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"Uniqueness Properties of Barrier Type Skorokhod Embeddings and Perkins Embedding with General Starting Law"

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# UNIQUENESS PROPERTIES OF BARRIER-TYPE SKOROKHOD EMBEDDINGS AND PERKINS EMBEDDING WITH GENERAL STARTING LAW

ANNEMARIE GRASS

Abstract. Since its formulation in 1961 the *Skorokhod embedding problem* - that is to represent a given probability measure as a standard Brownian motion $B$ stopped at a specific stopping time - has gained more and more fame until now even being considered a classical problem in probability theory. With new solution emerging every couple of years, the theory recently enjoyed (partial) unification in [BCH17].

A major inspiration for a lot of publications regarding the Skorokhod embedding problem was the idea of Root to identify solutions as hitting times of certain subsets of $\mathbb{R}_+ \times \mathbb{R}$. These solutions do not only have a nice geometric interpretation but also come with a uniqueness property.

[BCH17] gives the tools to identify a lot of known solutions as Root solutions in the sense that these solutions can be represented as hitting times of a process $(A, B)$, where the choice of the process $A$ will depend on the additional properties of the specific solutions.

The first objective of this thesis is to show rigorously that the uniqueness property of the original Root solution carries over to this more general setup - even if considering a generalized Skorokhod embedding problem where we embed in processes different from a Brownian Motion.

Another development in the theory around the Skorokhod embedding problem (often with regard to application in financial mathematics) is to embed into a Brownian motion started according to a nontrivial initial distribution.

In the second part of this thesis we will consider a specific solution to the Skorokhod embedding problem proposed by Perkins in [Per86].

For a simplified case, that is, where the initial and the terminal solution do not share any mass, we will by using methods given in [BCH17] to find a solution to the Perkins embedding with random starting which is given as a hitting time in sense of a generalized Root solution. We will also establish a uniqueness result in this context.

Zusammenfassung. 1961 hat Anatoli Skorokhod das heute als *Skorokhods Einbettungsproblem (Skorokhod embedding problem)* bekannte Problem formuliert: ein vorgegebenes Wahrscheinlichkeitsmaß als die Verteilung einer zufällig gestoppten Brown'schen Bewegung $B$ darzustellen. Das Problem erfreut sich seit dem wachsender Beliebtheit und wird heutzutage als ein klassisches Problem in der Wahrscheinlichkeitstheorie anerkannt. Über die Jahre sind zahlreiche neue und alternative Lösungen entwickelt und veröffentlich worden, und vor kurzem wurde in [BCH13] die Löesungtheorie sogar (teilweise) vereinheitlicht.

Für viele Forscherinnen und Forscher inspirierend war die Idee von Root, Skorokhods Einbettungsproblem mittels einer Treffzeit einer spezifischen Teilmenge von $\mathbb{R}_+ \times \mathbb{R}$ zu lösen. Solche Lösungen haben einerseits eine schöne geometrische Interpretation und besitzen außerdem eine Eindeutigkeitseigenschaft.

Mittels der in [BCH13] entwickelten Methoden können viele bekannte Lösungen des Skorokhod'schen Einbettungsproblems als Root-Lösungen in jenem Sinne verstanden werden, dass sie als Treffzeiten von Prozessen $(A, B)$ dargestellt werden können. Die Wahl des Prozesses $A$ hängt dann von den zusätzlichen Eigenschaften der entsprechenenden Lösung ab.

Das erste Ziel dieser Arbeit ist es rigoros zu zeigen, dass sich die Eindeutigkeitseigenschaft der originalen Root Lösung auch auf diese verallgemeinerten Root Lösungen überträgt, sogar wenn wir ein allgemeineres Einbettungsproblem betrachten, welches sich nicht auf die Brown'sche Bewegung beschränkt.

Ein anderer Aspekt rund um das Skorokhod'sche Einbettungsproblem, welcher in den letzten Jahren vor allem in Hinblick auf Anwendungen in der modellfreien Finanzmathematik interessant geworden ist, ist das Einbetten von Verteilungen in eine Brown'sche Bewegung, welche mittels einer nichttrivialen Anfangverteilung gestartet wurde. Im zweiten Teil dieser Arbeit werden wir uns mit einer spezifischen, 1986 von Edwin Perkins in [Per86] vorgestellten Lösung beschäftigen.

Für jenen Fall, in dem Anfang- und Endverteilung keine gemeinsame Masse tragen, werden wir mit Hilfe der Methoden in [BCH13] eine Lösung der Perkins Einbettung mit nichttrivialer Anfangsverteilung finden, welche als Treffzeit im Sinne einer verallgemeinerten Root Lösung gegeben ist. Wir werden auch die Eindeutigkeit dieser Lösung entsprechend des 1. Teiles zeigen.

Acknowledgement. First and foremost, I am infinitely grateful to my advisor Mathias Beiglböck. Without him I would not be aware of the beauty of the Skorkhod embedding problem. Without his encouragement I would not even have considered pursuing a master degree in mathematics. Without his guidance and the resources he provided I would have never been able to finish it, and without his constant support for my passion for climbing and our countless bouldering session I would be a lot unhappier and unhealthier.

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Contents

## 1. Introduction

1.1. **Skorokhod Embedding Problem.** Originally formulated as follows, given a centered probability measure $\mu$ on the real line with finite second moment, the *Skorokhod embedding problem* ($\text{SEP}_\mu$)is to find a stopping time $\tau$ such that

$$B_\tau \sim \mu, \quad \mathbb{E}[\tau] < \infty, \tag{$\text{SEP}_\mu$}$$

where $(B_t)_{t\geq 0}$ denotes a standard Brownian motion. We call the law of $B_\tau$ the law *generated by* $\tau$ and say $\tau$ *embeds* $\mu$. We will also call $\tau$ a *solution* to the ($\text{SEP}_\mu$). (Demanding integrability of the stopping time allows exclusion of trivial solutions.)

It was soon discovered that the assumptions on $\mu$ of finite second moment can be weakened to finite first moment. As it is then no longer possible to demand integrability of the stopping time we will instead ask for *minimality* of the stopping time.

**Definition 1.1.** *A stopping time $\tau$ solving the ($SEP_\mu$) is* minimal *if for any stopping time $\tau'$ s.t. $B_{\tau'} \sim \mu$, $\tau' \leq \tau$ implies $\tau' = \tau$ a.s.*

Note that minimality of the solution is equivalent to $\mathbb{E}[\tau] < \infty$ if $\mu$ does have a finite second moment. (In this case $\mathbb{E}[B_\tau^2] = \mathbb{E}[\tau]$ and for any $\tau' \leq \tau$ with $B_{\tau'} \sim \mu$ we then have $\mathbb{E}[\tau'] = \mathbb{E}[B_{\tau'}^2] = \mathbb{E}[B_{\tau'}^2] = \mathbb{E}[\tau]$, therefore $\tau' = \tau$ a.s., see [Mon72].)

The first to prove that it is always possible to construct such a stopping time was Skorokhod himself (in the finite second moment case) in [Sko61] (1961). The rise of fame of the problem started with the translation of Skorokhods book into English [Sko65], 1965. Numerous solutions to the ($\text{SEP}_\mu$) have been published since, featuring different optimality properties and introducing new approaches like potential theory, Markov theory, optimal transport, etc.

21 solutions alone can be found in the survey by Obłój [Obł04], listing all solution up to 2004. However there are many more recent contributions, especially those introducing the optimal transport approach.

1.2. **Root's Solution.** One of the earliest follow-up solutions was given by Root [Roo69]. He provides a solutions to the ($\text{SEP}_\mu$) which is given as a hitting time of a so called *barrier*.

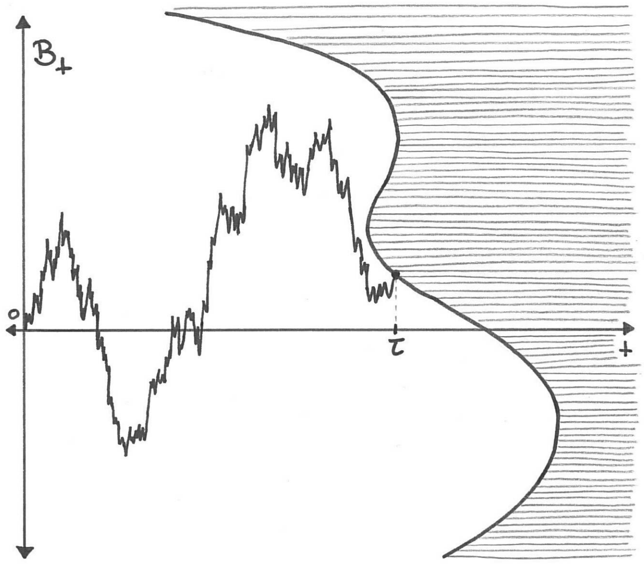
**Definition 1.2.** *A* barrier *(sometimes calles* right barrier*) is a set $R \subseteq \mathbb{R}^2$ such that if $(t,x) \in R$ then for all $s \geq t$ we have $(s,x) \in R$.*

Root then states that a barrier $R \subseteq \mathbb{R}_+ \times \mathbb{R}$ can be found such that

$$\tau_R := \inf\{t \geq 0 : (t, B_t) \in R\}$$

is a stopping time solving the ($\text{SEP}_\mu$).

We will call $\tau_R$ a *barrier type solution* or *Root solution*, call the law of $B_{\tau_R}$ the law generated by $R$, and say that $R$ embeds $\mu$. With this in mind it also makes sense to call the barrier $R$ a solution to ($\text{SEP}_\mu$).




Root's proof was in no way constructive, so finding an explicit barrier given a distribution is (with the exception of a few trivial cases) still quite difficult. Only recently it was established that the barriers can be described using PDE techniques, see e.g. [GMO15], [CP15] or [CW13].

Nevertheless the theoretical properties of Root solutions proved to be immensely useful and it also gives an intuitive geometric description of possible solutions. This served as an inspiration for subsequent work on the $(\text{SEP}_\mu)$and continues to inspire researchers.

