The Picard-Lindelöf theorem for smooth PDE

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Lewy yields: no PLT for smooth PDE

Lewy (1957) and Mizohata (1962) showed that a general Picard-Lindelöf theorem (PLT) for normal PDE

$$\begin{cases} \partial_t^d y(t,x) = F\left[t, x, (\partial_x^\alpha y)_{|\alpha|\le L}, (\partial_t^\gamma y)_{|\gamma|<d}\right], \\ \partial_t^j y(t_0,x) = y_0^j(x)\ j = 0,\dots,d-1, \end{cases} \qquad (CP)$$

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E.g. Mizohata proved the existence of smooth $F \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{C})$ such that $\partial_t y + it\partial_x y = F(t,x)$ has no solution $y \in \mathcal{C}^\infty(V, \mathbb{C})$ in any open $V \subseteq \mathbb{R}^2$.

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Do we have a contraction with PDE? E.g. for $\partial_t y = c \cdot \partial_x y$, we have:

$$\left\| \int_0^t c \cdot \partial_x u \, \mathrm{d}s - \int_0^t c \cdot \partial_x v \, \mathrm{d}s \right\|_i = \left\| \int_0^t c \cdot \partial_x (u - v) \, \mathrm{d}s \right\|_i \le$$
$$\le \alpha \cdot \|c\|_i \cdot \|\partial_x (u - v)\|_i \le \alpha \cdot \|c\|_i \cdot \|u - v\|_{i+1}$$

Loss of derivatives (in the Ekeland sense)

Graded Fréchet space: $\left(\mathcal{F}, (\|-\|_i)_{i\in\mathbb{N}}\right)$ Hausdorff, complete TVS, topology defined by seminorms $\|-\|_i \le \|-\|_{i+1}$ $\forall i \in \mathbb{N}$

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Definition

Let $\left(\mathcal{F},(\|-\|_i)_{i\in\mathbb{N}}\right)$ Fréchet space, X a closed subset of $\mathcal{F}$, $y_0\in X$, $L\in\mathbb{N}$. We say that $P: X\longrightarrow X$ *is a contraction with L loss of derivatives (LOD) starting from y_0* if P is continuous and

1. For all i, $n\in\mathbb{N}$ there exist $\alpha_{i,n}\in\mathbb{R}_{>0}$ such that

$$\left\|P^{n+1}(y_0)-P^n(y_0)\right\|_i \le \alpha_{i,n}\left\|P(y_0)-y_0\right\|_{i+nL}$$

2. For all $i\in\mathbb{N}$ the following *Weissinger* condition holds:

$$\sum_{n=0}^{+\infty}\alpha_{i,n}\left\|P(y_0)-y_0\right\|_{i+nL}<+\infty. \tag{W}$$

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1. For all i, $n \in \mathbb{N}$ there exist $\alpha_{i,n} \in \mathbb{R}_{>0}$ such that
$$\left\|P^{n+1}(y_0) - P^n(y_0)\right\|_i \le \alpha_{i,n} \left\|P(y_0) - y_0\right\|_{i+nL}$$
2. For all $i \in \mathbb{N}$ the following *Weissinger* condition holds:
$$\sum_{n=0}^{+\infty} \alpha_{i,n} \left\|P(y_0) - y_0\right\|_{i+nL} < +\infty. \tag{W}$$

Suff. cond. for (1): $\left\|P^n(u) - P^n(v)\right\|_i \le \alpha_{i,n} \left\|u - v\right\|_{i+nL} \quad \forall i, n \in \mathbb{N}\, \forall u, v \in X$

Banach with loss of derivatives

Theorem

If P is a contraction with L LOD from y_0, then $(P^n(y_0))_{n\in\mathbb{N}}$ is a Cauchy sequence, hence $\bar{y} := \lim_{n\to+\infty} P^n(y_0)$ is a fixed point of P. Moreover, for all i, $n \in \mathbb{N}$ we have that

$$\left\|\bar{y} - P^n(y_0)\right\|_i \le \sum_{j=n}^{+\infty} \alpha_{i,j} \left\|P(y_0) - y_0\right\|_{i+jL}.$$

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The proof is essentially "the usual one". Thereby, the key ideas are:

1. for PDE having a fixed point depends on the initial condition y_0 and on the LOD L

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The proof is essentially "the usual one". Thereby, the key ideas are:

1. for PDE having a fixed point depends on the initial condition y_0 and on the LOD L
2. We must have a suff. cond. for the convergence of Picard iterations

