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# ANALYSIS 1

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June 21, 2007

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CHAPTER I

**SEQUENCES, SERIES, AND  
SUBSETS OF  $\mathbb{R}$**

## §0. REVIEW OF REAL AND COMPLEX NUMBERS

In this section we briefly recall some of the basic definitions and properties concerning the sets of real and complex numbers (cf. [SS07]). This material is part of the prerequisites and is presented here without proofs. It merely should remind the reader of facts used in due course without further mentioning and serves to fix some notation.

### 0.1. Field axioms *(Körperaxiome)* for $\mathbb{R}$ :

#### Axioms of addition

(A1) Law of associativity *(Assoziativgesetz)*: for all  $x, y, z \in \mathbb{R}$

$$(x + y) + z = x + (y + z).$$

(A2) Law of commutativity *(Kommutativgesetz)*: for all  $x, y \in \mathbb{R}$

$$x + y = y + x.$$

(A3) Existence of zero: there exists a number  $0 \in \mathbb{R}$  such that

$$x + 0 = x \quad \forall x \in \mathbb{R}.$$

(A4) Existence of the additive inverse: for each  $x \in \mathbb{R}$  there exists a number  $-x \in \mathbb{R}$  such that

$$x + (-x) = 0.$$

#### Axioms of multiplication

(M1) Law of associativity: for all  $x, y, z \in \mathbb{R}$

$$(xy)z = x(yz).$$

(M2) Law of commutativity: for all  $x, y \in \mathbb{R}$

$$xy = yx.$$

(M3) Existence of a unit: there exists a number  $1 \in \mathbb{R}$  such that

$$x \cdot 1 = x \quad \forall x \in \mathbb{R}.$$

(M4) Existence of the multiplicative inverse: for each  $x \in \mathbb{R}$  with  $x \neq 0$  there exists a number  $x^{-1} \in \mathbb{R}$  such that

$$xx^{-1} = 1.$$

**Distributive law** *(Distributivgesetz)*

(D) For all  $x, y, z \in \mathbb{R}$

$$x(y + z) = xy + xz.$$

Recall some of the immediate consequences of the above axioms: the uniqueness of the zero element  $0 \in \mathbb{R}$ , the unit  $1 \in \mathbb{R}$ , as well as of the additive and multiplicative inverses  $-x$  and  $x^{-1}$  for any nonzero element  $x \in \mathbb{R}$ . Furthermore, there are no zero divisors (*Nullteiler*) in  $\mathbb{R}$ , i.e., for all  $x, y \in \mathbb{R} \setminus \{0\}$  we have  $xy \neq 0$ . Integer powers  $x^n$  ( $n \in \mathbb{N}$ ) of a real number  $x$  are always well-defined — as the  $n$ -fold product  $x \cdots x$ ; finite sums and products are well-defined and obey extended versions of the commutative and distributive law, thus leading to standard notation involving the symbols  $\sum$  and  $\prod$  as, for example, in

$$\sum_{j=1}^n a_j \quad \text{and} \quad \prod_{j=1}^n a_j \quad a_1, \dots, a_n \in \mathbb{R}.$$

**0.2. The complex number field  $\mathbb{C}$ :** The field axioms for  $\mathbb{C}$  are of course precisely the same with  $\mathbb{R}$  replaced by  $\mathbb{C}$  everywhere in the statements. (All axioms listed above are just instances of the abstract field axioms for the set  $\mathbb{R}$ .) Alternatively, we may identify  $\mathbb{C}$  as a set with  $\mathbb{R} \times \mathbb{R}$  equipped with operations of addition and multiplication for ordered pairs  $(x_j, y_j) \in \mathbb{R}^2$  ( $j = 1, 2$ ) as follows:

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1) \cdot (x_2, y_2) &:= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned}$$

Then we can prove that the field axioms are satisfied with  $(0, 0)$  as the zero element and  $(1, 0)$  as the unit. Recall that the element  $i := (0, 1)$  has the property that  $i^2 = (-1, 0)$  and that every complex number  $z$  can be written as

$$z = x + iy,$$

where  $x$  and  $y$  are uniquely determined real numbers. We call  $x$  the *real part* (*Realteil*) and  $y$  the *imaginary part* (*Imaginärteil*) of  $z$ , denoted by  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$ .

As an additional structure on  $\mathbb{C}$  there is *complex conjugation* (*komplexe Konjugation*) $^-: \mathbb{C} \rightarrow \mathbb{C}$ , defined by  $z = x + iy \mapsto \bar{z} := x - iy$ . As a map complex conjugation is a field automorphism (*Körperautomorphismus*) (i.e., isomorphism onto itself) with the additional property of involution (*Involution*), that is  $\bar{\bar{z}} = z$  holds for all  $z \in \mathbb{C}$ .

Note that by embedding  $\mathbb{R}$  as a subfield into  $\mathbb{C}$ ,  $x \mapsto x + i0$ , we may identify  $0 \in \mathbb{R}$  with  $0 \in \mathbb{C}$  as well as  $1 \in \mathbb{R}$  with  $1 \in \mathbb{C}$ .

**0.3.  $\mathbb{R}$  as an ordered field:** There exists a relation  $>$  on  $\mathbb{R}$  with the following properties:

(O1) Trichotomy (*Trichotomie*): for each  $x \in \mathbb{R}$  precisely one of the following holds

$$x > 0 \quad \text{or} \quad x = 0 \quad \text{or} \quad -x > 0.$$

Elements  $x$  satisfying  $x > 0$  are called *positive* (*positiv*). If  $-x > 0$  we also write  $x < 0$  and call  $x$  *negative* (*negativ*). If  $x$  non-positive we thus have  $x = 0$  or  $x < 0$ , which we summarize by writing  $x \leq 0$ . Similarly, we use the notation  $x \geq 0$  for a non-negative element  $x \in \mathbb{R}$ .

(O2) For all  $x, y \in \mathbb{R}$ : If  $x > 0$  and  $y > 0$  then  $x + y > 0$ .

(O3) For all  $x, y \in \mathbb{R}$ : If  $x > 0$  and  $y > 0$  then  $xy > 0$ .

DEFINITION: For  $x, y \in \mathbb{R}$  we henceforth write

$$\begin{aligned} x > y & \quad \text{if } x - y > 0, \\ x < y & \quad \text{if } y > x, \\ x \geq y & \quad \text{if } x > y \text{ or } x = y, \\ x \leq y & \quad \text{if } y - x \geq 0. \end{aligned}$$

From (O1) we obtain that for  $x, y \in \mathbb{R}$  precisely one of the relations  $x < y$ ,  $x = y$ , or  $x > y$  holds. (Thus the *maximum* and the *minimum* of two real numbers is well-defined.) We list a few more simple consequences for elements  $x, y, a \in \mathbb{R}$ :

transitivity (*Transitivität*):  $x < y$  and  $y < z$  implies  $x < z$   
 if  $x < y$  then  $x + a < y + a$   
 if  $x < y$  and  $a > 0$  then  $xa < ya$   
 $x < y \iff -x > -y$   
 if  $x \neq 0$  then  $x^2 > 0$   
 $y > x > 0 \iff x^{-1} > y^{-1} > 0$ .