## 2. Loynes Argument

Following the publication of Root's solution, a uniqueness property of barrier type solutions were established in Loynes [Loy70], that is, for a given centered probability measure with finite second moment, the barrier and therefore the barrier type solution of the corresponding $(\text{SEP}_\mu)$ is essentially unique. This is a consequence of the crucial argument that the union of barriers generating the same law also generates this law:

**Proposition 2.1.** *Let $R$ and $S$ be barriers with corresponding stopping times $\tau_R$ and $\tau_S$ both generating the same law. Then $R \cup S$ also generates this law and $\tau_{R\cup S} = \tau_R \wedge \tau_S$.*

For the classical case discussed so far, that is, $\tau_R$ and $\tau_S$ are Root solutions to the classical $(\text{SEP}_\mu)$ as defined below Definition 1.2, a proof of this proposition can be found in [Loy70]. As we would now like to consider more general processes we will investigate these statements in a setup not restricted to Brownian Motion. In the next chapter we will formulate and prove a generalization of Proposition 2.1 and derive uniqueness of barrier type solutions.

## 3. Leaving the grounds of Brownian motion

Previously developed theory around the $(\text{SEP}_\mu)$ often depends on very particular properties of Brownian motion. In [BCH17] this approach is dropped and possibilities for a much more general $(\text{SEP}_\mu)$ are opened. It is suggested, that instead of Brownian motion a greater class of (right) continuous Markov processes might be considered and details are carried out for continuous Feller processes.

3.1. **Generalized Skorokhod embedding problem.**
Let us define a more general Skorokhod embedding problem:
Let $Z = (Z_t)_{t\geq 0}$ be a real valued stochastic process defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We also want to consider random starting: Assume $Z_0 \sim \lambda$ where $\lambda$ is a probability measure on the real line.
Now given a probability measure $\mu$ on the real line, the *generalized Skorokhod embedding problem* $(\mathrm{GSEP}^Z_{\lambda,\mu})$ is to find a stopping time $\tau$, such that

$$Z_\tau \sim \mu \text{ and } \tau \text{ is minimal.} \tag{$\mathrm{GSEP}^Z_{\lambda,\mu}$}$$

In case $Z = B$ we will write $(\mathrm{SEP}_{\lambda,\mu})$ for the Skorokhod embedding problem with initial distribution $\lambda$.
Note that minimality of the stopping time is defined analogously to the previous setting, that is, we call $Z_\tau$ the law generated by $\tau$ and a stopping time $\tau$ is *minimal*, if for any stopping time $\tau'$ generating the same law, $\tau' \leq \tau$ implies $\tau' = \tau$ a.s.

3.2. **Barrier type solutions to the $\mathrm{GSEP}^Z_{\lambda,\mu}$.**
In [BCH17] it was observed that it is possible to identify some known solutions to $(\mathrm{SEP}_{\lambda,\mu})$ as barrier type solutions, if considered in the right phase space.

More generally this means that a stochastic process $(Y_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and a barrier $R \subseteq \mathbb{R}^2$ can be found, such that

$$\tau_R := \inf\{t \geq 0 : (Y_t, Z_t) \in R\}$$

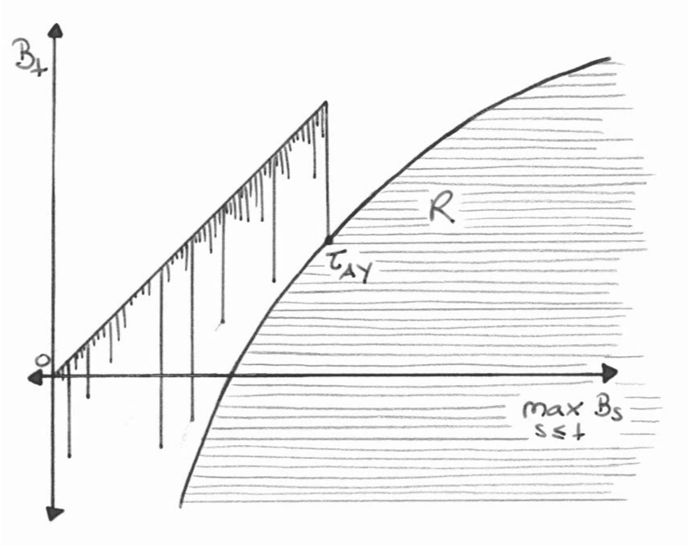
is a stopping time solving the $(\mathrm{GSEP}^Z_{\lambda,\mu})$. Again we want to allow random starting, so we will consider a joint initial distribution $(Z_0, Y_0) \sim \sigma$.
We will call $\tau_R$ a $(Y, Z)$*-barrier type solution* and now call the law of $Y_{\tau_R}$ the law generated by $\tau_R$ or by $R$ respectively.
Note, that by choosing $Z_t = B_t$ and $Y_t = t$ we arrive at our previous setting of Root solutions.

**Example 3.1** (The Azéma-Yor embedding)**.** *Another (by now well know) solution to the ($SEP_\mu$)was given in 1979 by Azéma and Yor* [AY79]. *This solution, denoted by $\tau_{AY}$, is found via construction of an increasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}$, such that*

$$\tau_{AY} = \inf\{t \geq 0 : B_t \leq \Psi\Big(\max_{s\leq t} B_s\Big)\}$$

*is a stopping time solving the ($SEP_\mu$). This implies, as pointed out in* [BCH17], *that the Azéma-Yor solution can be identified as the hitting time of a barrier type set, if considered in the phase space* $(\max_{s\leq t} B_s, B_t)$. *That is, there exists a barrier $R \subseteq \mathbb{R}^2$ such that*

$$\tau_{AY} = \inf\Big\{t \geq 0 : \Big(\max_{s\leq t} B_s, B_t\Big) \in R\Big\}.$$




3.3. **Uniqueness of barrier type solutions.**
We now want to establish that in this general setting we can still guarantee uniqueness of this barrier type solution. To this end, we will show that if two barriers generate the same law, so does their union.

The proof of Loynes was carried out for *closed* barriers. As for a barrier $R$ we have

$$\inf\{t \geq 0 \big| (t, B_t) \in R\} = \inf\{t \geq 0 \big| (t, B_t) \in \overline{R}\}$$

(where $\overline{R}$ denotes the topological closure of $R$ with respect to the Euclidean metric) this setting allows to simply assume every barrier to be a closed set.
This assumption is unfortunately no longer true in our more general setup as the following easy (if somewhat artificial) example illustrates:
Consider the processes $Z_t := Z_0$ for $t \geq 0$ where $Z_0$ is standard normally distributed and $Y_t := t$. Define the barrier $R := ([1, 2) \times \mathbb{Q}) \cup ([2, \infty) \times \mathbb{R})$. Then $\tau_R \in \{1, 2\}$, however $P[\tau_R = 1] = P[Z_1 \in \mathbb{Q}] = 0$.
Taking the topological closure yields $\overline{R} = [1, \infty) \times \mathbb{R}$ and therefore $\tau_{\bar{R}} = 1 \neq 2 = \tau_R$.

As it will be essential to know that we are in fact inside the barrier when we stop we will include this as an assumption in our next proposition. However, we will soon see how this does not restrict most applications of this proposition.

**Proposition 3.2.** *Let $(Y_t)_{t \geq 0}$ be a stochastic process and let $R$, $S$ be two barriers, such that*

*(i)* $\tau_R := \inf\{t \geq 0 : (Y_t, Z_t) \in R\}$ *and* $\tau_S := \inf\{t \geq 0 : (Y_t, Z_t) \in S\}$ *are stopping times both generating the same law* $\mu$,

*(ii)* $P[(Y_{\tau_R}, Z_{\tau_R}) \in R] = 1$ *and* $P[(Y_{\tau_S}, Z_{\tau_S}) \in S] = 1$.

*Then $R \cup S$ also generates $\mu$ and the corresponding stopping time is given by*

$$\tau_{R \cup S} = \tau_R \wedge \tau_S.$$

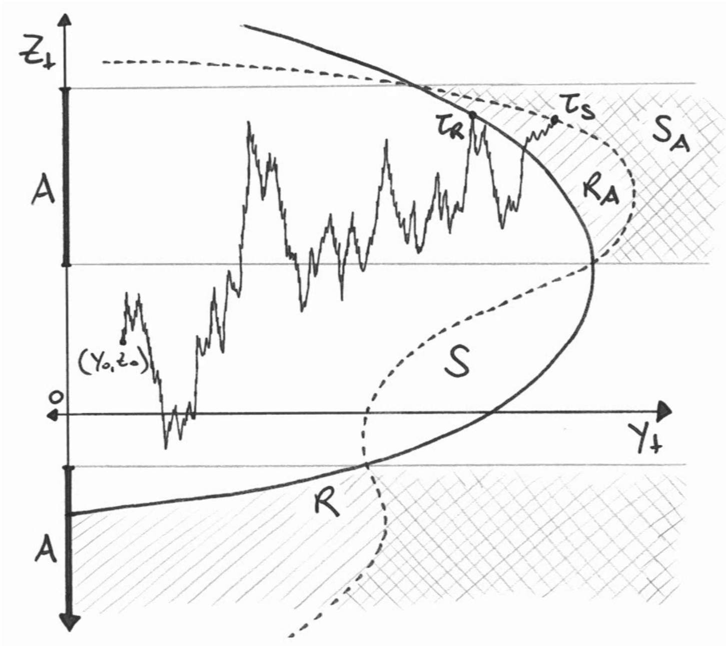
*Proof.* Consider the set of those $y$-points, where 'the barrier $R$ comes sooner', that is

$$A := \{z \in \mathbb{R} : \inf\{y \in \mathbb{R} : (y, z) \in R\} < \inf\{y \in \mathbb{R} : (y, z) \in S\}\},$$

and define

$$R_A := \{(x, y) \in R : y \in A\}, \quad R_{A^c} := \{(x, y) \in R : y \in A^c\},$$
$$S_A := \{(x, y) \in S : y \in A\}, \quad S_{A^c} := \{(x, y) \in S : y \in A^c\}.$$

Note that $S_A \subseteq R_A$ as well as $R_{A^c} \subseteq S_{A^c}$.




