R}^n} |\partial^\alpha(\mu^b_\varepsilon - b^n_\varepsilon \mu(b_\varepsilon|\,.\,|)(x)| = O(b^{-q}_\varepsilon)$ $(\varepsilon \to 0)$.*
- *(iv) $\forall q \in \mathbb{N}$ $\int \mu^b_\varepsilon(x)\,dx = 1 + O(b^{-q}_\varepsilon)$ $(\varepsilon \to 0)$.*
- *(v) $\forall q \in \mathbb{N}\ \forall \alpha \in \mathbb{N}^n$, $|\alpha| > 0$ $\int x^\alpha \mu^b_\varepsilon(x)\,dx = O(b^{-q}_\varepsilon)$ $(\varepsilon \to 0)$.*

Proof. Since μ is even, μ^b_ε is smooth. All the other properties have been proved for the special case $b_\varepsilon = \varepsilon^{-1}$ in [20, Sec. 3], and the arguments employed there carry over in a straightforward way to the present setting. □

Theorem 23. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let μ^b_ε as in Lemma 22. Set $\Omega_\varepsilon := \left\{x \in \Omega \mid d(x, \Omega^c) \ge \varepsilon,\ |x| \le \frac{1}{\varepsilon}\right\}$ and fix some $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi = 1$ on $\overline{B^{\mathrm{E}}_1(0)}$, $0 \le \chi \le 1$ and $\chi = 0$ on $\mathbb{R}^n \setminus B^{\mathrm{E}}_2(0)$. Also, take $\kappa_\varepsilon \in \mathcal{D}(\Omega_\varepsilon)$ such that $\kappa_\varepsilon = 1$ on a neighborhood L_ε of Ω_ε. Then the map*

$$\iota^b_\Omega : T \in \mathcal{D}'(\Omega) \mapsto \left[\left((\kappa_\varepsilon \cdot T) * \mu^b_\varepsilon\right)(-)\right] \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}}). \tag{3.14}$$

satisfies:

- *(i) $\iota^b : \mathcal{D}'(-) \longrightarrow {}^\rho\mathcal{GC}^\infty(\mathrm{c}((-)), {}^\rho\widetilde{\mathbb{R}})$ is a sheaf-morphism of real vector spaces: If $\Omega' \subseteq \Omega$ is another open set and $T \in \mathcal{D}'(\Omega)$, then $\iota^b_\Omega(T)|_{\mathrm{c}(\Omega')} = \iota^b_{\Omega'}(T|_{\Omega'})$.*
- *(ii) ι_b preserves supports, hence is in fact a sheaf-monomorphism.*
- *(iii) Any $f \in \mathcal{C}^\infty(\Omega)$ can naturally be considered an element of $\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ via $[x_\varepsilon] \mapsto [f(x_\varepsilon)]$. Moreover, $\forall q \in \mathbb{N}_{>0}\ \forall x \in \mathrm{c}(\Omega)$: $\left|\iota^b_\Omega(f)(x) - f(x)\right| \le b^{-q}$.*
- *(iv) If $f \in \mathcal{C}^\infty(\Omega)$ and if $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$, then $\iota^b_\Omega(f) = f$. In particular, ι^b then provides a multiplicative sheaf-monomorphism $\mathcal{C}^\infty(-) \hookrightarrow {}^\rho\mathcal{GC}^\infty(c(-), \mathbb{R})$.*
- *(v) For any $T \in \mathcal{D}'(\Omega)$ and any $\alpha \in \mathbb{N}^n_0$, $\iota^b_\Omega(\partial^\alpha T) = \partial^\alpha \iota^b_\Omega(T)$.*
- *(vi) $\lim_{\varepsilon\to 0^+} \int_\Omega \iota^b_\Omega(T)_\varepsilon \cdot \varphi\,dx = \langle T, \varphi\rangle$ for all $\varphi \in \mathcal{D}(\Omega)$ and all $T \in \mathcal{D}'(\Omega)$.*
- *(vii) $\iota^b_{\mathbb{R}^n}(\delta)(0) = b^n$ and if $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$, then $\iota^b_{\mathbb{R}}(H)(0) = \frac{1}{2}$.*
- *(viii) The embedding ι^b does not depend on the particular choice of (κ_ε) and (if $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$) χ as above.*

Proof. We follow ideas from [48, Sec. 1.2] and [20]. Let $T \in \mathcal{D}'(\Omega)$ and let $[x_\varepsilon] \in \mathrm{c}(\Omega)$. Then there exists some $K \Subset \Omega$ such that $x_\varepsilon \in K$ for ε small. Also, we may assume that $K + B^{\mathrm{E}}_{2|\log(b_\varepsilon^{-1})|^{-1}}(0) \subseteq L \subseteq \Omega_\varepsilon$ for these ε, where $L \Subset \Omega$. Then by (i) of Lemma 22, for ε small we have

$$\iota^b_\Omega(T)_\varepsilon(x_\varepsilon) = (\kappa_\varepsilon \cdot T) * \mu^b_\varepsilon(x_\varepsilon) = T * \mu^b_\varepsilon(x_\varepsilon) = \langle T, \mu^b_\varepsilon(x_\varepsilon - \,.\,)\rangle. \tag{3.15}$$

Since $T \in \mathcal{D}'(\Omega)$, we have a seminorm estimate of the form

$$\forall \varphi \in \mathcal{D}_L(\Omega): \ |\langle T, \varphi\rangle| \le C \max_{|\beta|\le m} \sup_{x\in L} |\partial^\beta \varphi(x)|.$$

Together with Lemma 22 (i) and (ii) this implies that $(\partial^\alpha \iota^b_\Omega(T)_\varepsilon(x_\varepsilon)) \in \mathbb{R}^n_\rho$ for each α. Hence ι^b_Ω indeed maps $\mathcal{D}'(\Omega)$ into ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$.

To show (i), let $\Omega' \subseteq \Omega$ be open and let $[x_\varepsilon] \in c(\Omega')$. Then using the notations introduced before (3.15), we may suppose that $L \subseteq \Omega'_\varepsilon$, and so for ε small we have $\mu^b_\varepsilon \in \mathcal{D}(\Omega')$. Therefore, (3.15) implies for such ε:

$$\iota^b_\Omega(T)_\varepsilon(x_\varepsilon) = \langle T, \mu^b_\varepsilon(x_\varepsilon - \,.\,)\rangle = \langle T|_{\Omega'}, \mu^b_\varepsilon(x_\varepsilon - \,.\,)\rangle = \iota^b_{\Omega'}(T|_{\Omega'})_\varepsilon(x_\varepsilon).$$

Next we show (ii). Suppose first that $T|_{\Omega'} = 0$ for some open subset Ω' of Ω. Let $[x_\varepsilon] \in \mathrm{c}(\Omega')$ and pick $K \Subset \Omega'$ such that $x_\varepsilon \in K$ for ε small. As above, for ε small we have $\mathrm{supp}(\mu^b_\varepsilon(x_\varepsilon - \,.\,)) \subseteq \Omega'$, as well as $\iota^b_\Omega(T)_\varepsilon(x_\varepsilon) = \langle T, \mu^b_\varepsilon(x_\varepsilon - \,.\,)\rangle$, which therefore vanishes. Hence $\iota^b_\Omega(T)|_{\Omega'} = 0$, and thereby $\mathrm{supp}(\iota^b_\Omega(T)) \subseteq \mathrm{supp}(T)$.

Conversely, let $\Omega' \subseteq \Omega$ such that $\iota^b_\Omega(T)|_{\Omega'} = 0$ and let $\varphi \in \mathcal{D}(\Omega')$. Since $(\kappa_\varepsilon T) * \mu^b_\varepsilon \to T$ in $\mathcal{D}'(\Omega)$, in order to show $\langle T, \varphi\rangle = 0$ it suffices to demonstrate that $(\kappa_\varepsilon T) * \mu^b_\varepsilon \to 0$ as $\varepsilon \to 0$, uniformly on compact subsets of Ω'. Suppose this were not the case, then we could find some $L \Subset \Omega'$, some $c > 0$ and sequences $\varepsilon_k \downarrow 0$ and $x_k \in L$ such that $|(\kappa_\varepsilon T) * \mu^b_{\varepsilon_k}(x_k)| \ge c$ for all k. Fixing any $z \in \Omega'$ and setting $x_\varepsilon := x_k$ for $\varepsilon = \varepsilon_k$ and $x_\varepsilon = z$ otherwise then defines an element $[x_\varepsilon] \in \mathrm{c}(\Omega')$ with $\iota^b_\Omega(T)([x_\varepsilon]) \neq 0$, a contradiction.

Consequently, ι^b_Ω induces an injective sheaf morphism (again denoted by) $\iota^b : \mathcal{D}'(-) \longrightarrow {}^\rho\mathcal{GC}^\infty(\mathrm{c}(-), {}^\rho\widetilde{\mathbb{R}})$.

(iii): If $f \in \mathcal{C}^\infty(\Omega)$ then any derivative of f is uniformly (in ε) bounded on any (x_ε) (for $[x_\varepsilon] \in c(\Omega)$). Thus $f \in \mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$. Now let $[x_\varepsilon] \in \mathrm{c}(\Omega)$ and suppose first that f has compact support. By Lemma 22 (iv), for any $x \in \Omega$, $f(x) = \int f(x)\mu^b_\varepsilon(y)\,dy + n_\varepsilon$, where $n_\varepsilon = O(b_\varepsilon^{-q})$ for every $q > 0$. Thus for ε small and any $q \in \mathbb{N}$ we have by Taylor expansion

$$\begin{aligned}(\iota^b_\Omega(f)_\varepsilon - f)(x_\varepsilon) &= \int (f(x_\varepsilon - y) - f(x_\varepsilon))\mu^b_\varepsilon(y)\,dy + n_\varepsilon \\ &= \sum_{k=1}^{q-1} \int \frac{1}{k!}((-y\cdot D)^k f)(x_\varepsilon)\mu^b_\varepsilon(y)\,dy \\ &+ \frac{b_\varepsilon^{-q}}{q!} \int ((-y\cdot D)^k f)(x_\varepsilon - \theta_\varepsilon b_\varepsilon^{-1} y)\mu(|y|)\chi(b_\varepsilon^{-1}y|\log(b_\varepsilon^{-1})|)\,dy + n_\varepsilon,\end{aligned} \tag{3.16}$$

where $\theta_\varepsilon \in (0,1)$. Here, the first sum is $O(b_\varepsilon^{-q})$ by Lemma 22 (v), as is the second term since f is compactly supported, χ is globally bounded, and $\mu \in \mathcal{S}(\mathbb{R})$. If f is not compactly supported, pick $L \Subset \Omega$ such that $x_\varepsilon \in L$ for ε small and let $\varphi \in \mathcal{D}(\Omega)$ equal 1 in a neighborhood of L. Then (ii) implies that $f(x) = (\varphi f)(x)$, so the general case follows as well.

(iv): It suffices to observe that, by our assumption on b, (iii) implies that $\iota^b_\Omega(f)([x_\varepsilon]) = [f(x_\varepsilon)] = f(x)$ for any $f \in \mathcal{C}^\infty(\Omega)$ and any $x = [x_\varepsilon] \in \mathrm{c}(\Omega)$.

(v): As in the proof of (iv) we may assume that T has compact support. Then for ε small we have

$$\iota^b_\Omega(\partial^\alpha T)_\varepsilon = \partial^\alpha T * \mu^b_\varepsilon = \partial^\alpha (T * \mu^b_\varepsilon) = \partial^\alpha \iota^b_\Omega(T)_\varepsilon.$$

(vi): Pick $\zeta \in \mathcal{D}(\Omega)$ such that $\zeta \equiv 1$ in a neighborhood of $\mathrm{supp}(\varphi)$. Then

$$\mathrm{supp}(\iota^b_\Omega(T) - \iota^b_\Omega(\zeta T)) \cap \mathrm{supp}\varphi = \mathrm{supp}(T - \zeta T) \cap \mathrm{supp}\varphi = \emptyset,$$

so we may replace T by ζT, i.e., we may assume without loss of generality that $T \in \mathcal{E}'(\Omega)$. Then the result follows from the fact that $T * \mu^b_\varepsilon \to T$ in $\mathcal{D}'(\Omega)$ (by ((iii) (for $\alpha = 0$) in Lemma 22).

(vii): The first claim is immediate from $\iota^b_\mathbb{R}(\delta)_\varepsilon(0) = \mu^b_\varepsilon(0)$. To show the second, note first that

$$\iota^b_\mathbb{R}(H)_\varepsilon(0) - \int H(y)\mu^b_\varepsilon(-y)\,dy = \int H(y)(1 - \kappa_\varepsilon(y))\mu^b_\varepsilon(-y) = 0$$

for ε small by the support properties of κ_ε and χ. Furthermore, since $\int_0^\infty \mu(y)\,dy = 1/2$, we obtain

$$\begin{aligned}\Big|\int H(y)\mu^b_\varepsilon(-y)\,dy - \frac{1}{2}\Big| &= \Big|\int_0^\infty \mu(y)(\chi(b_\varepsilon^{-1}\log(b_\varepsilon^{-1})y) - 1)\,dy\Big| \\ &\le \int_{b_\varepsilon \log(b_\varepsilon^{-1})^{-1}}^\infty |\mu(y)|\,dy = O(b_\varepsilon^{-q})\end{aligned}$$

for any $q \in \mathbb{N}$, so the claim follows.

(viii): We first note that any two choices for either (κ_ε) or χ provide sheaf morphisms as in (i), (ii), hence it suffices to check that the resulting embeddings coincide on compactly supported distributions. For any such T we have $\kappa_\varepsilon T = T$ for ε small, so independence from the choice of (κ_ε) follows.

Now suppose that two different χ's have been chosen and denote the corresponding functions from (3.13) by μ^b_ε and $\bar\mu^b_\varepsilon$, and the resulting embeddings by ι^b and $\bar\iota^b$, respectively. Since $T \in \mathcal{E}'(\Omega)$, it satisfies a seminorm estimate of the form

$$\forall \varphi \in \mathcal{C}^\infty(\Omega):\ |\langle T, \varphi\rangle| \le C \max_{|\beta|\le m} \sup_{x \in L} |\partial^\beta \varphi(x)|.$$

for some $L \Subset \Omega$. Together with Lemma 22 (iii), this implies that, for any $[x_\varepsilon] \in c(\Omega)$ and ε small, we have

$$|(\iota^b_\Omega(T)_\varepsilon - \bar\iota^b_\Omega(T)_\varepsilon)(x_\varepsilon)| = |\langle T, (\mu^b_\varepsilon - \bar\mu^b_\varepsilon)(x_\varepsilon - \,.\,)\rangle| = O(b_\varepsilon^{-q})$$

for any $q \in \mathbb{N}$. □

Whenever we use the notation ι^b for an embedding, we assume that $b \in {}^\rho\widetilde{\mathbb{R}}$ satisfies the overall assumptions of Thm. 23 and of (iv) in that Theorem, and that ι^b has been defined as in (3.14) using a Colombeau mollifier μ.

Remark 24.

(i) In Def. 1, we introduced the asymptotic gauge $\mathcal{I}(\rho)$, and the whole construction depends on the fix infinitesimal net ρ only through this set $\mathcal{I}(\rho)$. A more general definition of asymptotic gauge is possible (see [42]). Anyhow, [42, Sec. 4.3] shows that an embedding of Schwartz's distribution having minimal properties necessarily implies that the asymptotic gauge must be generated by a single net, like for $\mathcal{I}(\rho)$.

(ii) Let δ, $H \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$ be the corresponding ι^b-embeddings of the Dirac delta and of the Heaviside function. Then $\delta(x) = 0$ if x is near-standard and $x^\circ \neq 0$ or if x is infinite. However, $\delta(x) = b\mu(bx)$, so that δ can be represented like in the first diagram of Fig. 3.1. E.g. we have $\delta\left(\frac{3}{2b}\right) < 0$, and $\frac{3}{2b}$ is a positive infinitesimal.

(iii) Analogously, we have $H(x) = 1$ if x is near-standard and $x^\circ > 0$ or if $x > 0$ is infinite; $H(x) = 0$ if x is near-standard and $x^\circ < 0$ or if $x < 0$ is infinite. But at infinitesimal points we can have different values, e.g. $H\left(\frac{3}{2b}\right) > 1$.

(iv) In [80], S. Łojasiewicz introduced the notion of a point value for distributions. He defined that $T \in \mathcal{D}'(\Omega)$ has the point value $c \in \mathbb{C}$ in $x_0 \in \Omega$ if

$$\lim_{\varepsilon \to 0} \langle T(x_0 + \varepsilon x), \varphi(x) \rangle = c \int \varphi(x)\, dx \qquad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{3.17}$$

Not every distribution has point values in arbitrary points — in fact, if it does, it already has to be a function of first Baire class ([80]). Conversely, a continuous function f clearly has point value $f(x)$ in any point x in its domain.

We show that if T has point value c at $x_0 \in \Omega$ then $\iota^b_\Omega(T)_\varepsilon(x_0) \to c$ as $\varepsilon \to 0$. I am still unable to prove this after trying for a while, here is why: We know from (3.17) that

$$\langle T(x_0 + b_\varepsilon^{-1} x), \varphi(x) \rangle = \langle T, b_\varepsilon^n \varphi(b_\varepsilon(y - x_0)) \rangle \to \langle c, \varphi \rangle \quad (\varepsilon \to 0)$$

(we could in fact choose any speed of convergence here instead of b_ε^{-1}). Since $\mathcal{D}'(\Omega)$ is Montel, this even holds uniformly for φ varying in a bounded subset of $\mathcal{D}(\Omega)$. Now by (3.15) we can write

$$\iota^b_\Omega(T)_\varepsilon(x_0) = \langle T, \mu^b_\varepsilon(x_0 - \,.\,) \rangle = \langle T, b_\varepsilon^n \varphi^\varepsilon(b_\varepsilon(y - x_0)) \rangle = \langle T(x_0 + b_\varepsilon^{-1} x), \varphi^\varepsilon(x) \rangle,$$

where $\varphi^\varepsilon(z) := \mu(|z|)\chi(-z b_\varepsilon^{-1} |\log(b_\varepsilon^{-1})|)$. By the above, if $\{\varphi^\varepsilon \mid \varepsilon > 0\}$ were bounded in $\mathcal{D}(\Omega)$ we would be done. Unfortunately, however, it is not, because the supports of the χ-terms in φ^ε are unbounded for $\varepsilon \to 0$. Maybe this way: w.l.o.g. T compactly supported. Then $T(x_0 + b_\varepsilon^{-1} x)$ is a net in $\mathcal{E}'$ that converges in $\mathcal{D}'$ and φ^ε is bounded in $\mathcal{E}$. But problem then is that the supports of $T(x_0 + b_\varepsilon^{-1} x)$ do not remain in a fixed compact set, so the net doesn't converge in $\mathcal{E}'$ …

We close this section considering the following natural problem: let us define two embeddings ι^b_Ω, ι^c_Ω as in (3.14) but using two different infinite positive numbers b, $c \in {}^\rho\widetilde{\mathbb{R}}$, so that for all $T \in \mathcal{E}'(\Omega)$ we have

$$\iota^b_\Omega(T) := \left[\left(T * \mu^b_\varepsilon\right)(-)\right],$$
$$\iota^c_\Omega(T) := \left[\left(T * \mu^c_\varepsilon\right)(-)\right].$$

In the following result, we establish when they are equal.

Theorem 25. *Let b, $c \in {}^\rho\widetilde{\mathbb{R}}$ be infinite positive numbers and let μ be a Colombeau mollifier. Then $\iota^b = \iota^c$ if and only if $b = c$ in ${}^\rho\widetilde{\mathbb{R}}$, i.e. iff they are equal as Robinson-Colombeau generalized number.*

Proof. If $\iota^b_{\mathbb{R}} = \iota^c_{\mathbb{R}}$, then $\iota^b_{\mathbb{R}}(\delta) = [b_\varepsilon^n \cdot \mu(b_\varepsilon \cdot -)] = \iota^c_{\mathbb{R}}(\delta) = [c_\varepsilon^n \cdot \mu(c_\varepsilon \cdot -)]$ and $\iota^b_{\mathbb{R}}(\delta) \in {}^\rho\mathcal{GC}^\infty(\mathrm{c}(\mathbb{R}), {}^\rho\widetilde{\mathbb{R}})$. Evaluating these GSF at $x = 0$ and considering the definition of negligible net, we get

$$\forall m \in \mathbb{N}: \ |b_\varepsilon^n \mu(0) - c_\varepsilon^n \mu(0)| = O(b_\varepsilon^{-m}),$$

that is $b^n = c^n$ since $\mu(0) = 1$. The conclusion follows by applying the smooth function $\sqrt[n]{-} \in \mathcal{C}^\infty(\mathbb{R}_{>0})$.

Vice versa, assume that $b = c$; it suffices to prove that

$$\left[T * \left(b_\varepsilon^{-1} \odot \mu - c_\varepsilon^{-1} \odot \mu\right)(-)\right] = 0 \quad \forall T \in \mathcal{E}'(\Omega).$$

And hence it suffices to prove that $\lim_{\varepsilon \to 0^+} \left(b_\varepsilon^{-1} \odot \mu - c_\varepsilon^{-1} \odot \mu\right) = 0$ in $\mathcal{D}'(\Omega)$. For each $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int \left(b_\varepsilon^{-1} \odot \mu - c_\varepsilon^{-1} \odot \mu\right) \varphi = \int \mu(t) \cdot \left[\varphi\left(\frac{t}{b_\varepsilon}\right) - \varphi\left(\frac{t}{c_\varepsilon}\right)\right]. \tag{3.18}$$

Therefore, for $K := \text{supp}(\varphi)$, $\lim_{\varepsilon \to 0^+} \sup_{t \in K} \left|\varphi\left(\frac{t}{b_\varepsilon}\right) - \varphi\left(\frac{t}{c_\varepsilon}\right)\right| = 0$. From this and (3.18) the conclusion follows. □

3.3. **Closure with respect to composition.** In contrast to the case of distributions, there is no problem in considering the composition of two GSF. This property opens new interesting possibilities, e.g. in considering differential equations $y' = f(y, t)$, where y and f are GSF. For instance, there is no problem in studying $y' = \delta(y)$ (see [81]).

Theorem 26. *Subsets $S \subseteq {}^\rho\widetilde{\mathbb{R}}^s$ with the trace of the sharp topology, and generalized smooth maps as arrows form a subcategory of the category of topological spaces. We will call this category ${}^\rho\mathcal{GC}^\infty$, the* category of GSF.

Proof. From Thm. 15 (iii) we already know that every GSF is continuous; we have hence to prove that these arrows are closed with respect to identity and composition in order to prove that we have a concrete subcategory of topological spaces and continuous maps.

If $T \subseteq {}^\rho\widetilde{\mathbb{R}}^t$ is a arbitrary object, then $f_\varepsilon(x) := x$ is the net of smooth functions that globally defines the identity 1_T on T. It is immediate that 1_T is generalized smooth.

To prove that arrows of ${}^\rho\mathcal{GC}^\infty$ are closed with respect to composition, let $S \subseteq {}^\rho\widetilde{\mathbb{R}}^s, T \subseteq {}^\rho\widetilde{\mathbb{R}}^t, R \subseteq {}^\rho\widetilde{\mathbb{R}}^r$ and $f : S \longrightarrow T$, $g : T \longrightarrow R$ be GSF, then $f(x) = [f_\varepsilon(x_\varepsilon)] \in T$ and $g(y) = [g_\varepsilon(y_\varepsilon)] \in R$ for every $x \in S$ and $y \in T$, where $f_\varepsilon \in \mathcal{C}^\infty(\Omega'_\varepsilon, \mathbb{R}^t)$ and $g_\varepsilon \in \mathcal{C}^\infty(\Omega''_\varepsilon, \mathbb{R}^r)$ are suitable nets of smooth functions as in Def. 12, and where Ω'_ε is open in $\mathbb{R}^s$ and Ω''_ε is open in $\mathbb{R}^t$. Of course, the idea is to consider $g_\varepsilon \circ f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^r)$, where $\Omega_\varepsilon := f_\varepsilon^{-1}(\Omega''_\varepsilon)$ (let us note that, even in the case where Ω''_ε does not depend by ε, generally speaking Ω_ε still depends on ε).

Take $x \in S$, so that $f(x) = [f_\varepsilon(x_\varepsilon)] \in T \subseteq \langle \Omega''_\varepsilon \rangle$ and hence $f_\varepsilon(x_\varepsilon) \in_\varepsilon \Omega''_\varepsilon$ and $x_\varepsilon \in \Omega_\varepsilon$ for ε small. If we take another representative $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$ we have $f(x') = f(x)$ since f is well-defined and, proceeding as before, we still have that $x'_\varepsilon \in \Omega_\varepsilon$ for ε sufficiently small. This proves that $S \subseteq \langle \Omega_\varepsilon \rangle$. Moreover, since $[f_\varepsilon(x_\varepsilon)] \in T$, we also have that $[g_\varepsilon(f_\varepsilon(x_\varepsilon))] \in R$ and $g(f(x)) = [g_\varepsilon(f_\varepsilon(x_\varepsilon))]$. It remains to show that the net $(g_\varepsilon \circ f_\varepsilon)$ defines a GSF (Def. 12) of the type $S \longrightarrow R$. To this end, let us consider any $(x_\varepsilon) \in (\Omega_\varepsilon)_M$ and any $\gamma \in \mathbb{N}^s$. We can write

$$\partial^\gamma(g_\varepsilon \circ f_\varepsilon)(x_\varepsilon) = p\left[\partial^{\alpha_1} f_\varepsilon(x_\varepsilon), \ldots, \partial^{\alpha_A} f_\varepsilon(x_\varepsilon), \partial^{\beta_1} g_\varepsilon(f_\varepsilon(x_\varepsilon)), \ldots, \partial^{\beta_B} g_\varepsilon(f_\varepsilon(x_\varepsilon))\right], \tag{3.19}$$

where p is a suitable polynomial not depending on x_ε. Every term $\partial^{\alpha_i} f_\varepsilon(x_\varepsilon)$ and $\partial^{\beta_j} g_\varepsilon(f_\varepsilon(x_\varepsilon))$ is ρ-moderate by (ii) of Def. 12. Since moderateness is preserved by polynomial operations, it follows that also $\partial^\gamma(g_\varepsilon \circ f_\varepsilon)(x_\varepsilon)$ is ρ-moderate. □

For instance, we can think of the Dirac delta as a map of the form $\delta : {}^\rho\widetilde{\mathbb{R}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, and therefore the composition e^δ is defined in $\{x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists z \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ \delta(x) \le \log z\}$, which of course does not contain $x = 0$ but only suitable non zero infinitesimals. On the contrary, $\delta \circ \delta : {}^\rho\widetilde{\mathbb{R}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$. Moreover, from the inclusion of ordinary smooth functions (Thm. 23) and the closure with respect to composition, it directly follows that every ${}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ is an algebra with pointwise operations for every subset $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. For an open subset $\Omega \subseteq \mathbb{R}^n$, the algebra ${}^\rho\mathcal{GC}^\infty(\mathrm{c}(\Omega), {}^\rho\widetilde{\mathbb{R}})$ contains the space $\mathcal{D}'(\Omega)$ of Schwartz distributions.

A classical way to define a GSF is to follow the original idea of classical authors (see [64, 78, 21]) to fix an infinitesimal or infinite parameter in a suitable ordinary smooth function. We will call this type of GSF of *Cauchy-Dirac type*; the next theorem specifies this notion and states that GSF are of Cauchy-Dirac type whenever the generating net (f_ε) is smooth in ε.