A (left) inverse function theorem

Theorem

Let $\left(\mathcal{F},(\|-\|_m)_{m\in\mathbb{N}}\right)$ *be a graded Fréchet space,* $f:\mathcal{F}\to\mathcal{F}$ *be continuous,* $y_0\in\mathcal{F}$*,* $r_i\in\mathbb{R}_{>0}$ *and* $L\in\mathbb{N}$*. Set* $F(x):=x-f(x)$ *and*
$B\left(y_0,(r_i)_i\right):=\{y\in\mathcal{F}\mid \|y-y_0\|_i<r_i\ \forall i\in\mathbb{N}\}$*. Assume that for all* $i,n\in\mathbb{N}$*:*

$$\exists\alpha_{in}\in\mathbb{R}_{>0}:\ \left\|F^{n+1}(y_0)-F^n(y_0)\right\|_i\le\alpha_{in}\left\|F(y_0)-y_0\right\|_{i+nL}\tag{1}$$

$$\sum_{n=0}^{+\infty}\alpha_{in}\cdot\|F(y_0)\|_{i+nL}<+\infty\tag{2}$$

Then

$$\exists x\in B\left(y_0,(r_i)_i\right):\ f(x)=y_0.$$

In particular, if at $x_0\in\mathcal{F}$ *instead of* (1)*, we assume the stronger*

$$\exists a\in[0,1)\,\forall u,v\in B\left(x_0,(r_i)_i\right)\,\forall i\in\mathbb{N}\,\exists\alpha_i<a:\ \|F(u)-F(v)\|_i\le\alpha_i\|u-v\|_{i+L},$$

then for $s_i:=r_{i+L}(1-\alpha_i)$*, we have*

$$\forall y\in B\left(f(x_0),(s_i)_i\right)\,\exists x\in B\left(x_0,(r_i)_i\right):\ f(x)=y$$

Norms of integral functions: problem

If $\|y\|_i = \max_{\substack{|\beta|\le i,\ 1\le h\le d\\ \beta\in\mathbb{N}^{1+s}}} \max_{(t,x)\in[0,\alpha]\times S} \left|\partial^\beta y^h(t,x)\right|$, the inequality

$\left\|\int_0^t f(s,-)\,\mathrm{d}s\right\|_i \le \alpha\cdot\|f\|_i$ doesn't hold, but we need it...

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Reduction to 1st order: $y^1 := y$, $y^{j+1} := \partial_t y^j$, $j = 1,\dots,d-1$, our normal problem is equivalent to

$$\begin{cases}\partial_t Y(t,x) = \bar{F}\left[t,x,Y^2,\dots,Y^d,\left(\partial_x^\alpha Y^1\right)_{|\alpha|\le L}\right],\\ Y(t_0,x) = Y_0(x),\end{cases}$$

where $Y(t,x) := (y^1(t,x),\dots,y^d(t,x))$, $Y_0(x) := (y_0^0(x),\dots,y_0^{d-1}(x))$ and $\bar{F}_j := y^{j+1}$ for $j = 1,\dots,d-1$, $\bar{F}^d := F\left[t,x,y^2,\dots,y^d,\left(\partial_x^\alpha y^1\right)_{|\alpha|\le L}\right]$

Spaces of separately regular functions

Definition

Let $T \times S \Subset \mathbb{R}^{1+s}$. The space $\mathcal{C}^{0,\infty}\left(T \times S, \mathbb{R}^d\right)$ contains functions y which are called *separately* $\mathcal{C}^{0,\infty}$ *regular* i.e.:

1. $y : T \times S \longrightarrow \mathbb{R}^d$
2. $\forall x \in S : \ y(-, x) \in \mathcal{C}^0\left(T, \mathbb{R}^d\right)$
3. $\forall t \in T : \ y(t, -) \in \mathcal{C}^\infty\left(S, \mathbb{R}^d\right)$.

This space is endowed with the norms $\|-\|_i$, $i \in \mathbb{N}$, defined by

$$\|y\|_i := \max_{\substack{1 \le h \le d \\ |\beta| \le i \\ \beta \in \mathbb{N}_0^{1+s}}} \max_{(t,x) \in T \times S} \left|\partial^\beta y^h(t,x)\right|, \ \mathbb{N}_0^{1+s} := \left\{\beta \in \mathbb{N}^{1+s} \mid \beta_1 = 0\right\}.$$

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$$\|y\|_{i}:=\max_{\substack{1\le h\le d\\ |\beta|\le i\\ \beta\in\mathbb{N}_{0}^{1+s}}}\ \max_{(t,x)\in T\times S}\left|\partial^{\beta}y^{h}(t,x)\right|,\ \ \mathbb{N}_{0}^{1+s}:=\left\{\beta\in\mathbb{N}^{1+s}\mid\beta_{1}=0\right\}.$$

Theorem

$\left(\mathcal{C}^{0,\infty}\left(T\times S,\mathbb{R}^{d}\right),\left(\|-\|_{i}\right)_{i\in\mathbb{N}}\right)$ *is a graded Fréchet space*

