A Picard-Lindelöf theorem for singular nonlinear PDE

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What do we need to prove a Picard-Lindelöf thm for PDE?

Cauchy problem for normal PDE:

$$\begin{cases} \partial_t^L y = G\left(t, x, \left(\partial_t^j \partial_x^\alpha y\right)_{\substack{j<L \\ |\alpha|+j\le L}}\right) & \\ \partial_t^j y(0,x) = y_j(x) & 0 \le j < L \end{cases} \qquad \text{(e.g. } G \in \mathcal{D}'\text{)}$$

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- "It rarely happens that the r.h.s. of the PDE is Lipschitz"

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- If h is an infinitesimal, the ODE $y'(1+y)h = -t$, $y(0) = 0$ has solution defined only in $\left[-\sqrt{h}, \sqrt{h}\right]$

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- We need a good notion of compact set for GF and Fréchet-like spaces of GF with complete norms

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7. $B_\rho(x)=\{y\in{}^\rho\widetilde{\mathbb{R}}^n\mid|y-x|<\rho\}$ for $\rho>0$ generate the *sharp topology* $\ni$ sharply open sets

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The idea: Set an infinitesimal $\sigma = [\sigma_\varepsilon]$ in a Gaussian density $N(x, \sigma)$: we get a net $(N(-, \sigma_\varepsilon))$ of smooth functions...

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See: Giordano, Kunzinger, Vernaeve, "Strongly internal sets and GSF", JMAA 2015

Some properties of GSF

- Schwartz's distributions are embedded and smooth functions preserved (depending on a Colombeau mollifier and on an infinite number to choose w.r.t. the properties we need)
- GSF are freely closed with respect to composition
- Derivatives can be computed using a GSF that works as a generalized incremental ratio
- One-dimensional integral calculus using primitives
- Classical theorems: intermediate value, (integral) mean value, Taylor's formulas, extreme value theorem, local and global inverse and implicit function theorems (see M. Kunzinger's talk)
- Generalized sheaf property in the sharp topology
- Multidimensional integration with generalized σ-additivity, monotone and dominated convergence theorems
- ODE: Banach fixed point, Picard-Lindelöf, maximal set of existence, Gronwall, flux, continuous dependence on initial conditions, rel. with classical solutions...
- Calculus of variations: Fundamental Lemma, second variation and minimizers, necessary Legendre condition, Jacobi fields, Conjugate points and Jacobi's theorem, Noether's theorem

Links to other theories of GF (Colombeau, Vernaeve, Aragona et al.)

Theorem (link to Colombeau)

Let $\Omega \subseteq \mathbb{R}^n$ *be an open set. Let* $\mathcal{G}^s(\Omega)$ *be the special Colombeau algebra. Let* $\mathcal{G}^s_{\text{maps}}(\Omega, \mathbb{R}^d) := \left\{ [u_\varepsilon(-)] : \widetilde{\Omega}_c \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d \mid [u_\varepsilon] \in \mathcal{G}^s(\Omega)^d \right\}$ *be the set of all the set-theoretical maps defined by a CGF. Then* $\mathcal{GC}^\infty(\widetilde{\Omega}_c, {}^\rho\widetilde{\mathbb{R}}^d) = \mathcal{G}^s_{\text{maps}}(\Omega, {}^\rho\widetilde{\mathbb{R}}^d)$.

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Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ *be a sharply open set and* $\widetilde{\mathcal{G}}^s(X)^d$ *be the differential algebra defined in Vernaeve, "Generalized analytic functions on generalized domains". Set* $\widetilde{\mathcal{G}}^s{}_{\text{maps}}(X)^d$ *as above. Then* $\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^d) = \widetilde{\mathcal{G}}^s{}_{\text{maps}}(X)^d$.

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Theorem (link to Aragona et al.)