Corollary 27. *Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^d$, $P \subseteq \mathbb{R}^p$ be open sets and $\varphi \in \mathcal{C}^\infty(P \times X, Y)$ be an ordinary smooth function. Let $p \in [P]$, and define $f_\varepsilon := \varphi(p_\varepsilon, -) \in \mathcal{C}^\infty(X, Y)$, then $[f_\varepsilon(-)] : [X] \longrightarrow [Y]$ is a GSF. In particular, if $f : [X] \longrightarrow [Y]$ is a GSF defined by (f_ε) and the net (f_ε) is smooth in ε, i.e. if*

$$\exists \varphi \in \mathcal{C}^\infty((0,1) \times X, Y) : \ f_\varepsilon = \varphi(\varepsilon, -) \quad \forall \varepsilon \in (0,1),$$

and if $[\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, then the GSF f is of Cauchy-Dirac type because $f(x) = \varphi([\varepsilon], x)$ for all $x \in [X]$. Finally, Cauchy-Dirac GSF are closed with respect to composition.

Proof. In fact, the map $x \in [X] \mapsto (p, x) \in [P] \times [X]$ is trivially generalized smooth and hence from the inclusion of smooth functions (Theorem 23) and the closure with respect to composition (Theorem 26) the conclusions follow. □

Example 28. The composition $\delta \circ \delta \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$ is given by $(\delta \circ \delta)(x) = b\mu\left[b^2\mu(bx)\right]$ and is an even function. If x is near-standard and $x^\circ \ne 0$, or x is infinite, then $(\delta \circ \delta)(x) = b$. Since $(\delta \circ \delta)(0) = 0$, by the intermediate value theorem (see Cor. 47 below), we have that $\delta \circ \delta$ takes any value in the interval $[0, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$. If $0 \le x \le \frac{1}{kb}$ for some $k \in \mathbb{N}_{>1}$, then x is infinitesimal and $(\delta \circ \delta)(x) = 0$ because $\delta(x) \ge b\mu\left(\frac{1}{k}\right)$ is an infinite number. If $x = \frac{k}{b}$ for some $k \in \mathbb{N}_{>0}$, then x is still infinitesimal but $(\delta \circ \delta)(x) = b$ because $\mu(bx) = 0$. A representation of $\delta \circ \delta$ is given in Fig. 3.2. Analogously, one can deal with $H \circ \delta$ and $\delta \circ H$.

The theory of GSF originates from the theory of Colombeau quotient algebras. In this well developed approach, strong analytic tools, including microlocal analysis, and an elaborate theory of pseudodifferential and Fourier integral operators have been developed over the past few years (cf. [16, 17, 86, 48, 56, 30] and references therein). In these quotient algebras, each generalized function generates a unique GSF defined on a subset of ${}^{(\varepsilon)}\widetilde{\mathbb{R}}$. On the other hand, Colombeau generalized functions are in general not closed with respect to composition because they cannot be defined on arbitrary domains $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. See [41] to have more details about the links between Colombeau algebras and GSF. See [111, 112, 113] for the Colombeau algebra in the framework of nonstandard analysis.

Remark 29. To do: relations between $\iota^b_\Omega(T) \cdot \iota^b_\Omega(f) = \iota^b_\Omega(T) \cdot f$ as GSF and $T \cdot f$ in $\mathcal{D}'$ for f smooth (using the equality in ${}^\rho\widetilde{\mathbb{R}}$)

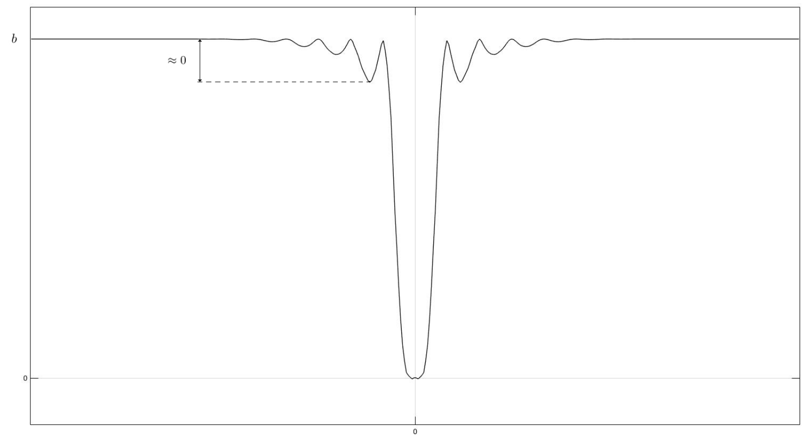

FIGURE 3.2. A representation of $\delta\circ\delta$

4. DIFFERENTIAL CALCULUS AND THE FERMAT-REYES THEOREM

In this section, we show how the derivatives of a GSF can be calculated using a form of incremental ratio. The idea is to prove the Fermat-Reyes theorem for GSF (see [36, 38, 66]). Essentially, this theorem shows the existence and uniqueness of another GSF serving as incremental ratio. This is the first of a long list of results showing the close similarities between ordinary smooth functions and GSF.

We recall that the *thickening* of an open set $\Omega\subseteq\mathbb{R}^n$ *along* $v\in\mathbb{R}^n$ is $\text{th}_v(\Omega):=\{(x,h)\in\mathbb{R}^{n+1}\mid[x,x+hv]_{\mathbb{R}^n}\subseteq\Omega\}$, and serves as a natural domain of a partial incremental ratio along v of any function defined on Ω. In order to prove the Fermat-Reyes theorem, it is simpler to define what *a* thickening of $U\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ along $v\in{}^\rho\widetilde{\mathbb{R}}^n$ is.

Definition 30. Let $U\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ and let $v\in{}^\rho\widetilde{\mathbb{R}}^n$, then we say that $T\subseteq{}^\rho\widetilde{\mathbb{R}}^{n+1}$ is *a (sharp) thickening of* U *along* v if

(i) $\forall x\in U:\ (x,0)\in T$

(ii) For all $(x,h)\in T$ there exist $a,\,b\in{}^\rho\widetilde{\mathbb{R}}_{>0}$, with $b<a$, such that:
   (a) $|h\cdot v|<b$
   (b) $B_a(x)\subseteq U$
   (c) $B_a(x)\times B_b(0)\subseteq T$.

Finally, we will say that T is *a large thickening of* U *along* v if the radii a, b in (ii) are real: $a,\,b\in\mathbb{R}_{>0}$.

Remark 31.

(i) Conditions (i) and (ii) implies that necessarily U is a sharply open set, whereas U is a large open set if T is a large thickening.

(ii) Let $(x,h)\in T$ and let the radii a, b be as in (ii), then for all $s\in[0,1]$ we have $|x+shv-x|\le|hv|<b<a$. Therefore $[x,x+hv]\subseteq B_a(x)\subseteq U$. This gives a connection with the classical definition of thickening and states that if $f:U\longrightarrow{}^\rho\widetilde{\mathbb{R}}$, we can consider the difference $f(x+hv)-f(x)$.

(iii) Condition (ii) of Def. 30 yields that T is a sharply open subset of ${}^{\rho}\widetilde{\mathbb{R}}^{2n}$; it is a large open subset in case T is a large thickening.
(iv) If T and $\bar{T}$ are two (large) thickenings of U along v, then also $T \cap \bar{T}$ is a (large) thickening of the same type. Finally, thickenings are also closed with respect to arbitrary non empty unions.

In the present setting, the Fermat-Reyes theorem is the following.

Theorem 32. *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a sharply open set, let $v = [v_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$, and let $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}})$ be a generalized smooth map generated by the net of smooth functions $f_\varepsilon \in \mathcal{C}^{\infty}(\Omega_\varepsilon, \mathbb{R})$. Then*

(i) If S is a thickening of U along v such that $S \subseteq \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle$, then there exists a thickening $T \subseteq S$ of U along v and a generalized smooth map $r \in {}^{\rho}\mathcal{GC}^{\infty}(T, {}^{\rho}\widetilde{\mathbb{R}})$, called the generalized incremental ratio *of f* along v*, such that*
$$f(x + hv) = f(x) + h \cdot r(x, h) \qquad \forall (x, h) \in T.$$
Moreover $r(x, 0) = \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon)\right]$ for every $x \in U$, and we can thus define $\frac{\partial f}{\partial v}(x) := r(x, 0)$, so that $\frac{\partial f}{\partial v} \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{i}\widetilde{\mathbb{R}})$.

(ii) Any two generalized incremental ratios of f coincide on the intersection of their domains.

If U is a large open set and S is a large thickening of U along v, then an analogous statement holds for a large thickening T of U along v.

Note that this result allows us to consider the partial derivative of f with respect to an arbitrary generalized vector $v \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$ which can be, e.g., near-standard or infinite.
Before proving this theorem, it is essential to show that GSF are uniquely determined by invertible elements.

Lemma 33. *Let $U \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$ be an open set in the sharp topology, and let $f \in {}^{\rho}\mathcal{GC}^{\infty}(U, {}^{\rho}\widetilde{\mathbb{R}}^{d})$ be a GSF. Then $f(x) = 0$ for every $x \in U$ if and only if $f(x) = 0$ for all $x \in U$ such that $|x|$ is invertible.*

Proof. Using Lem. 7, it is quite easy to prove that the group of invertible elements is dense in ${}^{\rho}\widetilde{\mathbb{R}}$ with respect to the sharp topology. This implies that the set of points in U whose every coordinate is invertible is dense in U. Clearly for any such point y, $|y|$ is invertible. Thus given any point $x \in U$ there exists a sequence (x_k) in U converging to x in the sharp topology and such that $|x_k|$ is invertible for each k. Since f is continuous with respect to the sharp topology (Thm. 15 (iii)), this yields $0 = f(x_k) \to f(x)$. □

To show the existence of thickenings, we also need the following result

Lemma 34. *Let (Ω_ε) be a net of open sets of $\mathbb{R}^n$, and let $v = [v_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^{n}$. Then*

(i) if $x \in \langle \Omega_\varepsilon \rangle$ then $(x, 0) \in \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle$.
(ii) If $(x, h) \in \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle$ then $x + thv \in \langle \Omega_\varepsilon \rangle$ for all $t \in [0, 1]$.
(iii) If $U \subseteq \langle \Omega_\varepsilon \rangle$ is sharply open, there exists a sharp thickening T of U along v such that $T \subseteq \langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle$.

The same properties hold if we consider the strongly internal sets $\langle \Omega_\varepsilon \rangle_{\mathrm{F}}$ and $\langle \mathrm{th}_{v_\varepsilon}(\Omega_\varepsilon) \rangle_{\mathrm{F}}$ in the Fermat topology. In (iii), U has to be supposed large open and the resulting thickening is large as well.

Proof. If $x \in \langle\Omega_\varepsilon\rangle$, then $x_\varepsilon \in \Omega_\varepsilon$ for ε small, and we also have $(x_\varepsilon, 0) \in \text{th}_{v_\varepsilon}(\Omega_\varepsilon)$ for the same ε. Now take $(x'_\varepsilon, z_\varepsilon) \sim_\rho (x_\varepsilon, 0)$, so that $x' := [x'_\varepsilon] \in \langle\Omega_\varepsilon\rangle$ and hence $B_\rho(x') \subseteq \langle\Omega_\varepsilon\rangle$ for some $\rho \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $\rho_\varepsilon < d(x'_\varepsilon, \Omega^c_\varepsilon)$ for all $\varepsilon \in I$ (see Thm. 9). The net $(z_\varepsilon) \sim_\rho 0$, so we also have $|z_\varepsilon v_\varepsilon| < \rho_\varepsilon$ for ε small. Thus for ε sufficiently small we can have both $x'_\varepsilon \in \Omega_\varepsilon$ and $|z_\varepsilon v_\varepsilon| < \rho_\varepsilon$, so that for all $s \in [0,1]_{\mathbb{R}}$ we have that $|x'_\varepsilon + s z_\varepsilon v_\varepsilon - x'_\varepsilon| \le |z_\varepsilon v_\varepsilon| < \rho_\varepsilon < d(x'_\varepsilon, \Omega^c_\varepsilon)$. Hence $x'_\varepsilon + s z_\varepsilon v_\varepsilon \in \Omega_\varepsilon$, i.e. $(x'_\varepsilon, z_\varepsilon) \in \text{th}_{v_\varepsilon}(\Omega_\varepsilon)$ for ε sufficiently small. This shows that $(x, 0) \in \langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$, implying (i).

To prove (ii), assume that $(x, h) \in \langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$ and $t \in [0,1]$. Therefore, $0 \le t_\varepsilon \le 1$ for ε small and some representative (t_ε) of t. Since $(x_\varepsilon, h_\varepsilon) \in_\varepsilon \text{th}_{v_\varepsilon}(\Omega_\varepsilon)$, we have that $x_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon \in \Omega_\varepsilon$ for ε small. If we take another representative $(y_\varepsilon) \sim_\rho (x_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon)$, then we can define $x'_\varepsilon := y_\varepsilon - t_\varepsilon h_\varepsilon v_\varepsilon$ so that $(x_\varepsilon, h_\varepsilon) \sim_\rho (x'_\varepsilon, h_\varepsilon)$. From $(x_\varepsilon, h_\varepsilon) \in_\varepsilon \text{th}_{v_\varepsilon}(\Omega_\varepsilon)$ we thus get that also $(x'_\varepsilon, h_\varepsilon) \in \text{th}_{v_\varepsilon}(\Omega_\varepsilon)$ for ε small. Therefore $x'_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon = y_\varepsilon \in \Omega_\varepsilon$ for ε small. This shows that $x + thv \in \langle\Omega_\varepsilon\rangle$.

Finally, in order to prove (iii), we assume that $U \subseteq \langle\Omega_\varepsilon\rangle$ is a sharply open subset. For all $x \in U \subseteq \langle\Omega_\varepsilon\rangle$, we have $(x, 0) \in \langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$ from (i), and hence Thm. 9 (iv) yields the existence of $c_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $B_{c_x}(x, 0) \subseteq \langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$. Since U is a neighborhood of x, there exists $a_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, $a_x < c_x$, such that $B_{a_x}(x) \subseteq U$. Choose $b_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $b_x < a_x$ and $a_x + b_x < c_x$. Because $v \in {}^\rho\widetilde{\mathbb{R}}^n$ is ρ-moderate, we have $|v| < \mathrm{d}\rho^{-N}$ for some $N \in \mathbb{N}$. Take $d_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $d_x < b_x \cdot \mathrm{d}\rho^N$ and define

$$T := \bigcup_{x \in U} B_{a_x}(x) \times B_{d_x}(0).$$

If $|h| < d_x$, then $|h \cdot v| < |v| \cdot d_x < b_x$ and hence T is a sharp thickening of U along v. We finally note that $(x', h) \in B_{a_x}(x) \times B_{d_x}(0)$ implies $|(x', h) - (x, 0)| \le |x' - x| + |h| < a_x + d_x < a_x + b_x < c_x$, so that $(x', h) \in B_{c_x}(x, 0) \subseteq \langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$. Therefore $T \subseteq \langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$.

Considering $\sim_{\mathrm{F}}$ instead of $\sim_\rho$ and radii in $\mathbb{R}_{>0}$, in the same way we can prove the analogous properties for strongly internal sets in the Fermat topology. $\square$

We can now prove the Fermat-Reyes theorem for GSF.

Proof of Theorem 32. Since U is sharply open, for any point $x \in U$ we can find a ball $B_{\rho_x}(x) \subseteq U$, $\rho_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. Define $a_x := \frac{\rho_x}{2}$ and $b_x := \frac{a_x}{2}$. Because $v \in {}^\rho\widetilde{\mathbb{R}}^n$ is ρ-moderate, we have $|v| < \mathrm{d}\rho^{-N}$ for some $N \in \mathbb{N}$. Take $d_x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $d_x < b_x \cdot \mathrm{d}\rho^N$ and set $T := \bigcup_{x \in U} B_{a_x}(x) \times B_{d_x}(0)$. Since for all $x \in U$ the pair $(x, 0)$ is an interior point of the given thickening S, we can assume to have chosen a_x and d_x so that $T \subseteq S \subseteq \langle\text{th}_{v_\varepsilon}(\Omega_\varepsilon)\rangle$. Let us note that if U is large open, then we can proceed as above but obtaining a_x, $d_x \in \mathbb{R}_{>0}$, so that T would be a large thickening.

Let us consider the net of smooth function $r_\varepsilon \in \mathcal{C}^\infty(\text{th}_{v_\varepsilon}(\Omega_\varepsilon))$ defined by $r_\varepsilon(y, h) := \int_0^1 \frac{\partial f_\varepsilon}{\partial v_\varepsilon}(y + thv_\varepsilon)\mathrm{d}t$ for all $\varepsilon \in I$. We calculate the partial derivative $\partial^\alpha r_\varepsilon(y_\varepsilon, h_\varepsilon)$ for $\alpha \in \mathbb{N}^{n+1}$ at an arbitrary point $(y, h) \in T$. For simplicity, set $\hat{\alpha} := (\alpha_1, \dots, \alpha_n)$,

$e_k := (0, \overset{k-1}{\ldots}, 0, 1, 0, \ldots, 0)$ and $v_\varepsilon =: (v_{1\varepsilon}, \ldots, v_{n\varepsilon}) \in \mathbb{R}^n$.

$$
\begin{aligned}
\partial^\alpha r_\varepsilon(y_\varepsilon, h_\varepsilon) &= \int_0^1 \frac{\partial^{|\alpha|}}{\partial h^{\alpha_{n+1}} \partial y^{\hat\alpha}} \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(y_\varepsilon + t h_\varepsilon v_\varepsilon) \right] \mathrm{d}t \\
&= \int_0^1 \sum_{k=1}^n \partial^{\hat\alpha + 2e_k} f_\varepsilon(y_\varepsilon + t h_\varepsilon v_\varepsilon) \cdot v_{k\varepsilon}^{1+\alpha_{n+1}} \cdot t^{\alpha_{n+1}} \, \mathrm{d}t \qquad (4.1) \\
&= \sum_{k=1}^n \partial^{\hat\alpha + 2e_k} f_\varepsilon(y_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon) \cdot v_{k\varepsilon}^{1+\alpha_{n+1}} \cdot t_\varepsilon^{\alpha_{n+1}},
\end{aligned}
$$

where we have used the mean value theorem for integrals to get $t_\varepsilon \in [0,1]_{\mathbb{R}}$ for all $\varepsilon \in I$. From $(y,h) \in T$, we get $y \in B_{a_x}(x)$ and $h \in B_{d_x}(0)$ for some $x \in U$. This gives $|y + thv - x| \le |y - x| + |hv| < a_x + b_x < \rho_x$, so that $y + thv \in B_{\rho_x}(x) \subseteq U$. From Def. 12 (ii) we hence have that $\left(\partial^{\hat\alpha+2e_k} f_\varepsilon(y_\varepsilon + t_\varepsilon h_\varepsilon v_\varepsilon)\right)$ is ρ-moderate. Since moderateness is preserved by polynomials, and $t_\varepsilon \in [0,1]_{\mathbb{R}}$ is moderate, from (4.1) we obtain that $(\partial^\alpha r_\varepsilon(y_\varepsilon, h_\varepsilon))$ is moderate. This proves that $r := [r_\varepsilon(-,-)]|_T : T \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is a GSF.

We have

$$
\begin{aligned}
h \cdot r(x,h) &= \left[h_\varepsilon \cdot \int_0^1 \frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon + t h_\varepsilon v_\varepsilon) \, \mathrm{d}t \right] \\
&= \left[\int_0^{h_\varepsilon} \frac{d}{ds} \left\{ f_\varepsilon(x_\varepsilon + s v_\varepsilon) \right\}(s) \, \mathrm{d}s \right] \\
&= [f_\varepsilon(x_\varepsilon + h_\varepsilon v_\varepsilon)] - [f_\varepsilon(x_\varepsilon)] = f(x + hv) - f(x).
\end{aligned}
$$

Of course $r(x,0) = \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon)\right]$, and this concludes the existence part.

To prove uniqueness, consider $(x,h) \in T \cap \bar T$, where T and $\bar T$ are two thickenings (along v) of the incremental ratios r, $\bar r$. Define $\rho(k) := r(x,k)$ and $\bar\rho(k) := \bar r(x,k)$ for $k \in B_b(0)$, where $(x,h) \in B_a(x) \times B_b(0) \subseteq T \cap \bar T$ by the definition of thickening. Since $k \in B_b(0) \mapsto (x,k) \in T \cap \bar T$ is a GSF, both ρ and $\bar\rho$ are still generalized smooth maps by the closure with respect to composition. Moreover

$$
k \cdot \rho(k) = k \cdot r(x,k) = f(x + kv) - f(x), \qquad (4.2)
$$

and analogously $k \cdot \bar\rho(k) = f(x + kv) - f(x) = k \cdot \rho(k)$. Therefore $\rho(k) = \bar\rho(k)$ for every $k \in B_b(0)$ which is invertible, and Lemma 33 yields $\rho = \bar\rho$. Since $h \in B_b(0)$ we get $\rho(h) = r(x,h) = \bar\rho(h) = \bar r(x,h)$. □

We will use the notation $\frac{\partial f}{\partial v}[-,-]_T \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$ (or simply $\frac{\partial f}{\partial v}[-,-]$ in case the domain is clear from the context) for the generalized smooth incremental ratio of a function $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ defined on the thickening T, so as to distinguish it from the derivative $\frac{\partial f}{\partial v} \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$. Since any partial derivative of a GSF is still a GSF, higher order derivatives $\frac{\partial^\alpha f}{\partial v^\alpha} \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ are simply defined recursively.

The following result shows that the concept of derivative defined using the Fermat-Reyes theorem coincides with the classical derivative of Schwarz's distributions.

Theorem 35. *In the assumptions of Thm. 23, let $\iota^b : \mathcal{D}' \longrightarrow {}^\rho\mathcal{GC}^\infty([-], {}^\rho\widetilde{\mathbb{R}})$ be the embedding of Schwartz's distributions. Then ι^b commutes with partial derivatives, i.e. $\partial^\alpha\left(\iota^b_\Omega(T)\right) = \iota^b_\Omega(\partial^\alpha T)$ for each $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}$.*

Proof. to write... □

to do: relations with derivatives of $\mathcal{C}^q$ functions

The following result follows from the analogous properties for the nets of smooth functions defining f and g.

Theorem 36. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be an open subset in the sharp topology, let $v \in {}^\rho\widetilde{\mathbb{R}}^n$ and $f,\, g : U \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be generalized smooth maps. Then*

(i) $\frac{\partial(f+g)}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v}$

(ii) $\frac{\partial(r\cdot f)}{\partial v} = r \cdot \frac{\partial f}{\partial v} \quad \forall r \in {}^\rho\widetilde{\mathbb{R}}$

(iii) $\frac{\partial(f\cdot g)}{\partial v} = \frac{\partial f}{\partial v} \cdot g + f \cdot \frac{\partial g}{\partial v}$

(iv) *For each $x \in U$, the map $\mathrm{d}f(x).v := \frac{\partial f}{\partial v}(x) \in {}^\rho\widetilde{\mathbb{R}}$ is ${}^\rho\widetilde{\mathbb{R}}$-linear in $v \in {}^\rho\widetilde{\mathbb{R}}^n$.*

Using the Fermat-Reyes theorem, it is also possible to give intrinsic proofs (i.e. without using nets of smooth functions that define given GSF), as in the following

Theorem 37. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $V \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be open subsets in the sharp topology and $g \in {}^\rho\mathcal{GC}^\infty(V,U)$, $f \in {}^\rho\mathcal{GC}^\infty(U,{}^\rho\widetilde{\mathbb{R}})$ be generalized smooth maps. Then for all $x \in V$ and all $v \in {}^\rho\widetilde{\mathbb{R}}^d$*

$$\frac{\partial\,(f\circ g)}{\partial v}(x) = \mathrm{d}f\,(g(x))\,.\frac{\partial g}{\partial v}(x)$$
$$(x) = \mathrm{d}f\,(g(x)) \circ \mathrm{d}g(x).$$

Proof. For h small (in the sharp topology), we can write

$$f\,[g(x+hv)] = f\left[g(x) + h\frac{\partial g}{\partial v}[x,h]\right]. \tag{4.3}$$

Set $u(x,h) := \frac{\partial g}{\partial v}[x,h] \in {}^\rho\widetilde{\mathbb{R}}^n$. Then (4.3) yields

$$f\,[g(x+hv)] = f(g(x)) + h \cdot \frac{\partial f}{\partial u(x,h)}\,[g(x),h]\,.$$

Therefore, the uniqueness of the smooth incremental ratio of $f \circ g$ in the direction v implies

$$\frac{\partial\,(f\circ g)}{\partial v}[x,h] = \frac{\partial f}{\partial u(x,h)}\,[g(x),h]\,.$$

For $h = 0$, we get

$$\frac{\partial\,(f\circ g)}{\partial v}(x) = \frac{\partial f}{\partial u(x,0)}\,(g(x)) = \mathrm{d}f(g(x)).u(x,0) = \mathrm{d}f(g(x)).\frac{\partial g}{\partial v}(x),$$

which is our conclusion. □

5. Integral calculus using primitives

In this section, we inquire existence and uniqueness of primitives F of a GSF $f \in {}^\rho\mathcal{GC}^\infty([a,b],{}^\rho\widetilde{\mathbb{R}})$. To this aim, we take the opportunity to talk of the derivative $F'(x)$ at boundary points $x \in [a,b]$, i.e. such that $x-a$ or $b-x$ is not invertible. Let us note explicitly, in fact, that the Fermat-Reyes Theorem 32 is stated only for sharply open domains. To this end, we need the following result.

Lemma 38. *Let $a,\, b \in {}^\rho\widetilde{\mathbb{R}}$ be such that $a < b$, then the interior $int([a,b])$ in the sharp topology is dense in $[a,b]$.*

Proof. Take representatives of a, b and $x \in [a,b]$ such that $a_\varepsilon < b_\varepsilon$ and $a_\varepsilon \le x_\varepsilon \le b_\varepsilon$ for ε small. Thm. 9 (ii) yields $\operatorname{int}([a,b]) = \langle (a_\varepsilon, b_\varepsilon) \rangle$. To prove the conclusion, it suffices to define

$$y_{k\varepsilon} := \begin{cases} x_\varepsilon & \text{if } a_\varepsilon + \rho_\varepsilon^k \le x_\varepsilon \le b_\varepsilon - \rho_\varepsilon^k \\ a_\varepsilon + \rho_\varepsilon^k & \text{if } x_\varepsilon < a_\varepsilon + \rho_\varepsilon^k \\ b_\varepsilon - \rho_\varepsilon^k & \text{if } x_\varepsilon > b_\varepsilon - \rho_\varepsilon^k \end{cases}$$

for any $k \in \mathbb{N}$ and $\varepsilon \in I$. We have $d(y_{k\varepsilon}, (a_\varepsilon, b_\varepsilon)^c) \ge \rho_\varepsilon^k$, so that $y_k \in \langle (a_\varepsilon, b_\varepsilon) \rangle$. Moreover, $|y_{k\varepsilon} - x_\varepsilon| < \rho_\varepsilon^k$ for all ε, and from this the desired limit condition follows. □

The following result shows that every GSF can have at most one primitive GSF up to an additive constant.

Theorem 39. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $(a,b) \subseteq X$. If $f'(x) = 0$ for all $x \in \operatorname{int}(a,b)$, then f is constant on (a,b). An analogous statement holds if we take any other type of interval (closed or half closed) instead of (a,b).*

Proof. By Lemma 16, we can assume that f is defined by a net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ defined on the whole of $\mathbb{R}$. From the Fermat-Reyes Theorem 32, we know that $f'(x) = [f'_\varepsilon(x_\varepsilon)]$ for every interior point $x = [x_\varepsilon] \in X$. For all x, $y \in \operatorname{int}(a,b) \subseteq U$, we can write

$$\begin{aligned} f(x) - f(y) = [f_\varepsilon(x_\varepsilon) - f_\varepsilon(y_\varepsilon)] &= \left[(y_\varepsilon - x_\varepsilon) \cdot \int_0^1 f'_\varepsilon(x_\varepsilon + s(y_\varepsilon - x_\varepsilon)) \, \mathrm{d}s \right] \\ &= (y - x) \cdot [f'_\varepsilon(x_\varepsilon + s_\varepsilon(y_\varepsilon - x_\varepsilon))] = (y-x) \cdot f'(x + s(y-x)), \end{aligned} \tag{5.1}$$

where $s_\varepsilon \in [0,1]_\mathbb{R}$ comes from the integral mean value theorem and $s := [s_\varepsilon] \in [0,1]$. Since x, $y \in \operatorname{int}(a,b)$, we have $x + s(y-x) \in \operatorname{int}(a,b)$ and hence $f'(x + s(y-x)) = 0$. Thereby, (5.1) yields $f(x) = f(y)$ as claimed. For a different type of interval, it suffices to consider Lemma 38 and sharp continuity of GSF (Thm. 15). □

Remark 40. From the Fermat-Reyes Thm. 32 and from Thm. 39, it follows that the function $i(x) := 1$ if $x \approx 0$ and $i(x) := 0$ otherwise cannot be a GSF on any large neighborhood of $x = 0$. This example stems from the property that different standard real numbers can always be separated by infinitesimal balls.