Recall that an *interval* (*Intervall*) in  $\mathbb{R}$  is a set which is of any of the following types: let  $a, b \in \mathbb{R}$  with  $a < b$  and define

the open bounded interval  $]a, b[ := \{x \in \mathbb{R} : a < x < b\}$ ,

the open half-bounded intervals  $] - \infty, b[ := \{x \in \mathbb{R} : x < b\}$ ,  $]a, \infty[ := \{x \in \mathbb{R} : a < x\}$ ,

the half-open intervals  $]a, b] := \{x \in \mathbb{R} : a < x \leq b\}$ ,  $[a, b[ := \{x \in \mathbb{R} : a \leq x < b\}$ ,

the closed half-bounded intervals  $] - \infty, b] := \{x \in \mathbb{R} : x \leq b\}$ ,  $[a, \infty[ := \{x \in \mathbb{R} : a \leq x\}$ ,

the closed bounded interval  $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ ,

and  $] - \infty, \infty[ = \mathbb{R}$ .

**0.4. Absolute value in  $\mathbb{R}$ :** For a real number  $x$  we define its *absolute value* (*Absolutbetrag* oder *Betrag*) by

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The basic properties of the map  $|\cdot| : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto |x|$  are

(i) for all  $x \in \mathbb{R}$  we have  $|x| \geq 0$  and  $|x| = 0 \iff x = 0$ .

(ii) for all  $x, y \in \mathbb{R}$ :  $|xy| = |x| \cdot |y|$

(iii) Triangle inequality (*Dreiecksungleichung*):  $|x + y| \leq |x| + |y|$  for all  $x, y \in \mathbb{R}$ .

Simple consequences are  $|-x| = |x|$ ,  $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$  whenever  $y \neq 0$ , the so-called reverse triangle inequality

$$|x - y| \geq |x| - |y|$$

(or equivalently,  $|x + y| \geq |x| - |y|$ ), and the following formulae for maximum and minimum

$$\max(x, y) = \frac{x + y + |x - y|}{2} \quad \min(x, y) = \frac{x + y - |x - y|}{2}.$$

**0.5. Countable and uncountable sets in  $\mathbb{R}$ :** Recall that a set  $X$  is *countable* (*abzählbar*) if there exists a bijective map  $F : X \rightarrow \mathbb{N}$ . For example,  $\mathbb{N}$ ,  $\mathbb{N} \times \mathbb{N}$ , and  $\mathbb{Q}$  are countable sets, but  $\mathbb{R}$  is not. Thus  $\mathbb{R}$  is an uncountable set. In fact, any interval  $]a, b[$  with  $a < b$  is uncountable. (cf. [SS07, 4.4].)

**0.6. Order completeness (*Ordnungsvollständigkeit*) and uniqueness of  $\mathbb{R}$ :**

**AXIOM OF ORDER COMPLETENESS:** Any non-empty subset of  $\mathbb{R}$  which is bounded above (resp. below) possesses a supremum (resp. infimum).

**THEOREM (DEDEKIND):** Up to isomorphism (of ordered fields)  $\mathbb{R}$  is the unique order complete ordered field with the rational numbers  $\mathbb{Q}$  as an ordered subfield<sup>1</sup>.

**0.7. Some consequences of the order completeness of  $\mathbb{R}$ :** Based on the order completeness one can prove the following important properties of the set of real numbers (cf. [SS07]):

**THE ARCHIMEDIAN PROPERTY:** If  $a > 0$  and  $y \in \mathbb{R}$  then there exists  $n \in \mathbb{N}$  such that  $na > y$ .

**DENSITY OF  $\mathbb{Q}$  AND  $\mathbb{R} \setminus \mathbb{Q}$  IN  $\mathbb{R}$ :** If  $x, y \in \mathbb{R}$  with  $x < y$ , then there exists a rational number  $q \in \mathbb{Q}$  such that  $x < q < y$  and an irrational number  $s \in \mathbb{R} \setminus \mathbb{Q}$  such that  $x < s < y$ .

**EXISTENCE OF ROOTS:** If  $a \in \mathbb{R}$  and  $a > 0$ , then for all  $n \in \mathbb{N}$  there exists a unique  $x \in \mathbb{R}$  such that  $x^n = a$ .

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<sup>1</sup>Richard Dedekind (1831–1916) [ˈrɪçəɪt ˈdeːdəkɪnt], German mathematician

<sup>2</sup>Archimedes (287–212 B.C.) (ὁ Ἀρχιμήδης), one of the greatest ancient Greek mathematicians, physicists, engineers ...



# §1. $\mathbb{N}$ AS A SUBSET OF $\mathbb{R}$ AND SOME CONSEQUENCES OF THE ARCHIMEDIAN PROPERTY

## 1.1. $\mathbb{N}$ as a subset of $\mathbb{R}$ :

Here we briefly discuss how the natural numbers can be characterized as a subset of the real number field by means of the following notion.

**DEFINITION:** A subset  $X \subseteq \mathbb{R}$  is said to be *inductive* if

- (i)  $0 \in X$
- (ii)  $x \in X \implies x + 1 \in X$ .

**THEOREM:** There exists a smallest inductive subset  $\mathcal{N} \subseteq \mathbb{R}$ . The set  $\mathcal{N}$  together with the *successor map*  $S: \mathcal{N} \rightarrow \mathbb{R}$ ,  $n \mapsto n + 1$  satisfies *Peano's axioms*<sup>1</sup>, that is

**(PA1)**  $0 \in \mathcal{N}$

**(PA2)**  $\forall n \in \mathcal{N}: S(n) \in \mathcal{N}$ , i.e.  $S(\mathcal{N}) \subseteq \mathcal{N}$

**(PA3)**  $\nexists n \in \mathcal{N}: S(n) = 0$ , i.e.  $0 \notin S(\mathcal{N})$

**(PA4)**  $S$  is injective, i.e.  $\forall n, m \in \mathcal{N}: S(n) = S(m) \implies n = m$

**(PA5)** *Induction axiom:* if  $\mathcal{M} \subseteq \mathcal{N}$  is inductive (as a subset of  $\mathbb{R}$ ) then  $\mathcal{M} = \mathcal{N}$ .

*Proof.* Define  $\mathcal{N}$  to be the intersection of all inductive subsets of  $\mathbb{R}$ . By construction  $\mathcal{N}$  is contained in every inductive subset of  $\mathbb{R}$ , hence it is the smallest such set.

Furthermore,  $\mathcal{N}$  is an inductive subset of  $\mathbb{R}$ , since both defining properties of inductive sets are preserved under intersection. Thus (PA1) and (PA2) are immediate.

(PA3): The set  $\mathcal{P} := \{x \in \mathcal{N} : x \geq 0\}$  is inductive, since  $0 \in \mathcal{P}$  and  $x \geq 0$  implies  $x + 1 \geq 0$ . Therefore  $\mathcal{P}$  must contain the smallest inductive set  $\mathcal{N}$ , which yields  $\mathcal{P} = \mathcal{N}$ . In other words, all elements of  $\mathcal{N}$  are non-negative. If  $n \in \mathcal{N}$  with  $S(n) = 0$  then  $n + 1 = 0$ . Hence  $n = -1 < 0$  and  $n$  belongs to  $\mathcal{N}$  — a contradiction  $\zeta$ .

(PA4) follows from the field axioms (in  $\mathbb{R}$ ), since  $n + 1 = m + 1$  implies  $n = m$ .

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<sup>1</sup>Giuseppe Peano (1858–1932) [dʒu'seppe pe'a:ɲo], Italian mathematician

(PA5):  $\mathcal{M}$  is an inductive set and must therefore contain the smallest inductive set  $\mathcal{N}$ , which implies  $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}$ . Hence  $\mathcal{M} = \mathcal{N}$ .  $\square$

By the uniqueness of the set theoretic construction of the natural numbers  $\mathbb{N}$  the set  $\mathcal{N}$  is isomorphic to  $\mathbb{N}$  (cf. [SS07]). Thus we obtain a model of  $\mathbb{N}$  inside  $\mathbb{R}$  in the sense of the following statement.

