Hyperfinite methods for generalized smooth functions

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The main aim of the proposed project is to develop hyperfinite methods for the solution of PDE in the framework of generalized smooth functions. The nonlinear theory of generalized smooth functions has recently emerged as a minimal extension of Colombeau's theory that allows for more general domains for generalized functions, resulting in the closure with respect to composition and a better behaviour on unbounded sets. By *hyperfinite methods*, we mean both the use of infinite integer Colombeau generalized numbers and the use of closed intervals with infinite boundary points. The former will be used to introduce a better notion of power series and hence a corresponding Cauchy-Kowalevski theorem which, in principle, will be applicable to any distributional PDE (not only to standard analytic ones). The latter will be used to define a Fourier transform applicable to any generalized smooth function (not only to those of tempered type). We plan to study the method of characteristics, to prove a Picard-Lindelöf theorem with hyperfinite iterations of contractions for a Cauchy problem with a normal PDE, i.e. of

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the form $\partial_t^k y(t,x) = G\left[t,x,\left(\partial_t^j\partial_x^\alpha y(t,x)\right)_{\substack{0\le j<k\\|\alpha|+j\le k}}\right]$, and to study operators defined by using hyperfinite methods. The proposal thus aims at both solving hard open problems like general existence theorems for nonlinear singular PDE, and at widely extending well known methods of mathematical analysis.

1 Aims and research objectives

The main objective of the present research project is to introduce hyperfinite methods into the study of generalized functions, in particular for solving nonlinear PDE. The framework is the category of generalized smooth functions (GSF), an extension of classical distribution theory which enables to model nonlinear singular problems, while at the same time sharing a number of fundamental properties with ordinary smooth functions, such as the closure with respect to composition and several non trivial classical theorems of the calculus, see [GiKuVe15, GioKun16a, GiKu16, GiKuSt16, LuGi16a, LeLuGi16]. One could describe GSF as a methodological restoration of Cauchy-Dirac's original conception of generalized function, see [Dir26, Laug89, KatTal12]. In essence, the idea of Cauchy and Dirac (but also of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view generalized functions as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy-Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties which nowadays are attributed to the Dirac delta, cf. [KatTal12]. Moreover, GSF are a minimal extension of Colombeau's theory of generalized functions (CGF), see [Col84, Col85, Col92, NePiSc98, Obe92, Pil94]. In fact, when the domain is of the form $\widetilde{\Omega}_c$, i.e. it is the set of compactly supported generalized points in the open set $\Omega\subseteq\mathbb{R}^n$, then the two spaces of generalized functions coincide, cf. [GiKuVe15]. Therefore, we expect that the methods envisaged in the present project will also exert a considerable impact on Colombeau's theory. For these reasons, the department of Mathematics of the University of Vienna, and in particular the research group of Prof. M. Kunzinger, seem the ideal place where to implement the present research project, because of the group's specific competences on generalized functions, PDE and functional analysis.

A first presentation of the project's main aims is the following:

(i) ***Basic hyperfinite methods in the ring $^\rho\widetilde{\mathbb{R}}$ of generalized numbers***

Problems and motivations: It is well known that, in the sharp topology on the ring of Colombeau generalized numbers $\widetilde{\mathbb{R}}$, the sequence $n\in\mathbb{N}_{>0}\mapsto\frac{1}{n}\in\widetilde{\mathbb{R}}$ does not converge to zero as $n\to+\infty$, because $\frac{1}{n}$ is never infinitesimal for $n\in\mathbb{N}_{>0}$. Analogously, a series $\sum_{n=0}^{+\infty}a_n$ in $\widetilde{\mathbb{R}}$ converges in the sharp topology if and only if $a_n\to 0$ as $n\to+\infty$, cf. [Ver11, PiScVa09, Ar-Fe-Ju05]. This is a general property of every ultrapseudometric space like $\widetilde{\mathbb{R}}$, see [Kob84]. As a consequence, we have that if the series $\sum_{n=0}^{+\infty}\frac{z^n}{n!}$ converges, then the generalized complex number $z\in\widetilde{\mathbb{C}}$ is necessarily an infinitesimal; analogously, e.g., $\sum_{n=0}^{+\infty}\frac{1}{2^n}$ do not converge in $\widetilde{\mathbb{R}}$, and if $\sum_{n=0}^{+\infty}a_nz^n$ converges and z is not infinitesimal, then a_n is infinitesimal for all $n\in\mathbb{N}$ sufficiently big.

The idea: We want to consider the set $\widetilde{\mathbb{N}}:=\left\{\text{int}\left([|x_\varepsilon|]\right)\mid[x_\varepsilon]\in\widetilde{\mathbb{R}}\right\}$ of *hyperfinite generalized number*, where $\text{int}(-)$ is the integer part function. The counter-intuitive properties mentioned above can be easily solved by considering $n\in\widetilde{\mathbb{N}}$ instead of $n\in\mathbb{N}$ (see below for details).

The plan: Let $^\rho\widetilde{\mathbb{R}}$ be the ring of generalized numbers, i.e. the ring defined exactly as $\widetilde{\mathbb{R}}$ (the quotient ring of moderate nets modulo negligible nets) but using an arbitrary net $\rho=(\rho_\varepsilon)\downarrow 0$ instead of the net (ε), see [GiLu15, GiKuSt16, LuGi16b]. Taking into account that $\widetilde{\mathbb{N}}$ is a directed set, we plan to study hyperfinite sequences, hyperfinite numerical series, hyperfinite power series, hyperfinite Riemann sums, and the hyperfinite Euler method for solving ODE. The basic method of derivation will be a careful generalization to ${}^{\rho}\widetilde{\mathbb{R}}$ of classical proofs in $\mathbb{R}$, possibly taking hints from the nonstandard approach to similar problems, see [LigRob75, Gol88].

Innovative features and deliverables: For example, we already proved that $\sum_{n\in\widetilde{\mathbb{N}}}\frac{x^n}{n!}=e^x$ for all $x\in{}^{\rho}\widetilde{\mathbb{R}}$ finite. See below for the definition of $\sum_{n\in\widetilde{\mathbb{N}}}a_n$. We thus expect to obtain more flexible and general notions of sequence and series, in particular of power series. The corresponding notion of *hyperanalytic* GSF will have a wider scope and better properties, since it is based on $\widetilde{\mathbb{N}}$ and hence it is different from the general notion of power series in non-Archimedean analysis, see [Kob84]. As already done for GSF, the generalization of classical proofs into the non-Archimedean setting of ${}^{\rho}\widetilde{\mathbb{R}}$ typically also leads to a smoother acceptance of this new theory outside the community of CGF.

(ii) ***The method of characteristics and hyperfinite Fourier transform***

Problems and the plan: GSF form a concrete category, i.e. they are closed with respect to composition. The classical method of characteristics, see e.g. [Eva10], seems thus extendible to the setting of GSF. The inverse and implicit function theorems for GSF, see [GiKu16, LeLuGi16], are helpful instruments in this generalization.

A different idea is based on the property that every GSF can be integrated on every closed interval $[a,b]$, a, $b\in{}^{\rho}\widetilde{\mathbb{R}}$, $a<b$, even if the boundary points are infinite numbers. This is a consequence of the extreme value theorem for GSF, see [GioKun16a, GiKuSt16]. We can hence consider a Fourier-like integral of the type $\mathcal{F}_k(f)(\omega):=(2\pi)^{-1/2}\int_{-k}^{k}e^{i\omega t}f(t)\,\mathrm{d}t\in{}^{\rho}\widetilde{\mathbb{R}}$ (1-dimentional case), where $k\in{}^{\rho}\widetilde{\mathbb{R}}$ is an infinite number and $f\in{}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}},{}^{\rho}\widetilde{\mathbb{R}})$ is an *arbitrary* GSF. What are the properties of this $\mathcal{F}_k$? Is it a convenient instrument to approach PDE?

Innovative features and deliverables: By generalizing the classical Picard-Lindelöf method, we already proved that every ODE has a local solution, [LuGi16a]. This and the above mentioned hyperfinite Euler method for ODE would give an innovative way to deal with the method of characteristics for non-smooth nonlinear PDE.

Concerning the domain-dependent Fourier transform $\mathcal{F}_k$, besides the usual elementary properties, suitably formulated, we already proved that $\mathcal{F}_k(\mathcal{F}_k^{-1}(f))(\omega)=f(\omega)$ for every f which is zero outside $[-k-\mathrm{d}\rho^a,k+\mathrm{d}\rho^a]$, for k infinite and for every ω finite. We recall that this result does not hold in the classical Colombeau's theory, see e.g. [Hor99]. For example, every CGF can be extended to a function of this type (i.e. a compactly supported GSF, see [GioKun16a, Thm. 28]) because they are defined only on finite points $\widetilde{\mathbb{R}}_c$. Therefore, this Fourier-like transform seems to have a broad range of applications and good properties.

(iii) ***Generalized smooth operators***

Problems and motivations: Like CGF, also GSF are defined by a net (f_ε) of smooth functions, see below. One could say that generalized numbers in ${}^{\rho}\widetilde{\mathbb{R}}$ are "dynamical numbers" (obtained by using ε-nets of real numbers) if compared to static numbers in $\mathbb{R}$; analogously, we could say that GSF are "dynamical smooth functions" if compared to static smooth functions in $\mathcal{C}^{\infty}(\mathbb{R}^n,\mathbb{R}^d)$. How to define and study a notion of *generalized smooth operator* as generated by nets of ordinary operators $T_\varepsilon:X_\varepsilon\longrightarrow Y_\varepsilon$, where X_ε and Y_ε are, e.g., real Banach spaces?