Norms of integral functions: solution

Lemma

Let $a,\ b \in \mathbb{R}_{>0}$, $K := [t_0 - a, t_0 + b] \times S =: T \times S \Subset \mathbb{R}^{1+s}$, $f \in \mathcal{C}^{0,\infty}(K, \mathbb{R}^d)$, *and assume that for every* $i \in \mathbb{N}$, *we have* $M_i \in \mathcal{C}^0(K, \mathbb{R})$ *such that*

$$\forall \beta \in \mathbb{N}_0^{1+s}\, \forall h = 1, \ldots, d\, \forall (t,x) \in K :\ |\beta| \le i \ \Rightarrow\ \left| \partial^\beta f^h(t,x) \right| \le M_i(t,x).$$

Set

$$\bar{M}_i(t,x) := \left| \int_{t_0}^{t} M_i(s,x)\, \mathrm{d}s \right| \quad \forall (t,x) \in K.$$

Then, we have

1. $\left\| \int_{t_0}^{(-)} f(s,-)\, \mathrm{d}s \right\|_i \le \max_{x \in S} \bar{M}_i(t_0 + \max(a,b), x)$.
2. $\left\| \int_{t_0}^{(-)} f(s,-)\, \mathrm{d}s \right\|_i \le \max(a,b) \cdot \|f\|_i$.

Lipschitz maps with LOD

Introduce the useful notation

$$G(t,x,y) := F\left[t, x, y^2(t,x), \ldots, y^d(t,x), \left(\partial_x^\alpha y^1(t,x)\right)_{|\alpha|\le L}\right] \in \mathbb{R}^d, \qquad (G)$$

for all $(t,x) \in T \times S$ and all $y \in \mathcal{C}^{0,\infty}(T \times S, \mathbb{R}^d)$.

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Definition

Let $K := T \times S \Subset \mathbb{R}^{1+s}$, $L \in \mathbb{N}$, $Y \subseteq \mathcal{C}^{0,\infty}\left(T \times S, \mathbb{R}^d\right)$. We say that *$G : K \times Y \to \mathbb{R}^d$* is *Lipschitz on Y with loss of derivatives* (LOD) *L and Lipschitz factors $(\Lambda_i)_{i\in\mathbb{N}}$* if

1. $\forall y \in Y : \ G(-,-,y) \in \mathcal{C}^{0,\infty}\left(K, \mathbb{R}^d\right)$
2. $\Lambda_i \in \mathcal{C}^0(K, \mathbb{R})$ for all $i \in \mathbb{N}$
3. If $i \in \mathbb{N}$, $\beta \in \mathbb{N}_0^{1+s}$, $|\beta| \le i$, $h = 1,\ldots,d$, u, $v \in Y$, $(t,x) \in K$, then

$$\left|\partial^\beta G^h(t,x,u) - \partial^\beta G^h(t,x,v)\right| \le \Lambda_i(t,x) \cdot \max_{\substack{k=1,\ldots,d \\ |\alpha|\le i+L}} \left|\partial_x^\alpha(u^k - v^k)(t,x)\right|$$

Smooth normal PDE are always Lipschitz

Theorem

Let $\emptyset \neq K := T \times S \Subset \mathbb{R}^{1+s}$, $L \in \mathbb{N}$, $\hat{L} := \mathit{Card}\{a \in \mathbb{N}^s \mid |a| \leq L\}$, $F \in \mathcal{C}^\infty\left(K \times \mathbb{R}^{d\cdot(d-1)} \times \mathbb{R}^{d\cdot\hat{L}}, \mathbb{R}^d\right)$, $y_0 \in \mathcal{C}^\infty(S, \mathbb{R}^s)$, $r_i \in \mathbb{R}_{>0}$ *for all* $i \in \mathbb{N}$. *Set*

$$Y := \left\{y \in \mathcal{C}^{0,\infty}(K, \mathbb{R}^d) \mid \|y - y_0\|_i \leq r_i \ \forall i \in \mathbb{N}\right\} \qquad (Y)$$

Then

1. Y *is closed in* $\left(\mathcal{C}^{0,\infty}\left(T \times S, \mathbb{R}^d\right), (\|-\|_i)_{i\in\mathbb{N}}\right)$;
2. *The function* G *is Lipschitz in* Y *with loss of derivatives* L *for some constant* $(\Lambda_i)_{i\in\mathbb{N}}$.

Picard-Lindelöf theorem for smooth normal PDE 1/2

Theorem (PLT, assumptions)

Let $T = [t_0 - a, t_0 + b]$, $S \Subset \mathbb{R}^s$, $y_0 \in \mathcal{C}^\infty(S, \mathbb{R}^s)$, M_i, $\Lambda_i \in \mathcal{C}^0(T \times S)$, $r_i \in \mathbb{R}_{>0}$ *for all* $i \in \mathbb{N}$.