Now assume that $Z_{\tau_S} \in A$. By definition of $\tau_S$ and Assumption *(ii)* we have $(Y_{\tau_S}, Z_{\tau_S}) \in S_A \subseteq R_A$. This means, $\tau_R \leq \tau_S$ and it is now impossible to have $(Y_{\tau_R}, Z_{\tau_R}) \in R_{A^c}$ (otherwise $Z_{\tau_S}$ would have stopped in $A^c$). This implies that we cannot have $Z_{\tau_R} \in A^c$ and therefore $P[Z_{\tau_S} \in A, Z_{\tau_R} \in A^c] = 0$.
As $Z_{\tau_R} \sim Z_{\tau_S}$ we have:

$$\begin{aligned} P[Z_{\tau_R} \in A] &= P[Z_{\tau_R} \in A, Z_{\tau_S} \in A] + P[Z_{\tau_R} \in A, Z_{\tau_S} \in A^c] \\ &= P[Z_{\tau_S} \in A, Z_{\tau_R} \in A] + P[Z_{\tau_S} \in A, Z_{\tau_R} \in A^c] = P[Z_{\tau_S} \in A], \end{aligned}$$

and altogether

$$0 = P[Z_{\tau_R} \in A, Z_{\tau_S} \in A^c] = P[Z_{\tau_S} \in A, Z_{\tau_R} \in A^c].$$

It is now clear that the sets $\Omega_1 := \{Z_{\tau_R} \in A, Z_{\tau_S} \in A\}$ where $\tau_R \leq \tau_S$ and $\Omega_2 := \{Z_{\tau_R} \in A^c, Z_{\tau_S} \in A^c\}$ where $\tau_S \leq \tau_R$ are disjoint and their union has full probability. Therefore $\tau_{R \cup S} = \tau_R \wedge \tau_S$ a.s.
That $\tau_{R\cup S}$ generates the same law is now obvious (due to decomposition into $\Omega_1$ and $\Omega_2$). □

For easier applications of Proposition 3.2 let us give a name to all barrier type solutions satisfying Assumption *(ii)* therein:

**Definition 3.3.** *A stopping time $\tau = \tau_R$ induced by a barrier $R \subseteq \mathbb{R}^2$ is called* strong $(Y, Z)$-barrier type solution *of the (GSEP$^Z_{\lambda,\mu}$) if it is a $(Y, Z)$-barrier type solution and $P[(Y_{\tau_R}, Z_{\tau_R}) \in R] = 1$.*

**Remark 3.4.** *(i) If a $(Y, Z)$-barrier type solution $\tau$ is induced by a closed set, that is $\tau = \tau_R$ for some closed barrier $R \subseteq \mathbb{R}^2$, then it is a strong $(Y, Z)$-barrier type solution.*

*(ii) In most applications the processes $(Y, Z)$ will be jointly Markov (e.g. continuous Feller processes as in* [BCH17]*). Considering sufficiently regular Markov processes (that is, Markov processes such that we have the Blumenthal-Getoor 0-1 law), a suitable alternative to topological closures for our purpose is the so called* fine closure *with respect to the process $(Y, Z)$. This essentially means*

*to add all points to our target set where 'we would stop anyway':*
*The* fine closure *of a set $R \subseteq \mathbb{R}^2$ with respect to a jointly Markov process $(Y, Z)$ will be denoted by $R^*$ and is defined as*

$$R^* := R \cup \{(t, x) \in \mathbb{R}^2 : P[\tau_R = 0 \big| (Y_0, Z_0) = (t, x)] = 1\}.$$

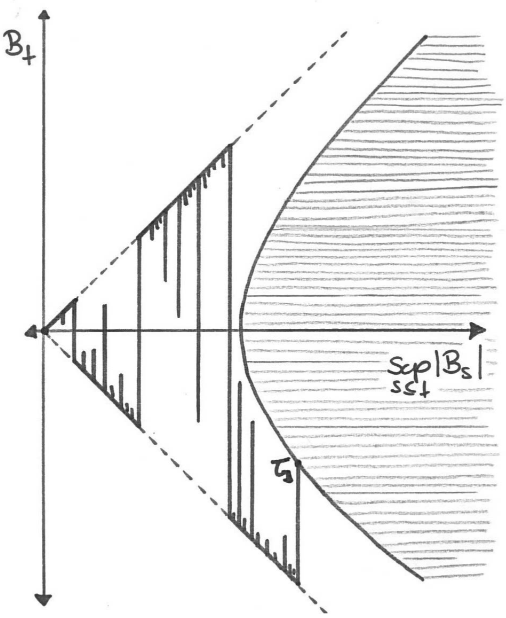
*By this definition follows $\tau_R = \tau_{R^*}$ a.s. and $(Y_{\tau_{R^*}}, Z_{\tau_{R^*}}) \in R^*$ by Blumenthal-Getoor. Moreover we also have $Y_{\tau_R} \sim Y_{\tau_{R^*}}$ and $Z_{\tau_R} \sim Z_{\tau_{R^*}}$.*
*For details on fine closures refer to* [CW05].
*We see that taking fine closures does not alter the stopping properties and we will therefore assume without loss of generality our $(Y, Z)$-barriers are always finely closed with respect to $(Y, Z)$. Moreover we see that in this case all $(Y, Z)$-barrier type solution are strong $(Y, Z)$-barrier type solution.*

Uniqueness of strong barrier-type solutions now follows as an easy corollary:

**Corollary 3.5.** *If $\tau$ is a strong $(Y, Z)$-barrier type solution to the (GSEP$^Z_{\lambda,\mu}$), then $\tau$ is a.s. unique.*

*Proof.* Let $R$ be the barrier associated to $\tau$, i.e. $\tau = \tau_R$. Assume there is another strong $(Y, Z)$-barrier type solution $\tilde{\tau} = \tau_S$ given by the barrier $S$. As $\tau_R$ and $\tau_S$ generate the same law, by Proposition 3.2 so does $\tau_{R\cup S} = \tau_R \wedge \tau_S$. Obviously $\tau_{R\cup S} \leq \tau_R$, therefore, by minimality of $\tau_R$, we have $\tau_{R\cup S} = \tau_S$ a.s. Analogously $\tau_{R\cup S} = \tau_R$ a.s. can be deduced and the result follows. □

**Example 3.6** (The Jacka embedding)**.** *In* [Jac88] *Jacka constructed a stopping time $\tau_J$ that maximizes the law of $\sup_{s \leq t} |B_s|$ over all solutions to the (SEP$_\mu$). With the methods established in* [BCH17]*, the Jacka solution was recovered and geometrically interpreted in the following way (see Theorem 6.6 therein):*




*Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a bounded, strictly increasing right-continuous function. There exists a stopping time $\tau_J$ which maximizes*

$$\mathbb{E}\Big[\varphi\Big(\sup_{s \leq \tau} |B_s|\Big)\Big]$$

*over all solutions to the (SEP$_\mu$). Moreover, there is a right barrier $R \subseteq \mathbb{R}_+ \times \mathbb{R}$ such that $\tau_J$ is of the form*

$$\tau_J = \inf\Big\{t \geq 0 : \Big(\sup_{s\leq t} |B_s|, B_t\Big) \in R\Big\}.$$

*An application of Corollary 3.5 now gives that the solution $\tau_J$ is independent of the choice of $\varphi$.*

We see that we now covered a more general version of the traditional setting of solutions given by barrier type sets.

### 3.4. **Inverse-, upwards-, and downwards barriers.**

We now want to extend the notion of *barrier type solutions*.

Soon after the publication of Root and the introduction of barriers and barrier type solution, Rost [Ros76] found another solution to the Skorokhod embedding problem given as a hitting of a specifically structured set, a so called *inverse barrier.*

**Definition 3.7.** *An* inverse barrier *(or sometimes also called* left barrier*) is a set $R \subseteq \mathbb{R}^2$ such that if $(t,x) \in R$ then for all $s \leq t$ we have $(s,x) \in R$.*

*If*

$$\tau := \tau_R := \inf\{t \geq 0 : (Y_t, Z_t) \in R\}$$

*is a stopping time solving the (GSEP$^Z_{\lambda,\mu}$), we call it a* $(Y,Z)$-inverse barrier type solution.

*If in addition $P[(Y_{\tau_R}, Z_{\tau_R}) \in R] = 1$, we call it* strong.

Uniqueness of strong inverse barrier type solution can be deduced in analogy to the discussion above.

**Corollary 3.8.** *If $\tau$ is a strong $(Y,Z)$-inverse barrier type solution to the (GSEP$^Z_{\lambda,\mu}$), then $\tau$ is a.s. unique.*

*Proof.* Let $R \subseteq \mathbb{R}^2$ be the inverse barrier generating $\tau$. If we consider the transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi((y,z)) = (-y,z)$. Then $\tilde{R} := \varphi(R)$ is a right barrier and for $(\tilde{Y}, \tilde{Z}) = \varphi((Y,Z)) = (-Y,Z)$ we have

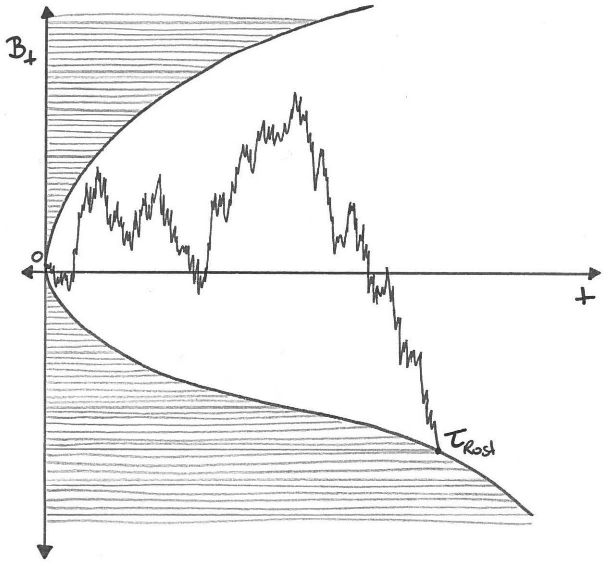
$$\tau = \inf\{t \geq 0 : (Y_t, Z_t) \in R\} = \inf\{t \geq 0 : (\tilde{Y}_t, \tilde{Z}_t) \in \tilde{R}\}.$$

We can therefore consider $\tau$ as a strong $(\tilde{Y}, \tilde{Z})$-barrier type solution and uniqueness follows by Corollary 3.5. □

**Example 3.9** (The Rost embedding)**.** *In* [Ros76] *it was show, that under the assumption $\mu(\{0\}) = 0$, there exists an inverse barrier $R \subseteq \mathbb{R}^2$ , such that*

$$\tau := \tau_R := \inf\{t \geq 0 : (t, B_t) \in R\}$$

*is a stopping time solving the (SEP$_\mu$). By Corollary 3.8, this solution is unique.*




Another notion now sometimes seen in the literature is that of so called *upwards-* or *downwards barriers*.