At interior points $x \in [a,b]$ in the sharp topology, the definition of derivative $f^{(k)}(x)$ follows from the Fermat-Reyes Theorem 32. At boundary points, we have the following

Theorem 41. *Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$ with $a < b$, and $f \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function. Then for all $x \in [a,b]$, the following limit exists in the sharp topology*

$$\lim_{\substack{y \to x \\ y \in \operatorname{int}([a,b])}} f^{(k)}(y) =: f^{(k)}(x).$$

Moreover, if the net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$ defines f and $x = [x_\varepsilon]$, then $f^{(k)}(x) = [f_\varepsilon^{(k)}(x_\varepsilon)]$ and hence $f^{(k)} \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$.

Proof. We have

$$\lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} f^{(k)}(y) = \lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} \left[f_\varepsilon^{(k)}(y_\varepsilon)\right] = [f_\varepsilon^{(k)}(x_\varepsilon)]$$

the last equality following by the sharp continuity of $[f_\varepsilon^{(k)}(-)]$ at every point $x \in [a,b] \subseteq \langle\Omega_\varepsilon\rangle$ (Thm. 15 (iii) and Lem. 38). □

We can now prove existence and uniqueness of primitives of GSF using the most natural statement

Theorem 42. *Let a, b, $c \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$ and $c \in [a,b] \subseteq U$. Let $f \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function. Then, there exists one and only one generalized smooth function $F \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$ such that $F(c) = 0$ and $F'(x) = f(x)$ for all $x \in [a,b]$. Moreover, if f is defined by the net $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ and $c = [c_\varepsilon]$, then $F(x) = \left[\int_{c_\varepsilon}^{x_\varepsilon} f_\varepsilon(s)\,\mathrm{d}s\right]$ for all $x = [x_\varepsilon] \in [a,b]$.*

Proof. Fix representatives (a_ε), (b_ε) and (c_ε) of a, b, c such that

$$a_\varepsilon \le c_\varepsilon \le b_\varepsilon \tag{5.2}$$

for ε small. By Lemma 16, we can assume that f is generated by a net $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$. Set

$$F_\varepsilon(x) := \int_{c_\varepsilon}^{x} f_\varepsilon(s)\,\mathrm{d}s \quad \forall x \in \mathbb{R}. \tag{5.3}$$

We want to prove that the net (F_ε) defines a GSF of type $[a,b] \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, and therefore we take $x \in [a,b]$ and $\alpha \in \mathbb{N}$. Choose a representative (x_ε) of x such that

$$a_\varepsilon \le x_\varepsilon \le b_\varepsilon \tag{5.4}$$

for ε small. If $\alpha > 0$, then $F_\varepsilon^{(\alpha)}(x_\varepsilon) = f_\varepsilon^{(\alpha-1)}(x_\varepsilon)$ and hence moderateness is clear since $x \in [a,b]$. For $\alpha = 0$ we have $F_\varepsilon(x_\varepsilon) = f_\varepsilon(\sigma_\varepsilon)\cdot(x_\varepsilon - c_\varepsilon)$, where

$$\sigma_\varepsilon \in [c_\varepsilon, x_\varepsilon] \cup [x_\varepsilon, c_\varepsilon] \quad \forall \varepsilon \in I \tag{5.5}$$

is obtained by the integral mean value theorem. For ε small, we have both (5.2) and (5.4), so that these inequalities and (5.5) yield $\sigma \in [a,b] \subseteq U$. Therefore $(f_\varepsilon(\sigma_\varepsilon))$ and $(F_\varepsilon(x_\varepsilon))$ are moderate. This proves condition Def. 12 (ii) for the net (F_ε), and we can hence set $F(x) := [F_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}$ for all $x = [x_\varepsilon] \in [a,b]$.
If $y \in \mathrm{int}([a,b])$, we can apply our differential calculus to the generalized smooth map $F|_{\mathrm{int}([a,b])} = [F_\varepsilon(-)]|_{\mathrm{int}([a,b])}$, obtaining $F'(y) = [f_\varepsilon(y_\varepsilon)] = f(y)$. From this, if $x \in [a,b]$, we get

$$F'(x) = \lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} F'(y) = \lim_{\substack{y\to x\\ y\in\mathrm{int}([a,b])}} f(y) = f(x)$$

because f is sharply continuous at $x \in [a,b] \subseteq U$. The uniqueness part follows from Theorem 39.

add citation to [120]; add some remarks on the generality of this integral □

Definition 43. Under the assumptions of Theorem 42, we denote by $\int_c^{(-)} f := \int_c^{(-)} f(s)\,\mathrm{d}s \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$ the unique generalized smooth function such that:

(i) $\int_c^c f = 0$

(ii) $\left(\int_c^{(-)} f\right)'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_c^x f(s)\,\mathrm{d}s = f(x)$ for all $x \in [a,b]$.

In Sec. 8, we develop a generalization of this concept of integration to GSF with several variables and to more general integration domains $M \subseteq {}^\rho\widetilde{\mathbb{R}}^d$.

Example 44.

(i) Since in ${}^\rho\widetilde{\mathbb{R}}$ we have infinitesimal and infinite numbers, our notion of definite integral also includes "improper integrals". Let e.g. $f(x) = \frac{1}{x}$ for $x \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ and $a = 1$, $b_\varepsilon = \rho_\varepsilon^{-q}$, $q > 0$. Then

$$\int_a^b f(s)\,\mathrm{d}s = \left[\int_1^{\rho_\varepsilon^{-q}} \frac{1}{s}\,\mathrm{d}s\right] = [\log \rho_\varepsilon^{-q}] - \log 1 = -q[\log \rho_\varepsilon], \tag{5.6}$$

which is, of course, a positive infinite generalized number. This apparently trivial result is strictly tied to the possibility to define GSF on given arbitrary domains, like $F \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$ in Thm. 42 where b is an infinite number as in (5.6), which is one of the key property to get the closure with respect to composition.

(ii) If p, $q \in {}^\rho\widetilde{\mathbb{R}}$, $p < 0 < q$ and both p, q are not infinitesimal, then $\int_p^q \delta(t)\,\mathrm{d}t \approx 1$.

Theorem 45. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ and $g \in {}^\rho\mathcal{GC}^\infty(Y, {}^\rho\widetilde{\mathbb{R}})$ be generalized smooth functions defined on arbitrary domains in ${}^\rho\widetilde{\mathbb{R}}$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$ with $a < b$ and c, $d \in [a,b] \subseteq X \cap Y$, then*

(i) $\int_c^d (f+g) = \int_c^d f + \int_c^d g$

(ii) $\int_c^d \lambda f = \lambda \int_c^d f \quad \forall \lambda \in {}^\rho\widetilde{\mathbb{R}}$

(iii) $\int_c^d f = \int_c^e f + \int_e^d f$ *for all* $e \in [a,b]$

(iv) $\int_c^d f = -\int_d^c f$

(v) $\int_c^d f' = f(d) - f(c)$

(vi) $\int_c^d f' \cdot g = [f \cdot g]_c^d - \int_c^d f \cdot g'$

Proof. This follows directly from (5.3) and the usual rules of the integral calculus. □

Theorem 46. *Let $f \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$ and $\varphi \in {}^\rho\mathcal{GC}^\infty(S,T)$ be generalized smooth functions defined on arbitrary domains in ${}^\rho\widetilde{\mathbb{R}}$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $[a,b] \subseteq S$, $\varphi(a) < \varphi(b)$, $[\varphi(a), \varphi(b)] \subseteq T$. Finally, assume that $\varphi([a,b]) \subseteq [\varphi(a), \varphi(b)]$. Then*

$$\int_{\varphi(a)}^{\varphi(b)} f(t)\,\mathrm{d}t = \int_a^b f[\varphi(s)] \cdot \varphi'(s)\,\mathrm{d}s.$$

Proof. Define

$$F(x) := \int_{\varphi(a)}^{x} f \quad \forall x \in [\varphi(a), \varphi(b)]$$

$$H(y) := \int_{\varphi(a)}^{\varphi(y)} f \quad \forall y \in [a,b]$$

$$G(y) := \int_a^y f[\varphi(s)] \cdot \varphi'(s)\,\mathrm{d}s \quad \forall y \in [a,b],$$

Each one of these functions is generalized smooth by Def. 43 of integral or by Thm. 26, because it can be written as composition of generalized smooth maps. We have $H(a) = G(a) = 0$, $H(y) = F[\varphi(y)]$ for every $y \in [a,b]$ and, by the chain rule (Prop. 37) $H'(y) = F'[\varphi(y)] \cdot \varphi'(y) = f[\varphi(y)] \cdot \varphi'(y) = G'(y)$, the last two equalities following by Def. 43 of integral. From the uniqueness Theorem 39, the conclusion $H = G$ follows. □

To do:

- relations with integral of continuous functions on a compact interval
- relations with primitives of distributions?

6. Some classical theorems for generalized smooth functions

It is natural to expect that several classical theorems of differential and integral calculus can be extended from the ordinary smooth case to the generalized smooth framework. Once again, we underscore that these faithful generalizations are possible because we don't have a priori limitations in the evaluation $f(x)$ for GSF. For example, one does not have similar results in Colombeau theory, where an arbitrary generalized function can be evaluated only at compactly supported points.
We start from the intermediate value theorem.

Corollary 47. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined on the subset $X \subseteq {}^\rho\widetilde{\mathbb{R}}$. Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $[a, b] \subseteq X$. Assume that $f(a) < f(b)$. Then*

$$\forall y \in {}^\rho\widetilde{\mathbb{R}} : \ f(a) \le y \le f(b) \ \Rightarrow \ \exists c \in [a, b] : \ y = f(c).$$

Proof. Let f be defined by the net $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. For small ε and for suitable representatives (a_ε), (b_ε), (y_ε), we have

$$a_\varepsilon < b_\varepsilon \quad , \quad f_\varepsilon(a_\varepsilon) \le y_\varepsilon \le f_\varepsilon(b_\varepsilon).$$

By the classical intermediate value theorem we get some $c_\varepsilon \in [a_\varepsilon, b_\varepsilon]$ such that $f_\varepsilon(c_\varepsilon) = y_\varepsilon$. Therefore $c := [c_\varepsilon] \in [a, b] \subseteq X$ and hence $f(c) = [f_\varepsilon(c_\varepsilon)] = [y_\varepsilon] = y$. □

Using this theorem we can conclude that no GSF can assume only a finite number of values on any nontrivial interval $[a, b] \subseteq X$, unless it is constant. For example, this provides an alternative way of seeing that the function i of Rem. 40 cannot be a generalized smooth map.
The solution $c \in [a, b]$ of the previous generalized smooth equation $y = f(x)$ need not even be continuous in ε. Indeed, let us consider the net of smooth functions depicted in Figure 6.1, where it is understood that, as ε approaches 0, the two waves at the extremes swing around the dashed rectilinear positions shown in the figure. Set $f(x) = \left[\int_0^1 f_\varepsilon(s)\,\mathrm{d}s - f_\varepsilon(x_\varepsilon)\right] \in {}^\rho\widetilde{\mathbb{R}}$ for $x \in [0, 1] \subseteq {}^\rho\widetilde{\mathbb{R}}$, and analyze the generalized smooth equation $f(x) = 0$. Let $\varepsilon_k = \frac{1}{k}$ be the "times" where the two waves of the net (f_ε) are rectilinear. At these times the solution $f(x_{\varepsilon_k}) = 0$ can be any point $x_{\varepsilon_k} \in [b, c]$. Assume that for $\varepsilon \in \left[\frac{1}{k}, \frac{1}{k} + \delta_k\right]$ only the wave on the left is rectilinear and for $\varepsilon \in \left[\frac{1}{k} - \delta_k, \frac{1}{k}\right]$ only the wave on the right is rectilinear (where $\delta_k \downarrow 0$ is sufficiently small). Therefore, in the first case, the solution must be $x_\varepsilon \in [c, 1]$ and in the second case $x_\varepsilon \in [0, b]$. The solution must jump at every time ε_k and the height of the jump must be at least $c - b$.

This example allows the drawing of the following general conclusion: if we think of our generalized numbers as solutions of smooth equations, then we are forced to

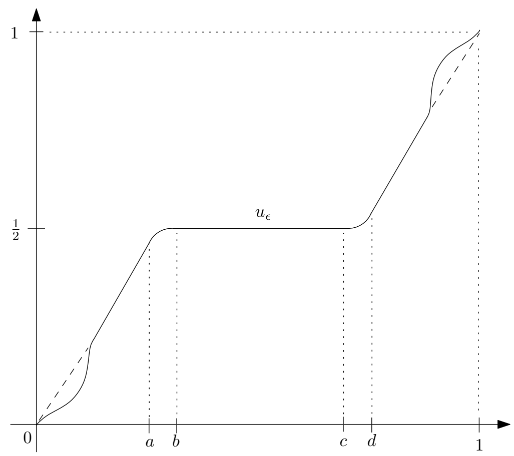

FIGURE 6.1. A net (f_ε) defining a discontinuous solution of a smooth equation.

consider a non totally ordered ring of scalars derived from discontinuous representatives. To put it differently: if we choose a ring of scalars with a total order or continuous representatives, we will not be able to solve every smooth equation, and the given ring can be considered, in some sense, incomplete. Of course, this doesn't mean that the study of better behaved subrings of ${}^\rho\widetilde{\mathbb{R}}$, useful for special purposes, is not interesting.

Theorem 48. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^d)$ be a generalized smooth function defined in the sharply open set $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}^n$, with $a < b$, such that $[a, b] \subseteq X$. Then*

(i) If $n = d = 1$, then $\exists c \in [a, b] : \ f(b) - f(a) = (b - a) \cdot f'(c)$.
(ii) If $n = d = 1$, then $\exists c \in [a, b] : \ \int_a^b f = (b - a) \cdot f(c)$.
(iii) If $d = 1$, then $\exists c \in [a, b] : \ f(b) - f(a) = \nabla f(c) \cdot (b - a)$.
(iv) Let $h := b - a$, then $f(a + h) - f(a) = \int_0^1 \mathrm{d}f(a + t \cdot h).h \, \mathrm{d}t$.

Proof. Using the usual notations, for small ε we have $a_\varepsilon < b_\varepsilon$ and

$$\exists c_\varepsilon \in [a_\varepsilon, b_\varepsilon] : \ f_\varepsilon(b_\varepsilon) - f_\varepsilon(a_\varepsilon) = (b_\varepsilon - a_\varepsilon) \cdot u'_\varepsilon(c_\varepsilon) \tag{6.1}$$

$$\exists c_\varepsilon \in [a_\varepsilon, b_\varepsilon] : \ \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon = (b_\varepsilon - a_\varepsilon) \cdot u_\varepsilon(c_\varepsilon) \tag{6.2}$$

From which the conclusions (i) and (ii) follow directly. The several variables and vector valued cases (iii), (iv) follow as usual by reduction to the one-variable and scalar valued case. □

Internal sets generated by a sharply bounded net of compact sets serve as compact subsets for our GSF since for them an extreme value theorem holds.

Lemma 49. *Let $\emptyset \neq K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be an internal set generated by a sharply bounded net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$ Let $\alpha : K \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be a map defined by $\alpha(x) = [\alpha_\varepsilon(x_\varepsilon)]$ for all $x \in K$, where $\alpha_\varepsilon : K_\varepsilon \longrightarrow \mathbb{R}$ are continuous maps. Then*

$$\exists m, M \in K \, \forall x \in K : \ \alpha(m) \leq \alpha(x) \leq \alpha(M).$$

Proof. Since $K \neq \emptyset$, for ε sufficiently small, let us say for $\varepsilon \in (0, \varepsilon_0]$, K_ε is non empty and, by our assumptions, it is also compact. Since each α_ε is continuous, for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\exists m_\varepsilon, M_\varepsilon \in K_\varepsilon \, \forall x \in K_\varepsilon : \ \alpha_\varepsilon(m_\varepsilon) \le \alpha_\varepsilon(x) \le \alpha_\varepsilon(M_\varepsilon).$$

Since the net (K_ε) is sharply bounded, both the nets (m_ε) and (M_ε) are moderate. Therefore $m = [m_\varepsilon]$, $M = [M_\varepsilon] \in K$. Take any $x \in [K_\varepsilon]$, then there exists a representative (x_ε) such that $x_\varepsilon \in K_\varepsilon$ for ε small. Therefore $\alpha(m) = [\alpha_\varepsilon(m_\varepsilon)] \le [\alpha_\varepsilon(x_\varepsilon)] = \alpha(x) \le \alpha(M)$. □

Corollary 50. *Let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the subset $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let $\emptyset \neq K = [K_\varepsilon] \subseteq X$ be an internal set generated by a sharply bounded net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$, then*

$$\exists m, M \in K \, \forall x \in K : \ f(m) \le f(x) \le f(M). \tag{6.3}$$

These results motivate the following

Definition 51. A subset K of ${}^\rho\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

(i) $K = [K_\varepsilon] \subseteq {}^\rho\widetilde{\mathbb{R}}^n$
(ii) (K_ε) is sharply bounded
(iii) $\forall \varepsilon \in I : \ K_\varepsilon \Subset \mathbb{R}^n$

If, in addition, $K \subseteq U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ then we write $K \Subset_{\mathrm{f}} U$. Finally, we write $[K_\varepsilon] \Subset_{\mathrm{f}} U$ if (ii), (iii) and $[K_\varepsilon] \subseteq U$ hold.

See [39] for a deep study of this type of compact sets in the case $\rho = (\varepsilon)$.

We close this section with the generalization of Taylor's theorem in different forms. In the following statement, $\mathrm{d}^k f(x) : {}^\rho\widetilde{\mathbb{R}}^{dk} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is the k-th differential of the GSF f, viewed as an ${}^\rho\widetilde{\mathbb{R}}$-multilinear map ${}^\rho\widetilde{\mathbb{R}}^d \times \overset{k}{\ldots} \times {}^\rho\widetilde{\mathbb{R}}^d \longrightarrow {}^\rho\widetilde{\mathbb{R}}$, and we use the common notation $\mathrm{d}^k f(x) \cdot h^k := \mathrm{d}^k f(x)(h, \ldots, h)$. Clearly, $\mathrm{d}^k f(x) \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^{dk}, {}^\rho\widetilde{\mathbb{R}})$. For these multilinear maps $A : {}^\rho\widetilde{\mathbb{R}}^p \longrightarrow {}^\rho\widetilde{\mathbb{R}}^q$, we set $|A| := [|A_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$, the generalized number defined by the norms of the operators $A_\varepsilon : \mathbb{R}^p \longrightarrow \mathbb{R}^q$.

Theorem 52. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. Let a, $b \in {}^\rho\widetilde{\mathbb{R}}^d$ such that the line segment $[a, b] \subseteq U$, and set $h := b - a$. Then, for all $n \in \mathbb{N}$ we have*

(i) $\exists \xi \in [a, b] : \ f(a + h) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!} \cdot h^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot h^{n+1}$.

(ii) $f(a + h) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!} \cdot h^j + \frac{1}{n!} \cdot \int_0^1 (1 - t)^n \, \mathrm{d}^{n+1} f(a + th) \cdot h^{n+1} \, \mathrm{d}t$.

Moreover, there exists $\rho \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that

$$\forall k \in B_\rho(0) \, \exists \xi \in [a, a + k] : \ f(a + k) = \sum_{j=0}^n \frac{\mathrm{d}^j f(a)}{j!} \cdot k^j + \frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} \tag{6.4}$$

$$\frac{\mathrm{d}^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} = \frac{1}{n!} \cdot \int_0^1 (1 - t)^n \, \mathrm{d}^{n+1} f(a + tk) \cdot k^{n+1} \, \mathrm{d}t \approx 0. \tag{6.5}$$

Formulas (i) and (ii) correspond to a plain generalization of Taylor's theorem for ordinary smooth functions with Lagrange and integral remainder, respectively. Dealing with generalized functions, it is important to note that this direct statement also includes the possibility that the differential $\mathrm{d}^{n+1}f(\xi)$ may be infinite at some point. For this reason, in (6.4) and (6.5), considering a sufficiently small increment k, we get more classical infinitesimal remainders $\mathrm{d}^{n+1}f(\xi)\cdot k^{n+1}\approx 0$.

Proof. Let $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d,\mathbb{R})$ be a net of smooth functions that defines f. We have $a+h=b\in[a,b]\subseteq U$ and U is sharply open, so by the Taylor formula applied to f_ε and by Theorem 32 we have

$$\begin{aligned} f(a+h) &= [f_\varepsilon(a_\varepsilon+h_\varepsilon)] \\ &= \left[\sum_{j=0}^{n}\frac{\mathrm{d}^j f_\varepsilon(a_\varepsilon)}{j!}h_\varepsilon^j + \frac{\mathrm{d}^{n+1}f_\varepsilon(\xi_\varepsilon)}{(n+1)!}h_\varepsilon^{n+1}\right] \\ &= \sum_{j=0}^{n}\frac{\mathrm{d}^j f(a)}{j!}h^j + \frac{\mathrm{d}^{n+1}f(\xi)}{(n+1)!}h^{n+1} \end{aligned}$$

for some $\xi_\varepsilon\in(a_\varepsilon,b_\varepsilon)$, and where $\xi=[\xi_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}$ so that $\xi\in[a,b]$. Analogously, we can prove (ii).

To prove the second part of the theorem, we start by considering a sharp ball $B_r(a)\subseteq U$, where $r=[r_\varepsilon]>0$. Set $H:=\left[\overline{B_{r_\varepsilon/2}(a_\varepsilon)}\right]$, and

$$K:=\max\left(\left|\mathrm{d}^{n+1}f(M)\right|,\left|\mathrm{d}^{n+1}f(m)\right|\right)\in{}^{\rho}\widetilde{\mathbb{R}},$$

where $\left|\mathrm{d}^{n+1}f(M)\right|$ and $\left|\mathrm{d}^{n+1}f(m)\right|$ are the maximum and the minimum values of $\left|\mathrm{d}^{n+1}f\right|=\left[\left|\mathrm{d}^{n+1}f_\varepsilon\right|\right]:U\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ on $H\subseteq U$, see Lem. 49. We hence have $\left|\mathrm{d}^{n+1}f(\xi)\right|\le K$ for all $\xi\in H$. Take any strictly positive number $P\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $P\ge K$ and any strictly positive infinitesimal $p\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ so that $\frac{p}{P}\approx 0$ and hence $\left(\frac{p}{P}\right)^{n+1}\le\frac{p}{P}$. Set $\rho:=\min\left(\frac{r}{2},\frac{p}{P}\right)$, then $\rho\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ since both r and $\frac{p}{P}$ are invertible. If $k\in B_\rho(0)$ then $[a,a+k]\subseteq H\subseteq U$. We can hence apply the previous (i) to get (6.4). Finally

$$\left|\frac{\mathrm{d}^{n+1}f(\xi)}{(n+1)!}k^{n+1}\right|\le\frac{K}{(n+1)!}\rho^{n+1}\le\frac{P}{(n+1)!}\cdot\left(\frac{p}{P}\right)^{n+1}\le\frac{P}{(n+1)!}\cdot\frac{p}{P}\approx 0.$$

□

The following definitions aellow to state Taylor formulas in Peano and in infinitesimal form. The latter has no remainder term thanks to the use of an equivalence relation that allows the introduction of a language of nilpotent infinitesimals, see e.g. [33] for a similar formulation. For simplicity, we only present the 1-dimentional case.

Definition 53. (i) Let $U\subseteq{}^{\rho}\widetilde{\mathbb{R}}$ be a sharp neighborhood of 0 and $P,Q:U\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ be maps defined on U. Then we say that

$$P(u)=o(Q(u))\quad\text{as }u\to 0$$

if there exists a function $R:U\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ such that

$$\forall u\in U:\ P(u)=R(u)\cdot Q(u)\quad\text{and}\quad\lim_{u\to 0}R(u)=0,$$

where the limit is taken in the sharp topology.

(ii) Let x, $y \in {}^\rho\widetilde{\mathbb{R}}$ and k, $j \in \mathbb{R}_{>0}$, then we write $x =_j y$ if there exist representatives (x_ε), (y_ε) of x, y, respectively, such that

$$|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^{\frac{1}{j}}). \tag{6.6}$$

We will read $x =_j y$ as *x is equal to y up to j-th order infinitesimals.* Finally, if $k \in \mathbb{N}_{>0}$, we set $D_{kj} := \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid x^{k+1} =_j 0\right\}$, which is called the *set of k-th order infinitesimals for the equality $=_j$*, and

$$D_{\infty j} := \left\{x \in {}^\rho\widetilde{\mathbb{R}} \mid \exists k \in \mathbb{N}_{>0} :\ x^{k+1} =_j 0\right\}$$

which is called the *set of infinitesimals for the equality $=_j$*.

Of course, the reformulation of Def. 53 (i) for the classical Landau's little-oh is particularly suited to the case of a ring like ${}^\rho\widetilde{\mathbb{R}}$, instead of a field. The intuitive interpretation of $x =_j y$ is that for particular (physics-related) problems one is not interested in distinguishing quantities whose difference $|x - y|$ is less than an infinitesimal of order j. The idea to take $\frac{1}{j}$ in (6.6) is to have that the greater is the order j of the infinitesimal error the greater the difference $|x - y|$ is allowed to be. This is a typical property in rings with nilpotent infinitesimals (see e.g. [33, 66]). The set D_{ki} represents the neighborhood of infinitesimals of k-th order for the equality $=_j$. Once again, the greater the order k, the bigger is the neighborhood (see Theorem 54 (viii) below).

Theorem 54. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}$. Let x, $\delta \in {}^\rho\widetilde{\mathbb{R}}$, with $\delta > 0$ and $[x - \delta, x + \delta] \subseteq U$. Let k, l, $j \in \mathbb{R}_{>0}$. Then*

(i) $\forall n \in \mathbb{N} :\ f(x + u) = \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r + o(u^n)$ *as* $u \to 0$.

(ii) *The definition of $x =_j y$ does not depend on the representatives of x, y.*

(iii) *$=_j$ is an equivalence relation on ${}^\rho\widetilde{\mathbb{R}}$.*

(iv) *If $x =_j y$ and $l \geq j$, then $x =_l y$.*

(v) *If $\forall^0 j \in \mathbb{R}_{>0} :\ x =_j y$, then $x = y$.*

(vi) *If $x =_j y$ and $z =_j w$ then $x + z =_j y + w$. If x and z are finite, then $x \cdot z =_j y \cdot w$.*

(vii) $\forall h \in D_{kj} :\ h \approx 0$.

(viii) $D_{mj} \subseteq D_{kj} \subseteq D_{\infty j}$ *if* $m \leq k$.

