The Grothendieck topos of generalized functions

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- We need infinitesimal and infinite numbers among our scalars

Robinson-Colombeau ring of generalized numbers

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7. $B_r(c) = \{x \in {}^\rho\widetilde{\mathbb{R}}^n \mid |x - c| < r\}$ for $r > 0$ generate the *sharp topology* $\ni$ sharply open sets

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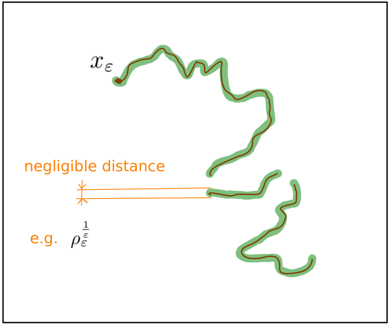

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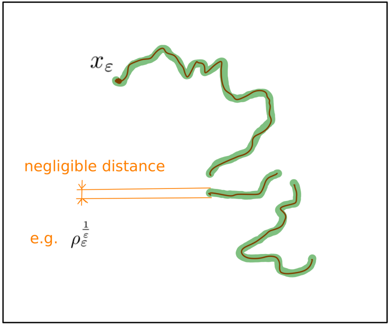

- Why arbitrary (ρ-moderate) representatives (x_ε)? It's necessary to have: intermediate value, mean value theorems

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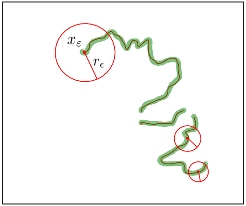

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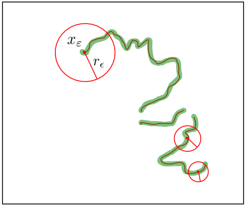

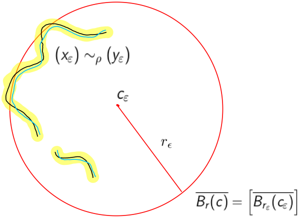

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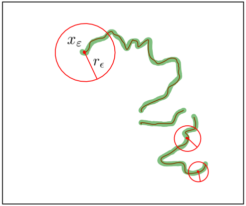

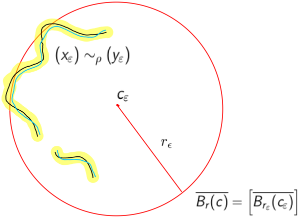

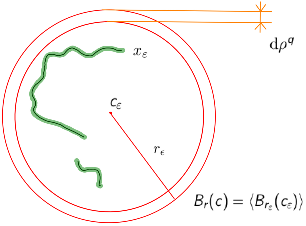

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2. There exists a net $(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^{(0,1]}$ such that for all $[x_\varepsilon] \in X$:

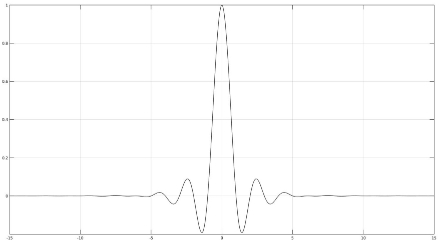
   2a. $f(x) = [f_\varepsilon(x_\varepsilon)]$

   2b. $\forall \alpha \in \mathbb{N}^n : \ (\partial^\alpha f_\varepsilon(x_\varepsilon))$ is $\rho -$ moderate

Some results on GSF

- Schwartz's distributions are embedded and smooth functions preserved (depending on a Colombeau mollifier and on an infinite number to choose w.r.t. the properties we need)
- GSF are freely closed with respect to composition: $\delta \circ \delta$, $H \circ \delta$, $\delta \circ H$...
- Derivatives can be computed using a unique GSF r working as an incremental ratio: $f(x+hv) = f(x) + h \cdot r(x,h)$, $r(x,0) = \frac{\partial f}{\partial v}(x)$
- One-dimensional integral calculus using primitives
- Classical theorems: intermediate value, (integral) mean value, Taylor's formulas, extreme value theorem, local and global inverse and implicit function theorems
- Multidimensional integration with generalized additivity and convergence theorems
- ODE: Banach fixed point, Picard-Lindelöf, maximal set of existence, Gronwall, flux, continuous dependence on initial conditions, rel. with classical solutions...
- Calculus of variations: Fundamental Lemma, second variation and minimizers, necessary Legendre condition, Jacobi fields, Conjugate points and Jacobi's theorem, Noether's theorem