The idea and the plan: We already have a promising definition of generalized smooth operator, and we used it to solve suitable ODE in non-infinitesimal neighbourhoods by using hyperfinite Picard-Lindelöf iterates, see [LuGi16a]. See below for a description of this idea. However, it is not yet clear when a given sharply closed space of GSF is closed under hyperfinite iterates. We therefore want to continue the study of these operators, e.g. characterizing the property of possessing hyperfinite iterates or proving that every GSF can be seen as a density of an integral generalized smooth operator. This is a first step in the study of ${}^{\rho}\widetilde{\mathbb{R}}$-modules of GSF, which are complete in the sharp topology defined by a countable family of ${}^{\rho}\widetilde{\mathbb{R}}$-valued norms, see e.g. [GioKun16a, Sec. 4, 7, 8] and below.

Innovative features and deliverables: This part of the research proposal could initiate a functional analysis of generalized smooth operators, which could expand into new approaches to unbounded operators, monotone operators, compact operators, ${}^{\rho}\widetilde{\mathbb{R}}$-Banach or Hilbert spaces, fixed point theory, etc.

(iv) ***General existence theorems for singular PDE***[2]

Problems and the plan: In this part of the project, we plan to propose two solutions for the important open problem of proving *general* existence theorems for singular PDE. The first idea is based on our new notion of hyperanalytic GSF. In fact, since we have a new notion of power series and of analytic function, a corresponding version of the Cauchy-Kowalevski theorem is expected to hold.

We already proved, see [GioLup17], a general existence theorem for the Cauchy problem for arbitrary nonlinear normal PDE, i.e. of the form $\partial_t^k y(t,x) = G\left[t, x, \left(\partial_t^j \partial_x^\alpha y(t,x)\right)_{\substack{0\le j<k \\ |\alpha|+j\le k}}\right]$, (where both G and y are GSF) by suitably generalizing the classical Banach fixed point theorem and the corresponding Picard-Lindelöf theorem. It is therefore natural to consider, as we already did for ODE in [LuGi16a], hyperfinite iterations of contractions and the use of the aforementioned method of characteristics to formulate sufficient conditions for the uniqueness of solutions of PDE. A full understanding of the classical Lewy counter-example [Lew57, Horm63] and its relations with GSF is also mandatory.

Innovative features and deliverables: We note that the classical example of smooth but not real analytic function, i.e. $f(x) = e^{-1/x}$ if $x > 0$ and $f(x) = 0$ otherwise, is identically zero in ${}^{\rho}\widetilde{\mathbb{R}}$ on every ball centred at $x = 0$ and with radius $\mathrm{d}\rho^a$ because $|e^{-1/\mathrm{d}\rho}| \le \mathrm{d}\rho^n$ for all $n \in \mathbb{N}$. It is hence real hyperanalytic in the new sense, using hyperfinite sequences and sharp neighbourhoods (see also below). Moreover, by the Schwartz-Paley-Wiener theorem, any Colombeau mollifier, cf. [GrKuObSt01, Del05, NePiSc98], is analytic and hence there are strong reasons to expect that the embedding of *any* distribution, by convolution with this mollifier, would be hyperanalytic in the new sense. This would give to this new Cauchy-Kowalevski theorem a remarkable range of applicability because it would be applicable to any PDE given by the embedding of a distribution. Finally, our existence result for Cauchy problem with normal PDE is one of the few general results in the field of PDE. Since in general finite iterations of contractions guarantee existence only in infinitesimal sharp time intervals, its extension to hyperfinite iterations would allow to include solutions on standard time intervals.

(v) ***Hyperfinite methods for GSF in nonstandard analysis***[3]

Problems and motivations: Since its inception, the use of hyperfinite sets in nonstandard analysis (NSA) has been of considerable importance. The classical work [BerRob66] of Bernstein and Robinson represents the main example. On the other hand, even if NSA allows to create frameworks which are formally more powerful than the corresponding ones in standard analysis (see e.g. [ObeTod98, Tod11, Tod13, TodVer08] and below), the matter of fact is that its acceptance in the corresponding mathematical community, and hence the publication of the associated works, faces non trivial problems. In

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the final part of the project, we therefore want to dedicate some months to develop the first results of this project in the framework of [Tod11, Tod13], i.e. of the valuation field ${}^\rho\mathbb{R}$, or directly in the hyperreal field ${}^*\mathbb{R}$, and to evaluate the interest of our research group in this further advancement.

The idea and the plan: Thanks to the use of powerful formal tools of NSA, like the transfer principle, the saturation principle, Loeb measure, etc., several results listed in the first part of the present proposal have been already established, see e.g. [Lind88, CapCut95, Gol88]. We plan to use these tools to start the theory of GSF using the field of scalars ${}^\rho\mathbb{R}$, instead of the ring ${}^\rho\widetilde{\mathbb{R}}$. In this way, our research group will have a more clear idea about the potentialities of this branch of the theory and can hence evaluate whether to prepare another FWF research proposal to pursue it further.

Innovative features and deliverables: Due to the better formal properties of the field of hyperreals ${}^*\mathbb{R}$ as compared to the ring ${}^\rho\widetilde{\mathbb{R}}$, we expect to obtain formally more powerful and simpler results. A classical example in this direction is the possibility to prove the Hahn-Banach theorem in the NSA counterpart of Colombeau's theory (see [TodVer08]), whereas this classical result does not hold in the standard theory.

The present research project is designed for three co-workers: the two applicants (P. Giordano and M. Kunzinger) and one PostDoc. The latter and the main applicant will work on this project full time for three years.

2 State of the art

2.1 State of the art in the research field

Generalized functions:

F. Colombeau's theory of generalized functions has been developed since 1984 and enables to perform certain non linear operations (up to polynomial growth) between embedded distributions, avoiding the difficulty of Schwartz's impossibility theorem. See e.g. [Ned-Pil06, GrKuObSt01, NePiSc98, Pil94, Obe92, Col92] for an introduction with applications. This theory makes it possible to find generalized solutions of some well-known PDE which do not have solutions in the classical space of distributions, see [Obe92], and has manifold applications, e.g. to the theory of elasticity, fluid mechanics and in the theory of shock waves (see e.g. [Col92, Obe92]), to differential geometry and relativity theory [GrKuObSt01, Kun04, SteVic06] and to quantum field theory [CoGs08].

It is also remarkable to compare this theory with the approach given in NSA [Tod13, Tod11, TodVer08, TodWol04, Tod99], where a strong simplification and generalization has been obtained. Moreover, in this NSA-based approach, the set of scalars of the spaces of generalized function is always an algebraically closed non-Archimedean Cantor complete field, whereas in Colombeau's theory we have zero divisors. Having a field of scalars, a version of the Hahn-Banach theorem holds, whereas this is not the case for the classical CGN, see [Ver10, TodVer08].

A new and fundamental step in the theory of generalized functions based on CGN, which presents several analogies with our present proposal, has first been achieved in [Ar-Fe-Ju05, Ar-Fe-Ju12]. In this work, the basic idea is to generalize the derivative as a limit of an incremental ratio taken with respect to the e-norm, [Ar-Fe-Ju05, Ar-Fe-Ju09] and with increments which are asymptotic to invertible infinitesimals of the form $[\varepsilon^r] \in \widetilde{\mathbb{R}}$, for $r \in \mathbb{R}_{\geq 0}$. This theory extends the usual classical notion of derivative and smoothness to set-theoretical functions on CGN, e.g. of the form $f : \widetilde{\mathbb{R}}^n \longrightarrow \widetilde{\mathbb{R}}^d$, and enables to prove that every CGF is infinitely differentiable in this new sense. Several important applications have already been achieved (see [Ar-Fe-Ju12]) and hence the theory promises to be very relevant. As explained in greater detail in [GiKuVe15, GiKuSt16, GiKu16], the notion of smoothness developed in [Ar-Fe-Ju05] includes functions like $i(x) = 1$ if x is infinitesimal and $i(x) = 0$ otherwise. This makes it impossible to prove classical theorems like the intermediate value one, whereas in our approach this theorem holds. The theory of GSF, on the other hand, while fully compatible with the approach in [Ar-Fe-Ju05], singles out a subclass of smooth functions with more favourable compatibility properties with respect to classical calculus and hence may be viewed as a refinement of that theory (cf. below).

It is also important to mention the approach to generalized solutions of PDEs by Dedekind order completion of spaces of piecewise smooth functions (see e.g. [Ang-Ros07, Ang04, ObeRos94]). In this approach, a general and type independent theory for the existence of solutions of a large class of systems of nonlinear PDEs is presented. The solutions may be assimilated with usual real measurable functions. The latter result is based on the characterization of the Dedekind order completion of sets of continuous functions in terms of spaces of Hausdorff continuous interval valued functions [Ang04]. Recently, the regularity of the solutions has been improved by introducing suitable uniform convergence structures on appropriate spaces of piecewise smooth functions; see [vdW09a, vdW09b] and references therein.