**COROLLARY:** We may identify  $\mathbb{N}$  with  $\mathcal{N} \subseteq \mathbb{R}$  and can henceforth consider  $\mathbb{N}$  as a subset of  $\mathbb{R}$ .

**REMARK:** In contrast with the totally ordered field  $\mathbb{R}$  its subset  $\mathbb{N}$  (as an ordered set with the order inherited from that on  $\mathbb{R}$ ) is *well-ordered*, i.e., any non-empty subset of  $\mathbb{N}$  possesses a minimum. To see this, let  $\emptyset \neq A \subseteq \mathbb{N}$ . If  $A$  is a finite subset, the minimum clearly exists (and can be found after finitely many comparisons of the elements in  $A$ ). If  $A$  is not finite, choose  $a \in A$  arbitrary and define  $B := \{x \in A : x \leq a\}$ ,  $C := A \setminus B$ . Then  $A = B \cup C$ , every element in  $C$  is greater than any element in  $B$ , and  $B$  is finite, thus has a minimum. By construction, the minimum of  $B$  is the minimum of  $A$  as well.

The following results are simple, but important, consequences of the Archimedean property.

## 1.2. THEOREM:

- (i) For all  $\varepsilon \in \mathbb{R}$  with  $\varepsilon > 0$  there exists  $n \in \mathbb{N}$ ,  $n \geq 1$ , such that  $\frac{1}{n} < \varepsilon$ .
- (ii) Let  $r \in \mathbb{R}$  with  $r \geq 0$ . If for all  $n \in \mathbb{N}$ ,  $n \geq 1$ , the inequality  $r < \frac{1}{n}$  holds, then  $r = 0$ .

*Proof.* (i): Application of the Archimedean property (with  $a = \varepsilon$  and  $y = 1$ ) gives that  $\exists n \in \mathbb{N}$  such that  $n\varepsilon > 1$  (note that this implies  $n > 0$ ). Hence we obtain  $\varepsilon > 1/n$ .

(ii): Let  $r \geq 0$ . If  $r > 0$  then (i) implies the existence of  $m \in \mathbb{N}$ ,  $m \geq 1$ , such that  $0 < 1/m < r$  — a contradiction  $\frac{1}{2}$ . Thus only  $r = 0$  is possible.  $\square$

**1.3. LEMMA:** Let  $x \in \mathbb{R}$ ,  $x \geq -1$ . Then we have

$$\forall n \in \mathbb{N}: \quad (1+x)^n \geq 1+nx \quad \text{Bernoulli's inequality}^2.$$

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<sup>2</sup>The Bernoulli family [ˈbɛrnɔli] was a family of Swiss mathematicians in the 17th and 18th century, who originally came from Holland.

*Proof.* By induction on  $n$ .

If  $n = 0$  we clearly have  $(1 + x)^0 = 1 \geq 1 + 0 \cdot x$ .

Suppose the statement holds for  $n$ . Since  $1 + x \geq 0$  we conclude

$$\begin{aligned} (1 + x)^{n+1} &= (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) = 1 + nx + x + nx^2 \\ &= 1 + (n + 1)x + \underbrace{nx^2}_{\geq 0} \geq 1 + (n + 1)x. \end{aligned}$$

□

**1.4. PROPOSITION:** Let  $b \in \mathbb{R}$ .

(i) If  $b > 1$  then  $\forall K \in \mathbb{R} \exists n \in \mathbb{N}: b^n > K$ .

(ii) If  $0 < b < 1$  then  $\forall \varepsilon > 0 \exists n \in \mathbb{N}: b^n < \varepsilon$ .

*Proof.* (i): Let  $x := b - 1$ , then  $x > 0$ . For all  $m \in \mathbb{N}$  Bernoulli's inequality gives

$$b^m = (1 + x)^m \geq 1 + mx.$$

Let  $K \in \mathbb{R}$ . By the Archimedian property there is some  $n \in \mathbb{N}$  such that  $nx > K - 1$ . Therefore using  $m = n$  in the above inequality we obtain

$$b^n \geq 1 + nx > 1 + K - 1 = K.$$

(ii): Put  $b_1 := 1/b$ , then  $b_1 > 1$ . Let  $\varepsilon > 0$ . Then (i) can be applied to  $b_1$  with  $K := 1/\varepsilon$ , i.e. there is some  $n \in \mathbb{N}$  such that  $b_1^n > K$ . Thus we have

$$b^n = \frac{1}{b_1^n} < \frac{1}{K} = \varepsilon.$$

□

**1.5. Geometric sums:** Let  $n \in \mathbb{N}$ . We define the function  $s_n: \mathbb{R} \rightarrow \mathbb{R}$  by

$$s_n(x) := \sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n \quad \forall x \in \mathbb{R}.$$

If  $x = 1$  we obtain

$$s_n(1) = \sum_{k=0}^n 1 = n + 1.$$

If  $x \neq 1$  we take the differences on both sides of the following equations

$$\begin{aligned} s_n(x) &= 1 + x + x^2 + \dots + x^{n-1} + x^n \\ x s_n(x) &= x + x^2 + x^3 + \dots + x^n + x^{n+1} \end{aligned}$$

to obtain

$$\underbrace{s_n(x) - x s_n(x)}_{(1-x)s_n(x)} = 1 - x^{n+1},$$

which in turn yields the following formula

$$(1.1) \quad \sum_{k=0}^n x^k = \frac{1 - x^{n+1}}{1 - x} \quad (n \in \mathbb{N}, x \in \mathbb{R} \setminus \{1\}).$$

**Observation:** We may rewrite (1.1) as

$$s_n(x) = \sum_{k=0}^n x^k = \frac{1}{1-x} - \frac{x^{n+1}}{1-x},$$

where the first term of the difference on the right-hand side is independent of  $n$ .

If  $|x| < 1$  the absolute value of the second term of the above difference has numerator  $\langle \text{Zähler} \rangle |x|^{n+1}$ , which becomes arbitrarily small when  $n$  gets large. To be more precise, for every  $\varepsilon_1 > 0$  Proposition 1.4.(ii) guarantees that there exists an  $N \in \mathbb{N}$  such that  $|x|^N < \varepsilon_1$ . In fact, the latter inequality then holds for all  $n \in \mathbb{N}$  with  $n \geq N$  as well:  $|x|^n < \varepsilon_1$ . Therefore we obtain that

$$\left| \frac{x^{n+1}}{1-x} \right| = \frac{|x|^{n+1}}{1-x} < \frac{\varepsilon_1}{1-x} \quad \forall n \geq N.$$

Let  $\varepsilon > 0$  be arbitrary and put  $\varepsilon_1 := \varepsilon(1-x)$ . Then the above inequality implies that

$$\left| s_n(x) - \frac{1}{1-x} \right| < \varepsilon \quad \forall n \in \mathbb{N}, n \geq N.$$

Thus we see that for arbitrary fixed  $x$  with  $|x| < 1$  the sum  $s_n(x)$  is approximately equal to  $1/(1-x)$  as  $n$  gets large, in the sense that the error can be made smaller than any given positive “tolerance” as soon as  $n$  is larger than an appropriately chosen “number of steps in the computation”.

## §2. SEQUENCES AND LIMITS

**2.1. DEFINITION:** A *sequence* (*Folge*) of real numbers is a map  $a: \mathbb{N} \rightarrow \mathbb{R}$ . Thus for every  $n \in \mathbb{N}$  a number  $a(n) \in \mathbb{R}$  is given.