Let $x_0 \in U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ *be a sharply open set. Let* $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}}^d)$, *then f is infinitely differentiable in the sense of Aragona, Fernandez, Juriaans, "Discontinuous Colombeau Differential Calculus", with the same derivatives.*

${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet spaces of GSF (1/5)

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   Rem: $\partial^\alpha f(x) := \lim_{\substack{y \to x \\ y \in int(K)}} \partial^\alpha f(y)$ for all $x \in K$

${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet spaces of GSF (1/5)

Neither $[0,1] = \{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid 0 \le x \le 1\}$ nor $[0,1] \cap \mathbb{R}$ are compact in the sharp topology

Definition

1. Let (A_ε) be a net of subsets of $\mathbb{R}^n$, then $[A_\varepsilon] := \{[a_\varepsilon] \mid \forall^0 \varepsilon : \; a_\varepsilon \in A_\varepsilon\}$ (*internal set* generated by (A_ε))
2. K is a *solid set in* ${}^{\rho}\widetilde{\mathbb{R}}^n$ if $\text{int}(K)$ is dense in K (in the sharp topology).
   Rem: $\partial^\alpha f(x) := \lim_{\substack{y \to x \\ y \in int(K)}} \partial^\alpha f(y)$ for all $x \in K$

Theorem (extreme value thm)

Let $f \in \mathcal{GC}^\infty(X, {}^{\rho}\widetilde{\mathbb{R}})$. Let $\emptyset \ne K = [K_\varepsilon] \subseteq X$ be an internal set generated by a net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$ such that $K \subseteq B_M(0)$ for some $M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, then

$$\exists m, M \in K \, \forall x \in K : \; f(m) \le f(x) \le f(M)$$

Definitions

We say that K is a *functionally compact* subset of ${}^{\rho}\widetilde{\mathbb{R}}^{n}$, and we write $K \Subset_{\text{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$, if there exists a net (K_ε) such that

1. $K = [K_\varepsilon] \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$
2. $\exists M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : \; K \subseteq B_M(0)$
3. $\forall \varepsilon : \; K_\varepsilon \Subset \mathbb{R}^n$

${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet spaces of GSF (2/5)

Definitions

We say that K is a *functionally compact* subset of ${}^{\rho}\widetilde{\mathbb{R}}^n$, and we write $K \Subset_{\text{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that

1. $K = [K_\varepsilon] \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$
2. $\exists M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : \; K \subseteq B_M(0)$
3. $\forall \varepsilon : \; K_\varepsilon \Subset \mathbb{R}^n$

Theorem

Let $K, H \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$ and $f \in \mathcal{GC}^\infty(U, {}^{\rho}\widetilde{\mathbb{R}}^d)$, then

1. *If $K \Subset_f U$ then $f(K) \Subset_f \mathbb{R}^d$*
2. *$[a, b] \Subset_f {}^{\rho}\widetilde{\mathbb{R}}$ and $K \times H \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$*
3. *If $K \cup H$ is internal, then $K \cup H \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$ (analogously for $K \cap H$ and $J \subseteq K$)*

Definition

Let $\emptyset \neq K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a solid set. Let $m \in \mathbb{N}$ and $f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d})$. Then

$$\|f\|_m := \max_{\substack{|\alpha| \le m \\ 1 \le i \le d}} \max \left(\left| \partial^\alpha f^i(M_{\alpha i}) \right|, \left| \partial^\alpha f^i(m_{\alpha i}) \right| \right) \in {}^{\rho}\widetilde{\mathbb{R}},$$

where $m_{\alpha i}$, $M_{\alpha i} \in K$ satisfy $\forall x \in K : \ \left| \partial^\alpha f^i(m_{\alpha i}) \right| \le \left| \partial^\alpha f^i(x) \right| \le \left| \partial^\alpha f^i(M_{\alpha i}) \right|$.