Analytic generalized functions and Cauchy-Kowalevski theorem for CGF

The theory of (real or complex) analytic CGF has been developed as a consequence of the definition of holomorphic CGF, see e.g. [Ar-Fe-Ju05, Ver08, PiScVa09] and references therein. The natural Cauchy-Riemann condition $(\partial_x + i\partial_y)\, u = 0$ is coupled with a study of analyticity by using classical $\widetilde{\mathbb{C}}$-valued power series which converge in the sharp topology. Due to the ultrapseudonorm property of the sharp norm, we have that the series $\sum_{n=0}^{+\infty} a_n z^n$ of $\widetilde{\mathbb{C}}$ converges if and only if $a_n z^n \to 0$ as $n \to +\infty$ (in the sharp topology). Although a notion of radius of convergence and related suitable properties have been established, the typical example of this type of convergent power series is $\sum_{n=0}^{+\infty} a_n\, \mathrm{d}\varepsilon^{P(n)} z^n$, where $P(n) \in \mathbb{R}[n]$ is a polynomial with positive leading coefficient, $(a_n)_{n\in\mathbb{N}}$ is a sharply bounded sequence, and $\mathrm{d}\varepsilon := [\varepsilon] \in \widetilde{\mathbb{R}}$. As said above, this notion of series doesn't enable to include the classical exponential function on compactly supported points.

A linear version of the Cauchy-Kowalevski theorem for CGF with holomorphic generalized coefficients has been proved in [Col85]. The existence proof is a classical ε-wise reduction to the standard Cauchy-Kowalevski theorem. We underscore here two limitations of this approach: (1) in Colombeau algebras one is forced to consider only bounded coefficients because of the moderateness condition (we always recall the classical ODE $y' = [1/\varepsilon]y$); (2) The classical obvious proof of uniqueness of classical holomorphic solutions does not work here since a holomorphic generalized function and all its partial derivatives may have the value 0 at a point of its domain, without being null in any standard neighbourhood of this point. In the present research proposal, the former limitation is solved by choosing a suitable gauge ρ (depending on the PDE we want to solve) for the ring of generalized numbers ${}^{\rho}\widetilde{\mathbb{R}}$, whereas the latter is again tied to the above mentioned notion of hyperfinite power series.

Hyperfinite methods in nonstandard analysis[4]

Hyperfinite methods first appeared in NSA, where infinite natural numbers are frequently used to approximate continuum spaces and problems by hyperfinite discrete ones. A standard example of use of hyperfinite methods in NSA is the Bernstein-Robinson solution of the problem raised by P.R. Halmos and K.T. Smith

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concerning invariant subspaces of polynomial compact operators; see [BerRob66, Ber67], and [Hal66] for a simpler proof in a standard setting. Hyperfinite sets are also used in NSA, e.g., in the approximation of Riemann integrals by hyperfinite sums and in the hyperfinite Euler method, see e.g. [Gol88].

The idea to fix an hyperreal infinitesimal $\rho \in {}^*\mathbb{R}$ and to consider the ring $\{x \in {}^*\mathbb{R} \mid \exists n \in \mathbb{N} : |x| < \rho^{-n}\}$ modulo the ideal $\{x \in {}^*\mathbb{R} \mid \forall n \in \mathbb{N} : |x| < \rho^{n}\}$ date back to [Rob73, LigRob75] for applications to asymptotic analysis (see also [Lux76, Pes91, TodWol04, TodVer08] for more recent works). Independently, J.F. Colombeau in [Col84] had the same algebraic idea, but without using the hyperreal field ${}^*\mathbb{R}$ and considering only the infinitesimal $\rho_\varepsilon = \varepsilon$. It is therefore clear that the trivial generalization ${}^\rho\widetilde{\mathbb{R}}$ owes its inception to both A. Robinson and J.F. Colombeau. On the other hand, the field ${}^\rho\mathbb{R}$ and the ring ${}^\rho\widetilde{\mathbb{R}}$ have very different formal properties. For example, ${}^\rho\mathbb{R}$ is real closed and totally ordered and, under a suitable form of the generalized continuum hypothesis, see [Tod13, TodVer08], a field isomorphism ${}^{\rho_1}\mathbb{R} \simeq {}^{\rho_2}\mathbb{R}$ for any pair of infinitesimal $\rho_1, \rho_2 \in {}^*\mathbb{R}$ holds. The ring ${}^\rho\widetilde{\mathbb{R}}$ is only partially ordered and the ring isomorphism ${}^{\rho_1}\widetilde{\mathbb{R}} \simeq {}^{\rho_2}\widetilde{\mathbb{R}}$ holds e.g. if both the infinitesimals ρ_i are strictly decreasing to zero, see [LuGi16b].

It is not completely clear why NSA finds difficulties in getting accepted by a large community of mathematicians, and hence why one frequently encounters difficulties in getting published in this field. The important problem of constructive and non-constructive models of the continuum has been analysed thoroughly in [ScBeOs01]. As proved by [Pal95, Palm97, Palm98], a key non constructive property of NSA is surely tied to the standard part map (and hence basic NSA tools to deal with hyperfinite sets, such as the full transfer principle and the Loeb measure have to be considered, generally speaking, non constructive).

Fourier transform for CGF

The Fourier transform for CGF has first been studied in [Col85, Das91, NedPil92]. The basic idea is to define the notion of integral of a tempered CGF on the whole of $\widetilde{\mathbb{R}}^n$. This is accomplished by multiplying the generalized function by a suitable damping measure, see [Hor99] for a general approach using this idea. This notion of Fourier transform shares several properties with the classical one, but it lacks the Fourier inversion theorem, which holds only at the level of association in the sense of generalized tempered distributions, see [Col85, Thm. 4.3.4] and [Sor98]. See also [Sor96] for a Paley-Wiener like theorem. Intuitively, one could say that the use of the multiplicative damping measure introduces a perturbation of infinitesimal frequencies that inhibit the inversion result. The only known possibility to obtain a strict Fourier inversion theorem is the approach used by [Nig16], where smoothing kernels are used as index set and therefore infinite dimensional calculus in convenient vector spaces is needed.

Operator theory for locally convex $\widetilde{\mathbb{C}}$-modules

The theory of locally convex $\widetilde{\mathbb{C}}$-modules is the key reference for the last part of the present project, see [Gar05a, Gar05b, Gar09, GarVer11]. Even if a general notion of operator defined by a net (T_ε) of classical operators has never been considered in these works, very general theorems like the closed graph and open mapping theorems, [Gar09], and the uniform boundedness principle, [Gar05a] have been proved. For the Hahn-Banach theorem in non-Archimedean valued fields, see [Ing52]. For a version of the Hahn-Banach theorem framed in subfields of $\widetilde{\mathbb{C}}$, see [May07]. The impossibility of a general Hahn-Banach theorem for $\widetilde{\mathbb{C}}$-functionals has been proved in [Ver10]. Therefore, note that the possibility to prove a general Hahn-Banach theorem for $\widetilde{\mathbb{C}}$-functionals defined by a suitable net (T_ε) of classical functionals is still an open problem.

2.2 State of the art in applicants' research

In this section, we will briefly introduce some of the key notions of the present research proposal.

Some basic notations we will use in the following are: nets in the variable $\varepsilon \in I := (0, 1]$ are written as (x_ε); if (x_ε) is a net of real numbers, $x = [x_\varepsilon]$ denotes the corresponding equivalence class with respect to the equivalence relation $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ iff $|x_\varepsilon - y_\varepsilon| = O(\rho_\varepsilon^m)$ for every $m \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

The ring of generalized numbers ${}^\rho\widetilde{\mathbb{R}}$

The role of the infinitesimal nets (ε^n), $n \in \mathbb{N}$, has been generalized in several ways, cf. [GioNig15, GiLu15] and references therein. In the work [GiLu15], we showed that our notion of *asymptotic gauge* is the most general one since it is able to include all the different types of Colombeau-like algebras: the special one $\mathcal{G}^{\mathrm{s}}$, the full one $\mathcal{G}^{\mathrm{e}}$, the NSA-based based algebra of asymptotic functions $\hat{\mathcal{G}}$, the diffeomorphism invariant algebras $\mathcal{G}^{\mathrm{d}}$, $\mathcal{G}^2$ and $\hat{\mathcal{G}}$, the Egorov algebra, and the algebra of non-standard smooth functions ${}^*\mathcal{C}^\infty(\Omega)$, see also [LuGi16b]. However, minimal consistency properties of the embeddings ι of distributions and σ of smooth functions, cf. [GiLu15, Thm. 4.12], imply that the simple initial choice of a different net (ρ_ε) instead of (ε) is necessarily the one that enables general results and good embedding properties. The definition of ${}^\rho\widetilde{\mathbb{R}}$ is therefore that of *ρ-moderate nets* ($\exists N \in \mathbb{N} :\ x_\varepsilon = O(\rho_\varepsilon^{-N})$) modulo *$\rho$-negligible nets* ($\forall n \in \mathbb{N} :\ x_\varepsilon = O(\rho_\varepsilon^n)$). The point of view of GSF is frequently that of a theory where ${}^\rho\widetilde{\mathbb{R}}$ acts as the ring of scalars for all the subsequent constructions. For example, sharp topology is preferably defined using the absolute value $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$ and the balls $B_r(x) := \{y \in {}^\rho\widetilde{\mathbb{R}}^d \mid |y - x| < r\}$, where $r > 0$ is a strictly positive generalized number, i.e. $r \in {}^\rho\widetilde{\mathbb{R}}_{\ge 0}$ and r is invertible. Different types of topologies are possible on ${}^\rho\widetilde{\mathbb{R}}^d$, in particular the Fermat one, where we take $r \in \mathbb{R}_{>0}$ as possible radii, see [GiKuVe15]. In this proposal, we use the notation $\mathrm{d}\rho := [\rho_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$.