We usually write sequences in indexed form, i.e. we set  $a_n := a(n)$  and denote the sequence by  $(a_n)_{n \in \mathbb{N}}$  or  $(a_0, a_1, a_2, \dots)$  or simply  $(a_n)$ . Occasionally we will encounter index sets other than  $\mathbb{N}$ , as for example in the sequence  $(a_k, a_{k+1}, \dots)$  starting with index  $k$ , also denoted by  $(a_n)_{n \geq k}$ ; sometimes  $k$  will be allowed to be a negative integer as well. (In fact, any countable set can serve as index set as long as it is totally ordered.)

### 2.2. EXAMPLES:

1) Let  $c \in \mathbb{R}$  and  $a_n = c$  for all  $n \in \mathbb{N}$ . This gives the constant sequence  $(c, c, \dots) = (c)_{n \in \mathbb{N}}$ .

2) Let  $a_n = \frac{1}{n}$  for all  $n \in \mathbb{N}$  with  $n \geq 1$ . Then we get  $\left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$ .

3)  $a_n = (-1)^n$  ( $n \in \mathbb{N}$ ) defines  $(a_n) = (1, -1, 1, -1, \dots)$ .

4)  $a_n = \frac{n}{n+1}$  gives the sequence  $\left(\frac{n}{n+1}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \dots\right)$ .

5)  $a_n = \frac{n}{2^n}$  describes the sequence  $\left(\frac{n}{2^n}\right) = \left(0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \dots\right)$ .

6) The *Fibonacci*<sup>1</sup> numbers: Let  $f_0 = 0$  and  $f_1 = 1$ . If  $n \geq 2$  define  $f_n$  inductively by

$$f_n = f_{n-1} + f_{n-2}.$$

Thus we obtain  $(f_n) = (0, 1, 1, 2, 3, 5, 8, 13, 21, \dots)$ .

7) Let  $x \in \mathbb{R}$  and put  $a_n = x^n$ . Then we obtain the sequence  $(a_n) = (1, x, x^2, x^3, \dots)$ .

8) Let  $x \in \mathbb{R}$  and define  $s_n = \sum_{k=0}^n x^k$  ( $= \sum_{k=0}^n a_k$  with  $a_k$  as in 7)). This gives the sequence of geometric sums  $(s_n) = (1, 1+x, 1+x+x^2, \dots)$ .

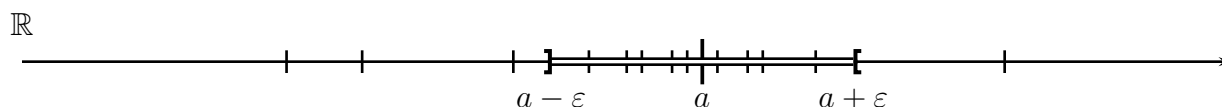
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<sup>1</sup>Fibonacci [fibo'nattfi], actually Leonardo Pisano (1170–1250 [?]) [leo'nardo pi'sa:no], an Italian mathematician, invented this series to solve a problem according to the breeding of rabbits.

With the following notion the subject of mathematical analysis really gets started.

**2.3. DEFINITION:** Let  $(a_n)$  be a sequence and  $a \in \mathbb{R}$ . The sequence  $(a_n)$  is said to be *convergent* (*konvergent*) to (wards)  $a$  if the following holds:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } |a_n - a| < \varepsilon \quad \forall n \geq N.$$



In this case  $a$  is called the *limit*<sup>2</sup> (*Grenzwert oder Limes*) of the sequence  $(a_n)$ . In symbols we describe this fact by  $a = \lim_{n \rightarrow \infty} a_n$ , briefly  $a = \lim a_n$ , or  $a_n \rightarrow a$  as  $n \rightarrow \infty$ , also  $a_n \rightarrow a$  ( $n \rightarrow \infty$ ).

Equivalently,  $a$  is the limit of  $(a_n)$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } a_n \in ]a - \varepsilon, a + \varepsilon[ \quad \forall n \geq N.$$

We then also say that the members  $a_n$  of the sequence eventually or finally lie in the interval  $]a - \varepsilon, a + \varepsilon[$  or the property  $|a - a_n| < \varepsilon$  holds for almost all  $n \in \mathbb{N}$ , i.e. for all but finitely many  $n$  (namely at most those with  $n < N$ ) the statement is true.

If a sequence  $(a_n)$  is not convergent (to any  $a \in \mathbb{R}$ ) it is said to be *divergent* (*divergent*).

A sequence converging to 0, i.e.  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ , is called a *zero sequence* or *null sequence* (*Nullfolge*).

**Note:** In the conditions stated above  $\varepsilon > 0$  is “given arbitrarily” and our task in showing convergence is to find  $N \in \mathbb{N}$  (which in general will depend on  $\varepsilon$ !) such that for all  $n \geq N$  the sequence element  $a_n$  belongs to the  $\varepsilon$ -neighborhood ( $\varepsilon$ -Umgebung)  $U_\varepsilon(a) := ]a - \varepsilon, a + \varepsilon[$  of  $a \in \mathbb{R}$ .

## 2.4. Examples:

1) The constant sequence  $(a_n) = (c)$  is convergent to  $c$ : for given  $\varepsilon > 0$  choose  $N = 0$ , then  $|a_n - c| = 0 < \varepsilon$  for all  $n \geq N$ .

2)  $\left(\frac{1}{n}\right)$  is a null sequence: let  $\varepsilon > 0$ ; by the Archimedian property there exists  $N \in \mathbb{N}$  such that  $N > 1/\varepsilon$ ; then we have for  $n \geq N$  that  $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ .

<sup>2</sup>The term “limit” (from the Latin *limes*, literally “border”) was first used by Isaac Newton.

3)  $((-1)^n)$  is divergent, which we prove by contradiction (*indirekt*): suppose there is  $a \in \mathbb{R}$  with  $(-1)^n \rightarrow a$  ( $n \rightarrow \infty$ ); let  $\varepsilon := 1/2$  and choose  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have  $|a_n - a| < \varepsilon = 1/2$ . Note that  $|a_{n+1} - a_n| = |(-1)^{n+1} - (-1)^n| = |(-1)^n(-1 - 1)| = 2$ . Hence we obtain

$$2 = |a_{n+1} - a_n| = |(a_{n+1} - a) + (a - a_n)| \underset{\text{[triangle inequ.]}}{\leq} |a_{n+1} - a| + |a - a_n| < \frac{1}{2} + \frac{1}{2} = 1$$

— a contradiction  $\nexists$ .

4)  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ , i.e.  $\left(\frac{n}{n+1}\right)$  is convergent to 1: let  $\varepsilon > 0$ ; choose  $N \in \mathbb{N}$  such that  $N > 1/\varepsilon$ , then we have for all  $n \geq N$

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{n - (n+1)}{n+1} \right| = \frac{1}{n+1} \leq \frac{1}{N} < \varepsilon.$$

5)  $\lim_{n \rightarrow \infty} \frac{n}{2^n} = 0$ : as a lemma we state that

$$\forall n \geq 4: n^2 \leq 2^n \quad \text{[proof by induction, exercise!]}.$$

Using the above result we have  $\frac{n}{2^n} \leq \frac{1}{n}$  if  $n \geq 4$ ; let  $\varepsilon > 0$  and choose  $N \in \mathbb{N}$  such that  $N \geq \max(4, 2/\varepsilon)$ , then we have for  $n \geq N$

$$\left| \frac{n}{2^n} - 0 \right| = \frac{n}{2^n} \leq \frac{1}{n} \leq \frac{1}{N} \leq \frac{\varepsilon}{2} < \varepsilon.$$

**2.5. Definition:** A sequence  $(a_n)$  is said to be *bounded (from) above* (*nach oben beschränkt*) (resp. *bounded (from) below* (*nach unten beschränkt*)), if there exists  $K \in \mathbb{R}$  such that for all  $n \in \mathbb{N}$  we have  $a_n \leq K$  (resp.  $a_n \geq K$ ). The sequence is *(bounded)* (*beschränkt*) if it is bounded above and below.