(ix) *D_{kj} is a subring of ${}^\rho\widetilde{\mathbb{R}}$. For all $h \in D_{kj}$ and all finite $x \in {}^\rho\widetilde{\mathbb{R}}$, we have $x \cdot h \in D_{kj}$.*

(x) *Let $n \in \mathbb{N}_{>0}$ and assume that j satisfies*

$$\forall z \in {}^\rho\widetilde{\mathbb{R}}\, \forall \xi \in [x - \delta, x + \delta] :\ z =_j 0 \quad \Longrightarrow \quad z \cdot f^{(n+1)}(\xi) =_j 0. \tag{6.7}$$

Then there exist $k \in \mathbb{R}_{>0}$ such that $k \leq n$ and

$$\forall u \in D_{kj} :\ f(x + u) =_j \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r.$$

(xi) *For all $n \in \mathbb{N}_{>0}$ there exist e, $k \in \mathbb{R}_{>0}$ such that $e \leq j$, $k \leq n$ and $\forall u \in D_{ke}$: $f(x + u) =_e \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r$.*

Proof. In order to prove (i) we set $P(u) = f(x+u) - \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r$, $Q(u) = u^n$ and $R(u) = u \cdot \int_0^1 \frac{f^{(n+1)}(x+tu)}{n!}(1-t)^n \,\mathrm{d}t$ for $u \in B_\delta(0)$. The segment $[x-u, x+u] \subseteq B_\delta(x) \subseteq U$, so Thm. 52 (ii) yields $P(u) = Q(u) \cdot R(u)$ for all $u \in U_x$. As in the previous proof, set

$$K := \max\left(\left|f^{(n+1)}(M)\right|, \left|f^{(n+1)}(m)\right|\right)$$

so that $\left|f^{(n+1)}(\xi)\right| \le K$ for all $\xi \in [x-\delta, x+\delta]$, then

$$|R(u)| \le |u| \cdot \left| \int_0^1 \frac{f^{(n+1)}(x+tu)}{n!}(1-t)^n \,\mathrm{d}t \right| \le |u| \cdot \frac{K}{(n+1)!}$$

which goes to 0 as $u \to 0$ in the sharp topology.

The proofs of (ii)-(ix) are simple. We only prove that D_{kj} is closed with respect to sums. Let x, $y \in D_{kj}$ so that

$$\left| \frac{x_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}} \right| \le M \quad , \quad \left| \frac{y_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}} \right| \le N \tag{6.8}$$

for ε small and for some M, $N \in \mathbb{R}_{>0}$. We can write

$$\begin{aligned}(x_\varepsilon + y_\varepsilon)^{k+1} &= \sum_{r=0}^{k+1} \binom{k+1}{r} x_\varepsilon^r y_\varepsilon^{k+1-r} \\ &= \sum_{r=0}^{k+1} \binom{k+1}{r} \left(x_\varepsilon^{k+1}\right)^{\frac{r}{k+1}} \left(y_\varepsilon^{k+1}\right)^{\frac{k+1-r}{k+1}},\end{aligned}$$

hence

$$\begin{aligned}\left| \frac{(x_\varepsilon + y_\varepsilon)^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}} \right| &\le \sum_{r=0}^{k+1} \binom{k+1}{r} \left| \frac{x_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}} \right|^{\frac{r}{k+1}} \left| \frac{y_\varepsilon^{k+1}}{\rho_\varepsilon^{\frac{1}{j}}} \right|^{\frac{k+1-r}{k+1}} \\ &\le \sum_{r=0}^{k+1} \binom{k+1}{r} M^{\frac{r}{k+1}} N^{\frac{k+1-r}{k+1}}\end{aligned}$$

which proves our claim.

In order to show (x), we first note that $x =_j y$ is equivalent to

$$\exists A \in \mathbb{R}_{>0} : \ |x-y| \le A \cdot \mathrm{d}\rho^{\frac{1}{j}}.$$

We again use the notation $K := \max\left(\left|f^{(n+1)}(M)\right|, \left|f^{(n+1)}(m)\right|\right)$ and set $p := \inf\left\{p \in \mathbb{R} \mid \exists B \in \mathbb{R}_{>0} : \ K \le B \mathrm{d}i^{-p}\right\}$ so that for any $r \in \mathbb{R}_{>0}$ we have

$$\begin{aligned}\left| f(x+u) - \sum_{r=0}^{n} \frac{f^{(r)}(x)}{r!} u^r \right| &\le \frac{K}{(n+1)!} \cdot |u|^{n+1} \\ &\le B \mathrm{d}\rho^{-p-r} \cdot |u|^{n+1}\end{aligned} \tag{6.9}$$

for some $B \in \mathbb{R}_{>0}$. Now, assume that $u \in D_{kj}$. Then

$$|u|^{k+1} \le A \cdot \mathrm{d}\rho^{\frac{1}{j}}$$

for some $A \in \mathbb{R}_{>0}$, so that

$$|u|^{n+1} \le A \cdot \mathrm{d}\rho^{\frac{n+1}{j(k+1)}}.$$

This and (6.9) yield

$$\left|f(x+u)-\sum_{r=0}^{n}\frac{f^{(r)}(x)}{r!}u^r\right|\le AB\cdot \mathrm{d}\rho^{\frac{n+1}{j(k+1)}-p-r},$$

which gives our conclusion if $\frac{n+1}{j(k+1)}-p-r\ge\frac{1}{j}$. There exists such an $r\in\mathbb{R}_{>0}$ if and only if $\frac{n+1}{j(k+1)}-p-\frac{1}{j}>0$, i.e. for $(k+1)(jp+1)<n+1$. If $jp+1\le 0$, then any $k>0$ satisfies this inequality. Otherwise, we need to prove the existence of $k<\frac{n-jp}{jp+1}$. Since $n\in\mathbb{N}_{>0}$, a sufficient condition for this is that $jp<1$. This gives a relation between the order of the equality $=_j$ and the order of infinity p of the derivative $f^{(n+1)}$ on the interval $[x-\delta,x+\delta]$. In order to prove that $jp<1$, we thus need to use the assumption (6.7). Set $z:=\mathrm{d}\rho^{\frac{s}{j}}$, where $1\le s<2$. Then $0\le z\le \mathrm{d}\rho^{\frac{1}{j}}$, i.e. $z=_j 0$, and (6.7) yields

$$\forall\xi\in[x-\delta,x+\delta]:\ z\cdot\left|f^{(n+1)}(\xi)\right|=_j 0 \tag{6.10}$$

Taking $\xi=M$ in (6.10) we obtain

$$\left|f^{(n+1)}(M)\right|\le C\cdot\mathrm{d}\rho^{-\frac{s-1}{j}}$$

for some $C\in\mathbb{R}_{>0}$. Analogously

$$\left|f^{(n+1)}(m)\right|\le D\cdot\mathrm{d}\rho^{-\frac{s-1}{j}}.$$

for some D that we may assume to be $\ge C$ so that $K\le D\mathrm{d}\rho^{-\frac{s-1}{j}}$, which implies $p\le\frac{s-1}{j}$ by definition of p. Therefore, $jp\le s-1<1$, which proves our claim.

To prove (xi), we only need to note that there always exists an $e\in\mathbb{R}_{>0}$ such that $ep<1$, $e\le j$, so that we can repeat the previous deduction with e instead of j. □

7. Sheaf properties

The aim of this section is to establish appropriate sheaf properties of generalized smooth functions. That this task is not entirely straightforward can be seen from the following example:

Example 55. Let $i:{}^{\rho}\widetilde{\mathbb{R}}\to{}^{\rho}\widetilde{\mathbb{R}}$ be as in Rem. 40, i.e., $i(x):=1$ if $x\approx 0$ and $i(x):=0$ otherwise. In this case, ${}^{\rho}\widetilde{\mathbb{R}}$ is the disjoint union of the sets $D_\infty=\{x\mid x\approx 0\}$ and its complement D_∞^{c}, both of which are open in the sharp topology. Moreover, $i|_{D_\infty}\equiv 1$ and $i|_{D_\infty^{\mathrm{c}}}\equiv 0$ are both generalized smooth functions. However, as we have seen in the remark following Cor. 47, i itself is *not* a GSF. This shows that ${}^{\rho}\mathcal{GC}^\infty$ is not a sheaf with respect to the sharp topology.

Trivially, if we introduce the space of (sharply) locally defined GSF by means of $f\in{}^{\rho}\mathcal{GC}^\infty_{\mathrm{loc}}(X,Y)$ if

(i) $f:X\to Y$, and

(ii) $\forall x\in X\,\exists r\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}:\ f|_{B_r(x)\cap X}\in{}^{\rho}\mathcal{GC}^\infty(B_r(x)\cap X,Y)$

then ${}^{\rho}\mathcal{GC}^\infty_{\mathrm{loc}}(\_,Y)$ is naturally a sheaf with respect to the sharp topology. By example 55, however, ${}^{\rho}\mathcal{GC}^\infty_{\mathrm{loc}}(X,Y)$ is strictly larger than ${}^{\rho}\mathcal{GC}^\infty(X,Y)$. This fact can be viewed as a necessary trade-off between the classical statement of locality for generalized functions, on the one hand, and the requirement to preserve classical theorems from smooth analysis on the other. In the above example, it is the validity of a mean value theorem in our setting that precludes the function i from qualifying as a GSF. Conversely, it follows that this result does not hold in ${}^{\rho}\mathcal{GC}^{\infty}_{\text{loc}}(X,Y)$. Any theory of generalized functions that is based on set-theoretical functions and includes actual infinitesimals has to face these dichotomies related to the total disconnectedness of its non-Archimedean ring or scalars.

A way to distinguish the function i of example 55 from GSFs is to observe that the latter preserve subpoints (see Def. 17), as explained in the following. Let $\hat{x}$ be a subpoint of x, so that we can write $\hat{x}_\varepsilon = x_{\bar{\varepsilon}(\varepsilon)}$ for all ε and for suitable representatives $\hat{x} = [\hat{x}_\varepsilon]$ and $x = [x_\varepsilon]$; if $f \in {}^{\rho}\mathcal{GC}^{\infty}(X,Y)$ and $x, \hat{x} \in X$, then

$$\left[f_{\bar{\varepsilon}(\varepsilon)}(\hat{x}_\varepsilon)\right] = \left[f_{\bar{\varepsilon}(\varepsilon)}(x_{\bar{\varepsilon}(\varepsilon)})\right] \text{ is a subpoint of } f(x) = [f_\varepsilon(x_\varepsilon)]\,.$$

Note that it is not that $f(\hat{x}) = [f_\varepsilon(\hat{x}_\varepsilon)]$ is a subpoint of $f(x)$, but that the GSF $\hat{f} := \left[f_{\bar{\varepsilon}(\varepsilon)}(-)\right]$ takes the subpoint x' into a subpoint $\hat{f}(\hat{x})$ of $f(x)$. We could rightly say that $\hat{f}$ is a subfunction of f. Below, we will precise these preliminary intuitive steps. On the contrary, the function i is not able to preserve the subpoints of a point that is continuously jumping from outside to inside of D. Indeed, let $x = [x_\varepsilon]$ be a point jumping between 0 and 1, so that $0 = \left[x_{\bar{\varepsilon}(\varepsilon)}\right]$ for some function $\bar{\varepsilon} : I \longrightarrow I$. Then, since $x_\varepsilon \not\to 0$ as $\varepsilon \to 0$, we have that $x \notin D_\infty$ and hence $i(x) = 0$. Let $j := i|_{D_\infty} = 1$, then at the constant subpoint $\hat{x} := 0 = \left[x_{\bar{\varepsilon}(\varepsilon)}\right]$, we have $\left[j_{\bar{\varepsilon}(\varepsilon)}(\hat{x}_\varepsilon)\right] = \left[j_{\bar{\varepsilon}(\varepsilon)}(0)\right] = 1$ which is not a subpoint of $i(x) = 0$. We can say that the function i is not able to glue together the two GSFs $i|_{D_\infty} = 1$ and $i|_{D^{\text{c}}_\infty} = 0$ because it behaves wrongly on these type of jumping points between D_∞ and D^{c}_∞. This idea can be further advanced to arrive at a new compatibility/coherence condition for a sheaf property of GSFs in the sharp topology. This condition has to talk of the behavior of a given set theoretical function $f : X \longrightarrow Y \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$ at "dynamic points" joining two possibly disjoint sharply open sets U_i, U_j over which $f|_{U_i}$ and $f|_{U_j}$ are GSFs. The idea is that if $x_i \in U_i$ and $x_j \in U_j$ are subpoints of x, so that we have $x_{i\varepsilon} = x_{\bar{\varepsilon}(\varepsilon)}$ and $x_{j\varepsilon} = x_{\hat{\varepsilon}(\varepsilon)}$, then $\left[(f|_{U_i})_{\bar{\varepsilon}(\varepsilon)}(x_{i\varepsilon})\right]$ and $\left[(f|_{U_j})_{\hat{\varepsilon}(\varepsilon)}(x_{j\varepsilon})\right]$ have to be subpoints of the generalized point $f(x) \in {}^{\rho}\widetilde{\mathbb{R}}^d$. To correctly formulate this new condition, we have to introduce some notions that will also be used to introduce the Groethendieck topos of sets of generalized functions (see Sec. 12). We start by generalizing the notion of moderate and negligible nets in case of a vector index $\varepsilon \in I^p$, even if for the sheaf property we will only use the one dimensional case.

Definition 56. Let $p \in \mathbb{N}$, $J \subseteq I^p$ such that $0 \in \mathbb{R}^p$ is a limit point of J and $(x_\varepsilon) : J \longrightarrow \mathbb{R}^n$.

(i) We say that (x_ε) is ρ-moderate if $\exists N \in \mathbb{N}\, \forall^0 \varepsilon \in J : \ |x_\varepsilon| \le \rho_{|\varepsilon|}^{-N}$.

(ii) We say that (x_ε) is ρ-negligible if $\forall n \in \mathbb{N}\, \forall^0 \varepsilon \in J : \ |x_\varepsilon| \le \rho_{|\varepsilon|}^{n}$.

(iii) If $(y_\varepsilon) : J \longrightarrow \mathbb{R}^n$, we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if $(x_\varepsilon - y_\varepsilon)$ is ρ-negligible. Equivalent classes modulo $\sim_\rho$ are still denoted as $[x_\varepsilon]$ (even if they depend on the subset J). Note that all the representatives are defined on the same subset $J \subseteq I^p$, so that we can define $\text{dom}(x) := J$.

(iv) The set of all the equivalence classes $x = [x_\varepsilon]$ whose $\text{dom}(x) = J$ is denoted by ${}^{\rho}\widetilde{\mathbb{R}}^n_J$.

Under the assumptions of Def. 56, let $K \subseteq I^q$ be another subset having $0 \in \mathbb{R}^q$ as a limit point. Let $\mu : J \longrightarrow K$ be a map such that $|\mu_\varepsilon| \le |\varepsilon|$ for all $\varepsilon \in J$. This includes the case of the map $\mu = \bar{\varepsilon}(-)$ defined just after Def. 17. If $x = [x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}^n_K$, we can define

$$x\circ\mu:=[x_{\mu_\varepsilon}]\in{}^\rho\widetilde{\mathbb{R}}^n_J. \tag{7.1}$$

This definition if independent from the representatives, because if $\bar{x}_\varepsilon=x_\varepsilon+z_\varepsilon$ for $\varepsilon\in J$ small and where $(z_\varepsilon)\sim_\rho 0$, then $\bar{x}_{\mu_\varepsilon}=x_{\mu_\varepsilon}+z_{\mu_\varepsilon}$ because $\mu_\varepsilon\to 0$; but $|z_{\mu_\varepsilon}|\le\rho^n_{|\mu_\varepsilon|}\le\rho^n_{|\varepsilon|}$ for $\varepsilon\in J$ small because $\rho_\varepsilon\downarrow 0$ and $|\mu_\varepsilon|\le|\varepsilon|$. Analogously, if $S\subseteq J$ and 0 is a limit point of S, then the definition

$$x|_S:=[(x_\varepsilon)|_S]$$

does not depend on the chosen representative. We can therefore introduce the following

Definition 57. We write

$$x'\xrightarrow{\ \mu\ }x \tag{7.2}$$

if

(i) $x'\in{}^\rho\widetilde{\mathbb{R}}^n_J$ and $x\in{}^\rho\widetilde{\mathbb{R}}^n_K$, where $J\subseteq I^p$, $K\subseteq I^q$ (for some p, $q\in\mathbb{N}$) are such that 0 is a limit point of both subsets.

(ii) $\mu:J\longrightarrow K$ and $|\mu_\varepsilon|\le|\varepsilon|$ for all $\varepsilon\in J$.

(iii) $x'=x\circ\mu$.

We can read (7.2) by saying that x' *is a subpoint of* x *through* μ.

Therefore, if $J=K=I$ and $x'\xrightarrow{\ \mu\ }x$, then x' is a subpoint of x also with respect to Def. 17.

Using the sets ${}^\rho\widetilde{\mathbb{R}}^n_J$ and ${}^\rho\widetilde{\mathbb{R}}^d_J$ and Def. 56, we can define what is a GSF $f=[f_\varepsilon(-)]$ for $\varepsilon\in J$ (see Def. 12). Analogously to the previous steps we did in case of points, we now have

$$f\circ\mu:=\left[(f_{\mu_\varepsilon}(-))_{\varepsilon\in J}\right]:X_\mu:=\left\{x'\in{}^\rho\widetilde{\mathbb{R}}^n_J\mid\exists x\in X:\ x'\xrightarrow{\ \mu\ }x\right\}\longrightarrow{}^\rho\widetilde{\mathbb{R}}^d_J,$$

and this definition does not depend on the defining net $f=\left[(f_\varepsilon(-))_{\varepsilon\in K}\right]:X\longrightarrow Y$, where $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n_K$ and $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d_K$. As above, we use the notation $\bar{f}\xrightarrow{\ \mu\ }f$ to denote that $\bar{f}$ is a GSF defined for $\varepsilon\in J\subseteq I^p$, f is defined for $\varepsilon\in K\subseteq I^q$, μ satisfies (ii) of Def. 57 and $\bar{f}=f\circ\mu$.

Using this language, we can now state that every GSF preserves subpoints:

Lemma 58. *Let* $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$, $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d$, $f\in{}^\rho\mathcal{GC}^\infty(X,Y)$ *and* $x\in X$. *Then*

$$x'\xrightarrow{\ \mu\ }x\quad\Rightarrow\quad(f\circ\mu)(x')\xrightarrow{\ \mu\ }f(x).$$

Proof. In fact, $x'=[x'_\varepsilon]=[x_{\mu_\varepsilon}]$ and hence $(f\circ\mu)(x')=[f_{\mu_\varepsilon}(x'_\varepsilon)]=[f_{\mu_\varepsilon}(x_{\mu_\varepsilon})]=f(x)\circ\mu$. □

We can also state the idea underlying the *generalized compatibility condition* ($X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$, $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d$):

$$\begin{gathered}\text{Let } x\in X \text{ and } x_i\in U_i\cap X \text{ for } i=1,2, \text{ where } x_i\xrightarrow{\ \mu_i\ }x.\\ \text{Assume that } f_i:=f|_{U_i\cap X}\in{}^\rho\mathcal{GC}^\infty(U_i\cap X,Y) \text{ for } i=1,2.\\ \text{Then } (f_i\circ\mu_i)(x_i)\xrightarrow{\ \mu_i\ }f(x)\ \forall i=1,2.\end{gathered} \tag{7.3}$$

Note immediately that asking this condition for any pair of sets U_i, U_j, i, j belonging to an arbitrary index set $\mathcal{I}$, is equivalent to ask it for any $i\in\mathcal{I}$. Moreover, by the previous Lem. 58, any GSF must satisfy this compatibility condition, so that (7.3) is a necessary condition to have a sheaf property in the sharp topology. If, for simplicity, $U_i = \langle \bar{U}_i \rangle$, we can intuitively describe this new compatibility condition by saying that the two sets U_1 and U_2 have a "dynamic intersection", i.e. that the same point x belongs for some ε_1 to $\bar{U}_1$ and for some ε_2 to $\bar{U}_2$. This idea is precisely formalized in the following result, whose proof is simple.

Theorem 59. *Set*

$$\mathcal{P}^d({}^\rho\widetilde{\mathbb{R}}^n) := \left\{ A \subseteq {}^\rho\widetilde{\mathbb{R}}^n \mid \forall x \in {}^\rho\widetilde{\mathbb{R}}^n \, \forall a \in A : \ a \text{ is a subpoint of } x \ \Rightarrow \ x \in A \right\},$$

and for A, $B \in \mathcal{P}^d({}^\rho\widetilde{\mathbb{R}}^n)$ set ("dynamical union and intersection")

$$A \cap^d B := \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^n \mid \exists a, b \text{ subpoints of } x : \ a \in A \text{ and } b \in B \right\}$$
$$A \cup^d B := \left\{ x \in {}^\rho\widetilde{\mathbb{R}}^n \mid \exists c \text{ subpoint of } x : \ c \in A \text{ or } c \in B \right\}.$$

Then $\left(\mathcal{P}^d({}^\rho\widetilde{\mathbb{R}}^n), \cap^d, \cup^d\right)$ is a lattice.

We note that the generalized compatibility condition (7.3) implies the classical one: Assume that $x \in X \cap U_1 \cap U_2$ and $f_i \in {}^\rho\mathcal{GC}^\infty(U_i \cap X, Y)$; set $x_i = x$, $\mu_i(\varepsilon) = \varepsilon$ in (7.3), so that it yields $(f_i \circ \mu_i)(x_i) = f(x) \circ \mu_i$, i.e. $f_1(x) = f_2(x) = f(x)$. This does not mean that it is better to assume the classical compatibility condition because it is weaker than the generalized one. In fact, as we will see, assuming only the classical condition we can prove the sheaf property only for particular coverings, such as those made of near-standard points and large open sets (see Cor. 67) or those made of increasing sequences of internal sets (see Thm. 65).

Condition (7.3) implies that the function f behaves well at the point x, even if the sets U_i are disjoint. For this reason, the function f obtained by gluing together locally defined GSFs will be defined on the following domain, which was introduced by [89].

Definition 60. For any $A \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, its *interleaved closure* is defined as

$$\text{interl}(A) := \{\sum_{j=1}^{m} e_{S_j} x_j \mid m \in \mathbb{N}, \{S_1, \ldots, S_m\} \text{ a partition of } (0,1], x_j \in A \ \forall j\},$$

where $e_{S_j} = \left[\chi_{S_j}(\varepsilon)\right]$ and χ_{S_j} is the characteristic function of $S_j \subseteq (0,1]$.

This notion can also be defined using a finite number of idempotent numbers, i.e. $e \in {}^\rho\widetilde{\mathbb{R}}$ such that $e^2 = e$, that sum up to 1, see [118]. In fact, if $e^2 = e$ then by contradiction we can firstly prove that for all $a \in \mathbb{R}_{>0}$ and for ε small, either $|e_\varepsilon| \le \rho_\varepsilon^a$ or $|e_\varepsilon - 1| \le \rho_\varepsilon^a$. Therefore, setting $S := \{\varepsilon \in I \mid |e_\varepsilon - 1| \le \rho_\varepsilon^a \ \forall a \in \mathbb{R}_{>0}\}$, we have that $e = e_S$.

Using the language of subpoints, we can give a characterization of the interleaved closure that will be applied in proving the sheaf property.

Definition 61. We say that $(x'_i \xrightarrow{\mu_i} x)_{i=1}^m$ *cover* x if

(i) $x'_i \xrightarrow{\mu_i} x$ for all $i = 1, \ldots, m$
(ii) $\bigcup_{i=1}^m \text{Im}(\mu_i) = \text{dom}(x)$, where $\text{Im}(\mu_i)$ is the image of the function μ_i.

We have the following

Lemma 62. *Let J be an arbitrary index set. For all $j \in J$, let $r_j = [r_{j\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, $x_j = [x_{j\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}^n$ and $x \in {}^{\rho}\widetilde{\mathbb{R}}^n$. Then $x \in \operatorname{interl}\left(\bigcup_{j\in J} B_{r_j}(x_j)\right)$ if and only if there exist*

$$
\begin{aligned}
& m \in \mathbb{N}, \\
& j_1, \dots, j_m \in J, \\
& S_i \xrightarrow{\mu_i} T_i := \operatorname{Im}(\mu_i) \xrightarrow{\eta_i} S_i \ \forall i = 1, \dots, m, \\
& x'_1 = [x'_{1\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}^n_{T_1}, \dots, x'_m = [x'_{m\varepsilon}] \in {}^{\rho}\widetilde{\mathbb{R}}^n_{T_m}
\end{aligned}
$$

such that

(i) $\mu_i \circ \eta_i = 1_{T_i}$ *for all $i = 1, \dots, m$, i.e. η_i is a right inverse of μ_i.j*

(ii) $\left(x'_i \xrightarrow{\mu_i} x\right)_{i=1}^m$ *cover x.*

(iii) $x'_{i\eta_i(\varepsilon)} \in B_{r_{j_i\varepsilon}}(x_{j_i\varepsilon})$ *for all $i = 1, \dots, m$ and all $\varepsilon \in T_i$.*

A short explanation of the statement: x'_i being a subpoint of x implies that all the values $x'_{i\varepsilon}$ of a representative can be obtained by a re-parametrization of x_ε, i.e. $x'_{ie} = x_{\mu_i(e)}$ for all $e \in \operatorname{dom}(x') = \operatorname{dom}(\mu_i)$; if we denote by the variable $\varepsilon = \mu_i(e)$ the values of μ_i, we could say that $x = (x_\varepsilon)_{\varepsilon\in I}$ and $x'_i = (x'_{ie})_{e\in\operatorname{dom}(\mu_i)}$. The right inverse η_i allows the opposite step $e = \eta_i(\varepsilon)$. This explain the statement of (iii).

Proof. Assume that $x \in \operatorname{interl}\left(\bigcup_{j\in I} B_{r_j}(x_j)\right)$, so that we can write

$$
x = \sum_{i=1}^m \bar{x}_i e_{S_i}, \tag{7.4}
$$

for some $m \in \mathbb{N}$, $\bar{x}_i \in \bigcup_{j\in I} B_{r_j}(x_j)$ and $\{S_1, \dots, S_m\}$ a partition of I. Therefore, for each i we have $\bar{x}_i \in B_{r_{j_i}}(x_{j_i})$ for some $j_i \in J$. Without loss of generality, we can assume that 0 is a limit point for each set S_i. Otherwise, we would have $(0, \delta) \cap S_i = \emptyset$ for some $\delta \in \mathbb{R}_{>0}$. This would imply $e_{S_i} = 0$ and hence we could avoid this summand in (7.4). Set $\mu_i(\varepsilon) := \varepsilon$ for all $\varepsilon \in S_i$ and $\eta_i(\varepsilon) := \varepsilon$ for all $\varepsilon \in T_i$. For a suitable choice of the representatives, we can write $x_{\mu_i(\varepsilon)} = x_\varepsilon = \bar{x}_{i\varepsilon} + z_\varepsilon$ for all $\varepsilon \in S_i$ by (7.4), and where $z = 0 \in {}^{\rho}\widetilde{\mathbb{R}}^n$. Therefore, setting $x'_{i\varepsilon} := \bar{x}_{i\varepsilon} + z_\varepsilon$ for all $\varepsilon \in S_i$, we have $x_{\mu_i(\varepsilon)} = x'_{i\varepsilon}$ and $x'_{i\varepsilon} = \bar{x}_{i\varepsilon} + z_\varepsilon \in B_{r_{j_i\varepsilon}}(x_{j_i\varepsilon})$ for $\varepsilon \in S_i$ small. This proves that $x'_i \xrightarrow{\mu_i} x$. Since $\bigcup_{i=1}^m \operatorname{Im}(\mu_i) = \bigcup_{i=1}^m S_i = I = \operatorname{dom}(x)$, we finally get (ii).