Generalized smooth functions as a category of smooth set-theoretical maps

If $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ are arbitrary subsets of generalized numbers, a GSF $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$ can be simply defined as a set-theoretical map $f : X \longrightarrow Y$ such that

$$\exists (f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)^I\ \forall [x_\varepsilon] \in X\ \forall \alpha \in \mathbb{N}^n :\ (\partial^\alpha f_\varepsilon(x_\varepsilon)) \text{ is } \rho-\text{moderate and } f(x) = [f_\varepsilon(x_\varepsilon)], \tag{1}$$

see [GiKuVe15, GiKuSt16]. If (1) holds, we say that the net (f_ε) *defines* f. If $X = \widetilde{\Omega}_c$, the set of compactly supported points in the open set $\Omega \subseteq \mathbb{R}^n$, then ${}^\rho\mathcal{GC}^\infty(\widetilde{\Omega}_c, {}^\rho\widetilde{\mathbb{R}})$ coincides exactly with the set-theoretical maps induced by all the CGF of the algebra $\mathcal{G}^{\mathrm{s}}(\Omega)$. The greater flexibility in the choice of the domains X leads to the closure of GSF with respect to composition, to the extreme value theorem on closed intervals bounded by infinite numbers, to purely infinitesimal solutions of ODE or also to inverses of given GSF, see [GiKuVe15, GiKuSt16, LuGi16a, GiKu16].

Classical theorems like the chain rule, existence and uniqueness of primitives, integration by change of variables, the intermediate value theorem, mean value theorems, the extreme value theorem, Taylor's theorem in several forms for the remainder, suitable sheaf properties, the local inverse and implicit function theorems, some global inverse function theorems, the Banach fixed point theorem, the Picard-Lindelöf theorem and several results in the classical theory of calculus of variations, hold for these GSF, see [GiKuVe15, GiKuSt16, GiKu16, LuGi16a, LeLuGi16]. One of the peculiar properties of GSF is that these extensions of classical theorems for smooth functions have very natural statements, formally similar to the classical ones but, e.g., where the sharp topology replaces the Euclidean one, where assumptions of invertibility in ${}^\rho\widetilde{\mathbb{R}}$ replace the property of being different from zero in $\mathbb{R}$ and where the strict order relation $>$ of ${}^\rho\widetilde{\mathbb{R}}$ replaces the usual $>$ of $\mathbb{R}$, mostly for topological properties. All this underscores the different philosophical approach as compared to [Ar-Fe-Ju05], which constitutes a more general approach (in [GiKuVe15, GiKu16] it is proved that every GSF is smoothly differentiable in the sense of [Ar-Fe-Ju05]), but where some of these classical theorems do not hold.

Particularly interesting for the present research proposal are the Fermat-Reyes theorem, [GiKuSt16], and the theory of compactly supported GSF, [GioKun16a]. The former states that given $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ defined on a sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $v = [v_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$, there exists another GSF $r \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$ defined on a sharp neighbourhood T of $U \times \{0\}$ such that $f(x + hv) = f(x) + h \cdot r(x, h)$ for all $(x, h) \in T$. Any two such functions r coincide in a sharp neighbourhood of $U \times \{0\}$. This allows to define intrinsically $\frac{\partial f}{\partial v}(x) := r(x, 0)$ and to compute derivatives as suitable incremental ratios. Clearly, one also has that $\frac{\partial f}{\partial v}(x) = \left[\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon)\right]$. The theory of compactly supported GSF is based on the notion of functionally compact set, i.e. sharply bounded internal sets $K = [K_\varepsilon] = \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in K_\varepsilon \text{ for } \varepsilon \text{ small}\} \subseteq B_r(0)$, for some $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, generated by a net $K_\varepsilon \Subset \mathbb{R}^n$ of compact sets. For example, every closed interval $[a, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$ is functionally compact. On this type of sets, GSF satisfy the extreme value theorem and hence on every closed interval they can be integrated $\int_a^b f \in {}^\rho\widetilde{\mathbb{R}}$. If $S \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, then $\text{ext}(S) := \{x \in {}^\rho\widetilde{\mathbb{R}}^n \mid |x - s| > 0 \ \forall s \in S\}$ is called *strong exterior of* S, and a GSF $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$ is called *compactly supported in* K, and we write $f \in \mathcal{GD}_K(X, Y)$, if f can be defined by a net $(f_\varepsilon) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ such that

$$\forall \alpha \in \mathbb{N}^n \, \forall [x_\varepsilon] \in \text{ext}(K) : \ [\partial^\alpha f_\varepsilon(x_\varepsilon)] = 0. \tag{2}$$

The spaces $\mathcal{GD}_K(U, {}^\rho\widetilde{\mathbb{R}}^d)$ share many properties with the classical spaces $\mathcal{D}_K$ of compactly supported smooth functions. In particular, they are sharply complete, their strict inductive limit is sharply complete, and their sharp topology can be defined using a countable family $\|f\|_i := \left[\max_{\substack{|\alpha| \le i \\ 1 \le k \le d}} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f^k_\varepsilon(x))|\right] \in {}^\rho\widetilde{\mathbb{R}}$, $i \in \mathbb{N}$, of ${}^\rho\widetilde{\mathbb{R}}$-valued norms, see [GioKun16a]. Concerning the previously mentioned proposal about a generalization of the Fourier transform, it is finally important to say that if $k \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ is an infinite number, i.e. $\lim_{\varepsilon \to 0} k_\varepsilon = +\infty$, and $K_\varepsilon := \{x \in \Omega \mid |x| \le k_\varepsilon\}$, then for all CGF $f \in {}^\rho\mathcal{GC}^\infty(\widetilde{\Omega}_c, {}^\rho\widetilde{\mathbb{R}}^d)$ there exists a GSF $\bar{f} \in \mathcal{GD}_K({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^d)$ compactly supported in $K := [K_\varepsilon]$ such that $\bar{f}|_{\widetilde{\Omega}_c} = f$. Therefore, since CGF, and hence also distributions, are defined only on finite points of $\widetilde{\Omega}_c$, they can be viewed as compactly supported GSF, i.e. as GSF which are zero with all their derivatives in the strong exterior of any infinite functionally compact set.

[5]To deal with partial derivatives at boundary points (note that the Fermat-Reyes theorem considers only interior points), one frequently considers *solid* functionally compact sets, i.e. functionally compact sets $K \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ such that the interior of K in the sharp topology is dense in K. If K is a solid functionally compact set then, as above, on the space ${}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$ we can introduce a countable family of ${}^\rho\widetilde{\mathbb{R}}$-valued norms that defines a Cauchy complete sharp topology.

We also recently proved a generalization of the Banach fixed point theorem and of the corresponding Picard-Lindelöf theorem that are applicable to any Cauchy problem with a normal generalized smooth PDE, see [GioLup17]. The basic idea to generalize these results of ODE theory is the classical notion of loss of derivatives: if $K \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ is a solid functionally compact set, and $y_0 \in X \subseteq {}^\rho\mathcal{GC}^\infty(K, {}^\rho\widetilde{\mathbb{R}}^d)$, then we say that $P : X \longrightarrow X$ is a *finite contraction on* X *with loss of derivatives* $L \in \mathbb{N}$ *starting from* y_0 if

$$\forall i \in \mathbb{N} \, \exists \alpha_i \in {}^\rho\widetilde{\mathbb{R}}_{>0} : \ \|P(u) - P(v)\|_i \le \alpha_i \cdot \|u - v\|_{i+L} \ \forall u, v \in X$$

[5]The following part is completely new.

and

$$\lim_{\substack{n,m\to+\infty\\ n\le m}} \alpha_{i+mL}^n \cdot \|P(y_0)-y_0\|_{i+mL}=0,$$

where the limit is taken with respect to the sharp topology and with n, $m\in\mathbb{N}$ (the use of finite natural numbers $\mathbb{N}$ instead of hyperfinite ones ${}^{\rho}\widetilde{\mathbb{N}}$ justifies the name *finite* contraction). We proved that if $\alpha_i\le\alpha_{i+1}$ and X is sharply Cauchy complete, then P is sharply continuous, $\exists\lim_{n\to+\infty}P^n(y_0)=:y$ and $P(y)=y$. Note explicitly that, in general, we don't have the uniqueness of the fixed point y, exactly because we can have a loss of $L>0$ derivatives. If $T\subseteq{}^{\rho}\widetilde{\mathbb{R}}$, $S\subseteq{}^{\rho}\widetilde{\mathbb{R}}^n$ are solid functionally compact sets, $Y\subseteq{}^{\rho}\mathcal{GC}^{\infty}(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ and the set-theoretical map $F:T\times S\times Y\longrightarrow{}^{\rho}\widetilde{\mathbb{R}}^d$ satisfies $F(-,-,y)\in{}^{\rho}\mathcal{GC}^{\infty}(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ for all $y\in Y$, then we say that F is *uniformly Lipschitz on* Y *with constants* $(\Lambda_i)_{i\in\mathbb{N}}\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}^{\mathbb{N}}$ *and loss of derivatives* $L\in\mathbb{N}$ *if*

$$\forall i\in\mathbb{N}\,\forall u,v\in Y:\ \|F(-,-,u)-F(-,-,v)\|_i\le\Lambda_i\cdot\|u-v\|_{i+L}.$$