Thus we have the following equivalence:

$$(2.1) \quad (a_n) \text{ is bounded} \iff \exists K > 0 \forall n \in \mathbb{N} : |a_n| \leq K.$$

Note that the constant  $K$  is independent of  $n$ .

**2.6. Proposition:** A convergent sequence is bounded.

*Proof.* Let  $a = \lim a_n$  and choose  $N \in \mathbb{N}$  such that  $|a_n - a| < 1$  holds when  $n \geq N$ . Then for all  $n \geq N$

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|.$$

Put  $K := \max(|a_0|, |a_1|, \dots, |a_{N-1}|, |a| + 1)$ , then  $|a_n| \leq K$  follows for all  $n \in \mathbb{N}$ .  $\square$

**2.7. Remark:** The converse statement of the above proposition is wrong. For example, the sequence  $((-1)^n)$  is bounded (since  $|a_n| = 1$  for all  $n$ ) but not convergent [cf. Example 2.4.3)].

## 2.8. Examples:

1) The sequence  $(f_n)$  of Fibonacci numbers [cf. Example 2.2.6)] is divergent.

*Proof.* We assert that  $\forall n \geq 5: f_n \geq n$ .

Indeed, we have  $f_5 = 5$  and then for  $n \geq 6$  inductively

$$f_{n+1} = f_n + f_{n-1} \underset{[\text{ind. hyp.}]}{\geq} n + (n-1) \geq n + (2-1) = n + 1.$$

The above assertion now implies that the sequence  $(f_n)$  is unbounded [Archimedean property of  $\mathbb{R}$ ] and hence cannot be convergent [by the negation of Proposition 2.6].  $\square$

2) Let  $x \in \mathbb{R}$  and consider the sequence  $(x^n)_{n \in \mathbb{N}}$ . The convergence properties depend on the value of  $x$ :

Case  $|x| > 1$ : Proposition 1.4.(i) implies that for every  $K \in \mathbb{R}$  there is an  $N \in \mathbb{N}$  such that  $\forall n \geq N$  we have  $|x|^n > K$ . Therefore  $(x^n)$  is not bounded, hence cannot be convergent.

Case  $|x| = 1$ : If  $x = -1$  then  $(x^n) = ((-1)^n)$  is not convergent [cf. Example 2.4.3)]; if  $x = 1$  then  $(x^n) = (1)_{n \in \mathbb{N}}$  is a constant sequence and thus convergent to 1.

Case  $|x| < 1$ : If  $x = 0$  then  $(x^n)_{n \geq 1} = (0)_{n \geq 1}$  has limit 0.

Finally, if  $0 < |x| < 1$  we can show that  $(x^n)$  also converges to 0: Let  $\varepsilon > 0$ . By Proposition 1.4.(ii) there is an  $N \in \mathbb{N}$  such that  $|x|^n < \varepsilon$  holds for all  $n \geq N$ . In other words,

$$|x^n - 0| = |x^n| = |x|^n < \varepsilon \quad \forall n \geq N,$$

which proves that  $\lim_{n \rightarrow \infty} x^n = 0$ .

**2.9. Proposition (Uniqueness of the limit):** If a sequence  $(a_n)$  converges to  $a \in \mathbb{R}$  and to  $b \in \mathbb{R}$ , then  $a = b$ .

(In particular, the notation  $\lim a_n = a$  is justified by this statement.)

*Proof.* (By contradiction.) Suppose  $a \neq b$  and put  $\varepsilon := \frac{|a-b|}{3}$ . Then  $\varepsilon > 0$  and we observe the following:

$$\lim a_n = a \implies \exists N_1 \text{ such that } \forall n \geq N_1: |a_n - a| < \varepsilon$$



as well as

$$\lim a_n = b \implies \exists N_2 \text{ such that } \forall n \geq N_2: |a_n - b| < \varepsilon.$$

Thus for  $n \geq N := \max(N_1, N_2)$  both of the above inequalities are valid and yield

$$|a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < 2\varepsilon = \frac{2}{3}|a - b|.$$

By assumption, we have  $|a - b| > 0$  and therefore obtain  $1 < 2/3$  — a contradiction  $\zeta$ .  $\square$

## 2.10. Basic operations with convergent sequences:

(i) **Sum and product:** If  $(a_n)$ ,  $(b_n)$  are convergent, then the sequences  $(a_n + b_n)$  and  $(a_n \cdot b_n)$  are also convergent and

$$(2.2) \quad \lim_{n \rightarrow \infty} (a_n + b_n) = \left( \lim_{n \rightarrow \infty} a_n \right) + \left( \lim_{n \rightarrow \infty} b_n \right)$$

$$(2.3) \quad \lim_{n \rightarrow \infty} (a_n b_n) = \left( \lim_{n \rightarrow \infty} a_n \right) \cdot \left( \lim_{n \rightarrow \infty} b_n \right).$$

*Proof.* Let  $a := \lim a_n$  and  $b := \lim b_n$ .

*Sum:* We have to show that  $a_n + b_n \rightarrow a + b$  ( $n \rightarrow \infty$ ).

Let  $\varepsilon > 0$ . Then  $\varepsilon/2 > 0$  as well, thus we have

$$\lim a_n = a \implies \exists N_1 \text{ such that } \forall n \geq N_1: |a_n - a| < \varepsilon/2$$

and

$$\lim b_n = b \implies \exists N_2 \text{ such that } \forall n \geq N_2: |b_n - b| < \varepsilon/2.$$

Therefore we obtain for  $n \geq N := \max(N_1, N_2)$

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

*Product:* We have to show that  $a_n b_n \rightarrow ab$  ( $n \rightarrow \infty$ ).

Since  $(a_n)$  is convergent it is bounded, thus  $\exists K_1 > 0$  such that  $|a_n| \leq K_1$  for all  $n \in \mathbb{N}$ . Define  $K := \max(K_1, |b|)$ , then  $K > 0$ .