Vice versa, assume that (i), (ii) and (iii) hold. Since the set T_i can intersect, we have to define a sort of finite partition of unity. For $\vec{\imath} = (i_1, \dots, i_k)$, where $1 \le i_1 \le \dots \le i_j \le m$, set

$$
\bar{T}_{\vec{\imath}} := \bigcap_{j=1}^k T_{i_j} \setminus \bigcup_{p=i_k+1}^m T_p,
$$

$$
\bar{x}_{\vec{\imath}\varepsilon} := \begin{cases} \frac{1}{k}\left(x'_{i_1\eta_{i_1}(\varepsilon)} + \dots + x'_{i_k\eta_{i_k}(\varepsilon)}\right) & \text{if } \varepsilon \in \bar{T}_{\vec{\imath}} \\ x_\varepsilon & \text{if } \varepsilon \in I \setminus \bar{T}_{\vec{\imath}} \end{cases}
$$

By (ii), it follows that the set of all the $\bar{T}_{\vec{\imath}}$ is a partition of I. Moreover, $x'_{i_j e} = x_{\mu_{i_j}(e)}$ for all $e \in S_{i_j}$. Therefore, (i) yields $x'_{i_j \eta_{i_j}(\varepsilon)} = x_\varepsilon$ for all $\varepsilon \in T_{i_j}$. Thus

$$\bar{x}_{\vec{\imath}\varepsilon} \cdot \chi_{\bar{T}_{\vec{\imath}}}(\varepsilon) = x_\varepsilon \quad \forall \varepsilon \in \bar{T}_{\vec{\imath}},$$

and hence $x = \sum_{\vec{\imath}} \bar{x}_{\vec{\imath}} \cdot e_{\bar{T}_{\vec{\imath}}}$. Finally, $\bar{x}_{\vec{\imath}\varepsilon} = x_\varepsilon = x'_{i_1 \eta_{i_1}(\varepsilon)} \in B_{r_{s\varepsilon}}(x_{s\varepsilon})$ for all $\varepsilon \in I$, where $s = j_{i_1}$. This proves that $x \in \text{interl}\left(\bigcup_{j \in I} B_{r_j}(x_j)\right)$. □

Theorem 63. *Let $\hat{X} \subseteq X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$, $f : X \longrightarrow Y$ be a set-theoretical map. Assume that $X \subseteq \bigcup_{j \in J} B_{r_j}(x_j)$, where $r_j \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ and $x_j \in \hat{X}$, and that $f_j := f|_{B_{r_j}(x_j) \cap X} \in {}^\rho\mathcal{GC}^\infty(B_{r_j}(x_j) \cap X, Y)$ for all $j \in J$ is sharply bounded, i.e.*

$$\forall j \in J \, \forall \alpha \in \mathbb{N}^n \, \exists N_j \in \mathbb{N} \, \forall x \in B_{r_j}(x_j) \cap X : \ |\partial^\alpha f_j(x)| \le \rho^{-N_j}. \tag{7.5}$$

We finally assume the generalized compatibility condition: If $x \in X$, $x' \in B_{r_j}(x_j) \cap X$, where $j \in J$ and $x' \xrightarrow{\mu} x$, then $(f_j \circ \mu)(x') \xrightarrow{\mu} f(x)$. Under these assumptions there exists

$$\bar{f} \in {}^\rho\mathcal{GC}^\infty\left(X \cap \text{interl}\left(\bigcup_{j \in J} B_{r_j}(x_j)\right), Y\right)$$

such that $\bar{f}|_X = f$. Therefore, $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$.

Proof. Let $f_j \in {}^\rho\mathcal{GC}^\infty(B_{r_j}(x_j) \cap X, Y)$ be defined by the net $(f_{j\varepsilon}) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ (see Lem. 16). Let $x_j = [x_{j\varepsilon}]$, $r_j = [r_{j\varepsilon}]$ and

$$x = [x_\varepsilon] \in X \cap \text{interl}\left(\bigcup_{j \in J} B_{r_j}(x_j)\right). \tag{7.6}$$

Since ${}^\rho\widetilde{\mathbb{R}}^n$ is not paracompact, we have to accomplished in $\mathbb{R}^n$ the construction of the partition of unity that we need to glue together all the $f_{j\varepsilon}$. Thus, fix $\varepsilon \in I$, so that we can take a locally finite refinement $(U_{i\varepsilon})_{i \in \mathbb{N}}$ of $\left(B^{\mathrm{E}}_{r_{j\varepsilon}}(x_{j\varepsilon})\right)_{j \in J}$. Take a base $(V_{i\varepsilon})_{i \in \mathbb{N}}$ of $\mathbb{R}^n$ such that $\overline{V}_{i\varepsilon} \Subset U_{i\varepsilon}$. Without loss of generality, we can assume that $(V_{i\varepsilon})_{i \in \mathbb{N}}$ is a locally finite refinement of $(U_{i\varepsilon})_{i \in \mathbb{N}}$ and that the order of both covering is n (see e.g. [24, Thm. 3.2.1]). By Lem. 62, there exist $m \in \mathbb{N}$, $\bar{\jmath}_1, \ldots, \bar{\jmath}_m \in J$ and $\mu_1, \ldots, \mu_m$ such that $\left(x_{\bar{\jmath}_i} \xrightarrow{\mu_i} x\right)_{i=1}^m$ cover x. Therefore, $\bigcup_{i=1}^m \text{Im}(\mu_i) = \text{dom}(x) = I$. Since $\varepsilon \in I$, we have $\varepsilon \in \text{Im}(\mu_{\hat{\imath}})$ for some $\hat{\imath} = 1, \ldots, m$, and hence

$$\mu_{\hat{\imath}}(\hat{\varepsilon}) = \varepsilon \tag{7.7}$$

for some $\hat{\varepsilon} \in \text{dom}(x_{\bar{\jmath}_{\hat{\imath}}})$. Since $x_{\bar{\jmath}_{\hat{\imath}}} \xrightarrow{\mu_{\hat{\imath}}} x$, this yields

$$x_{\bar{\jmath}_{\hat{\imath}}\hat{\varepsilon}} = (x \circ \mu_{\hat{\imath}})_{\hat{\varepsilon}} = x_\varepsilon. \tag{7.8}$$

Since $x_\varepsilon = x_{\bar{\jmath}_{\hat{\imath}}\hat{\varepsilon}} \in B^{\mathrm{E}}_{r_{\bar{\jmath}_{\hat{\imath}}\hat{\varepsilon}}}(x_{\bar{\jmath}_{\hat{\imath}}\hat{\varepsilon}})$ and the order of the covering $(V_{i\varepsilon})_{i \in \mathbb{N}}$ is n, there exist at most $n+1$ of its sets that contains the point x_ε. Let $h_1, \ldots, h_{n+1} \in \mathbb{N}$ be the $n+1$ indexes of these sets, so that

$$x_\varepsilon \in V_{h_1\varepsilon} \cup \ldots \cup V_{h_{n+1}\varepsilon} \subseteq U_{h_1\varepsilon} \cup \ldots \cup U_{h_{n+1}\varepsilon}. \tag{7.9}$$

Note that if there existed $i \in \mathbb{N} \setminus \{h_1, \ldots, h_{n+1}\}$ such that $x_\varepsilon \in \text{supp}(\chi_{i\varepsilon}) \subseteq U_{i\varepsilon}$, then we would have $x_\varepsilon \in U_{h_1\varepsilon} \cup \ldots \cup U_{h_{n+1}\varepsilon} \cup U_i$. But this is impossible since the order of the covering $(U_{i\varepsilon})_{i\in\mathbb{N}}$ is n. Now, let $\left\{h_{a_1},\dots,h_{a_{p_\varepsilon}}\right\}\subseteq\{h_1,\dots,h_{n+1}\}$ be a subset of indexes such that x_ε belongs to all the sets $U_{h_{a_k}\varepsilon}$, i.e. such that

$$\begin{aligned}&\forall i\in\{h_1,\dots,h_n\}\setminus\left\{h_{a_1},\dots,h_{a_{p_\varepsilon}}\right\}:\ x_\varepsilon\notin U_{i\varepsilon}\\&x_\varepsilon\in U_{h_{a_1}\varepsilon}\cap\ldots\cap U_{h_{a_{p_\varepsilon}}\varepsilon}.\end{aligned}\tag{7.10}$$

Note that $0\le p_\varepsilon\le n+1$ in general depends on ε since the entire construction depends on the point x_ε. Therefore from (7.9) and (7.10), we also get

$$x_\varepsilon\in V_{h_{a_1}\varepsilon}\cap\ldots\cap V_{h_{a_{p_\varepsilon}}\varepsilon}=:V_\varepsilon\subseteq U_{h_{a_1}\varepsilon}\cap\ldots\cap U_{h_{a_{p_\varepsilon}}\varepsilon}.=:U_\varepsilon,\tag{7.11}$$

and $\overline{V}_\varepsilon\Subset U_\varepsilon$. Choose $\eta_{i\varepsilon}\in\mathcal{D}(U_\varepsilon)$ with $\eta_{i\varepsilon}=1$ in V_ε. Then

$$\chi_{i\varepsilon}:=\frac{\eta_{i\varepsilon}}{\sum_{i\in\mathbb{N}}\eta_{i\varepsilon}}$$

is a partition of unity subordinate to $(U_{i\varepsilon})_{i\in\mathbb{N}}$, which is a refinement of $\left(B^{\mathrm{E}}_{r_{j\varepsilon}}(x_{j\varepsilon})\right)_j$. Therefore

$$\forall i\in\mathbb{N}\,\exists j_i\in J:\ U_{i\varepsilon}\subseteq B^{\mathrm{E}}_{r_{j_i\varepsilon}}(x_{j_i\varepsilon}).\tag{7.12}$$

The smooth function $\chi_{i\varepsilon}\cdot f_{j_i\varepsilon}\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)$ satisfies

$$\partial^\alpha\left(\chi_{i\varepsilon}\cdot f_{j_i\varepsilon}\right)=\partial^\alpha f_{j_i}\text{ on }V_{i\varepsilon}.\tag{7.13}$$

Note that not necessarily, the net $(\chi_{i\varepsilon})$ defines a GSF, but the previous relation says that the product $\chi_{i\varepsilon}\cdot f_{j_i\varepsilon}$ behaves well on $V_{i\varepsilon}$ (exactly as in the proof of Lem. 16). Set

$$\bar f_\varepsilon:=\sum_{i\in\mathbb{N}}\chi_{i\varepsilon}\cdot f_{j_i\varepsilon}\in\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R}^d)=\sum_{k=1}^{n+1}\chi_{h_k\varepsilon}\cdot f_{j_{h_k}\varepsilon}\quad\forall\varepsilon\in I.\tag{7.14}$$

We prove that the net $(\bar f_\varepsilon)$ defines a GSF of the type $X\cap\operatorname{interl}\left(\bigcup_{j\in J}B_{r_j}(x_j)\right)\longrightarrow Y$. Assume x as in (7.6). By (7.11) and (7.14), for all multi-index $\alpha\in\mathbb{N}^n$, we get

$$\partial^\alpha\bar f_\varepsilon(x_\varepsilon)=\sum_{k=1}^{p_\varepsilon}\partial^\alpha\left[\chi_{h_{a_k}\varepsilon}(x_\varepsilon)\cdot f_{j_{h_{a_k}}\varepsilon}(x_\varepsilon)\right].$$

We note that $x_\varepsilon\in V_{h_{a_k}\varepsilon}\subseteq U_{h_{a_k}\varepsilon}\subseteq B^{\mathrm{E}}_{s_k\varepsilon}(x_{s_k\varepsilon})$, where $s_k:=r_{j_{h_{a_k}}}$, for all $\varepsilon\in I$ and all $k=1,\dots,p_\varepsilon$ by construction (see (7.11) and (7.12)), so that this and (7.6) yield $x\in B_{s_k}(x_{s_k})\cap X$. By (7.13), (7.11) and $f_{j_{h_i}}\in{}^\rho\mathcal{GC}^\infty(B_{r_{h_i}}(x_{h_i})\cap X,{}^\rho\widetilde{\mathbb{R}}^d)$, there exists $M_{j_{h_i}}\in\mathbb{N}$ such that $\left|\partial^\alpha f_{j_{h_i}}(x_\varepsilon)\right|\le\rho_\varepsilon^{-M_{j_{h_i}}}$, $M_{j_{h_i}}\le N_{j_{h_i}}$ (see (7.5)), for ε small since, so that

$$\left|\partial^\alpha\bar f_\varepsilon(x_\varepsilon)\right|\le\sum_{k=1}^{p_\varepsilon}\rho_\varepsilon^{-M_{h_{a_k}}}\le\sum_{i=1}^{n+1}\rho_\varepsilon^{-N_i}\quad\forall^0\varepsilon.$$

This proves the first conclusion.

Now, we want to prove that $\bar f|_X=f$. It is only at this point that we use the new compatibility condition. Firstly note that

$$X\subseteq X\cap\operatorname{interl}\left(\bigcup_{j\in J}B_{r_j}(x_j)\right)$$

because $X\subseteq\bigcup_{j\in J}B_{r_j}(x_j)$. Let $f(x)=[y_\varepsilon]$. Using again the previous notations, we have

$$\begin{aligned}[][f_\varepsilon(x_\varepsilon)] &= \left[f_{\mu_{\hat{\imath}}(\hat{\varepsilon})}(x_{\mu_{\hat{\imath}}(\hat{\varepsilon})})\right] = \\ &= \left[\sum_{k=1}^{n+1}\chi_{h_k\mu_{\hat{\imath}}(\hat{\varepsilon})}(x_{\mu_{\hat{\imath}}(\hat{\varepsilon})})\cdot f_{h_k\mu_{\hat{\imath}}(\hat{\varepsilon})}(x_{\mu_{\hat{\imath}}(\hat{\varepsilon})})\right].\end{aligned}$$

But the compatibility condition yields $f_{h_k\mu_{\hat{\imath}}(\hat{\varepsilon})}(x_{\mu_{\hat{\imath}}(\hat{\varepsilon})})=(f(x)\circ\mu_{\hat{\imath}})_{\hat{\varepsilon}}$, so that

$$\begin{aligned}[][f_\varepsilon(x_\varepsilon)] &= \left[\sum_{k=1}^{n+1}\chi_{h_k\mu_{\hat{\imath}}(\hat{\varepsilon})}(x_{\mu_{\hat{\imath}}(\hat{\varepsilon})})\cdot (f(x)\circ\mu_{\hat{\imath}})_{\hat{\varepsilon}}\right] = \\ &= \left[(f(x)\circ\mu_{\hat{\imath}})_{\hat{\varepsilon}}\cdot\sum_{k=1}^{n+1}\chi_{h_k\mu_{\hat{\imath}}(\hat{\varepsilon})}(x_{\mu_{\hat{\imath}}(\hat{\varepsilon})})\right] = \\ &= \left[y_{\mu_{\hat{\imath}}(\hat{\varepsilon})}\cdot 1\right]=[y_\varepsilon]=f(x).\end{aligned}$$

□

This result yields the following

Corollary 64. *Let $X\subseteq{}^\rho\widetilde{\mathbb{R}}^n$ be a sharply open set, $Y\subseteq{}^\rho\widetilde{\mathbb{R}}^d$, and let $f:X\longrightarrow Y$ be a set-theoretical map. Suppose that $X\subseteq\bigcup_{j\in J}U_j$, where each U_j is a sharply open set, and that*

$$f_j:=f|_{U_j\cap X}\in{}^\rho\mathcal{GC}^\infty(U_j\cap X,Y)\quad\forall j\in J.$$

Finally, assume that all the GSFs f_j are compatible: : If $x\in X$, $x'\in U_j\cap X$, where $j\in J$ and $x'\xrightarrow{\ \mu\ }x$, then $(f_j\circ\mu)(x')\xrightarrow{\ \mu\ }f(x)$. Then $f\in{}^\rho\mathcal{GC}^\infty(X,Y)$.

Proof. For all $x\in X$ there exists $r_x\in{}^\rho\widetilde{\mathbb{R}}_{>0}$ and $j_x\in J$ such that $\overline{B_{r_x}(x)}\subseteq X\cap U_{j_x}$. Therefore, $X\subseteq\bigcup_{x\in X}B_{r_x}(x)$ and $f|_{B_{r_x}(x)\cap X}=f_{j_x}|_{B_{r_x}(x)\cap X}=f_{j_x}|_{B_{r_x}(x)}$ is a GSF. The function $f_{j_x}|_{B_{r_x}(x)}$ is sharply bounded on $B_{r_x}(x)\cap X\subseteq\overline{B_{r_x}(x)}$ because $\overline{B_{r_x}(x)}$ is functionally compact by (2.6). The conclusion hence follows by Thm. 63. □

As we mentioned above, if we assume the weaker classical compatibility condition, then we are able to prove a sheaf-like property on particular domains X. For example the following sheaf properties were established by H. Vernaeve in [115]:

Theorem 65.

(i) *Let $U\subseteq{}^\rho\widetilde{\mathbb{R}}^d$ be a union of an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of internal sets with A_{n+1} a sharp neighbourhood of A_n, for each n (hence, in particular, U is a sharply open set). If $u_n\in{}^\rho\mathcal{GC}^\infty(A_n)$ and $u_{n+1}|_{A_n}=u_n$, for each $n\in\mathbb{N}$, then there exists a unique $u\in{}^\rho\mathcal{GC}^\infty(U)$ such that $u|_{A_n}=u_n$, for each $n\in\mathbb{N}$.*

(ii) *For each $m\in\mathbb{N}$, let $U_m\subseteq{}^\rho\widetilde{\mathbb{R}}^d$ be a union of an increasing sequence $(A_{m,n})_{n\in\mathbb{N}}$ of internal sets with $A_{m,n+1}$ a sharp neighbourhood of $A_{m,n}$, for each n. Let $u_m\in{}^\rho\mathcal{GC}^\infty(U_m)$, for each $m\in\mathbb{N}$ such that $u_m|_{U_m\cap U_{\bar{m}}}=u_{\bar{m}}|_{U_m\cap U_{\bar{m}}}$, for each m, $\bar{m}\in\mathbb{N}$. Let $U=\mathrm{interl}(\bigcup_{m\in\mathbb{N}}U_m)$. Then there exists a unique $u\in{}^\rho\mathcal{GC}^\infty(U)$ such that $u|_{U_m}=u_m$, for each $m\in\mathbb{N}$.*

Analogously, the following sheaf property works in the Fermat topology and with domains consisting of near-standard points:

Lemma 66. *Let* $X\subseteq\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)^{\bullet}$ *be a Fermat open set,* $Y\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{d}$ *and let* $f:X\longrightarrow Y$ *be a set-theoretical map. Suppose that* $X\subseteq\bigcup_{x\in X}B_{r_x}(x)$*, where* $r_x\in\mathbb{R}_{>0}$ *for all* x*, and that*

$$f|_{B_{r_x}(x)\cap X}\in{}^{\rho}\mathcal{GC}^{\infty}(B_{r_x}(x)\cap X,Y)\quad\forall x\in X.$$

Then $f\in{}^{\rho}\mathcal{GC}^{\infty}(X,Y)$*.*

Proof. We firstly prove that the generalized compatibility condition always holds. Set in Thm. 63: $J=X=\hat{X}$ and $x_j=j$ for all $j\in J=X$. Let $x\in X$, $x'\in B_{r_z}(z)\cap X$ and $x'\xrightarrow{\mu}x$. Then $x'_\varepsilon=x_{\mu(\varepsilon)}$. But $\mu(\varepsilon)\to0$, so that $(x')^{\circ}=x^{\circ}$ and hence also $x\in B_{r_z}(z)$ because $r_z\in\mathbb{R}_{>0}$ and x is near-standard. Now let $f_z=[f_{z\varepsilon}(-)]$ and $f(x)=[y_\varepsilon]$. We have $f(x)\circ\mu=\left[y_{\mu(\varepsilon)}\right]$. But $x\in B_{r_z}(z)$, so $f(x)=f_z(x)=[f_{z\varepsilon}(x_\varepsilon)]=[y_\varepsilon]$ and hence $[f_{z\varepsilon}(x_\varepsilon)]\circ\mu=[y_\varepsilon]\circ\mu$, i.e. $\left[f_{z\mu(\varepsilon)}(x_{\mu(\varepsilon)})\right]=\left[y_{\mu(\varepsilon)}\right]$. But

$$(f_z\circ\mu)(x')=\left[f_{z\mu(\varepsilon)}(x'_\varepsilon)\right]=\left[f_{z\mu(\varepsilon)}(x_{\mu(\varepsilon)})\right]=\left[y_{\mu(\varepsilon)}\right]=f(x)\circ\mu.$$

We can now proceed as in Cor. (64), but with large balls, i.e. balls with real positive radius. ☐

Corollary 67. *Let* $X\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{n}$*,* $Y\subseteq{}^{\rho}\widetilde{\mathbb{R}}^{d}$ *and let* $f:X\longrightarrow Y$ *be a set-theoretical map. Suppose that* $X\subseteq\bigcup_{i\in I}U_i$*, where each* U_i *is a large open set, and that*

$$f|_{U_i\cap X}\in{}^{\rho}\mathcal{GC}^{\infty}(U_i\cap X,Y)$$

for all $i\in I$*.*

(i) *If* $X\subseteq\left({}^{\rho}\widetilde{\mathbb{R}}^{n}\right)^{\bullet}$*, then* $f\in{}^{\rho}\mathcal{GC}^{\infty}(X,Y)$*.*

(ii) *If* X *contains all the samples of its points and all points of* X *are finite, then* $f\in{}^{\rho}\mathcal{GC}^{\infty}(X,Y)$*.*

Proof. Property (i) follows directly from Thm. 66. Property (ii) follows by (i) and Thm. 19 applied to $f|_{X'}$, where $X':=\{x\in X\mid x\text{ is near-standard}\}$. ☐

8. Multidimensional integration

In this section we want to introduce integration of GSF over ordinary compact sets with respect to an arbitrary measure, and a corresponding push-forward measure by an arbitrary GSF. We will see later that it is more correct to talk of a push-forward integral instead of a push-forward measure, so we use this nomenclature since the beginning. This results in a notion of integration of GSF over multidimensional domains, the first trivial example being $\prod_{i=1}^{n}[a_i,b_i]$, where a_i, $b_i\in{}^{\rho}\widetilde{\mathbb{R}}$, $a_i\le b_i$ are (possibly infinite) generalized numbers.

The possibility to achieve theorems mirroring sigma-additivity or classical limit theorems for this notion of integral is strictly associated to the introduction of the notion of hyperseries and hyperlimit, i.e. of limits of ordinary sequences of generalized numbers $a=(a_n)_{n\in\mathbb{N}}:\mathbb{N}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ but for $n\to+\infty$ along infinite generalized natural number, i.e. for

$$n\in{}^{\rho}\widetilde{\mathbb{N}}:=\left\{[n_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}\mid n_\varepsilon\in\mathbb{N}\ \forall\varepsilon\right\}.$$

Mimicking nonstandard analysis, the numbers $n\in{}^{\rho}\widetilde{\mathbb{N}}$ are called *hypernatural* numbers. To glimpse the necessity to study ${}^{\rho}\widetilde{\mathbb{N}}$, it suffices to note that $\frac{1}{n}<\mathrm{d}\rho^{q}$ is always false for $n\in\mathbb{N}$ but it can be satisfied for suitable $n\in{}^{\rho}\widetilde{\mathbb{N}}$. Therefore, if

$\lim_{n\to+\infty} a_n = 0$ in the classical sense, i.e. for $n \in \mathbb{N}$ and with respect to the sharp topology, then necessarily a_n is infinitesimal for $n \in \mathbb{N}$ sufficiently large. This represents a severe limitation for this notion of limit. Analogously, if $\sum_{n=0}^{+\infty} a_n$ converges in ${}^\rho\widetilde{\mathbb{R}}$, then necessarily a_n must be infinitesimal for $n \in \mathbb{N}$ large. This is also clear from the fact that ${}^\rho\widetilde{\mathbb{R}}$ with the sharp topology is an ultrapseudo metric space, see e.g. [100], and hence a series in ${}^\rho\widetilde{\mathbb{R}}$ converges in the sharp topology if and only if its general term $a_n \to 0$ as $n \to +\infty$, $n \in \mathbb{N}$, in the sharp topology, see [65]. Clearly, this has negative consequences e.g. in dealing with power series and their connection with analytic GSF, see e.g. [115].

8.1. **Integration over ordinary compact sets.** In section 5, we already defined a notion of integral over intervals using the notion of primitive. This notion does not help if we want to define the integral $\int_D f$ of a GSF f over a domain $D \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ which is more general than an interval. In this case, it is natural to try an ε-wise definition of the type $\int_D f\,\mathrm{d}\mu := \left[\int_{D_\varepsilon} f_\varepsilon\,\mathrm{d}\mu\right] \in {}^\rho\widetilde{\mathbb{R}}$, where the net (f_ε) defines the GSF f and the net (D_ε) determines, in some way, the subset $D \subseteq {}^\rho\widetilde{\mathbb{R}}$, e.g. $D = [D_\varepsilon]$ in case of internal sets. In pursuing this idea, it is important to recall that the internal set (interval) $[0,1] = [[0,1]_\mathbb{R}]$ can also be defined by a net of finite sets. Indeed, if $\mathrm{int}(-)$ is the integer part function, and we set

$$\begin{aligned} N_\varepsilon &:= \mathrm{int}\left(\rho_\varepsilon^{1/\varepsilon}\right) \\ B_\varepsilon &:= \{\rho_\varepsilon^{1/\varepsilon}, 2\rho_\varepsilon^{1/\varepsilon}, \ldots, N_\varepsilon\rho_\varepsilon^{1/\varepsilon}\} \end{aligned} \tag{8.1}$$

then the Hausdorff distance $d_\mathrm{H}([0,1]_\mathbb{R}, B_\varepsilon) = \rho_\varepsilon^{1/\varepsilon}$ and hence $[0,1] = [B_\varepsilon]$ (see also [115, 41]). Consequently, if λ is the Lebesgue measure on $\mathbb{R}$, we have that the generalized number $[\lambda([0,1]_\mathbb{R})] = 1$, whereas $[\lambda(B_\varepsilon)] = 0$ and, in general, $\left[\int_{[0,1]_\mathbb{R}} f_\varepsilon\,\mathrm{d}\lambda\right] \neq \left[\int_{B_\varepsilon} f_\varepsilon\,\mathrm{d}\lambda\right] = 0$. Therefore, even the definition of integral over an interval cannot be easily accomplished by proceeding ε-wise, i.e. on defining nets.

We start by fist defining integration of GSF over ordinary compact sets with respect to an arbitrary measure μ, and then we greatly extend the family of integration domains by defining the push forward integral by an arbitrary GSF defined on a compact set.

Definition 68. Let $S \neq \emptyset$ be a compact subset of $\mathbb{R}^n$, and let $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^d)$ be such that $[S] \subseteq X$. Let $(\mathbb{R}^n, \mathcal{A}, \mu)$ be a measure space such that $S \in \mathcal{A}$ and $\mu(S)$ is finite. Then we define

$$\int_S f\,\mathrm{d}\mu := \left[\int_S f_\varepsilon\,\mathrm{d}\mu\right] \in {}^\rho\widetilde{\mathbb{R}}^d, \tag{8.2}$$

where (f_ε) is any net that defines f.

Remark.

(i) Note explicitly that we have defined the integral of f over S and not over the internal set $[S]$. Thus, we don't have to prove that the previous definition is independent from the net of subsets that define the internal set $[S]$. On the other hand, note that if λ is the Lebesgue measure on $\mathbb{R}$, then Thm. 42 and

Def. 43 of definite integral using primitives yields

$$\int_{[0,1]_{\mathbb{R}}} f \, \mathrm{d}\lambda = \int_0^1 f. \tag{8.3}$$

(ii) Since $[S] \subseteq X$ (therefore, $[S]$ is a functionally compact subset of X), we can apply the extreme value theorems Lem. 49, Cor. 50 to get

$$\exists m \in [S] \, \forall x \in [S] : \ |f(x)| \le |f(m)| .$$

Consequently

$$\left| \int_S f_\varepsilon \, \mathrm{d}\mu \right| \le \sup_{x \in S} |f_\varepsilon(x)| \cdot \mu(S) \le (|f_\varepsilon(m_\varepsilon)| + z_\varepsilon) \cdot \mu(S),$$

where $[z_\varepsilon] = 0$. This proves that if the net (f_ε) defines f, then the net $\left(\int_S f_\varepsilon \, \mathrm{d}\mu\right)$ is ρ-moderate.