We can prove that any PDE of the form

$$\partial_t y(t,x)=G\left[t,x,\partial_x y(t,x)\right], \tag{3}$$

where G is a GSF, defines a uniformly Lipschitz map on the space

$$Y=\left\{y\in{}^{\rho}\mathcal{GC}^{\infty}(T\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)\mid\|y-y_0\|_i\le r_i\ \forall i\in\mathbb{N}\right\}. \tag{4}$$

We finally proved the following generalization of the Picard-Lindelöf theorem: let $t_0\in{}^{\rho}\widetilde{\mathbb{R}}$, α, $r_i\in{}^{\rho}\widetilde{\mathbb{R}}_{>0}$ and $T_\alpha:=[t_0-\alpha,t_0+\alpha]$. Let $y_0\in{}^{\rho}\mathcal{GC}^{\infty}(S,H)$, where $H\subseteq{}^{\rho}\widetilde{\mathbb{R}}^d$ is a sharply closed set such that $\overline{B_r(y_0(x))}\subseteq H$ for all $x\in S$. Define Y_α as in (4), but using T_α instead of T, and assume that F is uniformly Lipschitz on Y_α with constants $(\Lambda_i)_{i\in\mathbb{N}}$ and loss of derivatives L. Finally, assume that

$$\begin{aligned}
&\Lambda_i\le\Lambda_{i+1}\quad\forall i\in\mathbb{N}\\
&\|F(-,-,y)\|_i\le M_i(y)\quad\forall y\in Y_\alpha\\
&\alpha\cdot M_i(y)\le r_i\quad\forall i\in\mathbb{N}\\
&\lim_{\substack{n,m\to+\infty\\ n\le m}}\alpha^{n+1}\cdot\Lambda_{i+mL}^n\cdot\|F(-,-,y_0)\|_{i+mL}=0
\end{aligned}$$

Then there exists a solution $y\in{}^{\rho}\mathcal{GC}^{\infty}(T_\alpha\times S,{}^{\rho}\widetilde{\mathbb{R}}^d)$ of the Cauchy problem

$$\begin{cases}\partial_t y(t,x)=F(t,x,y) & \forall(t,x)\in T_\alpha\times S\\ y(0,x)=y_0(x) & \forall x\in S\end{cases}$$

Note explicitly that this is only an existence result and nothing is stated about the uniqueness of the solution. The proofs are essentially a generalization of the classical Banach fixed point theorem and Picard-Lindelöf theorem to the GSF framework. Therefore, the importance of this result lies on treating Cauchy problems for normal PDE (i.e. of the form (3) in case of first order PDE) as a generalization of ODE. A generalization of these results to k-th order PDE is a work in progress.

3 Work program

In this section, we describe the methods we plan to employ in carrying out the research program sketched above. For each one of the four parts of the research project, we will also give a (subjective) judgement of its feasibility. Of course, this quantitative judgement of feasibility will be justified and will also be used to quantify and support the project's time planning.

3.1 Basic hyperfinite methods in the ring ${}^{\rho}\widetilde{\mathbb{R}}$ of generalized numbers

In this part of the project, we have to lay the foundations for the use of hyperfinite numbers ${}^{\rho}\widetilde{\mathbb{N}} := \left\{\text{int}\left([|x_\varepsilon|]\right) \mid [x_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{R}}\right\}$ for subsequent more advanced topics. We therefore start to prove classical properties of limits of *hyperfinite sequences* $s : {}^{\rho}\widetilde{\mathbb{N}} \longrightarrow {}^{\rho}\widetilde{\mathbb{R}}$: algebraic properties, relations with continuous functions, squeeze theorem, monotonous sequences, Cauchy sequences, relations with limits of classical sharply convergent sequences in ${}^{\rho}\widetilde{\mathbb{R}}^{\mathbb{N}}$, etc.

The definition of *hyperfinite series* applies to *ordinary* sequences $(a_n)_{n\in\mathbb{N}}$ of ${}^{\rho}\widetilde{\mathbb{R}}$:

$$\sum_{n\in\widetilde{\mathbb{N}}} a_n := \left[\sum_{n=0}^{N_\varepsilon} a_n\right], \tag{5}$$

where $N = [N_\varepsilon] \in {}^{\rho}\widetilde{\mathbb{N}}$. Note explicitly that, in order to define (5), we don't need to give a meaning to terms a_n for $n \in {}^{\rho}\widetilde{\mathbb{N}}$ because $N_\varepsilon \in \mathbb{N}$ for each ε. For example, in a power series we do not need to define x^n for $x \in {}^{\rho}\widetilde{\mathbb{R}}$ and $n \in {}^{\rho}\widetilde{\mathbb{N}}$. On the other hand, the sequence of partial sums $N \in {}^{\rho}\widetilde{\mathbb{N}} \mapsto \sum_{n=0}^{N} a_n \in {}^{\rho}\widetilde{\mathbb{R}}$ is defined on ${}^{\rho}\widetilde{\mathbb{N}}$ and can hence possibly converge to any value in ${}^{\rho}\widetilde{\mathbb{R}}$, similarly to the sharp convergence of $n \in {}^{\rho}\widetilde{\mathbb{N}}_{>0} \mapsto \frac{1}{n} \in {}^{\rho}\widetilde{\mathbb{R}}$ to 0. We then plan to extend to hyperfinite series classical convergence criteria.

Once the notion (5) has been introduced and studied, it is natural to open the examination of hyperfinite power series, their radius of convergence, algebraic operations on them, composition and their differentiation and integration. Note explicitly that in this setting the notion of radius of convergence gives only a *sufficient* condition for the convergence of a hyperfinite series: if the series converges at an invertible $x_0 \in {}^{\rho}\widetilde{\mathbb{R}}$, then it also converges at any x with $|x| < |x_0|$. The convergence of the zero hyperfinite power series for the function $f(x) = e^{1/x}$ if $x > 0$ and $f(x) = 0$ otherwise in any ball $B_{\mathrm{d}\rho^a}(0)$, $a \in \mathbb{R}_{>0}$, shows that a supremum of these radii of convergence need not exist in this non-Archimedean setting.

Hyperfinite sequences and sums will also be used to prove properties of hyperfinite Riemann sums for GSF, and to formalize a hyperfinite Euler method with an invertible infinitesimal step size.

To underscore the differences with respect to the standard notion of series used for CGF, we finally want to recover classical examples of analytic elementary functions. We recall that we already proved that $\sum_{n\in\widetilde{\mathbb{N}}} k^n = \frac{1}{1-k}$ for all $k \in (0,1) \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ and that $\sum_{n\in\widetilde{\mathbb{N}}} \frac{x^n}{n!} = e^x$ for all finite $x \in {}^{\rho}\widetilde{\mathbb{R}}$.

Risks: The plan is systematic and reasonably certain. Several properties could also be proved ε-wise. For these reasons, we do not expect problems in this part of the project.

Subjective assessment of feasibility: Because the risks are really minimal, we assess as *very high* the feasibility of this unit of the proposal.

3.2 The method of characteristics and hyperfinite Fourier transform

Method of characteristics

As presented by [Eva10], the key instruments adopted to develop the method of characteristics are: free composition of sufficiently regular functions and their calculus, implicit and inverse function theorems. Since all these results are available for GSF, and with formally very similar statements (cf. [GiKuVe15, GiKuSt16, GiKu16, LeLuGi16]), we strongly believe that this method can be fully carried out for GSF.

Hyperfinite Fourier transform

The idea of defining a domain-dependent Fourier-like transform on a functionally compact unbounded interval $[-k,k]$ is not related to the hypernatural numbers ${}^{\rho}\widetilde{\mathbb{N}}$. However, we already mentioned that GSF behave on functionally compact sets like if they were compact sets: e.g. the extreme value theorem, the existence of definite integrals and a full theory of compactly supported GSF hold for this type of sets. Therefore, we can say that GSF behave on functionally compact sets like ordinary smooth functions operate on compact sets. We can hence note the analogy between ${}^{\rho}\widetilde{\mathbb{N}}$, which contains also infinite numbers but which act as finite ones, and functionally compact sets. This provides a strong motivation for our use of the adjective "hyperfinite" also in the case of this Fourier transform.

In general, we define

$$\mathcal{F}_K(f)(\omega) := (2\pi)^{-n/2}\int_K f(x)e^{-ix\cdot\omega}\,\mathrm{d}x,$$

where $K \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ is a functionally compact set, $\omega \in {}^{\rho}\widetilde{\mathbb{R}}^h$ and $f \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}\widetilde{\mathbb{R}})$ is *any* GSF. Clearly, the fact that $\mathcal{F}_K : {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}\widetilde{\mathbb{R}}) \longrightarrow {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}^n, {}^{\rho}\widetilde{\mathbb{R}})$ is one of the key properties of this hyperfinite Fourier transform. The plan to advance this idea is the following:

(i) We already proved several basic properties in one dimension, and we hence plan to generalize these derivations, and different other classical results, to several dimensions. For example, we have that $\mathcal{F}_K(\tau_s f) = e^{is(-)}\mathcal{F}_{K+s}(f)$, $\mathcal{F}_K(f)'(\omega) = -i\mathcal{F}_K(t\cdot f(t))$, $\mathcal{F}_K(g(-.))(\omega) = \mathcal{F}_K(g)(-\omega)$, etc. For a full understanding of the ideas presented below, here we mention that if $K = [-k,k] \subseteq {}^{\rho}\widetilde{\mathbb{R}}$ and $f(k) = f(-k) = 0$, then $\mathcal{F}_K(f')(\omega) = i\omega\mathcal{F}_K(f)(\omega)$. As we will see below, the applicability of this weak version of the derivative property is related to the possibility to view any CGF as a compactly supported function which is hence zero at the boundary points of the interval $[-k,k]$, see Sec. 2.2.