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$$\Lambda_{i,0} := 1, \qquad \Lambda_{i,n+1}(t,x) := \int_{t_0}^t \Lambda_i(s,x) \cdot \Lambda_{i+L,n}(s,x)\,\mathrm{d}s$$

$$\bar{\Lambda}_{i,n} := \max_{x\in S} \Lambda_{i,n}(t_0 + \max(a,b), x) \qquad \bar{M}_i(t,x) := \int_{t_0}^t M_i(s,x)\,\mathrm{d}s.$$

Finally, assume:

1. $\left|\partial^\beta G(t,x,u)\right| \le M_i(t,x)\ \forall u \in Y$, $i \in \mathbb{N}$, $(t,x) \in T \times S$, $\beta \in \mathbb{N}_0^{1+s}$, $|\beta| \le i$
2. $\max_{x\in S} \bar{M}_i(t_0 + \max(a,b), x) \le r_i$ *for all* $i \in \mathbb{N}$
3. $\sum_{n=0}^{+\infty} \bar{\Lambda}_{i,n} \cdot \|P(y_0) - y_0\|_{i+nL} < +\infty$

Picard-Lindelöf theorem for smooth normal PDE 2/2

Theorem (PLT, conclusions)

Then, there exists a smooth solution $y \in Y \cap \mathcal{C}^\infty(T \times S, \mathbb{R}^d)$ *of the problem*

$$\begin{cases} \partial_t y(t,x) = F\left[t, x, y^2, \ldots, y^d, \left(\partial_x^\alpha y^1\right)_{|\alpha| \le L}\right], \\ y(t_0, x) = y_0(x), \end{cases}$$

given by $y = \lim_{n \to +\infty} P^n(y_0)$ *in* $\left(\mathcal{C}^{0,\infty}\left(T \times S, \mathbb{R}^d\right), \left(\|-\|_i\right)_{i \in \mathbb{N}}\right)$, *which satisfies*

$$\forall i, m \in \mathbb{N} : \ \|y - P^m(y_0)\|_i \le \sum_{n=m}^{+\infty} \bar{\Lambda}_{i,n} \cdot \|P(y_0) - y_0\|_{i+nL} .$$

Corollary of PLT

Corollary

Set $M_i := \|F\|_i$*, where the norm is taken in* $C_i := \bigcup_{|\alpha|\le i+L} \overline{B_{r_i+L}\left(\partial_x^\alpha y_0(T\times S)\right)}$*. Finally, assume that the following conditions are fulfilled:*

$$0 < \max(a,b) \le \inf_{i\in\mathbb{N}} \frac{r_i}{M_i}$$

$$\sum_{n=0}^{+\infty} \bar{\Lambda}_{i,n} \cdot \left\|P\left(y_0\right) - y_0\right\|_{i+nL} < +\infty$$

Then, there exists a smooth solution $y \in Y \cap \mathcal{C}^\infty([t_0-a, t_0+b]\times S, \mathbb{R}^d)$*.*

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where $\alpha := \max(a,b)$ and $i_c(t,x) := \sum_{j=0}^{d-1}\frac{y_0^j(x)}{j!}(t-t_0)^j$, for the initial problem (CP).

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3. If both F and y_0 are analytical, then $\prod_{k=0}^{n-1}\Gamma_{i+kL}\cdot\|P(i_c)-i_c\|_{i+nL}\le C_i^{i+nL}\cdot(i+nL)!$ and we can prove that (W) always holds.

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Study of Weissinger condition

1. Every smooth PDE is always Lipschitz w.r.t. constants $(\Gamma_i)_{i\in\mathbb{N}}$ (Thm. Lip.)
2. Condition (W) for the reduced problem is equivalent to

$$\sum_{n=0}^{+\infty}\prod_{k=0}^{n-1}\Gamma_{i+kL}\cdot\frac{\alpha^n}{(nd)!}\cdot\|P(i_c)-i_c\|_{i+nL}<+\infty$$

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5. If F and y_0 satisfy $\prod_{k=0}^{n-1}\Gamma_{i+kL}\cdot\|P(i_c)-i_c\|_{i+nL}\sim n^{i+nL}$ for $n\to+\infty$ (by Borel's lemma such F, y_0 always exist), then F, y_0 are smooth but not analytic

What we get more with generalized smooth functions

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5. These results cannot be improved because of Lewy, Mizohata, De Giorgi, Colombini, Spagnolo...

Some differences between ODE and PDE

1. PDE are related to contractions with LOD, whereas $L = 0$ for ODE
2. In the PLT for smooth normal PDE, existence depends on the initial condition y_0
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This is still a work in progress: comments are welcome!

Contacts and references

References:

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`Thank you for your attention!`