(iii) Now, let us assume that $(\bar{f}_\varepsilon)$ is another net defining f. By the extreme value theorems Lem. 49, Cor. 50, it is not hard to prove that

$$\forall n \in \mathbb{N} \, \forall^0 \varepsilon : \ \sup_{x \in S} \left| f_\varepsilon(x) - \bar{f}_\varepsilon(x) \right| \le \rho_\varepsilon^n .$$

Therefore, for ε small, we have

$$\left| \int_S f_\varepsilon \, \mathrm{d}\mu - \int_S \bar{f}_\varepsilon \, \mathrm{d}\mu \right| \le \sup_{x \in S} \left| f_\varepsilon(x) - \bar{f}_\varepsilon(x) \right| \cdot \mu(S) \le \rho_\varepsilon^n \mu(S).$$

This finally shows that (8.2) is well-defined.

We will use this notion of integral as a sort of starting point for a natural notion of push forward measure by a GSF $\varphi \in {}^\rho\mathcal{GC}^\infty(S, {}^\rho\widetilde{\mathbb{R}}^n)$.

8.2. Push-forward measure and push-forward integral.

Definition 69. Let $S \ne \emptyset$ be a compact subset of $\mathbb{R}^n$. Let $(\mathbb{R}^n, \mathcal{A}, \mu)$ be a measure space such that $S \in \mathcal{A}$ and $\mu(S)$ is finite. Finally, let $\varphi \in {}^\rho\mathcal{GC}^\infty(S, {}^\rho\widetilde{\mathbb{R}}^n)$. Then

(i) We set $\mathcal{A} \cap S := \{A \cap S \mid A \in \mathcal{A}\}$, so that $(S, \mathcal{A} \cap S, \mu|_{\mathcal{A} \cap S})$ is a measure space, and $\mathrm{k}(\mathcal{A} \cap S) := \{C \in \mathcal{A} \cap S \mid C \text{ is a closed subset of } \mathbb{R}^n\}$.

(ii) We set

$$\varphi(\mathcal{A} \cap S) : = \left\{ B \subseteq \varphi(S) \mid \varphi^{-1}(B) \in \mathcal{A} \cap S \right\}$$
$$(\varphi_* \mu)(B) : = \mu\left(\varphi^{-1}(B)\right) \quad \forall B \in \varphi(\mathcal{A} \cap S).$$

We first observe that if $\varphi \in \mathcal{C}^\infty(S, \mathbb{R}^n)$ is an ordinary smooth function, then the previous definition yields the classical push-forward measure induced by φ. Moreover, since any GSF is a set-theoretical map, the usual proof gives that $(\varphi(S), \varphi(\mathcal{A} \cap S), \varphi_* \mu)$ is an *ordinary* measure space. We also note explicitly that in this proof is essential to consider $\varphi(S)$ and not $\varphi([S])$ in the definition of the σ-algebra $\varphi(\mathcal{A} \cap S)$. On the contrary, we would have problems like $\varphi^{-1}(\varphi([S])) = [S] \notin \mathcal{A} \cap S$. We could comment this fact by saying that "$\varphi_* \mu$ is too standard for our needs". For example, if $\varphi(t) := a + (b - a)t$ for all $t \in [0,1]_{\mathbb{R}}$, where a, $b \in {}^\rho\widetilde{\mathbb{R}}$, $a < b$, then $\varphi([0,1]_{\mathbb{R}}) \subset [0,1]$ because it does not contain a huge amount of generalized points. On the other hand, it is also clear that we do not have to apply the classical Lebesgue theory here because this would result in the integration of ordinary integrable functions and not of GSF. For example, the compatibility with the definition of definite integral by using primitive do not ensue from the Lebesgue approach. For all of these reasons, we started the present section by saying that the name *push-forward integral* is more appropriate than push-forward measure in this context. We therefore need another approach, as in the following

Definition 70. In the assumptions of Def. 69, for all $f \in {}^{\rho}\mathcal{GC}^{\infty}(\varphi(S), {}^{\rho}\widetilde{\mathbb{R}}^{d})$ and all $B \in \varphi(\mathcal{A} \cap S)$ such that $\varphi^{-1}(B) \in \mathrm{k}(\mathcal{A} \cap S)$, we define

$$\int_B f \, \mathrm{d}(\varphi_* \mu) := \int_{\varphi^{-1}(B)} f \circ \varphi \, \mathrm{d}\mu \in {}^{\rho}\widetilde{\mathbb{R}}^{d}. \tag{8.4}$$

We immediately note that if $\varphi(t) = a + (b - a)t$ for all $t \in [0, 1]_{\mathbb{R}}$ and for $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$, $a < b$, then we can take $B = \varphi([0, 1]_{\mathbb{R}})$ to get

$$\int_{\varphi([0,1]_{\mathbb{R}})} f \, \mathrm{d}(\varphi_* \lambda) = \int_{[0,1]_{\mathbb{R}}} f \circ \varphi \, \mathrm{d}\lambda = \int_a^b f \tag{8.5}$$

by (8.3) and the change of variable Thm. 46. Note that a, b could also be infinite generalized numbers, and in that case the interval $[a, b]$ is not contained in any ordinary compact sets.

Before proving some properties of this notion of integral, we have to comment this definition with the following

Remark 71.

(i) Every $C \in \mathrm{k}(\mathcal{A} \cap S)$ is a compact subset of $\mathbb{R}^n$, so that (8.4) is a particular case of Def. 68.

(ii) We have

$$\int_B f \, \mathrm{d}(\varphi_* \mu) = \left[\int_{\varphi^{-1}(B)} f_\varepsilon \circ \varphi_\varepsilon \, \mathrm{d}\mu \right], \tag{8.6}$$

but note that the integration domain on the right hand side is not $\varphi_\varepsilon^{-1}(B)$ because $\varphi^{-1}(B) \in \mathrm{k}(\mathcal{A} \cap S)$ is the ordinary compact set over which we integrate the smooth function $f_\varepsilon \circ \varphi_\varepsilon$, as in Def. 68.

(iii) The notation used in (8.4) may seem misleading because we have just stated that we are not integrating, in the Lebesgue sense, with respect to the push forward measure $\varphi_* \mu$. However, this impression is not correct, firstly because f is a GSF and secondly for the following observation. If the net (f_ε) defines f and $\varphi \in \mathcal{C}^{\infty}(S, \mathbb{R}^n)$ is an ordinary smooth function, by Def. 68 and the change of variable formula for the ordinary push-forward measure, we obtain

$$\int_B f \, \mathrm{d}(\varphi_* \mu) = \left[\int_{\varphi^{-1}(B)} f_\varepsilon \circ \varphi \, \mathrm{d}\mu \right] = \left[\int_{\varphi(\varphi^{-1}(B))} f_\varepsilon \, \mathrm{d}(\varphi_* \mu) \right] = \\ = \left[\int_B f_\varepsilon \, \mathrm{d}(\varphi_* \mu) \right].$$

Note that $\varphi\left(\varphi^{-1}(B)\right) = B$ because $B \subseteq \varphi(S)$. Consequently, if also f is an ordinary smooth function, we have that Def. 70 generalizes the classical notion of integral with respect to the push-forward measure $\varphi_* \mu$.

(iv) Let 1_A be the characteristic function of $A \in \mathcal{A} \cap S$, and let $\iota := \iota_{\mathbb{R}^n}^{\mathrm{d}\rho^{-1}}$ be the embedding defined on $\Omega = \mathbb{R}^n$ by the infinite generalized number $\mathrm{d}\rho^{-1}$. Assume that φ is injective, then from (vi) of Thm. 23 we get the following $\approx$

has to be checked better!

$$\int_{\varphi(A)} \iota\left(1_A\right) \mathrm{d}(\varphi_* \mu) = \int_{\varphi^{-1}(\varphi(A))} \iota\left(1_A\right) \circ \varphi \, \mathrm{d}\mu \approx \int_A 1_A \, \mathrm{d}\mu = \mu(A).$$

(v) If B is a functionally compact subset of ${}^\rho\widetilde{\mathbb{R}}^n$ and $\varphi^{-1}(B) \in \mathrm{k}(\mathcal{A} \cap S)$, then also $[\varphi^{-1}(B)] \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^n$ because $\varphi^{-1}(B) \Subset \mathbb{R}^n$. Therefore, the extreme value theorem Cor. 50 yields

$$\left| \int_B f \, \mathrm{d}(\varphi_* \mu) \right| \le \max_{x \in [\varphi^{-1}(B)]} |f(\varphi(x))| \cdot \mu(\varphi^{-1}(B)) \le \max_{y \in B} |f(y)| \cdot (\varphi_* \mu)(B).$$

Before proving limit properties of this notion of integral, as hinted above, we need the following section.

8.3. **Hyperfinite limits and series.** We start by defining the set of hypernatural numbers in ${}^\rho\widetilde{\mathbb{R}}$ and the set of ρ-moderate nets of natural numbers

Definition 72. We set

(i) ${}^\rho\widetilde{\mathbb{N}} := \left\{ [n_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}} \mid n_\varepsilon \in \mathbb{N} \quad \forall \varepsilon \right\}$

(ii) $\mathbb{N}_\rho := \left\{ (n_\varepsilon) \in \mathbb{R}_\rho \mid n_\varepsilon \in \mathbb{N} \quad \forall \varepsilon \right\}$.

Therefore, $n \in {}^\rho\widetilde{\mathbb{N}}$ if and only if there esists $(x_\varepsilon) \in \mathbb{R}_\rho$ such that $n = [\mathrm{int}(|x_\varepsilon|)]$. Clearly, $\mathbb{N} \subset {}^\rho\widetilde{\mathbb{N}}$. Unfortunately, the integer part function $\mathrm{int}(-)$ is not well-defined on ${}^\rho\widetilde{\mathbb{R}}$. In fact, if $x = 1 = \left[1 - \rho_\varepsilon^{1/\varepsilon}\right] = \left[1 + \rho_\varepsilon^{1/\varepsilon}\right]$, then $\mathrm{int}\left(1 - \rho_\varepsilon^{1/\varepsilon}\right) = 0$ whereas $\mathrm{int}\left(1 + \rho_\varepsilon^{1/\varepsilon}\right) = 1$, for ε sufficiently small.

On the contrary, the nearest integer function is well defined on ${}^\rho\widetilde{\mathbb{N}}$.

Theorem 73. *Let $(n_\varepsilon) \in \mathbb{N}_\rho$ and $(x_\varepsilon) \in \mathbb{R}_\rho$ such that $[n_\varepsilon] = [x_\varepsilon]$. Let $\mathrm{rpi} : \mathbb{R} \longrightarrow \mathbb{N}$ be function rounding to the nearest integer towards $+\infty$. Then $\mathrm{rpi}(x_\varepsilon) = n_\varepsilon$ for ε small. The same result holds using $\mathrm{rni} : \mathbb{R} \longrightarrow \mathbb{N}$, the function rounding to the nearest integer towards $-\infty$.*

Proof. It is known that $\mathrm{rpi}(x) = \lfloor x + \frac{1}{2} \rfloor$, where $\lfloor - \rfloor$ is the floor function. For ε small, we have $\rho_\varepsilon < \frac{1}{2}$ and, since $[n_\varepsilon] = [x_\varepsilon]$, for the same ε we can also have $n_\varepsilon - \rho_\varepsilon < x_\varepsilon < n_\varepsilon + \rho_\varepsilon$. Therefore $n_\varepsilon - \rho_\varepsilon + \frac{1}{2} < x_\varepsilon + \frac{1}{2} < n_\varepsilon + \rho_\varepsilon + \frac{1}{2}$. But $n_\varepsilon \le n_\varepsilon - \rho_\varepsilon + \frac{1}{2}$ and $n_\varepsilon + \rho_\varepsilon + \frac{1}{2} < n_\varepsilon + 1$ because $\rho_\varepsilon < \frac{1}{2}$. Therefore $\lfloor x_\varepsilon + \frac{1}{2} \rfloor = n_\varepsilon$. We can analogously proceed using the function $\mathrm{rni}(-)$. ☐

We can hence define

Definition 74. The nearest integer function $\mathrm{nint}(-)$ satisfies the following properties:

(i) $\mathrm{nint} : {}^\rho\widetilde{\mathbb{N}} :\longrightarrow \mathbb{N}_\rho$

(ii) If $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{N}}$, then $\mathrm{nint}\left([x_\varepsilon]\right) = (\mathrm{rpi}(x_\varepsilon))$.

In other words, if $x \in {}^\rho\widetilde{\mathbb{N}}$, then $x = [\mathrm{nint}(x)_\varepsilon]$ and $\mathrm{nint}(x)_\varepsilon \in \mathbb{N}$ for all ε.

We first consider the notion of hyperlimit because that of hyperseries, as in the classical situation, is a particular case.

As we will see clearly in Example 78.(i), a key point in the definition of hyperlimit is to consider *two* gauges. Moreover, in several situations, we start from an ordinary sequence $a : \mathbb{N} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ defined only on $\mathbb{N}$ and not on the whole ${}^\rho\widetilde{\mathbb{N}}$.

Definition 75. Let ρ, σ be two gauges (see Def. 1). Let $a:\mathbb{N}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ be a sequence of ρ-generalized numbers. Let $[a_{n,\varepsilon}]=a_n$, $n\in\mathbb{N}$, be a representative of a_n and finally let $l=[l_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}$. Then we say that

$$l \text{ is the hyperlimit of } a:{}^{\sigma}\widetilde{\mathbb{N}}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$$

if

$$\forall q\in\mathbb{N}\,\exists M\in{}^{\sigma}\widetilde{\mathbb{N}}\,\forall(n_\varepsilon)\in\mathbb{N}_\sigma:\ [n_\varepsilon]\ge M\ \Rightarrow\ \forall^0\varepsilon:\ |a_{n_\varepsilon,\varepsilon}-l_\varepsilon|<\rho_\varepsilon^q. \tag{8.7}$$

Remark 76.

(i) Since $(n_\varepsilon)\in\mathbb{N}_\sigma$, $[n_\varepsilon]\ge M\in{}^{\sigma}\widetilde{\mathbb{N}}$, we have that $n:=[n_\varepsilon]\in{}^{\sigma}\widetilde{\mathbb{N}}$, and we cannot consider the evaluation a_n because $a=(a_n)_{n\in\mathbb{N}}$ is defined only on $\mathbb{N}$. For this reason, we consider the term $a_{n_\varepsilon,\varepsilon}$, where $n_\varepsilon\in\mathbb{N}$.

(ii) The part of (8.7) "$\forall(n_\varepsilon)\in\mathbb{N}_\sigma:\ [n_\varepsilon]\ge M$" could be read as "for every representative (n_ε) of $n\in{}^{\sigma}\widetilde{\mathbb{N}}_{\ge M}$ such that $n_\varepsilon\in\mathbb{N}$". However, because of Thm. 73, we only have one of such representative, which is $\text{nint}(n)\in\mathbb{N}_\sigma$. Therefore, (8.7) is equivalent to

$$\forall q\in\mathbb{N}\,\exists M\in{}^{\sigma}\widetilde{\mathbb{N}}\,\forall n\in{}^{\sigma}\widetilde{\mathbb{N}}_{\ge M}:\ \text{nint}(n)=:(n_\varepsilon)\ \Rightarrow\ \forall^0\varepsilon:\ |a_{n_\varepsilon,\varepsilon}-l_\varepsilon|<\rho_\varepsilon^q.$$

(iii) In a hyperlimit, we are considering ${}^{\sigma}\widetilde{\mathbb{N}}$ as an ordered set directed by $\le$:

$$n,m\in{}^{\sigma}\widetilde{\mathbb{N}}\ \Rightarrow\ n\vee m=[\max(\text{nint}(n)_\varepsilon,\text{nint}(m)_\varepsilon)]\in{}^{\sigma}\widetilde{\mathbb{N}}.$$

Whereas on ${}^{\rho}\widetilde{\mathbb{R}}$ we are considering the sharp topology, because it is Hausdorff. In fact, we easily have that if l, λ are hyperlimits of $a:{}^{\sigma}\widetilde{\mathbb{N}}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$, then

$$|l_\varepsilon-\lambda_\varepsilon|\le|l_\varepsilon-a_{n_\varepsilon,\varepsilon}|+|a_{n_\varepsilon,\varepsilon}-\lambda_\varepsilon|\le 2\rho_\varepsilon^q<\rho_\varepsilon^{q-1}$$

for all q and for ε small. So $l=\lambda\in{}^{\rho}\widetilde{\mathbb{R}}$. We will therefore use the notations

$$l={}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_n$$

or simply $l=\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}a_n$ if $\sigma=\rho$.

(iv) Def. 75 does not depend on the chosen representatives $[a_{n,\varepsilon}]=a_n$ and $[l_\varepsilon]=l$. In fact if $[z_{n\varepsilon}]=0$ for all $n\in\mathbb{N}$ and $[\zeta_\varepsilon]=0$, then

$$|a_{n_\varepsilon,\varepsilon}+z_{n_\varepsilon,\varepsilon}-l_\varepsilon-\zeta_\varepsilon|\le|a_{n_\varepsilon,\varepsilon}-l_\varepsilon|+|z_{n_\varepsilon,\varepsilon}-\zeta_\varepsilon|<\rho_\varepsilon^q$$

for ε small.

(v) Note that if condition (8.7) holds, then $l_\varepsilon<a_{n_\varepsilon,\varepsilon}+\rho_\varepsilon^q$ for ε small. Therefore, we necessarily have that $(l_\varepsilon)\in\mathbb{R}_\rho$.

(vi) The verbal expression "hyperlimit of $(a_n)_{n\in\mathbb{N}}$ from ${}^{\sigma}\widetilde{\mathbb{N}}$ to ${}^{\rho}\widetilde{\mathbb{R}}$" is partly misleading because the sequence $a=(a_n)_{n\in\mathbb{N}}:\mathbb{N}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ is defined only on $\mathbb{N}$. The extension of a to the whole ${}^{\sigma}\widetilde{\mathbb{N}}$ is tied to the implication

$$\text{If } n\in{}^{\sigma}\widetilde{\mathbb{N}} \text{ and } \left(a_{\text{nint}(n)_\varepsilon,\varepsilon}\right)\in\mathbb{R}_\rho \text{ then } a_n:=\left[a_{\text{nint}(n)_\varepsilon,\varepsilon}\right]\in{}^{\rho}\widetilde{\mathbb{R}}. \tag{8.8}$$

In fact, in this way a_n is well-defined because of Thm. 73; on the other hand we have defined an extension of the old sequence a because if $n\in\mathbb{N}$, then $\text{nint}(n)_\varepsilon=n$ for all ε and hence from (8.8) we get $a_n=[a_{n,\varepsilon}]$.

The notion of hyperseries is introduced in the following

Definition 77. Let σ, ρ be two gauges, let $a:\mathbb{N}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ be a sequence of ${}^{\rho}\widetilde{\mathbb{R}}$ and let $s\in{}^{\rho}\widetilde{\mathbb{R}}$. Then we say that s is the sum of the hyperseries with terms $a:{}^{\sigma}\widetilde{\mathbb{N}}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$ if s is the hyperlimit of the sequence $\left(N\in\mathbb{N}\mapsto\sum_{n=0}^{N}a_n\in{}^{\rho}\widetilde{\mathbb{R}}\right):{}^{\sigma}\widetilde{\mathbb{N}}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$. In that case, we write

$$s={}^{\rho}\lim_{N\in{}^{\sigma}\widetilde{\mathbb{N}}}\sum_{n=0}^{N}a_n=:{}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_n.$$

Explicitly written, this means

$$\forall q\in\mathbb{N}\,\exists M\in{}^{\sigma}\widetilde{\mathbb{N}}\,\forall(N_\varepsilon)\in\mathbb{N}_\sigma:\ [N_\varepsilon]\ge M\ \Rightarrow\ \forall^0\varepsilon:\ \left|\sum_{n=0}^{N_\varepsilon}a_{n,\varepsilon}-s_\varepsilon\right|<\rho_\varepsilon^q,$$

or

$$\forall q\in\mathbb{N}\,\exists M\in{}^{\sigma}\widetilde{\mathbb{N}}\,\forall N\in{}^{\sigma}\widetilde{\mathbb{N}}_{\ge M}:\Rightarrow\ \forall^0\varepsilon:\ \left|\sum_{n=0}^{\text{nint}(N)_\varepsilon}a_{n,\varepsilon}-s_\varepsilon\right|<\rho_\varepsilon^q.$$

Example 78.

(i) The following example strongly motivates the use of two gauges. Let ρ be a gauge and set $\sigma_\varepsilon:=\exp\left(-\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right)$, so that also σ is a gauge. We have

$${}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\frac{1}{\log n}=0\in{}^{\rho}\widetilde{\mathbb{R}}\quad\text{whereas}\quad\nexists\,{}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{1}{\log n}.$$

In fact, if $n_\varepsilon>1$, we have $0<\frac{1}{\log n}<\rho_\varepsilon^q$ if and only if $\log n_\varepsilon>\rho_\varepsilon^{-q}$, i.e. $n_\varepsilon>e^{\rho_\varepsilon^{-q}}$. We can thus take $M:=\left[\text{int}\left(e^{\rho_\varepsilon^{-q}}\right)+1\right]\in{}^{\sigma}\widetilde{\mathbb{N}}$ because $e^{\rho_\varepsilon^{-q}}<\exp\left(\rho_\varepsilon^{-\frac{1}{\rho_\varepsilon}}\right)$ for ε small.

Vice versa, by contradiction, if $\exists\,{}^{\rho}\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{1}{\log n}=:l\in{}^{\rho}\widetilde{\mathbb{R}}$, then by the definition of hyperlimit from ${}^{\rho}\widetilde{\mathbb{N}}$ to ${}^{\rho}\widetilde{\mathbb{R}}$ we would get the existence of $M\in{}^{\rho}\widetilde{\mathbb{N}}$ such that

$$\forall(n_\varepsilon)\in\mathbb{N}_\rho:\ [n_\varepsilon]\ge M\ \Rightarrow\ \forall^0\varepsilon:\ \frac{1}{\log n_\varepsilon}-\rho_\varepsilon<l_\varepsilon<\frac{1}{\log n_\varepsilon}+\rho_\varepsilon.\tag{8.9}$$

We have to explore two possibilities: if l is not invertible, then $l_{\varepsilon_n}=0$ for some sequence $(\varepsilon_n)\downarrow 0$. Therefore from (8.9) for $n_\varepsilon=M_\varepsilon$, we get

$$\frac{1}{\log M_{\varepsilon_n}}<l_{\varepsilon_n}+\rho_{\varepsilon_n}=\rho_{\varepsilon_n}$$

hence $M_{\varepsilon_n}>e^{-\frac{1}{\rho_{\varepsilon_n}}}$ in contradiction with $M\in{}^{\rho}\widetilde{\mathbb{R}}$. If l is invertible, $\mathrm{d}\rho^p<|l|$ for some $p\in\mathbb{N}$. Setting

$$q:=\min\{p\in\mathbb{N}\mid\mathrm{d}\rho^p<|l|\}+1,$$

we get that $|l_{\bar\varepsilon_n}|<\rho_{\bar\varepsilon_n}^q$ for some sequence $(\bar\varepsilon_n)_n\downarrow 0$. Therefore

$$\frac{1}{\log M_{\bar\varepsilon_n}}<l_{\bar\varepsilon_n}+\rho_{\bar\varepsilon_n}\le|l_{\bar\varepsilon_n}|+\rho_{\bar\varepsilon_n}<\rho_{\bar\varepsilon_n}^q+\rho_{\bar\varepsilon_n}$$

and hence $M_{\bar\varepsilon_n}>\exp\left(\frac{1}{\rho_{\bar\varepsilon_n}^q+\rho_{\bar\varepsilon_n}}\right)$ which is in contradiction with $M\in{}^{\rho}\widetilde{\mathbb{R}}$ because $q\ge 1$.

(ii) For all $k\in\mathbb{N}_{>0}$, we have $\lim_{n\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{1}{n^k}=0$. In fact, for all $n\in{}^{\rho}\widetilde{\mathbb{N}}_{>0}$, we have $0<\frac{1}{n^k}<\mathrm{d}\rho^q$ if and only if $n^k>\mathrm{d}\rho^{-q}$, i.e. $n>\mathrm{d}\rho^{-\frac{q}{k}}$. Thus, it suffices to take $M_\varepsilon:=\mathrm{int}\left(\rho_\varepsilon^{-\frac{q}{k}}\right)+1$ in the definition of hyperlimit. Analogously, we can treat rational functions having degree of denominator greater or equal to that of the numerator.

(iii) For all $k\in{}^{\rho}\widetilde{\mathbb{R}}_{<1}$ we have

$$
{}^{\rho}\!\!\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}}k^n=\frac{1}{1-k}. \tag{8.10}
$$

In fact, ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}}k^n={}^{\rho}\lim_{N\in{}^{\rho}\widetilde{\mathbb{N}}}\sum_{n=0}^{N}k^n={}^{\rho}\lim_{N\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{1-k^{N+1}}{1-k}$. But $k^{N+1}<\mathrm{d}\rho^q$ if and only if $(N+1)\log k<q\log\mathrm{d}\rho$. Since $0<k<1$, we have $\log k<0$ and we obtain $N>q\frac{\log\mathrm{d}\rho}{\log k}-1$. It suffices to take $M_\varepsilon:=\mathrm{int}\left(q\frac{\log\rho_\varepsilon}{\log k}\right)$ in the definition of hyperlimit.

(iv) For all $x\in{}^{\rho}\widetilde{\mathbb{R}}$ finite, we have ${}^{\rho}\sum_{n\in{}^{\rho}\widetilde{\mathbb{N}}}\frac{x^n}{n!}=e^x$. We have $|x|<M\in\mathbb{R}_{>0}$ because x is finite. For all $N\in\mathbb{N}$ and all ε, we have

$$
\sum_{n=0}^{N}\frac{x_\varepsilon}{n!}=e^{x_\varepsilon}-\sum_{n=N+1}^{+\infty}\frac{x_\varepsilon}{n!}. \tag{8.11}
$$

Now, take $N\in\mathbb{N}$ such that $\frac{M}{N+1}<\frac{1}{2}$. We have $\left|\sum_{n=N+1}^{+\infty}\frac{x_\varepsilon}{n!}\right|\le\sum_{n>N}\frac{M^n}{n!}$ and for all $n\ge N$

$$
\frac{M^{n+1}}{(n+1)!}\le\frac{M}{N+1}\frac{M^n}{n!}<\frac{1}{2}\frac{M^n}{n!}<\ldots<\frac{1}{2^k}\frac{M^{n-k}}{(n-k)!}<\ldots<\frac{1}{2^{n+1}}.
$$

Therefore $\left|\sum_{n=N+1}^{+\infty}\frac{x_\varepsilon}{n!}\right|\le\sum_{n>N}\frac{1}{2^n}$ and hence ${}^{\rho}\lim_{N\in{}^{\rho}\widetilde{\mathbb{N}}}\sum_{n=N+1}^{+\infty}\frac{x^n}{x!}=0$ by (8.10). This and (8.11) yields the conclusion.