(ii) We already proved that $\mathcal{F}_K(\mathcal{F}_K^{-1}(f))(\omega) = f(\omega)$ (the equality $=$ here is clearly that in ${}^{\rho}\widetilde{\mathbb{R}}$) for all compactly supported $f \in \mathcal{GD}_K({}^{\rho}\widetilde{\mathbb{R}}, {}^{\rho}\widetilde{\mathbb{R}})$ and all finite $\omega \in {}^{\rho}\widetilde{\mathbb{R}}$. It is interesting to note that the proof involves the use of *two* gauges ρ and σ and suitably connected infinite numbers. In fact, we first proved that if $\sigma_\varepsilon := \rho_\varepsilon^{1/\varepsilon}$, $h = \mathrm{d}\rho \in {}^{\sigma}\widetilde{\mathbb{R}}$, $k = \mathrm{d}\sigma \in {}^{\sigma}\widetilde{\mathbb{R}}$ and we set $x =_\rho y$ to denote $|x-y| \le \mathrm{d}\rho^n$ in ${}^{\sigma}\widetilde{\mathbb{R}}$ for all $n \in \mathbb{N}$, then $f(-\omega) =_\rho \mathcal{F}_h(\mathcal{F}_k(f))(\omega)$ for all finite $\omega \in {}^{\sigma}\widetilde{\mathbb{R}}$. Clearly, this relation $=_\rho$ on ${}^{\sigma}\widetilde{\mathbb{R}}$ entails the usual equality on ${}^{\rho}\widetilde{\mathbb{R}}$. We therefore plan to generalize this proof to several dimensions.

(iii) Finally, it is also important to study whether this hyperfinite Fourier transform commutes with the embedding of standard Schwartz functions.

Our procedure to apply $\mathcal{F}_K$ in the study of PDE is the following:

(a) We can start from a linear differential problem and assume that it has a solution $u \in {}^{\rho}\mathcal{GC}^{\infty}(\widetilde{\Omega}_c, {}^{\rho}\widetilde{\mathbb{R}})$.

(b) We can hence take any infinite number $k \in {}^{\rho}\widetilde{\mathbb{R}}$ and consider $K := \{x \in \Omega \mid |x| \le k\}$, $K/2 := \{x \in \Omega \mid |x| \le k/2\} \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$, and the (unique, see [GioKun16a, Thm. 24]) compactly supported function $\bar{u} \in \mathcal{GD}_{K/2}({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}})$ such that $\bar{u}|_{\widetilde{\Omega}_c} = u$. Since $\bar{u}(x) = 0$ for all $x \in \text{ext}(K)$, we have $\mathcal{F}_K(\partial_j \bar{u})(\omega) = i\omega_j \mathcal{F}_K(\bar{u})(\omega)$. As usual, this often allows to transform the differential problem into a simpler problem.

(c) We finally use the inversion theorem, noting that it applies at all $\omega \in \widetilde{\Omega}_c$ and to the compactly supported GSF $\bar{u}$, so that we can recover the initial CGF u.

(d) Vice versa, we can directly start from the $\mathcal{F}_K$-transformed problem and show that the inverse hyperfinite Fourier transform $\bar{u} := \mathcal{F}_K^{-1}(\bar{U})$ of the solution $\bar{U}$ of the transformed problem is a GSF which is compactly supported on K. This last property depends on the particular differential problem we are interested in.

Risks: Because of the results we already proved, we think that no reasonable risk is foreseeable in this unit of the proposal.
Subjective assessment of feasibility: Our assessment of feasibility for this section is therefore *very high*.

3.3 Generalized smooth operators

Definition of generalized smooth operators

To introduce our idea to define the concept of generalized smooth operator (GSO), we first need to fix two properties $\mathcal{S}$ and $\mathcal{O}$ that we want "to lift" from the static Archimedean universe associated with $\mathbb{R}$ to the dynamic non-Archimedean one associated with ${}^\rho\widetilde{\mathbb{R}}$. For example $\mathcal{S}(X)$ could be "X is a Fréchet space" or "X is a Banach space", and $\mathcal{O}(T)$ could be "T is a Weissinger contraction" or "T is a monotone operator", etc. Now, let $S_j \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $X_j \subseteq {}^\rho\mathcal{GC}^\infty(S_j, {}^\rho\widetilde{\mathbb{R}}^d)$, $j = 1, 2$; we say that $T : X_1 \longrightarrow X_2$ is a *generalized smooth operator of type* $\mathcal{S}$, $\mathcal{O}$ if $T : X_1 \longrightarrow X_2$ is a set-theoretical function and there exist (T_ε), $(X_{j\varepsilon})$, $(S_{j\varepsilon})$ such that for all $j = 1, 2$ the following conditions hold

(i) $\forall \varepsilon \in I : \; S_{j\varepsilon} \subseteq \mathbb{R}^n$ and $X_{j\varepsilon} \subseteq \left(\mathbb{R}^d\right)^{S_{j\varepsilon}}$.

(ii) For all $\varepsilon \in I$ both the properties $\mathcal{S}(X_{j\varepsilon})$ and $\mathcal{O}(T_\varepsilon)$ hold.

(iii) If $f \in {}^\rho\mathcal{GC}^\infty(S_j, {}^\rho\widetilde{\mathbb{R}}^d)$, then $f \in X_j$ if and only if there exists a net (f_ε) that defines f and such that $f_\varepsilon \in X_{j\varepsilon}$ for ε small.

(iv) If $f \in X_1$ and the net (f_ε) defines f and satisfies $f_\varepsilon \in X_{j\varepsilon}$ for all ε small, then the net $(T_\varepsilon(f_\varepsilon))$ defines $T(f)$.

In this way, condition (ii) represents problem-related properties we want to extend from $X_{j\varepsilon}$ and T_ε to X_j and T. Condition (iii) links the membership $f \in X_j$ to the ε-wise membership $f_\varepsilon \in X_{j\varepsilon}$. Finally, condition (iv) associates the values $T(f)$ with the net $(T_\varepsilon(f_\varepsilon))$. We can therefore say that (iii) and (iv) are more logical conditions, whereas (ii) is the technical one.

The first important aim of this part of the project is to check how this notion of GSO fits into the theory of locally convex $\widetilde{\mathbb{C}}$-modules and in what form the three pillars of functional analysis (closed graph and open mapping theorems, [Gar09], the uniform boundedness principle, [Gar05a], and the Hahn-Banach theorem framed in subfields of $\widetilde{\mathbb{C}}$, [May07]) hold for this type of operators. We also want to ensure that the counterexample of the Hahn-Banach theorem presented in [Ver10] does not apply to GSO. Finally, we plan to possibly prove a form of the Hahn-Banach theorem for generalized smooth ${}^\rho\widetilde{\mathbb{R}}$-functionals by substituting the use of supremums in classical proofs with maximums of suitable GSF on functionally compact sets.

Hyperfinite iterates

Generally speaking, Cauchy problems for ODE $y' = F(t,y)$, $y(t_0) = y_0$, where $F \in {}^\rho\mathcal{GC}^\infty(J \times U, {}^\rho\widetilde{\mathbb{R}})$, have a unique solution on a sharp neighbourhood of t_0. Sometimes this neighbourhood is infinitesimal, and a larger domain is not possible (e.g. let $F(t,y) = -\frac{t}{1+y}\,\mathrm{d}\rho^{-1}$, then the solution such that $y(0) = 0$ is $y(t) = \sqrt{1 - t^2\,\mathrm{d}\rho^{-1}} - 1$, which is defined in the infinitesimal interval $(-\sqrt{\mathrm{d}\rho}, \sqrt{\mathrm{d}\rho})$ and cannot be extended). To obtain a better result for particular but interesting F (a trivial example: ${}^\rho\widetilde{\mathbb{R}}$-linear ODE), a possible solution is to consider hyperfinite Picard-Lindelöf iterates, see [LuGi16a]. In other words, if X is a space of GSF and we take $T : X \longrightarrow X$ and $y \in X$, we want to consider the map $x \in {}^\rho\widetilde{\mathbb{R}} \mapsto T^{\langle n\rangle}(y)(x) := [(T(y) \circ \overset{n_\varepsilon}{\ldots} \circ T(y))\,(x_\varepsilon)]$ for $n \in \widetilde{\mathbb{N}}$. In our proof of the Picard-Lindelöf theorem with hyperfinite iterates, [LuGi16a], we formally assumed that F has the property that the iterates $T^{\langle n\rangle}(y)$ are always GSF. Clearly, a more useful sufficient condition has to be found. Our first idea is to solve the analogous problem for the hyperfinite iterates $f^{\langle n\rangle}$ of $f \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$. The Faà di Bruno formula yields that $\frac{d^k}{dx^k} f^{\langle n\rangle}(x) = P(f^{(k)}(y_k), f^{(k-1)}(y_{k-1}), \ldots, f'(y_1))$ for some $y_j \in {}^\rho\widetilde{\mathbb{R}}$ and for some polynomial $P = [P_\varepsilon(-)]$ of degree $\le n = [n_\varepsilon]$. If we thus assume that $|f^{(k)}(y)| \le -\log \mathrm{d}\rho$ for all derivatives $k \in \mathbb{N}$ and all points $y \in {}^\rho\widetilde{\mathbb{R}}$, then $\left| f_\varepsilon^{(k)}(y_\varepsilon)^{n_\varepsilon} \right| \le -\log(\rho_\varepsilon)^{n_\varepsilon} \le n_\varepsilon \cdot \rho_\varepsilon^{-2}$. We can hence prove that $\frac{d^k}{dx^k} f^{\langle n\rangle}(x)$ is ρ-moderate for all $n \in \widetilde{\mathbb{N}}$. We plan to find a similar condition for the iterates $T^{\langle n\rangle}(y)$.