Free composition and solution concept for DE



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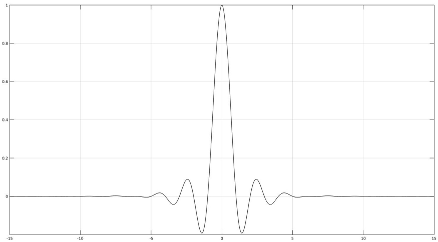

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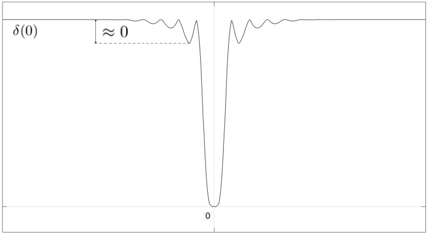

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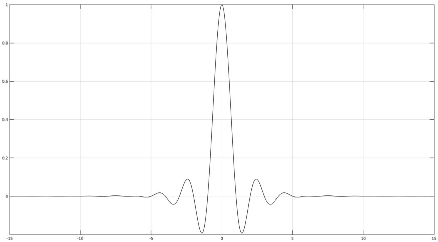

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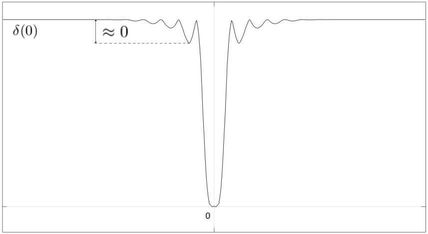

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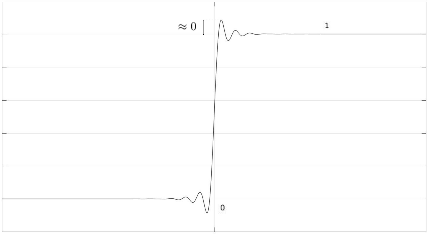

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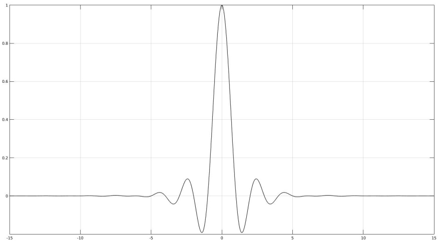

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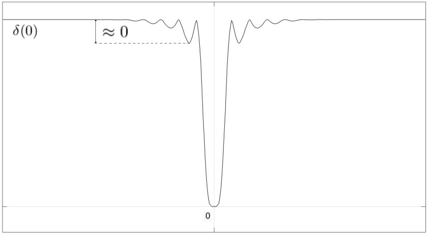

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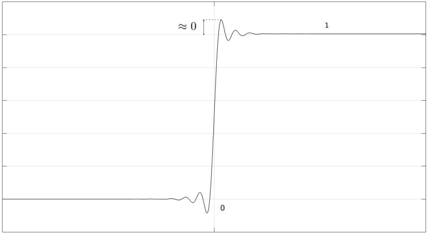

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Pointwise (i.e. strong) solution concept for DE:

$$F\left(x, (\partial^\alpha y(x))_{\alpha\in D}\right) = 0$$

$$\forall x \in U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$$

What do we need to prove a Picard-Lindelöf thm for PDE?