The following result allows us to obtain hyperlimits and hyperseries by proceeding ε-wise

Theorem 79. *Let $a:\mathbb{N}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$, with $a_n=[a_{n,\varepsilon}]$ for all $n\in\mathbb{N}$. Assume that for all ε*

$$
\exists\lim_{n\to+\infty}a_{n,\varepsilon}=:l_\varepsilon, \tag{8.12}
$$

and that $l:=[l_\varepsilon]\in{}^{\rho}\widetilde{\mathbb{R}}$. Then

$$
\exists\sigma\ \text{gauge}:\ l={}^{\rho}\lim_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_n.
$$

Proof. Let $q\in\mathbb{N}$, then assumption (8.12) yields

$$
\forall\varepsilon\,\exists\bar{M}_\varepsilon\in\mathbb{N}\,\forall n\in\mathbb{N}_{\ge M_\varepsilon}:\ |a_{n,\varepsilon}-l_\varepsilon|<\rho_\varepsilon^q.
$$

If the net $(\bar{M}_\varepsilon)$ is bounded, we can take $\sigma:=\rho$. Otherwise, we set $\sigma_\varepsilon:=\bar{M}_\varepsilon^{-1}$ and $M:=[\bar{M}_\varepsilon+1]\in{}^{\sigma}\widetilde{\mathbb{N}}$. For all $(n_\varepsilon)\in\mathbb{N}_\sigma$ with $[n_\varepsilon]\ge M$, for ε small we have $n_\varepsilon\ge\mathbb{N}_{\ge\bar{M}_\varepsilon}$ and hence $|a_{n_\varepsilon,\varepsilon}-l_\varepsilon|<\rho_\varepsilon^q$. $\square$

Analogously, we have

Theorem 80. *Let $a:\mathbb{N}\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}$, with $a_n=[a_{n,\varepsilon}]$ for all $n\in\mathbb{N}$. Assume that for all ε*

$$\exists\sum_{n=0}^{+\infty}a_{n,\varepsilon}=:s_{\varepsilon}$$

and that $s=[s_{\varepsilon}]\in{}^{\rho}\widetilde{\mathbb{R}}$. Then

$$\exists\sigma\ \text{gauge}:\ s={}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}a_n.$$

Note that assumptions l, $s\in{}^{\rho}\widetilde{\mathbb{R}}$ are necessary by Rem. 76.(v).

Clearly, one would like to take $\sigma=\rho$ in several cases. As we saw above, this is not always possible. Here we list some remarks that could help in obtaining this result

(i) Assume that the sequence $a=(a_n)_{n\in\mathbb{N}}$ is given by a GSF, i.e. $a\in{}^{\rho}\mathcal{GC}^{\infty}(\mathbb{N},{}^{\rho}\widetilde{\mathbb{R}})$. Then the inequality

$$l_{\varepsilon}-\rho_{\varepsilon}^{q}<a_{n,\varepsilon}<l_{\varepsilon}+\rho_{\varepsilon}^{q}$$

could be solved for n by using the inverse function theorem for GSF (see [40]) and suitable increasing/decreasing properties of the inverse function. For example, $-\rho_{\varepsilon}^{q}<\frac{1}{n}<\rho_{\varepsilon}^{q}$ can be solve by considering that the inverse of $n\mapsto\frac{1}{n}$ is decreasing.

(ii) If $a\in{}^{\rho}\mathcal{GC}^{\infty}(\mathbb{N},{}^{\rho}\widetilde{\mathbb{R}})$ is given by $a=\alpha\circ\beta$, where α and β are GSF and $\beta(\mathbb{N})$ is a functionally compact subset of ${}^{\rho}\widetilde{\mathbb{R}}$. Then the function a takes minimum and maximum values m, M at points in $\beta(\mathbb{N})$. This could be useful to solve

$$l_{\varepsilon}-\rho_{\varepsilon}^{q}<a_{n,\varepsilon}=\alpha_{\varepsilon}(\beta_{\varepsilon}(n))<l_{\varepsilon}+\rho_{\varepsilon}^{q}$$

since $m_{\varepsilon}\le a_{n,\varepsilon}\le M_{\varepsilon}$ for all $n\in\mathbb{N}$ and for ε small.

8.4. **Properties of push-forward integral.** Applying the notion of hyperseries, we can show that the push-forward integral is hyper σ-additive:

Theorem 81. *Let $S\ne\emptyset$ be a compact subset of $\mathbb{R}^n$. Let $(\mathbb{R}^n,\mathcal{A},\mu)$ be a measure space such that $S\in\mathcal{A}$ and $\mu(S)$ is finite. Assume that $\varphi\in{}^{\rho}\mathcal{GC}^{\infty}(S,{}^{\rho}\widetilde{\mathbb{R}}^n)$ and $f\in{}^{\rho}\mathcal{GC}^{\infty}(\varphi(S),{}^{\rho}\widetilde{\mathbb{R}}^d)$. Finally, let $(B_n)_{n\in\mathbb{N}}$ be a pairwise disjoint sequence of $\varphi(\mathcal{A}\cap S)$ such that*

$$\varphi^{-1}(B_n)\in\mathrm{k}(\mathcal{A}\cap S)$$

$$\varphi^{-1}\left(\bigcup_{n\in\mathbb{N}}B_n\right)\in\mathrm{k}(\mathcal{A}\cap S). \tag{8.13}$$

Then there exists a gauge σ such that

$$\int_{\bigcup_{n\in\mathbb{N}}B_n}f\,\mathrm{d}(\varphi_*\mu)={}^{\rho}\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\int_{B_n}f\,\mathrm{d}(\varphi_*\mu)\,. \tag{8.14}$$

Proof. We have

$$\int_{\bigcup_{n\in\mathbb{N}}B_n} f\,\mathrm{d}(\varphi_*\mu) = \left[\int_{\varphi^{-1}\left(\bigcup_{n\in\mathbb{N}}B_n\right)} f_\varepsilon\circ\varphi_\varepsilon\,\mathrm{d}\mu\right] = \left[\int_{\bigcup_{n\in\mathbb{N}}\varphi^{-1}(B_n)} f_\varepsilon\circ\varphi_\varepsilon\,\mathrm{d}\mu\right] = \\ = \left[\sum_{n=0}^{+\infty}\int_{\varphi^{-1}(B_n)} f_\varepsilon\circ\varphi_\varepsilon\,\mathrm{d}\mu\right]. \tag{8.15}$$

Note that the ε-series appearing in (8.15) converges for the σ-additivity of the measure μ. Moreover, as in Rem. 8.1.(ii), assumption (8.13) guaranties that the resulting net is ρ-moderate. We can hence use Thm. 80 to introduce the new gauge σ and have

$$\int_{\bigcup_{n\in\mathbb{N}}B_n} f\,\mathrm{d}(\varphi_*\mu) = {}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\left[\int_{\varphi^{-1}(B_n)} f_\varepsilon\circ\varphi_\varepsilon\,\mathrm{d}\mu\right] = \\ = {}^{\rho}\!\!\sum_{n\in{}^{\sigma}\widetilde{\mathbb{N}}}\int_{B_n} f\,\mathrm{d}(\varphi_*\mu)\,.$$

□

For push-forward integrals, Fubini's theorem can be stated as follows

Theorem 82. *For $i=1,2$, let $S_i\ne\emptyset$ be a compact subset of $\mathbb{R}^{n_i}$ and $(\mathbb{R}^{n_i},\mathcal{A}_i,\mu_i)$ be a measure space such that $S_i\in\mathcal{A}_i$ and $\mu_i(S_i)$ is finite. Assume that $\varphi_i\in{}^{\rho}\mathcal{GC}^\infty(S_i,{}^{\rho}\widetilde{\mathbb{R}}^{n_i})$ and $f\in{}^{\rho}\mathcal{GC}^\infty(\varphi_1(S_1)\times\varphi_2(S_2),{}^{\rho}\widetilde{\mathbb{R}}^{d})$. Then for all $B_i\in\varphi_i(\mathcal{A}_i\cap S_i)$ such that $\varphi_i^{-1}(B_i)\Subset\mathrm{k}(\mathcal{A}_i\cap S_i)$, we have*

$$\int_{B_1\times B_2} f\,\mathrm{d}(\varphi_1\times\varphi_2)_*(\mu_1\times\mu_2) = \int_{B_1}\left(\int_{B_2} f(y_1,y_2)\,\mathrm{d}(\varphi_{2*}\mu_2)(y_2)\right)\mathrm{d}(\varphi_{1*}\mu_1)(y_1) \\ = \int_{B_2}\left(\int_{B_1} f(y_1,y_2)\,\mathrm{d}(\varphi_{1*}\mu_1)(y_1)\right)\mathrm{d}(\varphi_{2*}\mu_2)(y_2), \tag{8.16}$$

where $\varphi_1\times\varphi_2:(x_1,x_2)\in S_1\times S_2\mapsto(\varphi_1(x_1),\varphi_2(x_2))\in{}^{\rho}\widetilde{\mathbb{R}}^{n_1+n_2}$ and $\mu_1\times\mu_2$ is the product measure on $(S_1\times S_2,\mathcal{A}_1\times\mathcal{A}_2)$.

Proof. Directly from (8.6), from the equality $(\varphi_1\times\varphi_2)^{-1}(B_1\times B_2)=\varphi_1^{-1}(B_1)\times\varphi_2^{-1}(B_2)$ and the classical Fubini's theorem. □

Integration over multidimensional intervals can be accomplished as follows: let $S:=[0,1]^n_{\mathbb{R}}\subseteq\mathbb{R}^n$ and a_i, $b_i\in{}^{\rho}\widetilde{\mathbb{R}}$, with $a_i<b_i$ for all $i=1,\dots,n$. Set

$$\varphi:(x_1,\dots,x_n)\in S\mapsto(a_1+(b_1-a_1)x_1,\dots,a_1+(b_n-a_n)x_n)\in\prod_{i=1}^n[a_i,b_i].$$

Denote both with $\mathrm{d}x=\mathrm{d}x_1\dots\mathrm{d}x_n$ and with λ_n the Lebesgue measure on $\mathbb{R}^n$. Then we can define

$$\int_{\prod_{i=1}^n[a_i,b_i]} f\,\mathrm{d}x := \int_{\varphi(S)} f\,\mathrm{d}(\varphi_*\lambda_n)$$

so that we have

$$\int_{\prod_{i=1}^n[a_i,b_i]} f\,\mathrm{d}x = \left[\int_{[0,1]^n_{\mathbb{R}}} f\left(a_1+(b_1-a_1)x_1,\dots,a_1+(b_n-a_n)x_n\right)\mathrm{d}x_1\dots\mathrm{d}x_n\right].$$

Using (8.5) and Fubini's Thm. 82, we get

$$\int_{\prod_{i=1}^{n}[a_i,b_i]} f\,\mathrm{d}x = \int_{a_1}^{b_1} \mathrm{d}x_1 \dots \int_{a_n}^{b_n} f(x_1,\dots,x_n)\,\mathrm{d}x_n.$$

The change of variable formula for the push-forward integral can be tackled in two ways. In the first one, we can consider a situation where we have

$$S_1 \Subset \mathbb{R}^m, \quad S_2 \Subset \mathbb{R}^n$$
$$\varphi_1 \in {}^{\rho}\mathcal{GC}^\infty(S_1,S_2), \quad \varphi_2 \in {}^{\rho}\mathcal{GC}^\infty(S_2,{}^{\rho}\widetilde{\mathbb{R}}^n).$$

Starting with a measure space $(S_1,\mathcal{A}_1,\mu_1)$, we can push-forward it firstly by φ_1 obtaining the measure space $(\varphi_1(S_1),\varphi_1(\mathcal{A}_1\cap S_1),\varphi_{1*}\mu_1)$; we can hence push-forward it by φ_2 obtaining $(\varphi_2(\varphi_1(S_1)),\varphi_2(\varphi_1(\mathcal{A}_1\cap S_1)),\varphi_{2*}(\varphi_{1*}\mu_1))$, or we can directly push-forward the initial measure space by the composition to get $((\varphi_2\circ\varphi_1)(S_1),(\varphi_2\circ\varphi_1)(\mathcal{A}_1\cap S_1),(\varphi_2\circ\varphi_1)_*\mu_1)$. Clearly, the final results coincide and a direct calculation yields

$$\int_B f\,\mathrm{d}((\varphi_2\circ\varphi_1)_*\,\mu_1) = \int_B f\,\mathrm{d}(\varphi_{2*}\,(\varphi_{1*}\mu_1))$$

for all $B\in(\varphi_2\circ\varphi_1)(\mathcal{A}_1\cap S_1)$ such that $(\varphi_2\circ\varphi_1)^{-1}(B)\in \mathrm{k}(\mathcal{A}_1\cap S_1)$.

A second approach to the change of variable formula is stated as follows

Theorem 83. *Let S be a non empty compact subset of $\mathbb{R}^n$ and let $\varphi\in{}^{\rho}\mathcal{GC}^\infty(S,{}^{\rho}\widetilde{\mathbb{R}}^n)$ be an injective GSF. Denote by λ_n the Lebesgue measure on $\mathbb{R}^n$ and set $\mathrm{d}\mu_n(x) := |\det(D\varphi)(x)|\,\mathrm{d}\lambda_n(x)$. Then for all $A\in \mathrm{k}(\mathcal{A}\cap S)$, we have*

$$\int_{\varphi(A)} f\,\mathrm{d}\,(\varphi_*\mu_n) = \int_A f(\varphi(x))\,|\det(D\varphi)(x)|\;\mathrm{d}\lambda_n(x) = \left[\int_{\varphi_\varepsilon(A)} f_\varepsilon\,\mathrm{d}\lambda_n\right]. \tag{8.17}$$

Proof. By Def. 70, we obtain

$$\int_{\varphi(A)} f\,\mathrm{d}\,(\varphi_*\mu_n) = \int_{\varphi^{-1}(\varphi(A))} f\circ\varphi\,\mathrm{d}\mu_n.$$

But $\varphi^{-1}(\varphi(A)) = A$ because φ is an injective map. Therefore (8.2) yields

$$\begin{aligned}\int_{\varphi(A)} f\,\mathrm{d}\,(\varphi_*\mu_n) &= \int_A f(\varphi(x))\,|\det(D\varphi)(x)|\;\mathrm{d}\lambda_n(x) = \\ &= \left[\int_A f_\varepsilon(\varphi_\varepsilon(x))\,|\det(D\varphi_\varepsilon)(x)|\;\mathrm{d}\lambda_n(x)\right] = \\ &= \left[\int_{\varphi_\varepsilon(A)} f_\varepsilon\,\mathrm{d}\lambda_n\right].\end{aligned}$$

□

We finally want to prove suitable versions of the monotone and dominated convergence theorems.

Theorem 84. *Let $S\ne\emptyset$ be compact subsets of $\mathbb{R}^n$. Let $(\mathbb{R}^n,\mathcal{A},\mu)$ be a measure space such that $S\in\mathcal{A}$ and $\mu(S)$ is finite. Assume that $\varphi\in{}^{\rho}\mathcal{GC}^\infty(S,{}^{\rho}\widetilde{\mathbb{R}}^n)$ and*

$B \in \varphi(\mathcal{A} \cap S)$ is such that $\varphi^{-1}(B) \in \mathrm{k}(\mathcal{A} \cap S)$. Let f, $f_n \in {}^\rho\mathcal{GC}^\infty(\varphi(S), {}^\rho\widetilde{\mathbb{R}}^d)$ be GSF defined by (f_ε), $(f_{n,\varepsilon}) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^I$ resp. such that for ε small

$$f_{n,\varepsilon}(x) \le f_{n+1,\varepsilon}(x) \quad \forall x \in \mathbb{R}^n \, \forall n \in \mathbb{N}$$
$$f_\varepsilon(x) = \lim_{n\to+\infty} f_{n,\varepsilon}(x) \quad \forall x \in \mathbb{R}^n.$$

Then there exists a gauge σ such that

$$\int_B f \,\mathrm{d}(\varphi_*\mu) = {}^\rho\!\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \int_B f_n \,\mathrm{d}(\varphi_*\mu)\,. \tag{8.18}$$

Proof. We have

$$\int_B f \,\mathrm{d}(\varphi_*\mu) = \left[\int_{\varphi^{-1}(B)} f_\varepsilon \circ \varphi_\varepsilon \,\mathrm{d}\mu\right] = \left[\int_{\varphi^{-1}(B)} \lim_{n\to+\infty} f_{n,\varepsilon}(\varphi_\varepsilon(x)) \,\mathrm{d}\mu(x)\right].$$

The classical monotone convergence theorem and Thm. 79 yield

$$\int_B f \,\mathrm{d}(\varphi_*\mu) = \left[\lim_{n\to+\infty} \int_{\varphi^{-1}(B)} f_{n,\varepsilon} \circ \varphi_\varepsilon \,\mathrm{d}\mu\right] = {}^\rho\!\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \left[\int_{\varphi^{-1}(B)} f_{n,\varepsilon} \circ \varphi_\varepsilon \,\mathrm{d}\mu\right] =$$
$$= {}^\rho\!\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \int_B f_n \,\mathrm{d}(\varphi_*\mu)$$

for a suitable gauge σ.

Analogously, we can prove the dominated convergence theorem □

Theorem 85. *Let $S \ne \emptyset$ be compact subsets of $\mathbb{R}^n$. Let $(\mathbb{R}^n, \mathcal{A}, \mu)$ be a measure space such that $S \in \mathcal{A}$ and $\mu(S)$ is finite. Assume that $\varphi \in {}^\rho\mathcal{GC}^\infty(S, {}^\rho\widetilde{\mathbb{R}}^n)$ and $B \in \varphi(\mathcal{A} \cap S)$ is such that $\varphi^{-1}(B) \in \mathrm{k}(\mathcal{A} \cap S)$. Let g, f, $f_n \in {}^\rho\mathcal{GC}^\infty(\varphi(S), {}^\rho\widetilde{\mathbb{R}}^d)$ be GSF defined by (g_ε), (f_ε), $(f_{n,\varepsilon}) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^I$ resp. such that for ε small*

$$|f_{n,\varepsilon}(x)| \le |g_\varepsilon(x)| \quad \forall x \in \mathbb{R}^n \, \forall n \in \mathbb{N}$$
$$f_\varepsilon(x) = \lim_{n\to+\infty} f_{n,\varepsilon}(x) \quad \forall x \in \mathbb{R}^n.$$

Then there exists a gauge σ such that

$$\int_B f \,\mathrm{d}(\varphi_*\mu) = {}^\rho\!\lim_{n\in{}^\sigma\widetilde{\mathbb{N}}} \int_B f_n \,\mathrm{d}(\varphi_*\mu)\,. \tag{8.19}$$

Example 86. a nonstandard probability space?

8.5. **Connections with discontinuous calculus and integration over membranes.** In [3], Aragona, Fernandez and Juriaans introduced a differential calculus at generalized points which is based on a specific form of convergence of difference quotients. The name *discontinuous calculus* is used in [3] to identify this approach to differentiation of set-theoretical functions defined at and valued on generalized points. In the remaining part of the present section, we restrict our attention to the case $\rho_\varepsilon = \varepsilon$, the gauge that is used in standard Colombeau theory (as well as in [3, 5]), and hence we use the customary notation ${}^\rho\widetilde{\mathbb{R}} = \widetilde{\mathbb{R}}$.

First, we recall the definition from [3]:

Definition 87. A map f from some sharply open subset U of $\widetilde{\mathbb{R}}^n$ to $\widetilde{\mathbb{R}}^m$ is called differentiable at $x_0 \in U$ in the sense of discontinuous calculus with derivative $A \in L(\widetilde{\mathbb{R}}^n, \widetilde{\mathbb{R}}^m)$ if

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - A(x - x_0)|_e}{|x - x_0|_e} = 0, \tag{8.20}$$

where

$$v : (x_\varepsilon) \in \mathbb{R}^n_{(\varepsilon)} \mapsto \sup\{b \in \mathbb{R} \mid |x_\varepsilon| = O(\varepsilon^b)\} \in (-\infty, \infty]$$
$$|-|_{\mathrm{e}} : x \in \widetilde{\mathbb{R}}^n \mapsto \exp(-v(x)) \in [0, \infty).$$

It is worth noting that $|-|_{\mathrm{e}} : \widetilde{\mathbb{R}} \longrightarrow \mathbb{R}_{\geq 0}$ induces an ultrametric on $\widetilde{\mathbb{R}}$ that generates exactly the sharp topology, see e.g. [4, 38] and references therein. However, we will not use this ultrametric structure in the present paper.

The following result shows compatibility of this notion with the derivative in the sense of GSF. For its proof, see [40].

Lemma 88. *Let U be sharply open in $\widetilde{\mathbb{R}}^n$, let $x_0 \in U$ and suppose that $f \in \mathcal{GC}^\infty(U, \widetilde{\mathbb{R}}^m)$. Then f is differentiable in the sense of discontinuous calculus in x_0 with derivative $Df(x_0)$.*

The function i of Rem. 40 is trivially everywhere differentiable an arbitrary number of times in the sense of the discontinuous calculus because it is locally constant in the sharp topology. We can therefore state that the main difference between the notion of smooth function in the sense of [3] and the GSF described in the present work lies in all the classical results that do not hold for [3].

In [5], a multidimensional integration theory of Colombeau generalized functions was introduced. As we will see, it is not a theory of integration over suitable subsets of $\widetilde{\mathbb{R}}^n$, but on convenient equivalence classes of nets of subsets of $\mathbb{R}^n$ as they are introduced in the following

Definition 89. Let Ω be an ordinary open set of $\mathbb{R}^n$ and let $M_\varepsilon \subseteq \Omega$ for all ε. Then we say that (M_ε) is a *pre-membrane* if

(i) $\exists K \Subset \Omega \, \forall^0 \varepsilon : \ M_\varepsilon \subseteq K$;

(ii) M_ε is Lebesgue measurable for all ε sufficiently small.

Moreover, if (M_ε) and (M'_ε) are pre-membranes, we say that (M_ε) is equivalent to (M'_ε) if there exists a net $(z_\varepsilon) \in \mathcal{C}^\infty(\Omega, \mathbb{R}^n)^I$ that defines the zero GSF $\mathrm{c}(\Omega) \longrightarrow \widetilde{\mathbb{R}}^n$ and such that

$$\forall^0 \varepsilon : \ (1_{\mathbb{R}^n} + z_\varepsilon)(M_\varepsilon) = M'_\varepsilon,$$

where $1_{\mathbb{R}^n}$ is the identity function on $\mathbb{R}^n$. A *membrane* $\mathrm{m}(M_\varepsilon)$ is an equivalence class of pre-membranes modulo this equivalence relation.

In [5], it is proved that if (M_ε) is equivalent to (M'_ε), then they identify the same internal set and the same generalized Lebesgue measure:

$$[M_\varepsilon] = [M'_\varepsilon] \subseteq \widetilde{\mathbb{R}}$$
$$[\lambda_n(M_\varepsilon)] = [\lambda_n(M'_\varepsilon)] \in \widetilde{\mathbb{R}}. \tag{8.21}$$

Since the space $\mathcal{G}(\Omega)^n$ of Colombeau generalized functions can be identified with $\mathcal{GC}^\infty(\mathrm{c}(\Omega), \widetilde{\mathbb{R}}^n)$, see [41], we can finally say that in [5] it is proved that the following definition

$$\int_{\mathrm{m}(M_\varepsilon)} f \, \mathrm{d}\lambda_n := \left[\int_{M_\varepsilon} f_\varepsilon \, \mathrm{d}\lambda_n \right] \in \widetilde{\mathbb{R}} \tag{8.22}$$

is correct, i.e. it is independent from the net (M_ε) representing the membrane $\mathrm{m}(M_\varepsilon)$ and the net (f_ε) that defines the GSF $f \in \mathcal{GC}^\infty(\mathrm{c}(\Omega), \widetilde{\mathbb{R}}^n)$.

To understand the relations between integration over membranes and the push-forward integral, we can start by considering example B_ε in (8.1). Since (8.21) does not hold if $M_\varepsilon = [0,1]_{\mathbb{R}}$ and $M'_\varepsilon = B_\varepsilon$, we have that these two nets define the same internal set $[0,1]$, but they generate two different membranes. We can thus state that [5] is not a theory of integration over subsets of ${}^\rho\widetilde{\mathbb{R}}^n$. Moreover, there does not exist $S \Subset \mathbb{R}^n$ and $\varphi \in {}^\rho\mathcal{GC}^\infty(S, {}^\rho\widetilde{\mathbb{R}})$ such that $\varphi_\varepsilon(S) = B_\varepsilon$. In fact, the number of connected components of B_ε is unbounded for $\varepsilon \to 0^+$ whereas if $\varphi_\varepsilon(S) = B_\varepsilon$ then it should be bounded by the number of connected components of S. This means that we can integrate over the membrane generated by (B_ε) but this does not correspond to some type of push-forward integral. On the other hand, membranes are always constraint by being generated by bounded nets, because of condition (i) of Def. 89. In Def. 70 there is not this limitation. Finally, in the hypotheses of Thm. 83, if $\varphi_\varepsilon(A) \subseteq K$ for some $K \Subset \mathbb{R}^n$ and for ε small, then (8.22) and (8.17) yield

$$\int_{\mathrm{m}(\varphi_\varepsilon(A))} f \,\mathrm{d}\lambda_n = \int_{\varphi(A)} f \,\mathrm{d}\left(\varphi_* \mu_n\right) = \int_A f(\varphi(x)) \left|\det(D\varphi)(x)\right| \,\mathrm{d}\lambda_n(x).$$

(i) Thm: What are the relations between $\int_M f \in \mathbb{R}$ for $f \in \mathcal{C}^0(\Omega)$, $M \Subset \Omega$ and $\int_M \iota^b_M(f) \in {}^\rho\widetilde{\mathbb{R}}$?
(ii) find the relations between this integral of GSF and the integration of compactly supported distributions introduced by Schwartz or better : Chinnaraman - Integrable distributions and phi-Fourier transform
(iii) state and prove [48, Thm. 1.2.63], i.e. $\int_\Omega \iota(T)(x)\iota(\varphi)(x)\,\mathrm{d}x = \langle T, \varphi\rangle$ in ${}^\rho\widetilde{\mathbb{R}}$ using the push-forward integral.

9. Generalized smooth functions as smooth functionals

(i) every GSF defines a continuous functional ${}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d) \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ acting as a density in integration
(ii) Thm 1.2.63 of [48]: the functional induced by a distribution
(iii) the functionals induced by a derivative $\partial^\alpha f$
(iv) Def. of **internal** smooth functional (in the diffeological sense):
  (i) $\varphi : V^\bullet \longrightarrow {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ is smooth iff $\varphi^\vee \in {}^\rho\mathcal{GC}^\infty(V^\bullet \times K, {}^\rho\widetilde{\mathbb{R}})$
  (ii) $T :^\rho \mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d) \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is smooth iff for all $\varphi : V^\bullet \longrightarrow {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ smooth, $T \circ \varphi \in {}^\rho\mathcal{GC}^\infty(V^\bullet, {}^\rho\widetilde{\mathbb{R}})$
(v) Def. of derivative of an internal smooth functional
(vi) Thm. Every derivative of an internal smooth functional is smooth
(vii) Thm. Every **internal** smooth functional is defined by a GSF using integration (we can try $f(x) = T(\delta_x)$ for each x in some domain)
(viii) what about preservation of $\mathcal{D}'$ by diffeomorphisms thinking both distributions and GSF as functionals?