Generalized smooth functions as functionals

Every GSF $f \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$ defines a functional via $T_f : \varphi \in \mathcal{GD}_{[a,b]}({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}) \mapsto \int_a^b f \cdot \varphi \in {}^\rho\widetilde{\mathbb{R}}$. The fundamental lemma of the calculus of variations, see [LeLuGi16, Lem. 37], entails that f is uniquely determined by this functional. All these functionals T_f are GSO in the sense of the definition given above, where $T_\varepsilon(\varphi_\varepsilon) = \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon \cdot \varphi_\varepsilon$, and where the net (φ_ε) satisfies the definition of compactly supported GSF for φ, see (2). The problem we want to solve in this unit of the research project is to investigate whether every *smooth operator* of this type is of the form T_f for some GSF f. To make precise the notion of infinite dimensional smooth operator that we plan to consider, we say that $\psi : V \longrightarrow \mathcal{GD}_{[a,b]}({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$ is smooth if $\psi^\vee := \psi(-)(-) \in {}^\rho\mathcal{GC}^\infty(V \times {}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$, where $V \subseteq {}^\rho\widetilde{\mathbb{R}}^v$ is a sharply open set. We then say that $T : \mathcal{GD}_{[a,b]}({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}) \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ is smooth if for all smooth $\varphi : V \longrightarrow \mathcal{GD}_{[a,b]}({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$, the composition $T \circ \varphi \in {}^\rho\mathcal{GC}^\infty(V, {}^\rho\widetilde{\mathbb{R}})$. This corresponds to the definition of smooth function in a diffeological space, see [Gio10a, Gio10b, Gio11a, Gio11b, GiWu16] and references therein. The idea to recover the density f from the generalized smooth functional T is hence to consider $f(x) := T(\delta_x)$, where δ_x is the Dirac delta centred at $x \in {}^\rho\widetilde{\mathbb{R}}$.

Risks: Introducing a meaningful definition is a very hard task in mathematics. Even though we believe that our first idea, mentioned above, is promising, it may happen that the results we want to achieve will force us to change the definition of GSO.
Solutions: The only possible method to face this risk is to focus the study on some important non trivial theorems and some meaningful examples, and to fine-tune the definition of GSO so as to derive these results and to include these desired cases. We are mainly thinking of Picard-Lindelöf contractions, integral functionals T_f defined by a GSF, the hyperfinite Fourier transform $\mathcal{F}_K$ and $e^A(f)(x) := \sum_{n\in\widetilde{\mathbb{N}}} \frac{A^n(f)(x)}{n!}$, where A is an operator acting on GSF f, and where $x \in {}^\rho\widetilde{\mathbb{R}}$ is such that $A(f)(x) \in {}^\rho\widetilde{\mathbb{R}}_c$.
Subjective assessment of feasibility: The foreseen risks are typical of research in mathematics. We also believe that the above mentioned definition of GSO is logically well structured and concretely grounded on the corresponding ε representatives. For this reason, we evaluate as *high* the feasibility of this part of the proposal.

3.4 General existence theorems for singular PDE

Cauchy-Kowalevski theorem

An important preliminary step in the analysis of the Cauchy problem

$$\begin{cases} \partial_t^k y = F\left(t, x, \left(\partial_t^j \partial_x^\alpha y\right)_{\substack{j<k \\ |\alpha|+j\le k}}\right) & \\ \partial_t^j y(0,x) = f_j(x) & 0 \le j < k \end{cases} \tag{6}$$

is to determine the gauge ρ (which will depend on F and all f_j) so that given smooth functions F and f_j could be embedded in ${}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ for some non trivial sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. We want to start by proving that given a finite number $F_1, \ldots, F_n \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ of smooth functions and two nets (a_ε), $(b_\varepsilon) \in \mathbb{R}^I$ such that $a_\varepsilon < b_\varepsilon$, there always exists an infinitesimal net $(\rho_\varepsilon) \downarrow 0$ (a gauge) such that the embeddings $\iota(F_j) \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$, where $a = [a_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$ and $b = [b_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$. This result will give strong indications on how to get a general result in several dimensions. Note that the case of a countable family $(F_n)_{n\in\mathbb{N}}$ implies $(\rho_\varepsilon) \equiv 0$ and hence the Schmieden-Laugwitz-Egorov model, see [GiKuSt16]. After this result, we will simply assume that F, f_j are arbitrary GSF with respect to a fixed gauge ρ.

We think it is highly promising to pursue an extension of the classical proof of the Cauchy-Kowalevski theorem (i.e. recursive calculation of coefficients of the power series of a given solution and the method of majorants) in the framework of GSF and hyperfinite series. In fact, this would greatly validate this notion of series and, moreover, it would represent a derivation method easily acceptable outside the community of CGF. Note also that the classical Kowalevski counterexample is still valid for hyperfinite power series and it hence implies the necessary limitations on the orders of derivatives in (6).

A subsequent important and related problem is to fathom the scope of the notion of analytic generalized smooth functions defined by a hyperfinite power series (*i.e.* a hyperanalytic GSF). As we mentioned above, the classical example $f(x) = e^{1/x}$ if $x \in \mathbb{R}_{>0}$ and $f(x) = 0$ otherwise results in a hyperanalytic GSF in all the sharp balls $B_{\mathrm{d}\rho^a}(0)$ for any $a \in \mathbb{R}_{>0}$. Another important example could be $\varphi(x) = \sum_1^\infty \frac{\cos(n!x)}{(n!)^n}$, which is not real analytic for each rational $x \in \mathbb{Q}$.

On the other hand, the classical construction of a Colombeau mollifier (i.e. a Schwartz function with all moments vanishing, see [GrKuObSt01]) starts by considering the inverse Fourier transform of a compactly supported function which is identically equal to 1 in a neighbourhood of the origin. By the Schwartz-Paley-Wiener theorem, this mollifier is hence an entire function. Therefore, there are strong reasons to explore whether the convolution of this mollifier with a compactly supported distribution yields a hyperanalytic GSF. We plan to perform the same study also using a compactly supported mollifier, see [NePiSc98, Del05, GiKuSt16].

Hyperfinite Picard-Lindelöf contractions for normal PDE[6]

As in the case of ODE, also the aforementioned existence theorem for Cauchy problems with normal PDE, see Sec. 2.2, usually yields an infinitesimal semi-amplitude $\alpha \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ for the solution in the time interval $[t_0 - \alpha, t_0 + \alpha]$. For example, this happens when all the Lipschitz constants Λ_i are bounded by an *infinite* generalized number $R \ge \Lambda_i$. As we already mentioned above, this is the most general obtainable result, even for ODE. To obtain sufficient conditions that guarantee an existence result on a classical finite interval, i.e. with semi-amplitude $\alpha \in \mathbb{R}_{>0}$, we need to generalize this Picard-Lindelöf theorem to hyperfinite

[6]This part is completely new.

contractions, i.e. we have to consider $P^{\langle N\rangle}(y_0)=\left[P_\varepsilon^{N_\varepsilon}(y_{0,\varepsilon})(-,-)\right]\in{}^\rho\mathcal{GC}^\infty(T\times S,{}^\rho\widetilde{\mathbb{R}}^d)$ for $N\in{}^\rho\widetilde{\mathbb{N}}$. The same step has already been accomplished for ODE, see [LuGi16a].

The hard problem of uniqueness of solutions for a Cauchy problem in PDE can be tackled using the method of characteristics for GSF that we want to develop in a previous part of this project. In fact, we can start with a solution given by this Picard-Lindelöf theorem for PDE and search for a sufficient condition on the PDE that allows to use the uniqueness of solutions of the characteristics ODE.

Moreover, a study of the relations between classical distributional solutions of linear PDE and solution given by GSF is mandatory. This can be realized using the characterization of distributions among GSF, as we already did in [LuGi16a] for the case of ODE. From this point of view, it is very important to fully clarify why the classical Lewy counterexample [Lew57, Horm63] cannot be reproduced in the setting of GSF.