Definition

Let $\emptyset \neq K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^n$ be a solid set. Let $m \in \mathbb{N}$ and $f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$. Then

$$\|f\|_m := \max_{\substack{|\alpha| \leq m \\ 1 \leq i \leq d}} \max\left(\left|\partial^{\alpha} f^i(M_{\alpha i})\right|, \left|\partial^{\alpha} f^i(m_{\alpha i})\right|\right) \in {}^{\rho}\widetilde{\mathbb{R}},$$

where $m_{\alpha i}$, $M_{\alpha i} \in K$ satisfy $\forall x \in K: \ \left|\partial^{\alpha} f^i(m_{\alpha i})\right| \leq \left|\partial^{\alpha} f^i(x)\right| \leq \left|\partial^{\alpha} f^i(M_{\alpha i})\right|$.

Theorem

1. $\|f\|_m = \left[\max_{\substack{|\alpha| \leq m \\ 1 \leq i \leq d}} \sup_{x \in K_{\varepsilon}} \left|\partial^{\alpha} f^i_{\varepsilon}(x)\right|\right] \in {}^{\rho}\widetilde{\mathbb{R}}$
2. $\|f\|_m \geq 0, \qquad \|f\|_m = 0$ *if and only if* $f = 0$
3. $\forall c \in {}^{\rho}\widetilde{\mathbb{R}}: \ \|c \cdot f\|_m = |c| \cdot \|f\|_m$
4. $\|f + g\|_m \leq \|f\|_m + \|g\|_m, \qquad \|f \cdot g\|_m \leq 2^m \cdot \|f\|_m \cdot \|g\|_m$

Definition

Let $\emptyset \neq K \Subset_{\text{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a solid set. Let $f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d})$, $m \in \mathbb{N}$, $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, then

1. ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d}) := \left({}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d}), (\| - \|_{m})_{m\in\mathbb{N}}\right)$ ${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet space
2. $B_{r}^{m}(f) := \left\{g \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d}) \mid \|f - g\|_{m} < r\right\} \quad \forall m \in \mathbb{N}\, \forall r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$
3. Let $V \subseteq {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d})$, we say that V is a *sharply open set in* ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d})$ if $\forall v \in V\, \exists m \in \mathbb{N}\, \exists r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : \ B_{r}^{m}(v) \subseteq V$

${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet spaces of GSF (5/5)

Theorem

Let $\emptyset \neq K \Subset_f {}^{\rho}\widetilde{\mathbb{R}}$ *be a solid set. Then we have:*

1. *Sharply open sets form a* T_2 *topology on* ${}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ *such that pointwise addition and multiplication by* ${}^{\rho}\widetilde{\mathbb{R}}$*-scalars are continuous*
2. *If* $f, g \in B_r^m(0)$ *and* $t \in [0,1]$*, then* $tf + (1-t)g \in B_r^m(0)$
3. *If* $t \in {}^{\rho}\widetilde{\mathbb{R}}$ *and* $|t| \leq 1$*, then* $t \cdot B_r^m(0) \subseteq B_r^m(0)$
4. *For all* $\forall f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)\, \exists t \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : \ f \in t \cdot B_r^m(0|_K)$
5. *Let* $\varphi \in \mathcal{D}_K(\Omega)$*,* $K \Subset \Omega \subseteq \mathbb{R}^n$*, then* $\varphi \in {}^{\rho}\mathcal{GF}([K], {}^{\rho}\widetilde{\mathbb{R}})$ *and* $\|\varphi\|_m \in \mathbb{R}$ *is the usual m-norm of* φ
6. *let* $\Omega \subseteq \mathbb{R}^n$ *be an open set and* $J = [J_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}$ *be an infinite number. Set* $K_\varepsilon := \{x \in \Omega \mid |x| \leq J_\varepsilon\}$ *and* $K := [K_\varepsilon]$*. Then for all* $f \in {}^{\rho}\mathcal{GC}^{\infty}(\widetilde{\Omega}_c, {}^{\rho}\widetilde{\mathbb{R}}^d)$ *there exists* $\bar{f} \in {}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ *defined by* $(\bar{f}_\varepsilon)$ *such that* $\bar{f}|_{\widetilde{\Omega}_c} = f$*,* $\bar{f}_\varepsilon|_{\mathbb{R}^n \setminus K_\varepsilon} = 0$ *for all* ε
7. *The* ${}^{\rho}\widetilde{\mathbb{R}}$*-Fréchet space* ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ *is Cauchy complete*