**Definition 3.10.**

- *An* upwards barrier *is a set* $R \subseteq \mathbb{R}^2$ *such that if* $(t, x) \in R$ *then for all* $y \geq x$ *we have* $(t, y) \in R$.
- *A* downwards barrier *is a set* $R \subseteq \mathbb{R}^2$ *such that if* $(t, x) \in R$ *then for all* $y \leq x$ *we have* $(t, y) \in R$.

When considering these kind of barriers the roles of the processes $Z$ and $Y$ are usually interchanged in the following way: We still consider hitting times of the process $(Y, Z)$ as defined above, but we imbed in the process $Y$ instead of the process $Z$. We can say we consider solutions to $(\mathrm{GSEP}^Y_{\lambda,\mu})$:

Let $R \subseteq \mathbb{R}^2$ be either an upwards or downwards barrier and let

$$\tau := \tau_R := \inf\{t \geq 0 : (Y_t, Z_t) \in R\}$$

be a stopping time solving the $(\mathrm{GSEP}^Y_{\lambda,\mu})$, then $\tau$ is called $(Y, Z)$*-upwards* respectively *downwards barrier type solution.*

Again $\tau$ will be called a *strong solution*, if in addition $P[(Y_{\tau_R}, Z_{\tau_R}) \in R] = 1$.

**Proposition 3.11.** *Let* $(Y_t)_{t\geq 0}$ *and* $(Z_t)_{t\geq 0}$ *be stochastic processes and let* $R, S$ *be two upwards respectively downwards barriers, such that*

*(i)* $\tau_R := \inf\{t \geq 0 : (Y_t, Z_t) \in R\}$ *and* $\tau_S := \inf\{t \geq 0 : (Y_t, Z_t) \in S\}$ *are stopping times and* $Y_{\tau_R} \sim Y_{\tau_S}$,

*(ii)* $P[(Y_{\tau_R}, Z_{\tau_R}) \in R] = 1$ *and* $P[(Y_{\tau_S}, Z_{\tau_S}) \in S] = 1$.

*Then also* $Y_{\tau_{R\cup S}} \sim Y_{\tau_R} \sim Y_{\tau_S}$ *and* $\tau_{R\cup S} = \tau_R \wedge \tau_S$.

*Proof.* Let $R$, $S$ be upwards barriers. Consider the transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi((y, z)) := (z, y)$. Then $\tilde{R} := \varphi(R)$ and $\tilde{S} := \varphi(S)$ are right barriers and for $(\tilde{Y}, \tilde{Z}) := \varphi((Y, Z)) = (Z, Y)$ we have:

$$\tau_R := \inf\{t \geq 0 : (\tilde{Y}_t, \tilde{Z}_t) \in \tilde{R}\} \text{ and } \tau_S := \inf\{t \geq 0 : (\tilde{Y}_t, \tilde{Z}_t) \in \tilde{S}\}.$$

Therefore the result follows via application of Proposition 3.2.

If $R$, $S$ are downwards barriers, use the transformation $\varphi((y, z)) = (-z, y)$. □

Again, we can deduce a uniqueness result for strong upwards respectively downwards barriers:

**Corollary 3.12.** *If $\tau$ is a strong $(Y, Z)$-upwards - respectively downwards barrier type solution to the $(GSEP^{Y}_{\lambda,\mu})$, then $\tau$ is a.s. unique.*

*Proof.* Uniqueness follows from minimality due to Proposition 3.11. □

**Example 3.13** (Distribution-constrained optimal stopping problems)**.** *Let $(B_t)_{t\geq 0}$ be a Brownian motion started in* 0 *on some filtered probability space* $(\Omega, \mathcal{G}, (\mathcal{G})_{t\geq 0}, P)$ *satisfying the usual conditions. For a measure $\mu$ on $\mathbb{R}_+$ and a cost function*

$$c : C(\mathbb{R}_+) \times \mathbb{R}_+ \to \mathbb{R}$$

*the* distribution-constrained optimal stopping problem *is to solve the optimization problem*

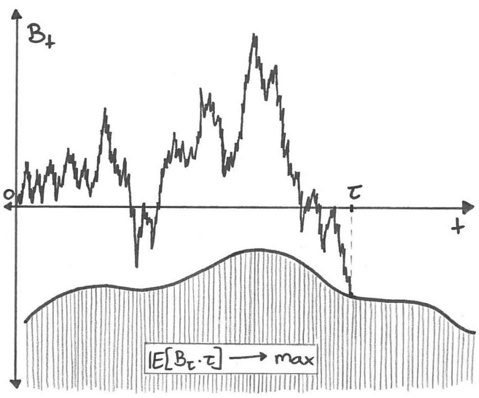
$$\max\{\mathbb{E}[c((B_t)_{t\leq\tau}, \tau)] : \tau \text{ is a stopping time on } (\Omega, \mathcal{G}, (\mathcal{G})_{t\geq 0}, P) \text{ such that } \tau \sim \mu\}.$$

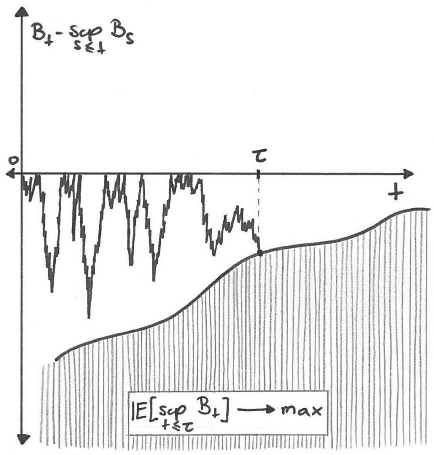
*The geometry of this problem is the subject of* [BEES16]*. Corollary 1.1. therein states, that if $\mu$ has finite first moment, an upper semi-continuous function $\beta : \mathbb{R}_+ \to [-\infty, \infty]$ can be found, such that*

$$\tau := \inf\{t > 0 : B_t \leq \beta(t)\} \sim \mu,$$

*that is, we can find a stopping time with distribution $\mu$ that is given as the hitting time of a* downwards barrier*.*

*Moreover, this $\tau$ is up to $P$-nullsets a solution of the distribution constrained optimal stopping problem for some cost functions $c$. Corollary 3.12 can now be used to deduce uniqueness of this solution (note, that the minimality assumption in the definition of $(GSEP^{Z}_{\lambda,\mu})$ used in the proof of Corollary 3.12 is replaced by the optimization condition).*



## 4. Perkins Embedding

In 1986, Edwin Perkins established a solution to the ($\text{SEP}_\mu$) which minimizes $\mathbb{E}[\max_{0\le t\le\tau} B_t]$ while simultaneously also maximizing $\mathbb{E}[\min_{0\le t\le\tau} B_t]$, see [Per86].
As most other solutions to the ($\text{SEP}_\mu$) are distinguished by one specific optimality property (Roots solution is minimizing $\mathbb{E}[\tau^2]$, Rosts solution is maximizing $\mathbb{E}[\tau^2]$, the Azema-Yor solution is maximizing $\mathbb{E}[\max_{0\le t\le\tau} B_t]$, etc. (see e.g. [Obł04], [Hob11],[BCH17])), Perkins solution is considered a notch more difficult to grasp thanks to this two-fold optimization property.

With the rising interest in robust finance, where model independent bounds for option prices are often found by solving a Skorokhod embedding problem (see e.g. [Hob11]) it became crucial to start considering the ($\text{SEP}_\mu$) with nontrivial starting law.

A first effort in giving a solution to Perkins' problem with a random initial distribution was taken by Hobson and Pedersen in [HP02]. However it was also observed, that considering a general starting law it is no longer possible to simultaneously optimize over the running minimum and the running maximum. For a brief discussion of the Hobson-Pedersen solution we refer to Remark 4.11 later in this chapter.

In this chapter we will use and extend the methods established in [BCH17] to suggest a solution to Perkins' problem with non-trivial starting (in case of mutually singular measures). We will also give a geometric description of this solution as well as the original Perkins solution in an extended barrier type setting and we will establish a Loynes-type uniqueness result in the spirit of Chapter 3.

### 4.1. **Terminology and techniques.**
We will adopt the underlying assumptions of [BCH17], that is we will live on a stochastic basis $\Omega = (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ which is rich enough to support a Brownian motion $B$ and a uniformly distributed $\mathcal{G}_0$-random variable independent of $B$.
When considering the ($\text{SEP}_{\lambda,\mu}$) we ask our target measure $\mu$ to have a finite second moment and we always assume that the starting distribution $\lambda$ and $\mu$ are *in convex order*, that is

$$\int f(x)\lambda(dx) \le \int f(x)\mu(dx) \text{ for any convex function } f:\mathbb{R}\to\mathbb{R}.$$

This will ensure the existence of a solution to ($\text{SEP}_{\lambda,\mu}$) with finite first moment.