Finally, the notion of morphism of gauges $f:\rho_1\longrightarrow\rho_2$ introduced in [LuGi16b] is the key notion to study PDE that can be solved in a space ${}^{\rho_1}\mathcal{GC}^\infty(T_1\times S_1,{}^{\rho_1}\widetilde{\mathbb{R}})$ but not in another ${}^{\rho_2}\mathcal{GC}^\infty(T_2\times S_2,{}^{\rho_2}\widetilde{\mathbb{R}})$. Particularly interesting is the common case where $(\rho_i(\varepsilon))_\varepsilon\downarrow 0$ (e.g. $\rho_1(\varepsilon)=\varepsilon$ and $\rho_2(\varepsilon)=e^{-1/\varepsilon}$), because in that case ${}^{\rho_1}\mathcal{GC}^\infty(T_1\times S_1,{}^{\rho_1}\widetilde{\mathbb{R}})\simeq{}^{\rho_2}\mathcal{GC}^\infty(T_2\times S_2,{}^{\rho_2}\widetilde{\mathbb{R}})$ and the unsolvable PDE is isomorphically transformed into a solvable PDE. See [LuGi16b] for details about the functorial properties of Colombeau's construction.

Risks: There is the possibility that the classical proof of the Cauchy-Kowalevski theorem cannot be extended to our context because of the existence of zero divisors in the ring of generalized numbers. Moreover, it could be difficult to find suitable conditions that guarantee $P^{\langle N\rangle}(y_0)$ is still a GSF for all $N\in{}^\rho\widetilde{\mathbb{N}}$.
Solutions: The usage of a classical ε-wise proof, see e.g. [Col85], is a secure alternative option to prove the Cauchy-Kowalevski theorem. Concerning the search for sufficient conditions, we plan to generalize to several variables the ideas we have already sketched above for the previous part of this proposal.
Subjective assessment of feasibility: We can consider as acceptable the risks we foresee for this part of the research proposal. The proposed solutions are solid and achievable, so that we can evaluate as *medium-high* the feasibility of this part of the proposal.

3.5 Hyperfinite methods for generalized smooth functions in nonstandard analysis[7]

Unfortunately, it is not feasible to include in the present proposal a complete reformulation of all the previous goals into the framework of NSA. In fact, even if some elementary results on hyperfinite sequences and series have already been studied, see e.g. [Lind88, CapCut95], the basic results of GSF, [GiKuVe15, GiKu16, GiKuSt16, GioKun16a, GioLup17, LuGi16a], have not been reformulated in a nonstandard setting. The work plan of this part of the project is therefore the following:

(i) Generalization of basic results on GSF (e.g. up to the Fermat-Reyes theorem) using the field ${}^\rho\mathbb{R}$;

(ii) Basic elementary results on hyperfinite sequences, series, power series, Euler method, definite integrals, etc. using the field ${}^\rho\mathbb{R}$;

(iii) Careful evaluation of the capabilities of our research group in a further development of this topic for a future FWF research proposal.

For this part of the project we plan to start a collaboration with Prof. M. Oberguggenberger of the University of Innsbruck, who is a renowned expert in NSA and its applications to generalized functions.

Risks: Thanks to the powerful tools of NSA to deal with hyperfinite sets, we do not foresee important risks for this part of the project. The only important concern we have to always consider is a careful time planning.

[7] This proposal is a resubmission. This part is completely new and elaborated in accordance with reviewers' indications.

Solutions: This part of the project is really feasible only if we plan to start a reformulation of GSF theory in a nonstandard setting, and we use this experience to evaluate the feasibility of a future research project. No more than 4 months are planned for this part of the project.
Subjective assessment of feasibility: For the reasons we have listed above, we assess as *very high* the feasibility of this part.

4 Scientific relevance, originality and expected benefits for potential users

The present research proposal takes place in the following international research frameworks:

- It fits well in the current threads of Austrian research, in particular those of the DIANA group of Prof. M. Kunzinger at the University of Vienna, who is also one of the main developers of the theory of GSF. Thanks to works like [GiKu16, GioKun16a, GiKuVe15, GrKuObSt01, SteVic06, Kun04, Hor99], the host institute is probably one of the best places where to perform this important further advancement of the theory of GSF. Moreover, the present research project would represent a solid way to continue the collaboration between the main applicant (P. Giordano) and other collaborators, such as L. Luperi Baglini.

- It also fits well into the research interests of the international community of Colombeau generalized functions. The main ideas of this research proposal have been presented and discussed during the recent workshop WING, June 29 – July 3 2016, at the University of Innsbruck, where they have sparked a sound interest and stimulated interesting and fruitful discussion.

Originality, innovations and benefits of the present proposal can be listed as follows:

- Our existence result for Cauchy problems with normal PDE would be of great interest for the study of PDE because it shows that PDE can essentially be treated using the usual well-known methods for ODE.

- The strengthening and the vast range of applicability of the planned extension of the Cauchy-Kowalevski theorem would be of considerable interest for the scientific community of research in PDE.

- Our notion of hyperfinite Fourier transform can be applied to any GSF and hence has a potentially huge range of applications in mathematics, physics and engineering. We also expect that our approach can suggest similar ideas in the study of oscillatory integrals. Although the latter is probably beyond what can be achieved within this proposal, we feel that it could lead to substantial contributions to microlocal analysis in algebras of generalized functions. We view it as a long-term project that we would like to pursue jointly with G. Hörmann, E. Nigsch, and M. Oberguggenberger.

- The fourth section of the project could stimulate the development of a functional analysis of generalized smooth operators, which could expand into new approaches to unbounded operators, monotone operators, compact operators, ${}^{\rho}\widetilde{\mathbb{R}}$-Banach or Hilbert spaces, fixed point theory, etc.

- The whole research project underlines the importance to consider a non-Archimedean ring of scalar as a basis for important parts of analysis and geometry. Therefore, we believe that it will be of particular interest for researcher in non-Archimedean mathematics.

- The present proposal confirms that the theory of GSF represents a versatile and powerful framework to deal with singularities and generalized functions in physics and engineering.

4.1 Importance for human resources

The articles originating from the present research project, together with past works, would constitute the basis for a monograph about generalized smooth functions. The work developed here will represent a good introduction of the post-doc collaborator into the community of Colombeau generalized functions and their applications, and could pave her/his way to a successful career as a researcher in Mathematics.

4.2 Ethical Issues

There are no ethical, security-related or regulatory aspects of the proposed research project.

5 Dissemination strategy and time planning

Dissemination strategy

Our strategy for the dissemination of the results of this proposal is addressed both to an internal and an international audience:

- From an internal point of view, we plan mini-workshops with the presentation of new results connected with the present proposal. These mini-workshops will also have the aim of embedding the new post-doc collaborator in the host research group.
- More typical internal public seminars addressed to all the interested colleagues are planned for presenting the milestones of this project.
- Contributions for two international conferences per year for the presentation of relevant results are planned. In particular, we are thinking of conferences in mathematical analysis, generalized functions and PDE. Of course, several articles for peer reviewed journals and the related dissemination by means of preprint-servers is planned.
- At the end of this project, we expect to have a sufficient amount of meaningful results to concretely start the writing of a monograph about GSF.
- [8]We would like to disseminate the results of the present project by a fruitful comparison with other settings for multiplication of distributions. For this reason, we plan to organize a joint workshop with researchers in fields such as distributions, ultradistributions, hyperfunctions, algebras of generalized functions and paracontrolled calculus and related applications. This could be realized, e.g., as a special session in one of the next ISAAC conferences (see http://mathisaac.org/).

Time planning

The research project is designed for three co-workers: two applicants (P. Giordano and M. Kunzinger) and one post-doc collaborator. The latter and the main applicant will work on this project at full time for three years. We plan to conclude the international search for the post-doc collaborator within a few months. For this reason, and for budgetary motivations this collaborator will be employed for 31 months, i.e. starting from the sixth month.

To estimate the total amount of work to be dedicated into each one of the three parts of this project, we note that

[8]This part is completely new and elaborated in accordance with reviewers' indications.

- The first part of the proposal (*Basic hyperfinite methods in the ring* ${}^{\rho}\widetilde{\mathbb{R}}$ *of generalized numbers*) could be considered the most elementary one, and it has been assessed with a *very high* feasibility. We therefore plan to accomplish its goals in 6 months.

- The second part (*General existence theorems for singular PDE*) is the longest one and it has been assessed with a *medium-high* feasibility. For these reason, we prefer to realize this part in 10 months.

- Because of their *very high* or *high* estimate of feasibility, we plan to conclude the remaining two parts of the proposal in 8 + 8 + 4 months.

The entire research project is hence planned to be concluded in 36 months and we can hence represent the time planning of the four different parts of this project in the following table:

	Time (months) $\longrightarrow$				
I	6				
II		10			
III			8		
IV				8	
V					4

6 Scientific environment

The ideal environment for realizing this project is the DIANA research group at Vienna University. This will enable us to closely collaborate with some of the leading scientists in the field, in particular with Professors Michael Grosser, Günther Hörmann, and Roland Steinbauer, as well as several PostDocs and PhD students. Moreover, we will cooperate with Prof. Michael Oberguggenberger from the University of Innsbruck, who is one of the architects of the field of algebras of generalized functions and an expert in nonstandard analysis.

We also endeavour to initiate a collaboration with the highly active research group of Professors J. Aragona and F. Juriaans in Sao Paolo, Brazil in the direction of differentiation theories for mappings on generic sets of generalized points, a field in which they are internationally leading. We believe that an exchange of ideas will be very fruitful for both research groups.

Personnel costs: In our view, work on the project goals can be pursued most effectively by funding one senior post-doc position for the main applicant Dr. P. Giordano, and one post-doc position for the duration of 35 months to a researcher with a strong background in PDE, functional analysis and possibly in NSA. Of course, we plan to create an international call to find the best candidate for this post-doc position.

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