10. Generalized smooth functions and differential equations

- only statements of Banach fixed point theorem and Picard-Lindelöf theorem

11. Applications to Mathematical Physics

(i) M manifold (or, more generally, a difffeological space)
(ii) $\varphi : U \longrightarrow \mathbb{R}^k$ is a *moderate observable* iff
 (i) $U \subseteq M$ is an open set and $\varphi : U \longrightarrow \mathbb{R}^k$ is a smooth map
 (ii) whenever we take a net of compact subsets (K_ε), $K_\varepsilon \Subset M$, such that $\bigcup_\varepsilon K_\varepsilon = U$, we have that $(\sup \varphi|_{K_\varepsilon}) \in \mathbb{R}^k_\rho$
(iii) Let (x_ε) be a net of M, then we say $(x_\varepsilon) \in M_\rho$ iff for every moderate observable $\varphi : U \longrightarrow \mathbb{R}^k$ the following implication holds

$$\left(\forall^0\varepsilon :\ x_\varepsilon \in U\right) \quad \Longrightarrow \quad (\varphi(x_\varepsilon)) \in \mathbb{R}^k_\rho.$$

(iv) Let (x_ε), $(y_\varepsilon) \in M_\rho$, then we say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ iff for every moderate observable $\varphi : U \longrightarrow \mathbb{R}^k$ the following conditions holds
 (i) $\left(\forall^0\varepsilon :\ x_\varepsilon \in U\right) \iff \left(\forall^0\varepsilon :\ y_\varepsilon \in U\right)$
 (ii) $\left(\forall^0\varepsilon :\ x_\varepsilon \in U\right) \quad \Longrightarrow \quad (\varphi(x_\varepsilon)) \sim_\rho (\varphi(y_\varepsilon))$
(v) Conjecture 1: Let $\rho_\varepsilon = \varepsilon$, let (M, h) be a Riemannian manifold and (x_ε) be a compactly supported net in (M, h), then $(x_\varepsilon) \in M_\rho$
(vi) Conjecture 2 (see [48, Lem. 3.2.6]): Let $\rho_\varepsilon = \varepsilon$, let (M, h) be a Riemannian manifold and (x_ε), (y_ε) be compactly supported nets in (M, h), then we have

$$(x_\varepsilon) \sim (y_\varepsilon) \iff (x_\varepsilon) \sim_\rho (y_\varepsilon).$$

In the following we are going to review two applications of GSF in a geometric context both of which rely on tools from semi-Riemannian geometry in the non-smooth setting. The first one is concerned with the Cauchy problem for the wave equation on Lorentzian manifolds while the second one deals with Lorentzian manifolds of low regularity which serve as general relativistic models of very short but violent bursts of gravitational radiation.

To begin with we review pseudo-Riemannian geometry in GSF. Since in all applications in geometry so far the freedom provided by the asymptotic gauge has not been put to use we set from now on $i = (\varepsilon)_\varepsilon$ and mainly use def by nets on fixed set to define generalized functions and sections.

insert discussion of pseudo-Riemannian geometry in GSF

11.1. The wave equation on non-smooth spacetimes.

Given a generalized Lorentzian metric g on M, we are interested in the initial value problem for the wave operator of g, i.e.

$$\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu = \sqrt{|\det g|}\frac{\partial}{\partial x^\mu}\left(\sqrt{|\det g|}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right). \tag{11.1}$$

We will first review the local existence result of [46] and then turn to its globalization achieved in [57]. To begin with we need to put some asymptotic conditions on g using some background Riemannian metric m on M. A generalized metric will be called *weakly singular* if the following condition (A) holds:

(A) $\forall K \Subset M$, $\forall k \in \mathbb{N}_0$ for all smooth vector fields $\eta_1, \ldots, \eta_k$, and for any representative $(g_\varepsilon)_\varepsilon$ we have:

$$\sup_K \|\mathscr{L}_{\eta_1} \ldots \mathscr{L}_{\eta_k} g_\varepsilon\|_m = O(\varepsilon^{-k}) = \sup_K \left\|\mathscr{L}_{\eta_1} \ldots \mathscr{L}_{\eta_k} g_\varepsilon^{-1}\right\|_m \quad (\varepsilon \to 0).$$

Now with a view to formulating the local Cauchy problem we consider a local foliation of M given by the level sets of some non-singular smooth function $t \in \mathcal{C}^\infty(U)$, where $U \subseteq M$ is open and relatively compact. We will suppose the level sets $\Sigma_\tau = \{q \in U : t(q) = \tau\}$ to be uniformly space-like with respect to ε with an additional uniform bound on the covariant derivative of the normal form which essentially is a condition on the second fundamental form of the level sets. More precisely, we demand:

(B) Each $p \in M$ possesses a neighborhood U on which there exists a *local time-function*, that is a smooth function t with uniformly timelike differential $dt =: \sigma$, i.e.,

$$g_\varepsilon^{-1}(\sigma, \sigma) \leq -C < 0 \quad \text{for some positive constant } C \tag{11.2}$$

and one (hence any) representative g_ε and all small ε. In addition, we have for all $K \Subset U$ that

$$\sup_K \|\nabla^\varepsilon \sigma\|_m = O(1) \quad (\varepsilon \to 0), \tag{11.3}$$

where ∇ is the covariant derivative of g.

We remark that (the regularizations) of several interesting non-smooth geometries such as conical spacetimes and impulsive gravitational waves in Rosen form satisfy conditions (A) and (B), see [57, Rem. 5.1].

To formulate the local Cauchy problem let $p \in \Sigma_0 =: \Sigma$ and write ξ for the normal vector field of the level sets Σ_τ defined form the normal covector field σ by $\sigma = g_\varepsilon(\xi_\varepsilon, .)$. Also we need the corresponding normalized versions $\widehat{\xi} = [(\widehat{\xi}_\varepsilon)_\varepsilon] = [(\xi_\varepsilon/V_\varepsilon)_\varepsilon]$ and $\widehat{\sigma} = [(\widehat{\sigma}_\varepsilon)_\varepsilon] = g(\widehat{\xi}, \cdot)$, where we have set $V_\varepsilon^2 = -g_\varepsilon(\xi_\varepsilon, \xi_\varepsilon)$. We now consider the initial value problem

$$\Box u = 0 \quad \text{on } U, \quad \text{and} \quad u = u_0, \quad \nabla_{\widehat{\xi}} u = u_1 \quad \text{on } \Sigma, \tag{11.4}$$

where the initial data u_0, u_1 are supposed to be in $\mathcal{G}(\Sigma)$ and look for a local solution $u \in \mathcal{G}$ on U or an open subset thereof.

The general strategy to solve differential equations in GSF is the following: First, solve the equation for fixed ε in the smooth setting and form the net $(u_\varepsilon)_\varepsilon$ of smooth solutions. This will be a candidate for a solution in $\mathcal{G}$, but particular care has to be taken to guarantee that the u_ε share a common domain of definition. hmm, this now changes, does it? In the second step, one shows that the solution candidate $(u_\varepsilon)_\varepsilon$ is a moderate net P: we need to prove less now, see Def. 13, i.e. "moderate at each point", hence obtaining existence of a solution $[(u_\varepsilon)_\varepsilon]$ in $\mathcal{G}$. Finally, to obtain uniqueness of solutions, one has to prove that changing representatives of the data leads to a solution that is still in the class $[(u_\varepsilon)_\varepsilon]$. Note that this amounts to an additional stability of the equation with respect to negligible perturbations of the initial data and right hand side.

Indeed in the present situation the classical theory of the wave equation on Lorentzian manifolds [28, 6] be more precise here provides us with a solution candidate under the following mild additional assumption:

(C) For each $p \in \Sigma$ there exists a neighborhood $V \subseteq U$ and a representative $(g_\varepsilon)_\varepsilon$ of the metric g on V such that V is, for each ε, an RCCSV-neighborhood in (M, g_ε) with $\Sigma \cap V$ a spacelike Cauchy hypersurface for V.

Here an RCCSV-neighborhood V (for *relatively compact causal with small volume*) is relatively compact and so small that $\mathrm{vol}(\overline{V}) \cdot \|K_\pm\|_{\mathcal{C}^0(\overline{V} \times \overline{V})} < 1$, where $K_\pm$ gives the failure by which the approximate fundamental solution constructed from the Hadamard coefficients deviates from a true fundamental solution, for details see e.g. [57, Sec. 2]. We may now state the local result.

Theorem 90 (Local existence)**.** *Let (M, g) be a generalized space-time with a weakly singular metric and assume that conditions (B) and (C) Then, for each $p \in \Sigma$ there exists an open neighborhood Ω such that for all compactly supported $u_0, u_1 \in \mathcal{G}(\Sigma \cap \Omega)$, the initial value problem (11.4) has a unique solution u in $\mathcal{G}(\Omega)$.*

The core of the proof of Th. 90 consists of higher order energy estimates (introduced in [119]) which eventually prove that the solution candidate is moderate P: to change and establish uniqueness.

We next extend Th. 90 to a global result. Analogous to the smooth situation we have to impose (a generalized version) of the key property of *global hyperbolicity*, which we define via a suitable metric splitting. cite bernal, sanchez?

Definition 91. Let g be a generalized Lorentz metric on the smooth $(n+1)$-dimensional manifold M. We say that (M, g) allows a *globally hyperbolic metric splitting* if there exists a $\mathcal{C}^\infty$-diffeomorphism $\psi\colon M \to \mathbb{R} \times S$, where S is an n-dimensional smooth manifold such that the following holds for the pushed forward generalized Lorentz metric $\lambda := \psi_* g$ on $\mathbb{R} \times S$:

(i) There is a representative $(\lambda_\varepsilon)_{\varepsilon \in I}$ of λ such that every λ_ε is a Lorentz metric and each slice $\{t_0\} \times S$ with arbitrary $t_0 \in \mathbb{R}$ is a (smooth, spacelike) Cauchy hypersurface for every λ_ε $(\varepsilon \in I)$.

(ii) We have the metric splitting of λ in the form

$$\lambda = -(b_\varepsilon) dt^2 + h,$$

where $h \in \Gamma_{\mathcal{G}}(\mathrm{pr}_2^*(T^0_2 S))$ is a t-dependent generalized Riemannian metric (see [57, Def. 4.2]) and $(b_\varepsilon) \in \mathcal{G}(\mathbb{R} \times S)$ is globally bounded and *locally uniformly positive*, i.e., for some (hence any) representative (β_ε) of (b_ε) and for every $K \subset\subset \mathbb{R} \times S$ we can find a constant $C > 0$ such that $\beta_\varepsilon(x) \geq C$ holds for small $\varepsilon > 0$ and $x \in K$.

(iii) For every $T > 0$ there exists a representative (h_ε) of h and a smooth complete Riemannian metric ρ on S which uniformly bounds h from below in the following sense: for all $t \in [-T, T]$, $x \in S$, $v \in T_x S$, and $\varepsilon \in I$

$$(h_\varepsilon)_t(v, v) \geq \rho(v, v).$$

check notation!

Non-trivial examples of such generalized space-times are e.g. Robertson-Walker space-times with a generalized warping function, see [57, Ex. 1].

To simplify notations we will suppress the diffeomorphism providing the metric splitting and assume that $M = \mathbb{R} \times S$ and $g = \lambda$ with S and λ as in the statement of Def. 91. Thus, the generalized space-time is represented by a family of globally hyperbolic space-times (M, g_ε) such that $S \cong \{0\} \times S$ is a Cauchy hypersurface for every g_ε $(\varepsilon \in I)$ and we consider the Cauchy problem

$$\Box u = 0 \quad \text{on } M, \quad \text{and} \quad u = u_0, \quad \nabla_{\widehat{\xi}} u = u_1 \quad \text{on } S. \tag{11.5}$$

Here the unit normal vector field of S is given by $\widehat{\xi} = \frac{1}{\sqrt{\beta}} \partial_t$ and the initial data u_0, u_1 are assumed to belong to $\mathcal{G}(S)$ and to have compact supports, e.g., arising by embedding distributional data from $\mathcal{E}'(S)$.

In this situation classical smooth theory (e.g. [57, Th. 2.7]) provides us with a solution candidate even without assuming condition (C). Also parts of condition (B) are consequences of the metric splitting: the existence of a suitable (in this case even global) foliation is a consequence of the globally hyperbolic metric splitting. Indeed we globally have $g_\varepsilon^{-1}(dt,dt) = -1/\beta_\varepsilon \leq -C < 0$ for some positive C, which implies (11.2). The second asymptotic boundedness condition (11.3) in (B), i.e., $\sup_K \|\nabla^\varepsilon dt\|_m = O(1)$, now simply reads

$$\|d\beta_\varepsilon\|_m = O(1) \qquad (\varepsilon \to 0) \tag{11.6}$$

uniformly on compact sets. Now the main result of this section is:

Theorem 92 (Global existence)**.** *Let (M,g) be a generalized space-time with a weakly singular metric admitting a globally hyperbolic metric splitting and assume that condition (11.6) holds. check Then the Cauchy problem (11.5) has a unique solution $u \in \mathcal{G}(M)$ for all compactly supported $u_0, u_1 \in \mathcal{G}(S)$.*

11.2. **Geodesics in impulsive gravitational waves.** Impulsive gravitational waves are exact spacetimes of general relativity that model short but intense bursts of gravitational radiation. Their simplest class, *impulsive pp-waves*, has been introduced by Penrose (see e.g. [90]) using the following distributional form of the metric

$$ds^2 = f(x,y)\delta(u)du^2 - 2dudv + dx^2 + dy^2. \tag{11.7}$$

Here t,x,y,z are the usual global coordinates in Minkowski space and the null coordinates u and v are given by $u = 1/\sqrt{2}(t-z)$ and $v = 1/\sqrt{2}(t+z)$, while f is some smooth function of the 'transversal variables' x and y which span the 'wave surface'. This form of the metric gives a clear insight to the situation: The spacetime is Minkowski space off the null hypersurface $\{u = 0\}$ which is the history of the plane wave impulse traveling with the speed of light in the z-direction and all curvature is concentrated there. This, however, comes at the price of introducing a distributional coefficient in the metric which leads out of the Geroch–Traschen class [31] of metrics ($W^{2,1}_{\text{loc}} \cap L^\infty_{\text{loc}}$), which guarantees existence and stability of the curvature in classical linear distribution theory. (We remark that this class is consistently contained in the generalized setting, see [110].) However, due to its simple geometrical structure the metric (11.7) nevertheless allows us to calculate the curvature as a distribution.

On the other hand impulsive pp-waves have also been described by a continuous metric [90] which in the special case of a plane wave, i.e., $f(x,y) = x^2 - y^2$ is of the form

$$ds^2 = (1+u_+)^2 dX^2 + (1-u_+)^2 dY^2 - 2dudV. \tag{11.8}$$

Here u_+ is the kink function $u_+ := 0$ ($u \leq 0$), and $u_+ = u$ ($u \geq 0$). Although both metrics (11.7) and (11.8) describe the same physical model the transformation relating them can only be given formally by

$$x = 1/\sqrt{2}\, X(1+u_+), \quad y = 1/\sqrt{2}\, Y(1-u_+), \tag{11.9}$$

$$v = V + \frac{X^2}{2}(u_+ + \theta(u)) + \frac{Y^2}{2}(u_+ - \theta(u)), \tag{11.10}$$

where θ is the Heaviside function. Clearly, a mathematically sound treatment of the transformation (11.9) is a delicate matter. However, this has been achieved in [73] using GSFP: CGF right? But of course (11.9) has a clear meaning also in the framework of GSF.... More precisely, the 'discontinuous coordinate change' could be shown to be the distributional limit of a 'generalized diffeomorphism', a concept further studied in [26, 27]. The key to these results was a nonlinear distributional analysis of the geodesics in the metric (11.7), and, in particular, an existence and uniqueness result for the geodesic equation in nonlinear generalized functions ([72]), which of course also is of independent interest since it allows for a clear geometric understanding of these distributional spacetimes.

This suggests that *the* key to understanding the physically more relevant but also more complicated classes of impulsive gravitational waves, such as *nonexpanding impulsive waves propagating on a background spacetime of constant curvature*

$$ds^2 = \frac{2\,d\eta\,d\bar\eta - 2\,d\mathcal{U}\,d\mathcal{V} + 2H(\eta,\bar\eta)\,\delta(\mathcal{U})\,d\mathcal{U}^2}{[\,1+\frac{1}{6}\Lambda(\eta\bar\eta - \mathcal{U}\mathcal{V})\,]^2}\,, \tag{11.11}$$

explain coordinates or their *expanding* counterpart whose distributional form can only be given formally ([91]) by

$$\text{insert } \delta^2\text{-monster} \tag{11.12}$$

one has to study the geodesic equation in a GSF-setting.

Indeed the methods to tackle this problem goes back to [72] and have subsequently been refined, see e.g. [96, 98]. In a first step one replaces the distributional metric by a generalized one which is represented by a line element containing a general regularization of the Dirac-delta, say a strict delta net $(\delta_\varepsilon)_\varepsilon$. E.g. in the simple case (11.7) we have $g = [(g_\varepsilon)_\varepsilon]$ with g_ε given by the line element

$$ds_\varepsilon^2 = f(x,y)\delta_\varepsilon(u)du^2 - 2dudv + dx^2 + dy^2. \tag{11.13}$$

Then one considers the geodesic equation for this generalized metric in GSF. Observe that these are nonlinear ODEs locally given by

$$\ddot x_\varepsilon^i(t) = \Gamma^i_{jk}(x(t))\dot x^j(t)\dot x^k(t) \tag{11.14}$$

use better notation and explain! Observe that such an equation due to its nonlinearity cannot even be written down in distributions and that only a nonlinear theory of generalized functions can provide a valid solution concept.

Now, according to the strategy explained in the previous subsection, one has to solve the smooth equations for fixed ε. In all the models introduced above it turns out that the key is to solve the 'transversal part' which in the simpler cases takes the form

$$D^{(N)}_{\dot x_\varepsilon}\dot x_\varepsilon = \frac{1}{2}\nabla_x f(x_\varepsilon)\delta_\varepsilon. \tag{11.15}$$

explain! Observe that for fixed ε the equations are smooth and hence a local solution is guaranteed to exist. However, the time of existence might depend on ε and in principle could even shrink to zero as $\varepsilon \to 0$. So the main objective here is to provide a result which guarantees that the time of existence is independent of ε, and large enough such that the solutions pass through the support of δ_ε since then they (re)enter the flat region 'behind' the wave and can be continued to all (positive) values of their parameter. To prove this, one employs a fixed point argument to the solution operator which typically results in a statement of the form:

Theorem 93 (Complete geodesics)**.** *The initial problem for the geodesic equation in the regularized spacetime (11.13) possess a global solution provided ε is small enough as compared to the initial data.*

Such a result provides the basis for proving global existence and uniqueness of geodesics in the generalized spacetimes modelling the distributional impulsive wave. It has been explicitly carried out in case of impulsive *pp*-waves [72], impulsive *Np*-waves [97], gyratonic *pp*-waves [92] and nonexpanding impulsive waves on a cosmological background [98].

However, observe that due to the fact that the smallness condition for ε depends on the data of the individual geodesic such results do not lead to a completeness result for the regularized spacetimes (11.13) for any $\varepsilon > 0$. A way to remedy this flaw within GSF has been found in [97]. Indeed the following definition of geodesic completeness for generalized spacetimes is very natural:

Definition 94. Let $g \in \mathcal{G}^2_0(M)$ be a generalized semi-Riemannian metric. Then the generalized space-time (M, g) is said to be *geodesically complete* if every geodesic γ is defined for all values of its parameter.

Now by the very definition geodesics completeness for the above generalized spacetimes follows from the respective global existence results.

12. The Groethendieck topos of sets of generalized functions

As we argued in the introduction, function spaces and Cartesian closedness are considered by many authors as important features for Mathematical Physics. Therefore, a good theory of nonlinear generalized functions for Mathematical Physics should strive for an extension to function spaces, possibly in a Cartesian closed category. As stated above, Colombeau's theory of generalized functions can be extended to any locally convex space E, (see [**?**, **?**]). The theory has also been extended to finite dimensional manifolds and to the study of differential geometry with singularities, see [71, 76, **?**, 75, 74, 70]. On the other hand, in [77] (p. 2) it is stated that: *"locally convex topology is not appropriate for non-linear questions in infinite dimensions"*, and indeed a different approach to infinite dimensional spaces is to embed smooth manifolds into a Cartesian closed category $\mathcal{C}$ (see [35] for a review of this type of approaches). A theory of generalized functions framed in a Cartesian closed category $\mathcal{C}$ embedding smooth manifolds could be applicable also to function spaces like $\mathcal{C}(M, N)$, where M, N are smooth manifolds (not necessarily compact). Similar lines of thought can be found in [**?**, 67, 68], but where generalized functions are seen as generalized smooth functionals, hence not following Cauchy-Dirac's original conception but Schwartz-Lawvere's conception instead.

The ideas used in this section arise from analogous ideas about diffeological spaces and Frölicher spaces (see [60, 7, 29, 35] and below). For these reasons, in this section we will not present the proofs of some elementary facts; these can easily be found in [77, 60, 29, 7, 15, 35, 32]. One of the main future aims of the present work is to generalize the construction $\langle - \rangle : (A_\varepsilon) \in \mathcal{P}(\mathbb{R}^n)^I \mapsto \langle A_\varepsilon \rangle \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ so as to obtain a functor $\langle - \rangle$ defined for nets of diffeological spaces. The first problem is hence the definition of a suitable domain and codomain of this functor. We will proceed as in [36, 35, 32]: we abstract from the category $\mathcal{O}\mathbb{R}^\infty$ of open sets and ordinary smooth functions introducing the notion of *category of figures* (but with *two* topologies instead of only one, as in [36, 35, 32]; in our case these topologies are the sharp and the Fermat ones). Intuitively, a category of figures represents the well known notion of smooth objects and smooth functions. Therefore, we firstly prove that generalized smooth maps form a category ${}^\rho\mathcal{GC}^\infty$ of figures. The category

Dlg of diffeological spaces is obtained from $\mathcal{O}\mathbb{R}^\infty$ by a general procedure ([35] but also [7]) that we will abstract in the notion of *Cartesian closure*. This construction allows to start from a category of figures $\mathcal{F}$ and to obtain a Cartesian closed category $\bar{\mathcal{F}}$ embedding $\mathcal{F}$. For example $\mathbf{Dlg} = \overline{\mathcal{O}\mathbb{R}^\infty}$. We thus set $\mathcal{G}\mathbf{Dlg} := \overline{{}^i\mathcal{G}\mathcal{C}^\infty}$ and prove that it is, like **Dlg** ([7]), a quasi-topos.

We hence start from the definition of category of figures (with two topologies) and of Cartesian closure.

Definition 95. *We say that* $\mathcal{F}$ *is a* category of figures with two topologies *if:*

(i) $\mathcal{F}$ is a category.
(ii) Every object $(U, \tau^{\mathrm{c}}, \tau^{\mathrm{f}}) \in \mathcal{F}$ is given by a set U and two topologies $\tau^{\mathrm{c}} \subseteq \tau^{\mathrm{f}}$.
(iii) Every arrow of $\mathcal{F}$ is continuous with respect to the finer topology τ^{f}, and contains all the constant maps $c : H \longrightarrow X$ and all the open subspaces $U \subseteq H$ (with the induced topologies) of every object $H \in \mathcal{F}$. The corresponding inclusion $i : U \hookrightarrow H$ is also an arrow of $\mathcal{F}$, i.e. $i \in \mathcal{F}_{UH} := \mathcal{F}(U, H)$.

In the following, we denote by $|-| : \mathcal{F} \longrightarrow \mathbf{Set}$ the forgetful functor $|(U, \tau^{\mathrm{c}}, \tau^{\mathrm{f}})| := U$, and with $\mathbf{Top}H^{\mathrm{f}}$ and $\mathbf{Top}H^{\mathrm{c}}$ the two topologies of H; $(U \prec H) = (U, \tau^{\mathrm{c}} \cap U, \tau^{\mathrm{f}} \cap U)$ is the subspace of H induced on the open set $U \in \mathbf{Top}H$ with the induced topologies. The remaining assumptions on $\mathcal{F}$ are the following:

(iv) The category $\mathcal{F}$ is closed with respect to restrictions to open sets of the finest topology, that is if $f \in \mathcal{F}_{HK}$, $U \in \mathbf{Top}H^{\mathrm{f}}$, $V \in \mathbf{Top}K^{\mathrm{f}}$ and finally $f(U) \subseteq V$, then $f|_U \in \mathcal{F}(U \prec H, V \prec K)$;
(v) Every topological space $H \in \mathcal{F}$ has the following "sheaf property": let H, $K \in \mathcal{F}$ be two objects of $\mathcal{F}$, $(H_i)_{i\in I}$ an open cover of H with respect to the coarsest topology $\mathbf{Top}H^{\mathrm{c}}$ and $f : |H| \longrightarrow |K|$ a map such that

$$\forall i \in I : f|_{H_i} \in \mathcal{F}(H_i \prec H, K),$$

then $f \in \mathcal{F}_{HK}$.

Generally speaking, we can think of $\mathcal{F}$ as a category of *types of figures*. Always considering the case $\mathcal{F} = \mathcal{O}\mathbb{R}^\infty$, we can also think of $\mathcal{F}$ as a category which represents *the well known notion of regular space and regular function*: with the Cartesian closure $\bar{\mathcal{F}}$, we want to extend this notion to a more general type of space (e.g. spaces of mappings). $\mathcal{F} = \mathbf{O}\mathbb{R}^n$, the category having as objects open sets $U \subseteq \mathbb{R}^u$ (with the induced topology), for some $u \in \mathbb{N}$ depending on U, and with hom-set the usual space $\mathcal{C}^n(U, V)$ of $\mathcal{C}^n$ functions between the open sets $U \subseteq \mathbb{R}^u$ and $V \subseteq \mathbb{R}^v$, is a category of figures. In general, what type of category $\mathcal{F}$ we have to choose depends on the setting we need: e.g., in case we want to consider manifolds with boundary, we have to take the analogue of the above mentioned category $\mathbf{O}\mathbb{R}^n$, but having as objects sets of the type $U \subseteq \mathbb{R}^u_+ = \{x \in \mathbb{R}^u \mid x_u \geq 0\}$.

The basic idea we will use to define a smooth space X, which faithfully generalizes the notion of manifold but includes also function spaces like $\mathbf{Man}(M, N)$, is to substitute the notion of chart by a family of mappings $d : H \longrightarrow X$ of type $H \in \mathcal{F}$. Indeed, for $\mathcal{F} = \mathcal{O}\mathbb{R}^\infty$ these mappings are of type $d : U \longrightarrow X$ with U open in some $\mathbb{R}^u$, thus they can be thought of as u-dimensional smooth figures on X. The idea is that X can be regarded as a support set together with the specification of all the finite-dimensional figures on the space itself.

Definition 96. If X is a set, then we say that $(\mathcal{D}, X)$ is an object of $\bar{\mathcal{F}}$ (or simply an $\bar{\mathcal{F}}$-object) if $\mathcal{D} = \{\mathcal{D}_H\}_{H\in\mathcal{F}}$ is a family with

$$\mathcal{D}_H \subseteq \mathbf{Set}(|H|, X) \quad \forall H \in \mathcal{F}.$$

We indicate by the notation $\mathcal{F}_{JH} \cdot \mathcal{D}_H$ the set of all the compositions $f \cdot d$ of functions $f \in \mathcal{F}_{JH}$ and $d \in \mathcal{D}_H$. The family $\mathcal{D}$ has finally to satisfy the following conditions:

(i) $\mathcal{F}_{JH} \cdot \mathcal{D}_H \subseteq \mathcal{D}_J$.

(ii) $\mathcal{D}_H$ contains all the constant maps $d : |H| \longrightarrow X$.

(iii) Let $H \in \mathcal{F}$, $(H_i)_{i\in I}$ an open cover of H and $d : |H| \longrightarrow X$ a map such that $d|_{H_i} \in \mathcal{D}_{(H_i \prec H)}$, then $d \in \mathcal{D}_H$.

Finally, we set $|(\mathcal{D}, X)| := X$ to denote the underlying set of the space $(\mathcal{D}, X)$.

Theorem 97. *${}^\rho\mathcal{GC}^\infty$ is a a category of figures with two topologies.*

Proof. to do □

Theorem 98. *The Cartesian closure $\mathcal{G}\mathbf{Dlg} := \overline{{}^i\mathcal{GC}^\infty}$ of ${}^\rho\mathcal{GC}^\infty$ is a quasi-topos.*

Proof. to do □

13. Future perspectives

Schwartz solved the problem "how to derive continuous functions?". Also Sebastiao e Silva (see XXX) solved the same problem without using nothing of functional analysis but using only a formal approach and arriving at an isomorphic solution. We solved the problem: "how to derive continuous functions obtaining set-theoretical functions valued in a non-Archimedean ring, composable, extending the usual classical theorems of calculus and such that, e.g., $\delta(0)$ is an infinite?". This second problem doesn't appear to have a trivial formal solution.

- it's a good framework for mathematical physics (see e.g. papers by Marsden on singular mechanics with Schwartz distributions)

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