This work relies heavily on arguments on a pathwise level, especially arguments carried out for individual paths. Therefore we will introduce and use the following terminology:

**Definition 4.1** (stopped paths)**.** *We define the set of* stopped paths

$$S := \{(f,s) : s \ge 0, f \in C([0,s],\mathbb{R})\}.$$

If given two paths, we want to run through them one after another, we speak of concatenated paths:

**Definition 4.2** (concatenation of paths)**.** *For two paths $(f,s),(g,t) \in S$ we define an operation of* concatenation $\oplus$ *by*

$$(f\oplus g)(r) := \begin{cases} f(r) & r\in[0,s] \\ f(s)-g(0)+g(r-s) & r\in(s,s+t]. \end{cases}$$

**Definition 4.3** (going paths)**.** *For any set of stopped paths $\Gamma \subseteq S$ we define the set of initial segments of this paths as*

$$\Gamma^{<} := \{(f, s) \in S : \exists(\tilde{f}, \tilde{s}) \in \Gamma \text{ such that } s < \tilde{s} \text{ and } f|_{[0,s]} = \tilde{f}|_{[0,s]}\}$$

*and call it the set of* going paths*.*

In [BCH17] a solution theory for so called *optimal Skorokhod embedding problems* was introduced. That is, among all stopping times solving the ($\mathrm{SEP}_{\lambda,\mu}$) we are looking for those stopping times satisfying the following optimality condition:

**Definition 4.4** (multifold optimal Skorokhod embedding problem)**.** *Let $n \in \mathbb{N}$ and consider a function*

$$\gamma = (\gamma_1, \dots, \gamma_n) : S \to \mathbb{R}^n.$$

*Let $Opt_\gamma^{(1)}$ be the set of all $\mathcal{G}$-stopping times $\tau$ on $\Omega$ solving the following optimization problem*

$$\inf\{\mathbb{E}[\gamma_1((B_t)_{t\leq\tau}, \tau)] : \tau \text{ solves } (\mathrm{SEP}_{\lambda,\mu})\}. \qquad (\mathrm{OptSEP}^{(1)}_{\lambda,\mu})$$

*For $j \in \{2, \dots, n\}$ define $Opt_\gamma^{(j)}$ as the set of all stopping times $\tau \in Opt_\gamma^{(j-1)}$ that solve the optimization problem*

$$\inf\{\mathbb{E}[\gamma_j((B_t)_{t\leq\tau}, \tau)] : \tau \in Opt_\gamma^{(j-1)}\}. \qquad (\mathrm{OptSEP}^{(j)}_{\lambda,\mu})$$

*We call* ($\mathrm{OptSEP}_{\lambda,\mu}$):= ($\mathrm{OptSEP}^{(n)}_{\lambda,\mu}$) *the* multifold optimal Skorokhod embedding problem*.*

The optimization problem ($\mathrm{OptSEP}^{(1)}_{\lambda,\mu}$) is *well posed* if for all stopping times solving the ($\mathrm{SEP}_{\lambda,\mu}$), $\mathbb{E}[\gamma_1((B_t)_{t\leq\tau}, \tau)]$ exists and $\mathbb{E}[\gamma_1((B_t)_{t\leq\tau}, \tau)] \in (-\infty, \infty]$ and if there is at least one stopping time $\tau$ so that $\mathbb{E}[\gamma_1((B_t)_{t\leq\tau}, \tau)] < \infty$.

We call ($\mathrm{OptSEP}_{\lambda,\mu}$) *well posed*, if ($\mathrm{OptSEP}^{(1)}_{\lambda,\mu}$) is well posed and for all $j \in \{2, \dots, n\}$ the problem ($\mathrm{OptSEP}^{(j)}_{\lambda,\mu}$) is well posed in the above sense (considering stopping times in $Opt_\gamma^{(j-1)}$).

In the multifold optimal Skorokhod embedding problem we consecutively optimize over functions of stopped paths. We will soon learn that it is interesting and useful to derive structural arguments about the sets of stopped paths satisfying these optimality conditions. In order to do this, we would like to have a strategy for dealing with different stopped paths stopping at the same value. Among those paths we would like to identify those that should be stopped and those, that should be allowed to continue - keeping in mind our optimization problem. This leads us to the notion of *stop-go* pairs, which by considering possible continuations of the paths gives a rule on how to decide on which one to stop and which one to allow to continue.

**Definition 4.5** (Stop-Go Pair)**.** *A pair of paths $((f, s), (g, t)) \in S \times S$ is considered a* stop-go pair *with respect to $\gamma$ (short: SG-pair), iff*

*(i)* $f(s) = g(t)$
*(ii)* $\mathbb{E}[\gamma(f \oplus (B_u)_{u\leq\sigma}, s + \sigma)] + \gamma(g, t) > \gamma(f, s) + \mathbb{E}[\gamma(g \oplus (B_u)_{u\leq\sigma}, t + \sigma)]$ *in the lexicographic ordering of $\mathbb{R}^n$ for every $(\mathcal{F}^B_t)_{t\geq0}$-stopping time $\sigma$ such that $\mathbb{E}[\sigma] \in (0, \infty)$, both sides are well defined and the left-hand side is finite in every component.*

*For the set of all* SG*-pairs we will write*

$$\mathsf{SG}_\gamma := \{((f,s),(g,t)) \in S \times S : ((f,s),(g,t)) \text{ is a stop-go pair with respect to } \gamma\},$$

*and for* $j \in \{1,\dots,n\}$ *we will call*

$$\mathbb{E}[\gamma_j(f \oplus (B_u)_{u\le\sigma}, s+\sigma)] + \gamma_j(g,t) \ge \gamma_j(f,s) + \mathbb{E}[\gamma_j(g \oplus (B_u)_{u\le\sigma}, t+\sigma)]$$

*the* $j$-th stop-go condition ($\mathrm{SGC}_j$)*.*

We now want to identify sets of stopped paths, such that it is not advantageous to stop any of these paths earlier in comparison to the other paths in this set. In other words, the stopping rule cannot be improved within this set.

**Definition 4.6** ($\gamma$-Monotonicity)**.** *A set of stopped paths* $\Gamma \subseteq S$ *is called* $\gamma$-monotone*, if*

$$\mathsf{SG}_\gamma \cap (\Gamma^< \times \Gamma) = \emptyset$$

The most important effort taken in [BCH17] was to establish the following existence and monotonicity result. Even though the main proofs were carried out for $n = 1$ (Chapter 3-5) and generalizations are provided for $n = 2$ (Chapter 6), we can generalize in precisely the same way to arbitrary $n \in \mathbb{N}$.

**Theorem 4.7** (Existence of a minimizer)**.** *Let* $\gamma : S \to \mathbb{R}^n$ *be lsc and bounded from below in the following sense:*

*For all* $j \in \{1,\dots,n\}$ *there exist some constants* $a_j, b_j, c_j \in \mathbb{R}_+$ *such that*

$$-\big(a_j + b_j t + c_j \max_{s\le t} B_s^2\big) \le \gamma_j((B_s)_{s\le t}, t)). \tag{4.1}$$

*Then optsep admits a minmizer* $\tau$*.*

**Theorem 4.8** (Monotonicity Principle)**.** *Let* $\gamma : S \to \mathbb{R}^n$ *be Borel measurable and assume that (OptSEP*$_{\lambda,\mu}$*) is well posed and that* $\tau$ *is a minimizer. Then there exists a* $\gamma$*-monotone Borel set* $\Gamma \subseteq S$ *such that*

$$P[((B_t)_{t\le\tau}, \tau) \in \Gamma] = 1.$$

*We say that* $\Gamma$ supports $\tau$*.*

We will see that the Monotonicity Principle is the key to the geometric approach to optimal Skorokhod embedding problems.

For the rest of the thesis we will make use of the following abbreviations:
For $(f,s) \in S$, we set

$$\overline{f} = \max_{r\le s} f(r) \quad \text{and} \quad \underline{f} = \min_{r\le s} f(r).$$

As it will sometimes be convenient to consider the running minimum respectively maximum up to a more recent timepoint $\tilde{s} \le s$ we will indicate this the following way:

$$\overline{f}_{\tilde{s}} = \max_{r\le \tilde{s}} f(r) \quad \text{and} \quad \underline{f}_{\tilde{s}} = \min_{r\le \tilde{s}} f(r).$$

4.2. **Perkins-embedding with deterministic starting.** Finding a geometric interpretation of the Perkins solution in the case of $\lambda = \delta_0$ is feasible with the methods established in [BCH17], see Theorem 6.8. therein.
We will give the following slight reformulation of this theorem in order to stress our interest in the specific structure of the target set.

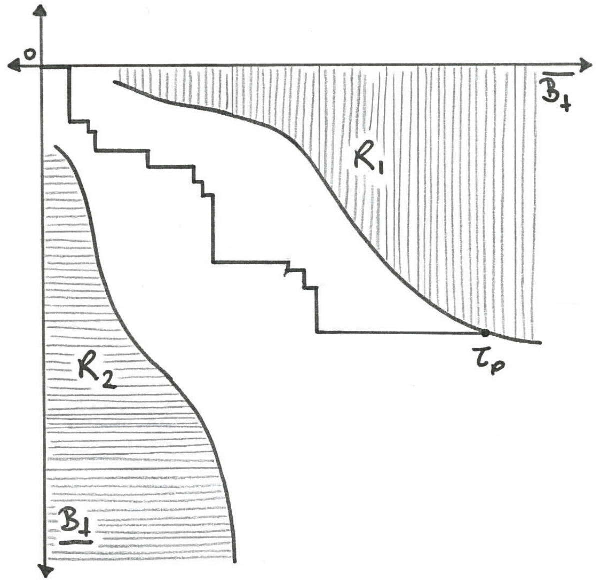
**Theorem 4.9** (The Perkins Embedding, cf. [Per86])**.** *Let $\lambda = \delta_0$ and assume $\mu(\{0\}) = 0$. Let $\varphi : \mathbb{R}_+^2 \to \mathbb{R}$ be a bounded function which is continuous and striclty increasing in both arguments. Then there exists a stopping time $\tau_P$ which minimizes*

$$\mathbb{E}[\varphi(\overline{B}_\tau, -\underline{B}_\tau)]$$

*over all solution to ($SEP_{\lambda,\mu}$) and which is of the form*

$$\tau_P = \inf\{t \geq 0 : (\overline{B}_t, \underline{B}_t) \in R\}.$$

*Here $R \subseteq \mathbb{R} \times \mathbb{R}_-$ can be represented as $R = R_1 \cup R_2$ with $R_1$ being an upwards barrier and $R_2$ being an inverse barrier (in the sense of Definitions 3.10, 3.7). Moreover the boundaries of $R_1$ and $R_2$ are both given by decreasing functions $\mathbb{R}_- \to \mathbb{R}$.*




Unfortunately, some of the arguments in this proof of this theorem cannot be extended to the random starting case.

Foremost it is no longer possible to optimize over the running minimum and the running maximum *simultaneously* without resorting to randomized stopping.

The stopping rule of Perkins' problem will only stop paths when they reach a new running extremum. This justifies the representation of $\tau_P$ as a hitting time of the process $(\overline{B}, \underline{B})$. Note, that since we do not allow $\mu$ to hold any mass in 0, we have $\tau_P = \tau > 0$ and by properties of Brownian Motion then $\overline{B}_\tau > \underline{B}_\tau$ a.s. These two facts imply that whenever we consider two paths stopped by this stopping rule at the same terminal value, either their running minima or their running maxima coincide. However, now it becomes very easy to decide on which one of those two paths to stop if we look at the other running extremum.