Banach fixed point thm with loss of derivatives (1/2)

A basic idea: consider e.g. $\partial_t y(t,x) = \alpha(t,x) \cdot \partial_x y(t,x) =: F(t,x,y)$, then for $(t,x) \in K \Subset_f {}^\rho\widetilde{\mathbb{R}}^{n+1}$, K solid, we have

$$\|F(t,x,y_1) - F(t,x,y_2)\|_i \le \max_{(t,x)\in K} \|\alpha(t,x)\|_i \cdot \max_{(t,x)\in K} \|\partial_x(y_1 - y_2)\|_i \le$$
$$\le M_i \cdot \|y_1 - y_2\|_{i+1}$$

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$$\le M_i \cdot \|y_1 - y_2\|_{i+1}$$

Definition

Let $\emptyset \ne K \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set, $y_0 \in X \subseteq {}^\rho\mathcal{GF}\left(K, {}^\rho\widetilde{\mathbb{R}}^d\right)$ and $L \in \mathbb{N}$. Then P is a *finite sharp contraction on X with loss of derivatives L starting from y_0* if

1. $P : X \longrightarrow X$ is a set-theoretical map
2. $\forall i \in \mathbb{N}\, \exists \alpha_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall u, v \in X: \ \|P(u) - P(v)\|_i \le \alpha_i \cdot \|u - v\|_{i+L}$, $\alpha_i \le \alpha_{i+1}$
3. For all $i \in \mathbb{N}$, we have $\lim_{\substack{n,m\to+\infty \\ n\le m}} \alpha^n_{i+mL} \cdot \|P(y_0) - y_0\|_{i+mL} = 0$ (sharp top.)
4. $\exists s \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall m \in \mathbb{N}: \ \alpha_{i+mL} < 1 - s$

Banach fixed point thm with loss of derivatives (2/2)

Theorem

Let K, X, y_0, L, P be as above. Assume that $X \subseteq {}^{\rho}\mathcal{GF}\left(K, {}^{\rho}\widetilde{\mathbb{R}}^d\right)$ is Cauchy complete. Then:

1. *P is sharply continuous*
2. *$\exists y \in X$ such that $\lim_{n\to+\infty} P^n(y_0) = y$*
3. *$P(y) = y$*

Banach fixed point thm with loss of derivatives (2/2)

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Hints from the proof.

1. $\left\| P^{n+1}(y_0) - P^n(y_0) \right\|_i \le \alpha_i \cdot \ldots \cdot \alpha_{i+nL} \cdot \|P(y_0) - y_0\|_{i+nL} \le \alpha_{i+nL}^n \|P^n(y_0) - y_0\|_{i+nL}$

Theorem

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2. $\|P^m(y_0) - P^n(y_0)\|_i \le \left\|P^m(y_0) - P^{m-1}(y_0)\right\|_i + \ldots + \left\|P^{n+1}(y_0) - P^n(y_0)\right\|_i \le \frac{\alpha^n_{i+(m-1)L} - \alpha^m_{i+(m-1)L}}{1-\alpha_{i+(m-1)L}} \|P(y_0) - y_0\|_{i+(m-1)L}$

Banach fixed point thm with loss of derivatives (2/2)

Theorem

Let K, X, y_0, L, P be as above. Assume that $X \subseteq {}^\rho\mathcal{GF}\left(K, {}^\rho\widetilde{\mathbb{R}}^d\right)$ is Cauchy complete. Then:

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2. $\|P^m(y_0) - P^n(y_0)\|_i \le \left\|P^m(y_0) - P^{m-1}(y_0)\right\|_i + \ldots + \left\|P^{n+1}(y_0) - P^n(y_0)\right\|_i \le \frac{\alpha^n_{i+(m-1)L} - \alpha^m_{i+(m-1)L}}{1-\alpha_{i+(m-1)L}} \|P(y_0) - y_0\|_{i+(m-1)L}$

No uniqueness because we have a loss of derivatives!