Let  $\varepsilon > 0$ . Using  $\frac{\varepsilon}{2K}$  in place of  $\varepsilon$  in the defining property of convergence for  $(a_n)$  and  $(b_n)$  we obtain

$$\exists M_1 \in \mathbb{N} \text{ such that } |a_n - a| < \varepsilon/(2K) \text{ for all } n \geq M_1 \text{ as well as}$$

$$\exists M_2 \in \mathbb{N} \text{ such that } |b_n - b| < \varepsilon/(2K) \text{ for all } n \geq M_2.$$

Therefore we have for all  $n \geq M := \max(M_1, M_2)$

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| = |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n| |b_n - b| + |a_n - a| |b| < K \frac{\varepsilon}{2K} + \frac{\varepsilon}{2K} K = \varepsilon. \end{aligned}$$

$\square$

(ii) **Linearity of the limit:** As a corollary to (i) we obtain: If  $\lambda, \mu \in \mathbb{R}$  and  $(a_n), (b_n)$  are convergent, then the linear combination  $\lambda \cdot (a_n)_{n \in \mathbb{N}} + \mu \cdot (b_n)_{n \in \mathbb{N}} := (\lambda a_n + \mu b_n)_{n \in \mathbb{N}}$  converges as well and

$$\lim_{n \rightarrow \infty} (\lambda a_n + \mu b_n) = \lambda \lim_{n \rightarrow \infty} a_n + \mu \lim_{n \rightarrow \infty} b_n.$$

*Proof.* Let  $a := \lim a_n$  and  $b := \lim b_n$ . The constant sequence  $(\lambda)$ , resp.  $(\mu)$ , converges to  $\lambda$ , resp.  $\mu$ , hence by (i)  $\lambda a_n \rightarrow \lambda a$  and  $\mu b_n \rightarrow \mu b$ ; furthermore, again by (i), we have the sum rule  $(\lambda a_n) + (\mu b_n) \rightarrow \lambda a + \mu b$ .  $\square$

(iii) **Quotient:** If  $(a_n), (b_n)$  are convergent and  $b := \lim b_n$  is nonzero, then there exists  $n_0 \in \mathbb{N}$  such that  $b_n \neq 0$  for all  $n \geq n_0$  and the sequence  $\left(\frac{a_n}{b_n}\right)_{n \geq n_0}$  converges with limit

$$(2.4) \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}.$$

*Proof.* Let  $a := \lim a_n$ .

We first show that  $\frac{1}{b_n} \rightarrow \frac{1}{b}$  ( $n \rightarrow \infty$ ).

Put  $\varepsilon' := |b|/2$  and note that  $\varepsilon' > 0$ . By convergence of  $(b_n)$  to  $b$ , there is an  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have

$$\frac{|b|}{2} = \varepsilon' > |b_n - b| \underset{[\text{reverse tri. inequ.}]}{\geq} |b| - |b_n|.$$

The outermost inequality implies  $|b_n| > |b|/2$  and thus  $b_n \neq 0$  for all  $n \geq n_0$ .

Let  $\varepsilon > 0$ , then, again by the basic convergence estimate of  $(b_n)$  applied to  $\varepsilon'' := |b|^2 \varepsilon / 2 > 0$ , we obtain the existence of  $N_1 \in \mathbb{N}$  such that

$$|b_n - b| < \varepsilon'' = \frac{|b|^2 \varepsilon}{2} \quad \forall n \geq N_1.$$

We conclude that for  $n \geq N := \max(n_0, N_1)$

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{b_n b} \right| = \frac{1}{|b_n|} \frac{1}{|b|} |b - b_n| \leq \frac{2}{|b|} \frac{1}{|b|} \frac{|b|^2 \varepsilon}{2} = \varepsilon.$$

Finally, the product rule (i) gives  $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \rightarrow a \cdot \frac{1}{b} = \frac{a}{b}$ .  $\square$

(iv) Example: Let  $a_n = \frac{3n^2 + 13n}{n^2 + 2}$  ( $n \in \mathbb{N}$ ).

We rewrite  $a_n$  (upon division of the numerator and the denominator by the highest order term with respect to  $n$ ) in the form

$$a_n = \frac{3 + \frac{13}{n}}{1 + \frac{2}{n^2}}$$

and observe that repeated use of (i) and (ii) yields  $3 + \frac{13}{n} = 3 + 13 \cdot \frac{1}{n} \rightarrow 3 + 13 \cdot 0 = 3$  and  $1 + \frac{2}{n^2} = 1 + 2 \cdot \frac{1}{n} \cdot \frac{1}{n} \rightarrow 1 + 2 \cdot 0 \cdot 0 = 1$ , hence by the quotient rule (iii)

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3n^2 + 13n}{n^2 + 2} = \lim_{n \rightarrow \infty} \frac{3 + \frac{13}{n}}{1 + \frac{2}{n^2}} = \frac{\lim 3 + \frac{13}{n}}{\lim 1 + \frac{2}{n^2}} = \frac{3}{1} = 3.$$

(v) If  $(a_n), (b_n)$  are convergent and  $a_n \leq b_n$  for almost all  $n$  (i.e. for all  $n \geq n_0$ ), then

$$\lim a_n \leq \lim b_n.$$

*Proof.* Put  $c_n := b_n - a_n$ , then  $c_n \geq 0$  for almost all  $n$  and  $c := \lim c_n = \lim b_n - \lim a_n$  by (ii). Thus it suffices to show that  $c \geq 0$ , since this implies  $\lim a_n \leq \lim b_n$ .

Suppose the contrary, that is  $c < 0$ . Let  $\varepsilon := -c$ , then  $\varepsilon > 0$  and there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$  we have

$$\varepsilon > |c_n - c| = |c_n - (-\varepsilon)| = |c_n + \varepsilon| \underset{[c_n \geq 0, \varepsilon > 0]}{=} c_n + \varepsilon.$$

But this implies that for all  $n \geq N$  we would have  $c_n < 0$  — a contradiction  $\zeta$ . □

(vi) **The Sandwich Lemma:** If  $(a_n), (b_n), (c_n)$  are sequences such that  $a_n \leq b_n \leq c_n$  holds for almost all  $n$  and  $a_n \rightarrow a, c_n \rightarrow a$  ( $n \rightarrow \infty$ ), then also  $(b_n)$  is convergent and  $\lim b_n = a$ .

*Proof.* Let  $\varepsilon > 0$  and choose  $N \in \mathbb{N}$  such that for all  $n \geq N$  the inequalities  $|a_n - a| < \varepsilon$  and  $|c_n - a| < \varepsilon$  hold. Then

$$a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon \quad \forall n \geq N$$

and upon subtracting  $a$  we obtain

$$-\varepsilon < b_n - a < \varepsilon \quad \forall n \geq N.$$

In other words, the inequality  $|b_n - a| < \varepsilon$  holds for all  $n \geq N$ , thus  $(b_n)$  converges to  $a$ . □

(vii) Example: Consider the sequence  $(b_n)$  given by

$$b_n := \sum_{k=n+1}^{2n} \frac{1}{k^2} = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}.$$

Note that  $n+1 \leq k \leq 2n$  implies  $\frac{1}{k^2} < \frac{1}{n^2}$  and therefore

$$0 < b_n < \underbrace{\frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2}}_{n \text{ terms}} \leq n \cdot \frac{1}{n^2} = \frac{1}{n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Hence we may apply the Sandwich lemma with  $a_n = 0$ ,  $c_n = 1/n$  and conclude that  $(b_n)$  is convergent with limit 0.

Remark: Note that the above example shows that strict inequalities  $a_n < b_n$  for all  $n$  do (in general) not imply a strict inequality for the respective limits, i.e.  $a_n < b_n \not\Rightarrow \lim a_n < \lim b_n$ , but of course  $\lim a_n \leq \lim b_n$ .

**2.11. Series:** Many sequences  $(s_m)$  in applications occur through summation over the first  $m$  members of a given sequence  $(a_n)$  of real numbers.