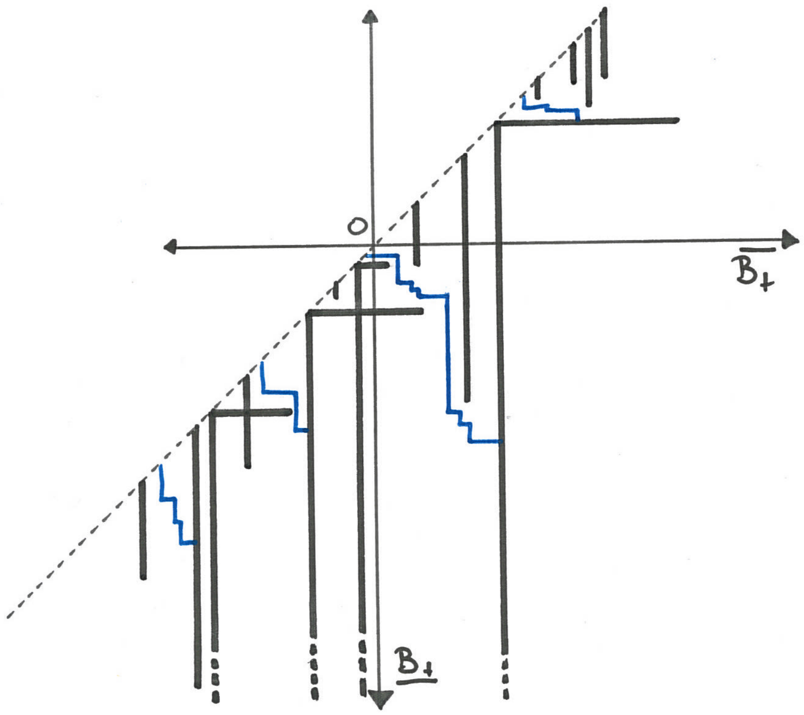
If we now allow for random starting, we can also encounter (among other problems) the following situation: Two paths $(f, s), (g, t) \in S$ stop at the same value, that is $f(s) = g(t)$, however - lets say $f$ - does so by reaching a new running minimum and $g$ by reaching a new running maximum. Then $\overline{f} < \overline{g}$ and $\underline{f} < \underline{g}$. It is therefore no longer obvious, which of those two paths should be stopped and we can see that the solution can no longer be identified as the hitting time of set serving both optimization problems simultaneously.

4.3. **Perkins-embedding with random starting.** By interpreting Perkins' problem with general starting law as an (OptSEP$_{\lambda,\mu}$) for a well chosen $\gamma$, we can use the methods and results in Subsection 4.1 to prove existence of a solution that is given as the hitting time of the process $(\overline{B}, \underline{B})$ of a specifically structured target set. As the difficulty of the problem increases when the two measures $\lambda$ and $\mu$ share mass, we will for now consider them to be mutually singular. This case still serves for a lot of applications, especially when considering discrete initial distributions and continuous target distributions, or whenever considering two measures with disjoint support.

The previous subsection ended by explaining why we can no longer optimize simultaneously over the running minimum and the running maximum. We will therefore decide, that from now on the running maximum is more important to us. However, it should be obvious that all following results and calculations work analogously if our choice fell on the other extremum.

**Theorem 4.10** (The Perkins Embedding with random starting)**.** *Let $\lambda$ and $\mu$ be mutually singular probability measures on $\mathbb{R}$ satisfying the usual convex ordering condition. Then there exists a stopping time $\tau_{RP}$ which minimizes $\mathbb{E}[\overline{B}_\tau]$ over all solutions of (SEP$_{\lambda,\mu}$) and maximizes $\mathbb{E}[\underline{B}_\tau]$ over all stopping times satisfying the former. Moreover there exists a set $R \subseteq \mathbb{R}^2$ such that*

$$\tau_{RP} = \inf\{t \geq 0 : (\overline{B}_t, \underline{B}_t) \in R\}.$$



*Proof.* Choose a function $\varphi : \mathbb{R} \to \mathbb{R}$ that is bounded and strictly increasing. We will consider the function $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) : S \to \mathbb{R}^4$ given by:

$$\begin{aligned}\gamma_1((f,s)) &:= \overline{f}\\ \gamma_2((f,s)) &:= -\underline{f}\\ \gamma_3((f,s)) &:= -\varphi(\overline{f})f(s)^2\\ \gamma_4((f,s)) &:= -\varphi(-\underline{f})f(s)^2.\end{aligned}$$

Then all $\gamma_j$ are bounded from below in the sense of (4.1) (choose $a = b = 0$ and $c := \sup_{x\in\mathbb{R}} |\varphi(x)|$ for all $j = 1, 2, 3, 4$) and (OptSEP$_{\lambda,\mu}$) is well posed. Thus Theorem 4.7 guarantees the existence of a minimizer $\tau_{RP}$ which by Theorem 4.8 is supported by a $\gamma$-monotone Borel set $\Gamma \subseteq S$.
The stop-go conditions amount to the following:
Let $((f, s), (g, t)) \in S \times S$ such that $f(s) = g(t)$, then

$$\mathbb{E}[\overline{f} \vee (f(s) + \overline{B}_\sigma)] + \overline{g} \geq \overline{f} + \mathbb{E}[\overline{g} \vee (g(s) + \overline{B}_\sigma)] \tag{SGC$_1$}$$

$$\mathbb{E}[\underline{f} \wedge (f(s) + \underline{B}_\sigma)] + \underline{g} \leq \underline{f} + \mathbb{E}[\underline{g} \wedge (g(s) + \underline{B}_\sigma)] \tag{SGC$_2$}$$

$$\begin{aligned}\mathbb{E}[\varphi(\overline{f} \vee (f(s) + \overline{B}_\sigma))(f(s) + B_\sigma)^2] + \varphi(\overline{g})g(t)^2 \leq&\\ \varphi(\overline{f})f(s)^2 + \mathbb{E}[\varphi(\overline{g} \vee (g(t) + \overline{B}_\sigma))(g(t) + B_\sigma)^2]& \qquad \text{(SGC}_3)\\ \mathbb{E}[\varphi(-\underline{f} \wedge (f(s) + \underline{B}_\sigma))(f(s) + B_\sigma)^2] + \varphi(-\underline{g})g(t)^2 \leq&\\ \varphi(-\underline{f})f(s)^2 + \mathbb{E}[\varphi(-\underline{g} \wedge (g(t) + \underline{B}_\sigma))(g(t) + B_\sigma)^2]& \qquad \text{(SGC}_3)\end{aligned}$$

Note, that $\tau_{RP} > 0$ a.s. since $\lambda$ and $\mu$ are mutually singular. Therefore by standard properties of Brownian motion we may assume that for all $(f, s) \in \Gamma$ we have $\overline{f} > \underline{f}$.

To legitimate that $\tau_{RP}$ is given as the hitting time of the process $(\overline{B}, \underline{B})$ we will start by showing that $\tau_{RP}$ will only stop Brownian motion when it is reaching a new running minimum or running maximum:

Consider a path which stops somewhere between its current running extrema, that is a path $(f, s) \in S$ such that $\underline{f} < f(s) < \overline{f}$. Let $r$ be the time where $f$ hits its last new extremum. As $f$ does not reach a new extremum at time $s$ we can consider the initial segment of this path up to a time point $\tilde{s} < r$ such that $f(\tilde{s}) = f(s)$ and either $\underline{f}_{\tilde{s}} = \underline{f}$ or $\overline{f}_{\tilde{s}} = \overline{f}$.
Let $\tilde{f} := f|_{[0,\tilde{s}]}$, then we claim that $((\tilde{f}, \tilde{s}), (f, s)) \in \mathsf{SG}_\gamma$.

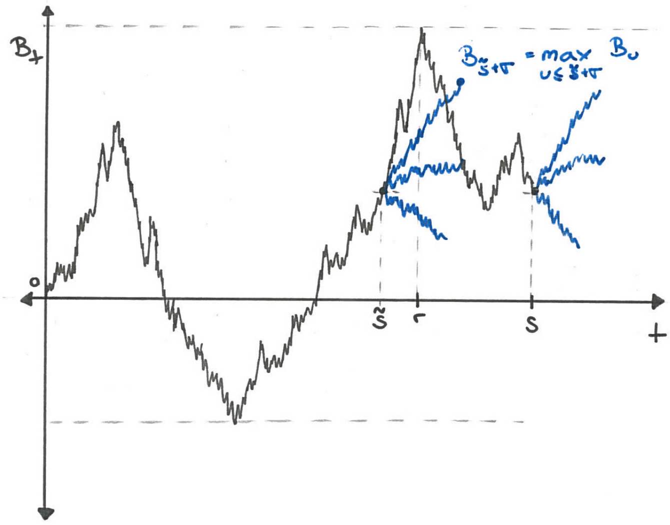
1. Case: $\underline{f}_{\tilde{s}} = \underline{f}$ and $\overline{f}_{\tilde{s}} < \overline{f}$, that is the last new extremum hit was a new maximum.
Assume $\overline{f}_{\tilde{s}} < f(\tilde{s}) + \overline{B}_\sigma$, then $\overline{f}_{\tilde{s}} \vee (f(\tilde{s}) + \overline{B}_\sigma) = f(\tilde{s}) + \overline{B}_\sigma$ and (SGC$_1$) reads

$$\begin{aligned}&\mathbb{E}[f(\tilde{s}) + \overline{B}_\sigma] + \overline{f} \geq \overline{f}_{\tilde{s}} + \overline{f} && \text{if} \quad \overline{f} \geq f(s) + \overline{B}_\sigma\\ &\mathbb{E}[f(\tilde{s}) + \overline{B}_\sigma] + \overline{f} \geq \overline{f}_{\tilde{s}} + \mathbb{E}[f(s) + \overline{B}_\sigma] && \text{if} \quad \overline{f} < f(s) + \overline{B}_\sigma\end{aligned}$$

and we see that in both cases a strict inequality holds due to our assumptions.
On the other hand, if $\overline{f}_{\tilde{s}} \geq f(\tilde{s}) + \overline{B}_\sigma$, then $\overline{f}_{\tilde{s}} \vee (f(\tilde{s}) + \overline{B}_\sigma) = \overline{f}_{\tilde{s}}$ as well as $\overline{f} \vee (f(s) + \overline{B}_\sigma) = \overline{f}$ and this leads to an equality in (SGC$_1$). Since $\underline{f}_{\tilde{s}} = \underline{f}$ we also have an equality in (SGC$_2$) and therefore jump to our third condition (SGC$_3$):

$$\mathbb{E}[\varphi(\overline{f}_{\tilde{s}})(f(\tilde{s}) + B_\sigma)^2] + \varphi(\overline{f})f(s)^2 \leq \varphi(\overline{f}_{\tilde{s}})f(\tilde{s})^2 + \mathbb{E}[\varphi(\overline{f})(f(s) + B_\sigma)^2]$$

Again, due to our assumptions $f(\tilde{s}) = f(s)$, $\overline{f}_{\tilde{s}} < \overline{f}$ and as $\varphi$ is strictly increasing, a strict inequality holds here and now obviously $((\tilde{f}, \tilde{s}), (f, s)) \in \mathsf{SG}_\gamma$.