□

Uniformly Lipschitz condition (1/2)

In our normal PDE, we set $\partial_t^L y = G\left(t, x, \left(\partial_t^j \partial_x^\alpha y\right)_{\substack{j<L \\ |\alpha|+j\le L}}\right) =: F(t,x,y)$, where $G : T \times S \times {}^\rho\widetilde{\mathbb{R}}^{\hat{L}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ and $\hat{L} := \text{card}\left\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n \mid j < L,\ |\alpha| + j \le L\right\}$.

Uniformly Lipschitz condition (1/2)

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Definition

Let $\emptyset \ne T \Subset_f {}^\rho\widetilde{\mathbb{R}}$, $\emptyset \ne S \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be solid sets, and $Y \subseteq {}^\rho\mathcal{GF}\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$, $L \in \mathbb{N}$. We say that *F is uniformly Lipschitz on Y with constants $(\Lambda_i)_{i\in\mathbb{N}}$ and loss of derivatives L* if:

1. $F : T \times S \times Y \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$ is a set-theoretical map;
2. $\forall y \in Y :\ F(-,-,y) \in {}^\rho\mathcal{GF}\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$;
3. $\forall i \in \mathbb{N}\, \forall u, v \in Y :\ \|F(-,-,u) - F(-,-,v)\|_i \le \Lambda_i \cdot \|u - v\|_{i+L}$
4. $\forall i \in \mathbb{N} :\ \Lambda_i \le \Lambda_{i+1}$

Reduction to 1st order in time

Introducing a new variable for each *t-derivative*, we can consider only the case $\partial_t y = G\left(t, x, (\partial_x^a y)_{|a|\le L}\right)$, where $G : T \times S \times {}^\rho\widetilde{\mathbb{R}}^{d\cdot\hat{L}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$.

E.g. $\partial_t^2 y = G(t, x, \partial_t y, \partial_t\partial_x y, \partial_x y, \partial_x^2 y)$ introducing $y_1 = y$ and $y_2 = \partial_t y_1$ becomes

$$\begin{cases}\partial_t y_2 = G(t, x, y_1, \partial_x y_1, \partial_x y_2, \partial_x y_1, \partial_x^2 y_1)\\ \partial_t y_1 = y_2\end{cases}$$

Uniformly Lipschitz condition (2/2)

Reduction to 1st order in time

Introducing a new variable for each *t*-derivative, we can consider only the case $\partial_t y = G\left(t, x, (\partial_x^a y)_{|a|\le L}\right)$, where $G : T \times S \times {}^\rho\widetilde{\mathbb{R}}^{d\cdot\hat{L}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$.

E.g. $\partial_t^2 y = G(t, x, \partial_t y, \partial_t\partial_x y, \partial_x y, \partial_x^2 y)$ introducing $y_1 = y$ and $y_2 = \partial_t y_1$ becomes

$$\begin{cases} \partial_t y_2 = G(t, x, y_1, \partial_x y_1, \partial_x y_2, \partial_x y_1, \partial_x^2 y_1) \\ \partial_t y_1 = y_2 \end{cases}$$

Theorem

If $y_0 \in {}^\rho\mathcal{GC}^\infty(S, {}^\rho\widetilde{\mathbb{R}}^d)$ with $\|y_0\|_i \le s_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ and $r_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ for all $i \in \mathbb{N}$, then for all $H \Subset_f {}^\rho\widetilde{\mathbb{R}}^d$, the function F is uniformly Lipschitz with loss of derivatives L on the space

$$Y := \{y \in {}^\rho\mathcal{GC}^\infty(T \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N}\}$$