**DEFINITION:** The sequence  $(s_m)_{m \in \mathbb{N}}$  of *partial sums* (*Partialsummen*) is defined by

$$s_m := \sum_{k=0}^m a_k = a_0 + a_1 + \dots + a_m \quad (m \in \mathbb{N})$$

and is called (infinite) *series* (*(unendliche) Reihe*), usually denoted by  $\sum_{k=0}^{\infty} a_k$ . If  $(s_m)$  is convergent, then  $\lim s_m$  is called the *sum* of the series and we write

$$\sum_{k=0}^{\infty} a_k = \lim_{m \rightarrow \infty} s_m = \lim_{m \rightarrow \infty} \sum_{k=0}^m a_k.$$

**REMARK:** (i) We will also consider series with summation starting at  $n_0 \in \mathbb{N}$ , that is

$$\sum_{k=n_0}^{\infty} a_k \text{ with corresponding partial sums } (s_m)_{m \geq n_0}.$$

(ii) Every sequence can be interpreted as a series: Let  $(c_n)$  be an arbitrary sequence and define  $a_k := c_k - c_{k-1}$  ( $k \geq 1$ ) and  $a_0 := c_0$ . Then we obtain

$$c_n = \sum_{k=0}^n a_k.$$

EXAMPLES: 1) Let  $a_k = \frac{1}{k(k+1)}$  ( $k \geq 1$ ), then the corresponding series is  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ .

Observe that  $a_k = \frac{k}{k+1} - \frac{k-1}{k}$ , hence

$$s_n = \sum_{k=1}^n a_k = \sum_{k=1}^n \left( \frac{k}{k+1} - \frac{k-1}{k} \right) = \left( \frac{1}{1+1} - \frac{1-1}{1} \right) + \left( \frac{2}{2+1} - \frac{2-1}{2} \right) + \dots + \left( \frac{n-1}{n} - \frac{n-2}{n-1} \right) + \left( \frac{n}{n+1} - \frac{n-1}{n} \right) = \frac{n}{n+1}$$

is convergent to 1, that is  $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \rightarrow \infty} s_n = 1$ .

2) *The geometric series:* Let  $x \in \mathbb{R}$  and consider  $\sum_{k=0}^{\infty} x^k$ .

Recall that we have already determined the values of the partial sums ( $s_m$ ) in 1.5, where we have obtained

$$s_m = m + 1 \quad \text{if } x = 1, \quad s_m = \frac{1 - x^{m+1}}{1 - x} \quad \text{if } x \neq 1.$$

Case  $x = 1$ :  $s_m = m + 1$  is unbounded, hence ( $s_m$ ) is divergent.

Case  $x = -1$ :  $s_{2n} = \frac{1 - (-1)^{2n}}{2} = 1$  and  $s_{2n+1} = \frac{1 - (-1)^{2n+1}}{2} = 0$ , hence ( $s_m$ ) is not convergent (argue as in Example 2.4.3)).

Case  $|x| > 1$ :  $s_m = \frac{1}{1-x} - \frac{x^{m+1}}{1-x}$ ; as observed in Example 2.8.2), the sequence ( $x^{m+1}$ ) is unbounded and thus ( $s_m$ ) as well. Therefore the series is not convergent.

Case  $|x| < 1$ :  $s_m = \frac{1}{1-x} - \frac{x^{m+1}}{1-x}$ , where  $x^{m+1} \rightarrow 0$  [cf. Example 2.8.2)]. Applying the rules for the computation of limits in 2.10 yields  $\lim s_m = \frac{1}{1-x}$ , that is

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (|x| < 1).$$

3) As special cases of 2) consider  $x = \pm \frac{1}{2}$  in the geometric sum, then we obtain

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{1 - \frac{1}{2}} = 2$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

**PROPOSITION:** If  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  are convergent series and  $\lambda, \mu \in \mathbb{R}$ , then

$\sum_{n=0}^{\infty} (\lambda a_n + \mu b_n)$  is convergent and

$$\sum_{n=0}^{\infty} (\lambda a_n + \mu b_n) = \lambda \sum_{n=0}^{\infty} a_n + \mu \sum_{n=0}^{\infty} b_n.$$

*Proof.* Apply the limit rules 2.10 to the partial sums. □

## 2.12. Infinite or improper limits:

**DEFINITION:** A sequence  $(a_n)$  is said to have (*improper or*) *infinite limit*  $+\infty$  (*ist uneigentlich konvergent oder bestimmt divergent gegen  $+\infty$* ), if for all  $K \in \mathbb{R}$  there is an  $N \in \mathbb{N}$  such that  $a_n > K$  for all  $n \geq N$ . We then denote this fact by  $\lim_{n \rightarrow \infty} a_n = +\infty$ .

A sequence  $(a_n)$  is said to have (*improper or*) *infinite limit*  $-\infty$ , if  $(-a_n)$  has infinite limit  $+\infty$ .

Note that a sequence with infinite limit is necessarily unbounded, namely not bounded above if the improper limit is  $+\infty$  and not bounded below if the improper limit is  $-\infty$ .

**EXAMPLES:** 1)  $\lim_{n \rightarrow \infty} n = +\infty$                       2)  $\lim_{n \rightarrow \infty} (-n^2) = -\infty$

3) Let  $a_n = (-1)^n n$  ( $n \in \mathbb{N}$ ), then  $(a_n)$  is unbounded hence not convergent. But  $(a_n)$  is not improperly convergent either, since  $a_{2n} \rightarrow +\infty$  and  $a_{2n+1} \rightarrow -\infty$ .

4) Let  $a_n = n$  for even  $n$ , and  $a_n = 0$  for odd  $n$ . Then  $(a_n)$  is unbounded, divergent, and also not improperly convergent.

5) Some rules for operations with improper limits: if  $(a_n)$  is convergent to  $a \in \mathbb{R}$  and  $(b_n)$ ,  $(c_n)$  have improper limit  $+\infty$ , then

$$\begin{aligned} \lim(a_n + b_n) &= \lim(b_n + a_n) = +\infty \\ \lim(c_n + b_n) &= \lim(b_n + c_n) = +\infty \\ \lim(a_n - b_n) &= \lim(-b_n + a_n) = -\infty \\ \text{if } a > 0 \text{ then } \quad \lim(a_n b_n) &= \lim(b_n a_n) = +\infty \\ &\quad \lim(c_n b_n) = \lim(b_n c_n) = +\infty. \end{aligned}$$

**WARNING:** There are no general limit relations for differences of sequences with infinite limit  $+\infty$  or products of a zero sequence with an improperly convergent sequence. For example,

$$\lim n = +\infty, \lim n^2 = +\infty \quad \text{yield} \quad \lim(n - n) = 0, \text{ but } \lim(n - n^2) = -\infty$$

and

$$\lim \frac{1}{n} = 0, \lim \frac{1}{n^2} = 0 \quad \text{give} \quad \lim \left(n \cdot \frac{1}{n}\right) = 1, \text{ but } \lim \left(n \cdot \frac{1}{n^2}\right) = 0.$$

**PROPOSITION:** (i) Let  $(a_n)$  have infinite limit  $+\infty$  or  $-\infty$ , then there exists  $n_0 \in \mathbb{N}$  such that  $\left(\frac{1}{a_n}\right)_{n \geq n_0}$  is well-defined and  $\lim \frac{1}{a_n} = 0$ .

(ii) If  $(a_n)$  is a zero sequence with  $a_n > 0$  (resp.  $a_n < 0$ ) for all  $n \in \mathbb{N}$ , then  $\lim \frac{1}{a_n} = +\infty$  (resp.  $\lim \frac{1}{a_n} = -\infty$ )

*Proof.* (i) It suffices to consider the case  $\lim a_n = +\infty$ . If we put  $K = 0$ , then by definition there is an  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have  $a_n > K = 0$ . Thus  $\left(\frac{1}{a_n}\right)_{n \geq n_0}$  is well defined. Note that in addition  $\frac{1}{a_n} > 0$  ( $n \geq n_0$ ).