Cauchy problem for normal PDE:

$$\begin{cases} \partial_t^L y = G\left(t, x, \left(\partial_t^j \partial_x^\alpha y\right)_{\substack{j<L \\ |\alpha|+j\le L}}\right) & \qquad\qquad (\text{e.g. } G \in \mathcal{D}') \\ \partial_t^j y(0,x) = y_j(x) & 0 \le j < L \end{cases}$$

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- "It rarely happens that the r.h.s. of the PDE is Lipschitz"

What we need to prove a Picard-Lindelöf theorem for PDE

Non-classical point of view

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  - We need GF defined only on infinitesimal domains: Colombeau , GSF
- We need a good notion of compact set for GF and Fréchet-like spaces of GF with complete norms

${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet spaces of GSF (1/5)

Neither $[0,1] = \{x \in {}^{\rho}\widetilde{\mathbb{R}} \mid 0 \le x \le 1\}$ nor $[0,1] \cap \mathbb{R}$ are compact in the sharp topology

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Theorem (extreme value thm)

Let $f \in \mathcal{GC}^\infty(X, {}^{\rho}\widetilde{\mathbb{R}})$. Let $\emptyset \ne K = [K_\varepsilon] \subseteq X$ be an internal set generated by a net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$ such that $K \subseteq B_M(0)$ for some $M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$, then

$$\exists m, M \in K \, \forall x \in K : \; f(m) \le f(x) \le f(M)$$

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Example: $\mathbb{R}^n \subseteq K = [a,b]^n \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$, where a, $b \in {}^{\rho}\widetilde{\mathbb{R}}$, $a < 0 < b$ and a, b are infinite numbers

${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet spaces of GSF (2/5)

Definitions

We say that K is a *functionally compact* subset of ${}^{\rho}\widetilde{\mathbb{R}}^{n}$, and we write $K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$, if there exists a net (K_ε) such that

1. $K = [K_\varepsilon] \subseteq {}^{\rho}\widetilde{\mathbb{R}}^{n}$
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Theorem

Let $K, H \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$ and $f \in \mathcal{GC}^\infty(U, {}^{\rho}\widetilde{\mathbb{R}}^d)$, then

1. *If $K \Subset_f U$ then $f(K) \Subset_f \mathbb{R}^d$*
2. *$[a, b] \Subset_f {}^{\rho}\widetilde{\mathbb{R}}$ and $K \times H \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$*
3. *If $K \cup H$ is internal, then $K \cup H \Subset_f {}^{\rho}\widetilde{\mathbb{R}}^n$ (analogously for $K \cap H$ and $J \subseteq K$)*

Definition

Let $\emptyset \neq K \Subset_{\mathrm{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n}$ be a solid set. Let $m \in \mathbb{N}$ and $f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^{d})$. Then

$$\|f\|_m := \max_{\substack{|\alpha| \le m \\ 1 \le i \le d}} \max\left(\left|\partial^{\alpha} f^{i}(M_{\alpha i})\right|, \left|\partial^{\alpha} f^{i}(m_{\alpha i})\right|\right) \in {}^{\rho}\widetilde{\mathbb{R}},$$

where $m_{\alpha i}$, $M_{\alpha i} \in K$ satisfy $\forall x \in K : \; \left|\partial^{\alpha} f^{i}(m_{\alpha i})\right| \le \left|\partial^{\alpha} f^{i}(x)\right| \le \left|\partial^{\alpha} f^{i}(M_{\alpha i})\right|$.

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Theorem

1. $\|f\|_m = \left[\max_{\substack{|\alpha|\le m\\ 1\le i\le d}} \sup_{x\in K_{\varepsilon}} \left|\partial^{\alpha} f^{i}_{\varepsilon}(x)\right|\right] \in {}^{\rho}\widetilde{\mathbb{R}}$
2. $\|f\|_m \ge 0$, $\qquad \|f\|_m = 0$ *if and only if* $f = 0$
3. $\forall c \in {}^{\rho}\widetilde{\mathbb{R}}: \ \|c \cdot f\|_m = |c| \cdot \|f\|_m$
4. $\|f + g\|_m \le \|f\|_m + \|g\|_m$, $\qquad \|f \cdot g\|_m \le 2^m \cdot \|f\|_m \cdot \|g\|_m$

Definition

Let $\emptyset \neq K \Subset_{\text{f}} {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set. Let $f \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, $m \in \mathbb{N}$, $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, then

1. ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d) := \left({}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d), (\| - \|_m)_{m\in\mathbb{N}}\right)$ ${}^\rho\widetilde{\mathbb{R}}$-Fréchet space
2. $B_r^m(f) := \left\{ g \in {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d) \mid \|f - g\|_m < r \right\} \quad \forall m \in \mathbb{N}\, \forall r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$
3. Let $V \subseteq {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, we say that V is a *sharply open set in* ${}^\rho\mathcal{GF}(K, {}^\rho\widetilde{\mathbb{R}}^d)$ if $\forall v \in V\, \exists m \in \mathbb{N}\, \exists r \in {}^\rho\widetilde{\mathbb{R}}_{>0} :\ B_r^m(v) \subseteq V$

Theorem

Let $\emptyset \ne K \Subset_f {}^{\rho}\widetilde{\mathbb{R}}$ be a solid set. Then we have:

1. *Sharply open sets form a T_2 topology on ${}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ such that pointwise addition and multiplication by ${}^{\rho}\widetilde{\mathbb{R}}$-scalars are continuous*
2. *If $f, g \in B_r^m(0)$ and $t \in [0,1]$, then $tf + (1-t)g \in B_r^m(0)$*
3. *If $t \in {}^{\rho}\widetilde{\mathbb{R}}$ and $|t| \le 1$, then $t \cdot B_r^m(0) \subseteq B_r^m(0)$*
4. *For all $\forall f \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)\, \exists t \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : \ f \in t \cdot B_r^m(0|_K)$*
5. *Let $\varphi \in \mathcal{D}_K(\Omega)$, $K \Subset \Omega \subseteq \mathbb{R}^n$, then $\varphi \in {}^{\rho}\mathcal{GF}([K], {}^{\rho}\widetilde{\mathbb{R}})$ and $\|\varphi\|_m \in \mathbb{R}$ is the usual m-norm of φ*
6. *The ${}^{\rho}\widetilde{\mathbb{R}}$-Fréchet space ${}^{\rho}\mathcal{GF}(K, {}^{\rho}\widetilde{\mathbb{R}}^d)$ is Cauchy complete*

Banach fixed point thm with loss of derivatives (1/2)

A basic idea: consider e.g. $\partial_t y(t,x) = \alpha(t,x)\cdot\partial_x y(t,x) =: F(t,x,y)$, then for $(t,x)\in K \Subset_{\text{f}} {}^{\rho}\widetilde{\mathbb{R}}^{n+1}$, K solid, we have

$$\|F(t,x,y_1)-F(t,x,y_2)\|_i \le \max_{(t,x)\in K}\|\alpha(t,x)\|_i \cdot \max_{(t,x)\in K}\|\partial_x(y_1-y_2)\|_i \le$$
$$\le M_i\cdot\|y_1-y_2\|_{i+1}$$

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Definition

Let $\emptyset\ne K\Subset_{\text{f}} {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set, $y_0\in X\subseteq {}^\rho\mathcal{GF}\left(K,{}^\rho\widetilde{\mathbb{R}}^d\right)$ and $L\in\mathbb{N}$. Then P is a *finite sharp contraction on X with loss of derivatives L starting from y_0* if

1. $P: X\longrightarrow X$ is a set-theoretical map
2. $\forall i\in\mathbb{N}\,\exists\alpha_i\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall u,v\in X:\ \|P(u)-P(v)\|_i\le\alpha_i\cdot\|u-v\|_{i+L}$, $\alpha_i\le\alpha_{i+1}$
3. For all $i\in\mathbb{N}$, we have $\lim_{\substack{n,m\to+\infty\\ n\le m}}\alpha_{i+mL}^n\cdot\|P(y_0)-y_0\|_{i+mL}=0$ (sharp top.)
4. $\exists s\in{}^\rho\widetilde{\mathbb{R}}_{>0}\,\forall m\in\mathbb{N}:\ \alpha_{i+mL}<1-s$

Banach fixed point thm with loss of derivatives (2/2)

Theorem

Let K, X, y_0, L, P be as above. Assume that $X \subseteq {}^\rho\mathcal{GF}\left(K, {}^\rho\widetilde{\mathbb{R}}^d\right)$ is Cauchy complete. Then:

1. *P is sharply continuous*
2. *$\exists y \in X$ such that $\lim_{n\to+\infty} P^n(y_0) = y$*
3. *$P(y) = y$*

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No uniqueness because we have a loss of derivatives!