Picard-Lindelöf theorem for PDE

Theorem

Let $t_0 \in {}^\rho\widetilde{\mathbb{R}}$, α, $r_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ $\forall i \in \mathbb{N}$. *Set* $T_\alpha := [t_0 - \alpha, t_0 + \alpha]$. *Let* $S \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ *be a solid set and* $H \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ *be a sharply closed set. Let* $y_0 \in {}^\rho\mathcal{GF}(S, H)$ *such that* $\overline{B_{r_0}(y_0(x))} \subseteq H$ *for all* $x \in S$. *Set*

$$Y_\alpha := \left\{ y \in {}^\rho\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N} \right\}$$

and assume that F *is uniformly Lipschitz on* Y_α *with constants* $(\Lambda_i)_{i\in\mathbb{N}}$ *and loss of derivatives* L. *Finally assume that for all* $y \in Y_\alpha$ *and all* $i \in \mathbb{N}$:

$$\|F(-,-,y)\|_i \le M_i(y), \quad \alpha \cdot M_i(y) \le r_i$$

$$\lim_{n,m\to\infty, n\le m} \alpha^{n+1} \cdot \Lambda^n_{i+mL} \cdot \|F(-,-,y_0)\|_{i+mL} = 0$$

$$\exists s \in {}^\rho\widetilde{\mathbb{R}}_{>0} \, \forall m \in \mathbb{N} : \ \alpha \cdot \Lambda_{i+mL} < 1 - s.$$

Then there exists a solution $y \in Y_\alpha$ *of the Cauchy problem*

$$\begin{cases} \partial_t y(t,x) = G\left(t, x, (\partial_x^a y)_{|a|\le L}\right) = F(t,x,y) \\ y(t_0, x) = y_0(x) \end{cases}$$

Corollary

Using the previous notation, let $T := [t_0 - \beta, t_0 + \beta]$, $\hat{L} := \text{card}\,\{a \in \mathbb{N}^n \mid |a| \le L\}$. *Assume that* $\|y_0\|_i \le s_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ *for all* $i = 1, \ldots, \hat{L}$. *Set* $H := \overline{B_{r_0+s_0}(0)} \subseteq {}^\rho\widetilde{\mathbb{R}}^d$, $D := \prod_{i=0}^{\hat{L}} \overline{B_{r_i+s_i}(0)}$, $M_i := \|G|_{T\times S\times D}\|$. *Let* $(\Lambda_i)_{i\in\mathbb{N}}$ *be the Lipschitz constants for* G *as stated in Thm. at slide 16. Finally, assume that* $\alpha \in (0, \beta]$ *and*

$$\exists R \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall i \in \mathbb{N}: \ \Lambda_i \le R$$

$$\exists p \in \mathbb{R}_{>0}: \ \alpha < \min\left(\frac{\mathrm{d}\rho^p}{R}, \frac{r_i}{M_i}\right) \tag{1}$$

$$\lim_{n,j\to\infty} \mathrm{d}\rho^{np} \cdot \left\| G(-,-,(\partial_x^a y_0)_{|a|\le L}) \right\|_j = 0. \tag{2}$$

Then there exists a solution in $Y_\alpha = \left\{ y \in {}^\rho\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N} \right\}$.

Corollaries of Picard-Lindelöf (1/2)

Corollary

Using the previous notation, let $T := [t_0 - \beta, t_0 + \beta]$, $\hat{L} := \text{card}\{a \in \mathbb{N}^n \mid |a| \le L\}$. *Assume that* $\|y_0\|_i \le s_i \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ *for all* $i = 1, \dots, \hat{L}$. *Set* $H := \overline{B_{r_0+s_0}(0)} \subseteq {}^{\rho}\widetilde{\mathbb{R}}^d$, $D := \prod_{i=0}^{\hat{L}} \overline{B_{r_i+s_i}(0)}$, $M_i := \|G|_{T\times S\times D}\|$. *Let* $(\Lambda_i)_{i\in\mathbb{N}}$ *be the Lipschitz constants for* G *as stated in Thm. at slide 16. Finally, assume that* $\alpha \in (0, \beta]$ *and*

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Then there exists a solution in $Y_\alpha = \left\{ y \in {}^{\rho}\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i\ \forall i \in \mathbb{N} \right\}$.