*(In this figure we see that possible continuation of $B_{\tilde{s}}$ are more likely to increase the running maximum while continuing $B_s$ leaves the current running maximum unchanged.)*

2. Case: $\overline{f}_{\tilde{s}} = \overline{f}$ and $\underline{f}_{\tilde{s}} > \underline{f}$, that is the last new extremum hit was a new minimum. Now (SGC$_1$) as well as (SGC$_3$) will always exhibit an equality and conditions (SGC$_2$) and (SGC$_3$) can be treated analogously to the previous case.

As now due to $\gamma$-monotonicity $((\tilde{f}, \tilde{s}), (f, s)) \notin \Gamma^< \times \Gamma$ it follows that $\Gamma \cap \{(f, s) \in S : \underline{f} < f(s) < \overline{f}\} = \emptyset$, that is, when stopped we will almost surely have reached a new running minimum or a new running maximum.

Summing up, we now know that $\tau_{RP}$ stops paths of Brownian motion only when they reach a new running minimum or a new running maximum.

We will denote the set of stopped paths satisfying this condition by

$$\tilde{S} := \{(f, s) \in S : f(s) = \underline{f} \text{ or } f(s) = \overline{f}\}.$$

This justifies to consider the phase space $(\overline{B}, \underline{B})$. All possible paths will lie in the diagonal half plane $D := \{(x, y) \in \mathbb{R} : y \le x\}$.

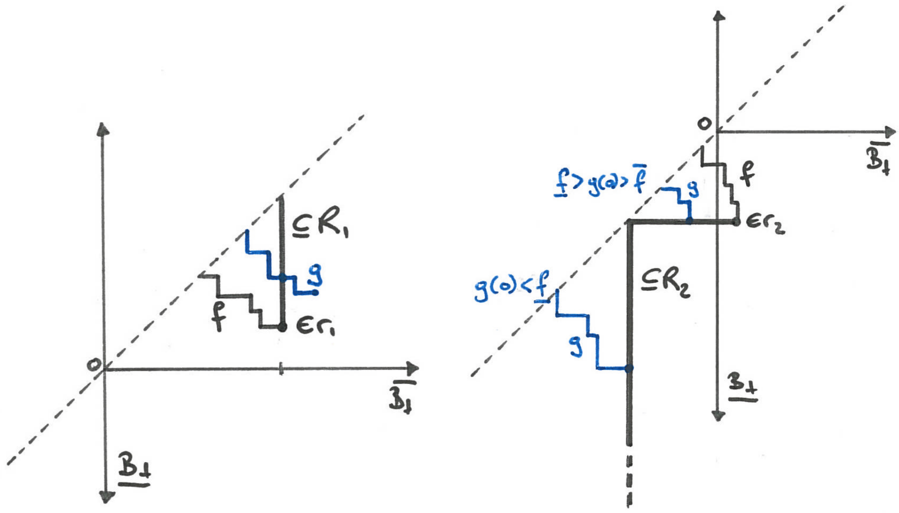
We propose that there are two sets of points $r_1, r_2 \subseteq D$, such that $R = R_1 \cup R_2$, where

$R_1 := \big\{\{x\} \times [y, x] : (x, y) \in r_1\big\}$, and $R_2 := \big\{\big([y, x] \times \{y\}\big) \cup \big(\{y\} \times [y, \infty)\big) : (x, y) \in r_2\big\}$.

Let us legitimate this target set structure:
Assume we know that there is a path $(f, s) \in \Gamma$ such that $s > 0$ and $f(s) = \overline{f}$, that is we stop at a new running maximum. We claim that it is impossible for a trajectory of the process $(\overline{B}, \underline{B})$ to *traverse* the $\{\overline{f}\} \times [\underline{f}, \overline{f})$ line-segment and then be stopped.

Consider a path $(g, t) \in \tilde{S}$ such that $g(0) \in (\underline{f}, \overline{f}]$, $\overline{g} \ge \overline{f}$ and $\underline{g} > \underline{f}$. Then there has to exist a timepoint $\tilde{t} \le t$ such that $g(\tilde{t}) = \overline{g}_{\tilde{t}} = \overline{f} = f(s)$ and note that still $\underline{g}_{\tilde{t}} > \underline{f}$ has to hold. However, this situation equals the 1. case of above discussion, hence again $((g, \tilde{t}), (f, s)) \in \mathsf{SG}_\gamma$. By $\gamma$-monotonicity it now follows that $(g, \tilde{t}) \notin \Gamma^<$,

therefore $(g,t) \notin \Gamma$ and our claim is proven.
It becomes clear that by choosing $r_1 := \{(\overline{f}, \underline{f}) : (f,s) \in \Gamma \text{ and } f(s) = \overline{f}\}$ the definition of $R_1$ as above is reasonable.




Now assume we know that there is a path $(f,s) \in \Gamma$ such that $s > 0$ and $f(s) = \underline{f}$, that is we stop at a new running minimum. We claim that it is impossible for a trajectory of the process $(\overline{B}, \underline{B})$ to *traverse* either the $[\underline{f}, \overline{f}) \times \{\underline{f}\}$ or the $\{\underline{f}\} \times [\underline{f}, \infty)$ line-segment and then be stopped.

Consider a path $(g,t) \in \tilde{S}$ such that $\overline{g} \in [\underline{f}, \overline{f})$ and $\underline{g} \leq \underline{f}$ and lets first assume $g(0) \in [\underline{f}, \overline{f})$.

Then again there has to exist a time point $\tilde{t} \leq t$ such that $g(\tilde{t}) = \underline{g}_{\tilde{t}} = \underline{f} = f(s)$ while $\overline{g}_{\tilde{t}} < \overline{f}$. Again $((g,\tilde{t}),(f,s)) \in \mathsf{SG}_\gamma$ as in the 1. Case above.

Now assume $g(0) < \underline{f}$. It is then possible to find a timepoint $\tilde{t} < t$ such that $g(\tilde{t}) = \overline{g}_{\tilde{t}} = \underline{f} = f(s)$ and still $\overline{g}_{\tilde{t}} < \overline{f}$. We remember that (SGC$_1$) exhibits an equality if $\overline{g}_{\tilde{t}} \geq g(\tilde{t}) + \overline{B}_\sigma$, which in our setting is equivalent to $\overline{B}_\sigma = 0$. If on the other hand $\overline{B}_\sigma > 0$, we will have a strict inequality. However as trivial stopping times are excluded, the later will always happen with positive probability and thus taking the expectation (SGC$_1$) will always exhibit a strict inequality, implying that $((g,\tilde{t}),(f,s)) \in \mathsf{SG}_\gamma$.

This concludes the proof of $(g,t) \notin \Gamma$.
Again, in analogy to above let us take $r_2 := \{(\overline{f}, \underline{f}) : (f,s) \in \Gamma \text{ and } f(s) = \underline{f}\}$ to justify the definition of $R_2$.

Define the following two target sets

$$R_{\text{CL}} := R = R_1 \cup R_2,$$

$$R_{\text{OP}} := \big\{\{x\} \times (y,x] : (x,y) \in r_1\big\} \cup \big\{\big([y,x) \times \{y\}\big) \cup \big(\{y\} \times [y,\infty)\big) : (x,y) \in r_2\big\}.$$

and consider

$$\tau_{\text{CL}} := \inf\{t \geq 0 : (\overline{B}_t, \underline{B}_t) \in R_{\text{CL}}\} \leq \tau_{\text{OP}} := \inf\{t \geq 0 : (\overline{B}_t, \underline{B}_t) \in R_{\text{OP}}\}.$$

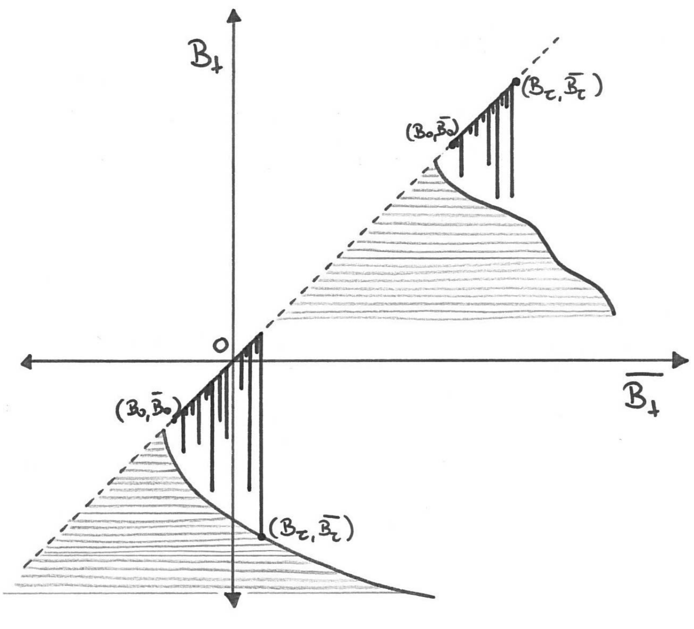
Note that since $\Gamma$ is Borel, the sets $r_1$ and $r_2$ are analytic sets since they are continuous images of Borel sets. This implies that $R_1$ and $R_2$ are analytic sets and we see that $\tau_{\text{CL}}$ and $\tau_{\text{OP}}$ are stopping times.

We would like to show, that $\tau_{\text{CL}} = \tau_{RP} = \tau_{\text{OP}}$ a.s. as our claim then follows. As $\Gamma \cap \{(f,s) \in S : \underline{f} < f(s) < \overline{f}\} = \emptyset$ it is obvious that $\tau_{\text{CL}} \leq \tau_{RP}$ a.s. by definition of $\tau_{\text{CL}}$.