Uniformly Lipschitz condition (1/2)

In our normal PDE, we set $\partial_t^L y = G\left(t, x, \left(\partial_t^j \partial_x^\alpha y\right)_{\substack{j<L\\|\alpha|+j\le L}}\right) =: F(t, x, y)$, where $G : T \times S \times {}^\rho\widetilde{\mathbb{R}}^{\hat{L}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ and $\hat{L} := \operatorname{card}\left\{(j,\alpha) \in \mathbb{N} \times \mathbb{N}^n \mid j < L,\ |\alpha| + j \le L\right\}$.

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Definition

Let $\emptyset \ne T \Subset_f {}^\rho\widetilde{\mathbb{R}}$, $\emptyset \ne S \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be solid sets, and $Y \subseteq {}^\rho\mathcal{GF}\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$, $L \in \mathbb{N}$. We say that *F is uniformly Lipschitz on Y with constants $(\Lambda_i)_{i\in\mathbb{N}}$ and loss of derivatives L* if:

1. $F : T \times S \times Y \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$ is a set-theoretical map;
2. $\forall y \in Y :\ F(-,-,y) \in {}^\rho\mathcal{GF}\left(T \times S, {}^\rho\widetilde{\mathbb{R}}^d\right)$;
3. $\forall i \in \mathbb{N}\, \forall u, v \in Y :\ \|F(-,-,u) - F(-,-,v)\|_i \le \Lambda_i \cdot \|u - v\|_{i+L}$
4. $\forall i \in \mathbb{N} :\ \Lambda_i \le \Lambda_{i+1}$

Uniformly Lipschitz condition (2/2)

Reduction to 1st order in time

Introducing a new variable for each t-derivative, we can consider only the case $\partial_t y = G\left(t, x, (\partial_x^a y)_{|a|\le L}\right)$, where $G : T \times S \times {}^\rho\widetilde{\mathbb{R}}^{d\cdot\hat{L}} \longrightarrow {}^\rho\widetilde{\mathbb{R}}^d$.

E.g. $\partial_t^2 y = G(t, x, \partial_t y, \partial_t\partial_x y, \partial_x y, \partial_x^2 y)$ introducing $y_1 = y$ and $y_2 = \partial_t y_1$ becomes

$$\begin{cases} \partial_t y_2 = G(t, x, y_2, \partial_x y_1, \partial_x y_2, \partial_x y_1, \partial_x^2 y_1) \\ \partial_t y_1 = y_2 \end{cases}$$

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Theorem

If $y_0 \in {}^\rho\mathcal{GC}^\infty(S, {}^\rho\widetilde{\mathbb{R}}^d)$ with $\|y_0\|_i \le s_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ and $r_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ for all $i \in \mathbb{N}$, then for all $H \Subset_f {}^\rho\widetilde{\mathbb{R}}^d$, the function F is uniformly Lipschitz with loss of derivatives L on the space

$$Y := \{y \in {}^\rho\mathcal{GC}^\infty(T \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N}\}$$

Picard-Lindelöf theorem for PDE

Theorem

Let $t_0 \in {}^\rho\widetilde{\mathbb{R}}$, α, $r_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ $\forall i \in \mathbb{N}$. Set $T_\alpha := [t_0 - \alpha, t_0 + \alpha]$. Let $S \Subset_f {}^\rho\widetilde{\mathbb{R}}^n$ be a solid set and $H \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be a sharply closed set. Let $y_0 \in {}^\rho\mathcal{GF}(S, H)$ such that $\overline{B_{r_0}(y_0(x))} \subseteq H$ for all $x \in S$. Set

$$Y_\alpha := \left\{ y \in {}^\rho\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i \ \forall i \in \mathbb{N} \right\}$$

and assume that F is uniformly Lipschitz on Y_α with constants $(\Lambda_i)_{i\in\mathbb{N}}$ and loss of derivatives L. Finally assume that for all $y \in Y_\alpha$ and all $i \in \mathbb{N}$:

$$\|F(-,-,y)\|_i \le M_i(y), \quad \alpha \cdot M_i(y) \le r_i$$

$$\lim_{n,m\to\infty, n\le m} \alpha^{n+1} \cdot \Lambda^n_{i+mL} \cdot \|F(-,-,y_0)\|_{i+mL} = 0$$

$$\exists s \in {}^\rho\widetilde{\mathbb{R}}_{>0} \, \forall m \in \mathbb{N}: \ \alpha \cdot \Lambda_{i+mL} < 1 - s.$$