The Lipschitz constant must be bounded by some $R \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ uniformly in $i \in \mathbb{N}$. E.g. we cannot solve $\partial_t y = \delta(y)$, $y(0) = 0$ or $\partial_t y = e^{\delta(0)y}$.

Corollaries of Picard-Lindelöf (2/2)

Corollary

Let us assume all the hypotheses of the previous Cor. but (1) *and* (2)*. If we also assume that*

$$M,\ r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$$
$$0 < M_i \le M$$
$$r \le r_i$$
$$\exists p \in \mathbb{R}_{>0}:\ \alpha < \min\left(\frac{\mathrm{d}\rho^p}{R}, \frac{r}{M}\right),$$

then there exists a solution in $Y_\alpha = \left\{ y \in {}^{\rho}\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N} \right\}$.

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1. The radii r_i have not to decrease at zero: $0 < r \le r_i$ for all $i \in \mathbb{N}$

Corollaries of Picard-Lindelöf (2/2)

Corollary

Let us assume all the hypotheses of the previous Cor. but (1) and (2). If we also assume that

$$
\begin{aligned}
&M,\ r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\\
&0 < M_i \le M\\
&r \le r_i\\
&\exists p \in \mathbb{R}_{>0}:\ \alpha < \min\left(\frac{\mathrm{d}\rho^p}{R}, \frac{r}{M}\right),
\end{aligned}
$$

then there exists a solution in $Y_\alpha = \left\{ y \in {}^{\rho}\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N} \right\}$.

1. The radii r_i have not to decrease at zero: $0 < r \le r_i$ for all $i \in \mathbb{N}$
2. The norms $M_i := \|G|_{T\times S\times D}\|$ must be uniformly bounded: $0 < M_i \le M$ for all $i \in \mathbb{N}$

Corollaries of Picard-Lindelöf (2/2)

Corollary

Let us assume all the hypotheses of the previous Cor. but (1) and (2). If we also assume that

$$M,\ r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$$
$$0 < M_i \le M$$
$$r \le r_i$$
$$\exists p \in \mathbb{R}_{>0} :\ \alpha < \min\left(\frac{\mathrm{d}\rho^p}{R}, \frac{r}{M}\right),$$

then there exists a solution in $Y_\alpha = \left\{ y \in {}^{\rho}\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i\ \forall i \in \mathbb{N} \right\}$.

1. The radii r_i have not to decrease at zero: $0 < r \le r_i$ for all $i \in \mathbb{N}$
2. The norms $M_i := \|G|_{T\times S\times D}\|$ must be uniformly bounded: $0 < M_i \le M$ for all $i \in \mathbb{N}$
3. The product $\alpha^n \cdot \Lambda_j^n$ must go to zero as $n \to \infty$ more quickly than $\left\| G(-,-,(\partial_x^a y_0)_{|a|\le L}) \right\|_j$ as $j \to \infty$. E.g. $\alpha^n \cdot \Lambda_j^n = O(\mathrm{d}\rho^{np})$ and $\left\| G(-,-,(\partial_x^a y_0)_{|a|\le L}) \right\|_j = O(\log \mathrm{d}\rho^{jq})$

Example: polynomial PDE

Let's consider e.g.

$$\begin{cases} \partial_t y(t,x) = G\left(t,x,y(t,x),\partial_x y(t,x),\partial_x^2 y(t,x)\right) \\ y(0,x) = \delta(x) \end{cases}$$

where $G \in \mathbb{R}[t,x,d_0,d_1,d_2]$ is a real polynomial. In the last Cor. we set β, γ arbitrary, $T := [-\beta,\beta]$, $S := [-\gamma,\gamma]$, $s_i := \mathrm{d}\rho^{-i-1}$, $0 < r \le r_i \le \bar{r}$ arbitrary. Since G is a real polynomial, we have

$$\exists M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\,\forall i \in \mathbb{N}: \ M_i := \left\| G|_{T\times S\times D} \right\|_i \le M$$

and we get that there exists a solution in

$$Y_\alpha = \{y \in {}^{\rho}\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N}\}$$

Contacts and references

References:

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`Thanks for your attention!`