To show that $\tau_{RP} \leq \tau_{\text{OP}}$ a.s. let us assume that this is not the case. Choose $\omega \in \Omega$ such that $((B_t(\omega))_{t \leq \tau_{RP}(\omega)}, \tau_{RP}(\omega)) \in \Gamma$ and assume $\tau_{\text{OP}}(\omega) < \tau_{RP}(\omega)$. Then there has to exist an $s \in [\tau_{\text{OP}}(\omega), \tau_{RP}(\omega))$ such that for $f = (B_t(\omega))_{t \leq s}$ we have $(\overline{f}, \underline{f}) \in R_{\text{OP}}$. As $s \leq \tau_{RP}(\omega)$ it follows, that $(f,s) \in \Gamma^{<}$, that is $(f,s)$ is a going path. By definition of $\tau_{\text{OP}}$ we can find find a point $(x,y) \in r_1$ such that $(\overline{f}, \underline{f}) \in \{x\} \times (y,x]$ or a point $(x,y) \in r_2$ such that $(\overline{f}, \underline{f}) \in \big([y,x) \times \{y\}\big) \cup \big(\{y\} \times [y,\infty)\big)$. However, we then find ourselves in the same situation of traversing line-segments as above. More precisely, by considering the path $(g,t) \in \Gamma$ corresponding to this $r_1$ respectively $r_2$ point we again find a SG-pair contradicting the $\gamma$-monotonicity of $\Gamma$. Hence $\tau_{RP} \leq \tau_{OP}$ a.s.

By standard properties of Brownian motion now $\tau_{\text{OP}} = \tau_{\text{CL}}$ a.s. which concludes the proof. □

**Remark 4.11** (Comparison to the Hobson-Pedersen solution)**.** *In* [HP02] *we find an alternative solution to the Perkins embedding with random starting law. The geometric idea of this construction can be derived from the Azema-Yor solution seen in Example 3.1. The Azema-Yor solution maximizes* $\mathbb{E}[\overline{B}_\tau]$ *over all solutions to the* $(SEP_\mu)$ *and we remember that it is given as the hitting time of the process* $(\overline{B}_t, B_t)$ *of a right barrier. In* [HP02] *we see that a solution to the Perkins embedding can be found in a similar fashion. As we now* minimize $\mathbb{E}[\overline{B}_\tau]$ *over all solutions to the* $(SEP_\mu)$ *instead of maximizing it, the barrier inducing the solution will be a* left *barrier. However, in* [HP02] *we see that in this case external randomization has to be introduced. That is, instead of only stopping when hitting the barrier, there is also the possibility of sometimes stopping somewhere on the diagonal according to a distribution independent of the Brownian motion and the initial distribution. Both possibilities of stopping are shown in the figure below.*



*We now want to point out that the significance in our solution found in Theorem 4.10 is that it does not rely upon external randomization.*

4.4. **Uniqueness.** In the previous section we have seen the Perkins solution with random starting be given as a hitting time of the process $(\overline{B}, \underline{B})$ of a specifically structured target set. As we have discussed uniqueness among barrier-type-solutions to the $(\text{GSEP}^{Z}_{\lambda,\mu})$ in the third chapter one might conjecture that a similar uniqueness result has to hold here. We see, however, that we cannot immediately apply the results of Chapter 3 and therefore want to investigate if our target set can somehow be seen as a barrier type set in the sense of this chapter, and if our Loynes type uniqueness result can be extended to this setting.

Barrier type arguments rely heavily on the consideration of *levels* reached by our process - in our current case by the Brownian Motion. The discussion in the previous section showed that there are two possibilities of stopping at a certain level. Considering the phase space of $(\overline{B}, \underline{B})$ we can either reach a level $b \in \mathbb{R}$ by attaining a new running maximum - which corresponds to somewhere hitting the vertical line $\{b\} \times \mathbb{R}$ - or by attaining a new running minimum - which corresponds to somewhere hitting the horizontal line $\mathbb{R} \times \{b\}$. The encoding of this concept of levels for our phase space $(\overline{B}, \underline{B})$ can therefore be done the following way:
Let us consider the set of all *levels* $\{A_b : b \in \mathbb{R}\}$, where for $b \in \mathbb{R}$ we define:

$$A_b := \{(x, b) : x \geq b\} \cup \{(b, y) : y \leq b\}.$$

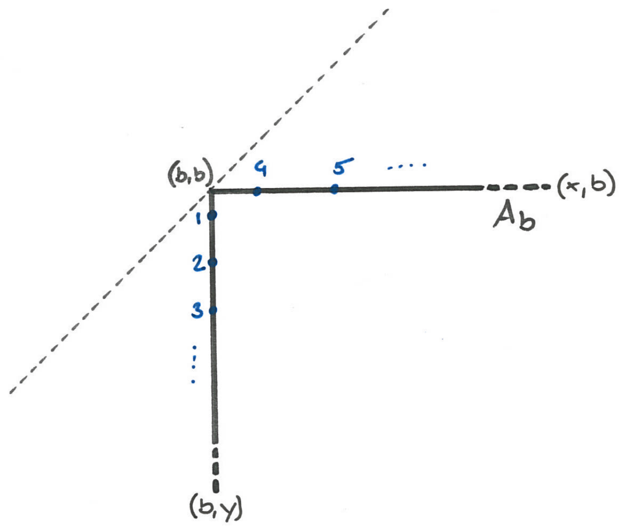
For each level we would like to determine whether we have stopped sooner or later, that is, we need some kind of (total) order on $A_b$.
For $(x_1, y_1), (x_2, y_2) \in A_b$ define the following relation which we will call *level-ordering*:

$$(x_1, y_1) \leq (x_2, y_2) :\Leftrightarrow \begin{cases} y_1 \geq y_2 & \text{if } x_1 = x_2 \\ x_1 < x_2 & \text{else.} \end{cases}$$

Defining the following space of all levels

$$A := \bigcup_{b \in \mathbb{R}} \{b\} \times A_b,$$

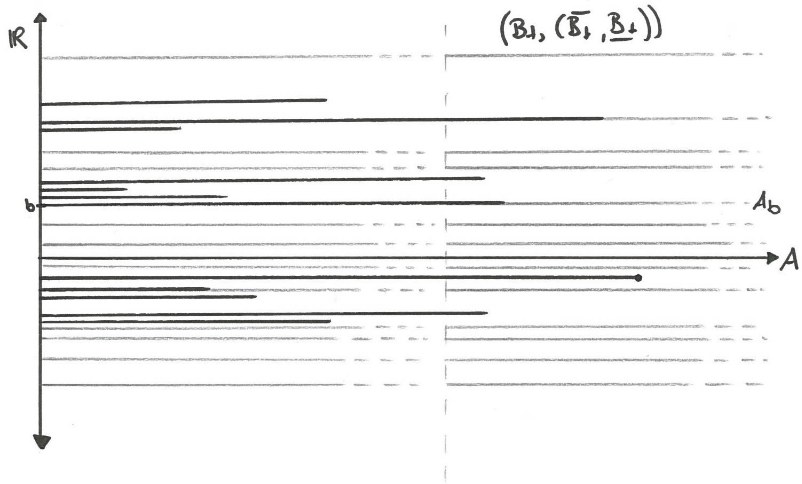
suggests considering the process $(B_t, (\overline{B}_t, \underline{B}_t))_{t \geq 0}$ on A.




It now becomes natural to represent the solution $\tau_{RP}$ found in Theorem 4.10 in the following way:

$$\tau_{RP} = \inf\{t \geq 0 : (B_t, (\overline{B}_t, \underline{B}_t)) \in \tilde{R}\} \quad \text{for an appropriate subset } \tilde{R} \in A.$$

The structural arguments concerning the target set $R$ of Theorem 4.10 show that together with the level-ordering defined above, the set $\tilde{R}$ will have an *inverse barrier* structure in the sense of Definition 3.7. We will call such a inverse barrier $\tilde{R}$ a *Perkins barrier* and assume it to be finely closed with respect to $(B, (\overline{B}, \underline{B}))$. A Loynes-type uniqueness result can now be established:




**Theorem 4.12.** *The Perkins-solution $\tau_{RP}$ is a.s. unique.*

*Proof.* Assume that there are two Perkins-barriers $R, S \subseteq A$ such that

$\tau_R = \inf\{t \geq 0 : (B_t, (\overline{B}_t, \underline{B}_t)) \in R\}$ respectively $\tau_S = \inf\{t \geq 0 : (B_t, (\overline{B}_t, \underline{B}_t)) \in S\}$

solve the $(SEP_\mu)$. Analogously to Proposition 3.2 and Corollary 3.5 the equality $\tau_R = \tau_S$ and therefore the uniqueness of the Perkins-Solution will follow due to minimality from the identity $\tau_{R\cap S} = \tau_R \wedge \tau_S$.
This identity is established analogously to Proposition 3.2 if the arguments are reinterpreted in this slightly different setup. For each $b \in \mathbb{R}$ we will by $R_b$ respectively $S_b$ denote the part of this barrier on the level $A_b$. As we now consider a form of inverse barrier, instead of levels where 'the barrier R comes sooner' we will consider the levels, where 'the barrier R holds out longer', that is

$$K := \{b \in \mathbb{R} : \sup R_b > \sup S_b\},$$

where the order is understood as the level-order defined above.
If now $B_{\tau_S} \in K$ respectively $B_{\tau_R} \in K^c$, we have $\tau_R \leq \tau_S$ respectively $\tau_S \leq \tau_R$ and it is impossible for $B_{\tau_R} \in K$ respectively $B_{\tau_S} \in K^c$. The proof can now be concluded in complete analogy to the proof of Proposition 3.2. □

Note that Theorem 4.12 also applies to the situation of Perkins' embedding with deterministic starting and simultaneous optimization of Theorem 4.9 since it is easy to see that modifying the sets $R_1$ and $R_2$ in Theorem 4.9 to fit the structure of the sets $R_1$ and $R_2$ described in the proof of Theorem 4.10 leaves the stopping time $\tau_P$ unchanged. Most importantly this implies the independence of the solution $\tau_P$ from the specific choice of $\varphi$ considered in Theorem 4.9.

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