Then there exists a solution $y \in Y_\alpha$ of the Cauchy problem

$$\begin{cases} \partial_t y(t,x) = G\left(t, x, (\partial^a_x y)_{|a|\le L}\right) = F(t,x,y) \\ y(t_0, x) = y_0(x) \end{cases}$$

Corollary

Using the previous notation, let $T := [t_0 - \beta, t_0 + \beta]$, $\hat{L} := \text{card}\{a \in \mathbb{N}^n \mid |a| \le L\}$. *Assume that* $\|y_0\|_i \le s_i \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ *for all* $i = 1, \dots, \hat{L}$. *Set* $H := \overline{B_{r_0+s_0}(0)} \subseteq {}^\rho\widetilde{\mathbb{R}}^d$, $D := \prod_{i=0}^{\hat{L}} \overline{B_{r_i+s_i}(0)}$, $M_i := \|G|_{T\times S\times D}\|_i$. *Let* $(\Lambda_i)_{i\in\mathbb{N}}$ *be the Lipschitz constants for* G *as stated in Thm. at slide 18. Finally, assume that* $\alpha \in (0, \beta]$ *and*

$$\exists R \in {}^\rho\widetilde{\mathbb{R}}_{>0}\, \forall i \in \mathbb{N} :\ \Lambda_i \le R$$

$$\exists p \in \mathbb{R}_{>0} :\ \alpha < \min\left(\frac{\mathrm{d}\rho^p}{R}, \frac{r_i}{M_i}\right) \tag{1}$$

$$\lim_{n,j\to\infty} \mathrm{d}\rho^{np} \cdot \left\| G(-,-,(\partial_x^a y_0)_{|a|\le L}) \right\|_j = 0. \tag{2}$$

Then there exists a solution in $Y_\alpha = \left\{ y \in {}^\rho\mathcal{GF}(T_\alpha \times S, H) \mid \|y - y_0\|_i \le r_i\ \forall i \in \mathbb{N} \right\}$.

Corollaries of Picard-Lindelöf (1/2)

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The Lipschitz constant must be bounded by some $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ uniformly in $i \in \mathbb{N}$. E.g. we cannot solve $\partial_t y = \delta(y)$, $y(0) = 0$ or $\partial_t y = e^{\delta(0)y}$.

Example: polynomial PDE

Let's consider e.g.

$$\begin{cases} \partial_t y(t,x) = G\left(t, x, y(t,x), \partial_x y(t,x), \partial_x^2 y(t,x)\right) \\ y(0,x) = \delta(x) \end{cases}$$

where $G \in \mathbb{R}[t, x, d_0, d_1, d_2]$ is a real polynomial. In the last Cor. we set β, γ arbitrary, $T := [-\beta, \beta]$, $S := [-\gamma, \gamma]$, $s_i := \mathrm{d}\rho^{-i-1}$, $0 < r \le r_i \le \bar{r}$ arbitrary. Since G is a real polynomial, we have

$$\exists M \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}\, \forall i \in \mathbb{N} : \; M_i := \left\| G|_{T \times S \times D} \right\|_i \le M$$

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Subpoints and their first properties

Definition

Let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$, then we say that x' is a subpoint of x if $\exists J \subseteq (0,1]:\ 0$ is a limit point of J (we write $J \subseteq_0 (0,1]$) and $x' = x|_J =: \left[(x_\varepsilon)_{\varepsilon \in J}\right]$.

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  $f \in {}^\rho\mathcal{OGC}^\infty(U,V)$ is the extension of f to all the subpoints
- $\{Sp(U) \mid U \in {}^\rho\mathcal{OGC}^\infty,\ U \subseteq {}^\rho\widetilde{\mathbb{R}}^n\}$ is a base for a topology of sets of subpoints in ${}^\rho\widetilde{\mathbb{R}}^n$; If $\bar{S}$ is open in this topology, then
  $Pt(\bar{S}) := \bigcup\{U \in {}^\rho\mathcal{OGC}^\infty \mid Sp(U) \subseteq \bar{S}\} \in {}^\rho\mathcal{OGC}^\infty$

Strong compatibility

The classical sheaf property on open sets does not hold for GSF: $i(h) := 1$ if h is infinitesimal and $i(h) := 0$ otherwise

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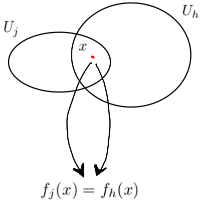

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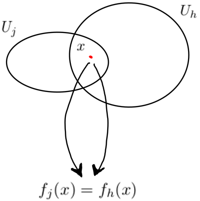

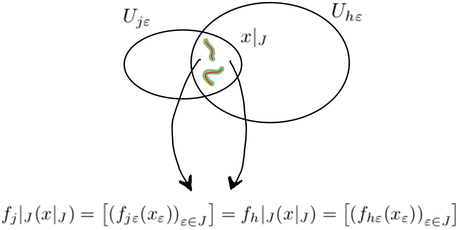

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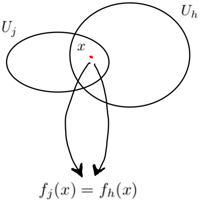
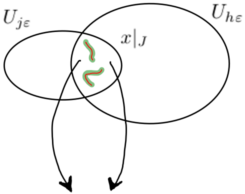
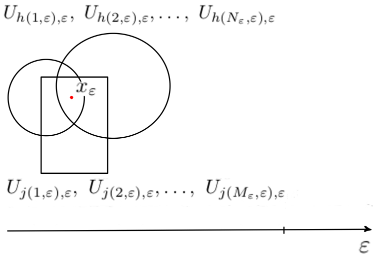




$$f_j|_J(x|_J) = \left[(f_{j\varepsilon}(x_\varepsilon))_{\varepsilon\in J}\right] = f_h|_J(x|_J) = \left[(f_{h\varepsilon}(x_\varepsilon))_{\varepsilon\in J}\right]$$



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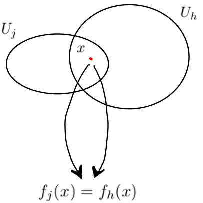

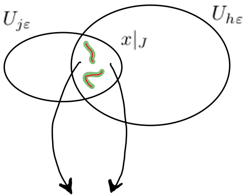

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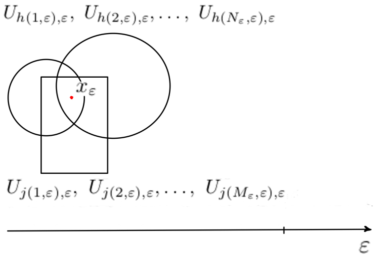

For all $p = 1, \dots, N_\varepsilon$, $q = 1, \dots, M_\varepsilon$:

$$f_j|_J(x|_J) = \left[\left(f_{j(p,\varepsilon),\varepsilon}(x_\varepsilon)\right)_{\varepsilon\in J}\right] =$$

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Strong compatibility condition

Sheaf property for GSF

Theorem

Assume that the family $f_i \in {}^\rho\mathcal{OGC}^\infty(U_i, Y)$ for $i \in H$ is strongly compatible, then there exists one and only one $f \in {}^\rho\mathcal{OGC}^\infty(\bigcup_{i\in H} U_i, Y)$ such that $f|_{U_i} = f_i$.

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Does this mean that GSF are a sheaf w.r.t. the topology of subpoints?

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- In the topos ${}^\rho\mathcal{TGC}^\infty$ we can define a language of nilpotent infinitesimals in order to prove properties such as: tangent vectors are infinitesimal lines on the space, vector fields are infinitesimal transformations, Lie brackets are their commutator... All this in finite and infinite dimensional spaces such as ${}^\rho\mathcal{GC}^\infty(X, Y)$

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- Translate "concrete sheaves" into a definition of *generalized diffeological space* which is more near to the usual one

Future developments

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- Start Analytical mechanics

Contacts and references

References:

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Thank you for your attention!