Let  $\varepsilon > 0$ . Putting  $K = 1/\varepsilon$  we obtain some  $N \in \mathbb{N}$  such that  $a_n > K = 1/\varepsilon$  whenever  $n \geq N$ . Therefore we have for  $n \geq \max(n_0, N)$  the inequality  $0 < \frac{1}{a_n} < \varepsilon$ , which shows convergence of  $\left(\frac{1}{a_n}\right)$  to 0.

(ii) is left as an exercise. □

**EXAMPLE:**  $\lim \frac{n}{2^n} = 0$  [cf. Example 2.4.5)] and  $\frac{n}{2^n} > 0$ , hence (ii) of the above Proposition implies that  $\left(\frac{2^n}{n}\right)$  has infinite limit  $+\infty$ .

**REMARK:** If  $(a_n), (b_n)$  satisfy  $a_n \leq b_n$  for almost all  $n$  and  $\lim a_n = +\infty$ , then  $\lim b_n = +\infty$  follows directly.

### §3. COMPLETENESS OF $\mathbb{R}$ AND CONVERGENCE PRINCIPLES

In this section we investigate important consequences of order completeness of  $\mathbb{R}$  [cf. Section 0] for sequences of real numbers. We recall the statement of the

**3.1. Axiom of order completeness:** A non-empty subset of  $\mathbb{R}$  which is bounded above (resp. below) possesses a supremum (resp. infimum).

We define the key notions for sequences which allow us to deduce strong methods for convergence tests of sequences.

**3.2. Definition:** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers.

(i) If  $(n_k)_{k \in \mathbb{N}}$  is a sequence of natural numbers (i.e.  $n_k \in \mathbb{N}$  for all  $k$ ) satisfying  $n_0 < n_1 < n_2 < \dots$  (i.e.  $n_k < n_{k+1}$  for all  $k$ ) then the sequence  $(a_{n_k})_{k \in \mathbb{N}} = (a_{n_0}, a_{n_1}, a_{n_2}, \dots)$  is called a *subsequence* (*Teilfolge*) of the sequence  $(a_n)$ .

(ii) A real number  $a$  is called *cluster point* (*Häufungswert*) of the sequence  $(a_n)$ , if there exists a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  converging to  $a$ .

**3.3. Proposition:** A real number  $a$  is a cluster point of the real sequence  $(a_n)$  if and only if every  $\varepsilon$ -neighborhood of  $a$  contains infinitely many members of  $(a_n)$ , that is

$$(3.1) \quad \forall \varepsilon > 0 \forall N \in \mathbb{N} \exists m \geq N : a_m \in U_\varepsilon(a) = ]a - \varepsilon, a + \varepsilon[$$

(recall that  $a_m \in U_\varepsilon(a)$  is equivalent to  $|a_m - a| < \varepsilon$ ).

*Proof.* • We show the ‘only if’-part (i.e. *necessity*):  $a$  is cluster point  $\Rightarrow$  (3.1).

Let  $(a_{n_k})_{k \in \mathbb{N}}$  be a subsequence with  $\lim_{k \rightarrow \infty} a_{n_k} = a$  and  $\varepsilon > 0$  arbitrary. We can choose  $k_0 \in \mathbb{N}$  such that  $a_{n_k} \in U_\varepsilon(a)$  holds for all  $k \geq k_0$ .

Let  $N \in \mathbb{N}$ , then there exists  $k_1 \geq k_0$  such that  $n_{k_1} \geq N$  [since  $\dots < n_{k-1} < n_k < n_{k+1} < \dots$  by definition of a subsequence]. Therefore, if we put  $m := n_{k_1}$ , then  $a_m = a_{n_{k_1}} \in U_\varepsilon(a)$  by construction.



- We show the ‘if’-part (i.e. *sufficiency*): (3.1)  $\Rightarrow a$  is cluster point.

Putting  $\varepsilon := 1$  and  $N := 0$  we obtain an  $n_0 \in \mathbb{N}$  such that  $a_{n_0} \in U_1(a)$ . Then construct a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  with the property

$$\forall k \in \mathbb{N}, k \geq 1, \exists n_k \in \mathbb{N} : n_k > n_{k-1} \text{ and } a_{n_k} \in U_{\frac{1}{k+1}}(a)$$

inductively: For the basic case  $k = 1$ , (3.1) with  $\varepsilon := 1/2$  and  $N := n_0 + 1$  gives some  $n_1 \geq N > n_0$  such that  $a_{n_1} \in U_{\frac{1}{2}}(a)$ . If  $a_{n_1}, \dots, a_{n_k}$  have been defined, then applying (3.1) with  $\varepsilon := 1/(k+2)$  and  $N := n_k + 1$  we obtain  $n_{k+1} \geq N > n_k$  such that  $a_{n_{k+1}} \in U_{\frac{1}{k+2}}(a)$ .

We assert that  $(a_{n_k})_{k \in \mathbb{N}}$  converges to  $a$ : Let  $\varepsilon > 0$ . Choose  $k_0 \in \mathbb{N}$  such that  $\frac{1}{k_0+1} < \varepsilon$  [cf. 1.2.(i)], then we have for all  $k \geq k_0$  that

$$|a_{n_k} - a| < \frac{1}{k_0 + 1} < \varepsilon.$$

□

**3.4. Examples:** 1) If  $a_n = (-1)^n$  then  $(a_n)$  has 1 and  $-1$  as cluster points, since the subsequences  $(a_0, a_2, a_4, \dots)$  (with even index only) and  $(a_1, a_3, a_5, \dots)$  (with odd index only) converge to these values:

$$\lim_{k \rightarrow \infty} a_{2k} = \lim 1 = 1 \quad \lim_{k \rightarrow \infty} a_{2k+1} = \lim(-1) = -1.$$

2)  $\left( (-1)^n + \frac{1}{n} \right)_{n \geq 1}$  also has cluster points 1 and  $-1$ , since

$$\lim_{k \rightarrow \infty} a_{2k} = \lim \left( 1 + \frac{1}{2k} \right) = 1 + 0 = 1 \quad \lim_{k \rightarrow \infty} a_{2k+1} = \lim \left( -1 + \frac{1}{2k+1} \right) = -1 + 0 = -1.$$

3)  $(n)$  has no cluster points, since every subsequence is unbounded, hence divergent.

4) Let  $a_n = n$ , if  $n$  is even, and  $a_n = \frac{1}{n}$ , if  $n$  is odd. The sequence  $(a_n)$  is unbounded, since for example  $a_{2k} \rightarrow \infty$  ( $k \rightarrow \infty$ ). But  $(a_n)$  has 0 as cluster point, because the subsequence  $(a_{2k+1})_{k \in \mathbb{N}} = \left( \frac{1}{2k+1} \right)$  converges to 0.

**3.5. Remark:** If  $(a_n)$  is a sequence with limit  $a$ , then  $a$  is the only cluster point of  $(a_n)$  [cf. the Exercises].

**3.6. Theorem (of Bolzano-Weierstraß<sup>1</sup>):** Every bounded sequence of real numbers has a cluster point (that is, possesses at least one cluster point).

*Proof.* Let  $(a_n)$  be a bounded sequence, then there exists  $K \in \mathbb{R}$  such that  $|a_n| \leq K$  holds for all  $n \in \mathbb{N}$ . Consider the subset

$$A := \{x \in \mathbb{R} : a_n > x \text{ holds for at most finitely many } n\} \subseteq \mathbb{R}.$$

Since no  $a_n$  can be larger than  $K$ , we have  $K \in A$ , thus  $A$  is nonempty.

$A$  is bounded from below: If  $x < -K$ , then  $x \notin A$  [since  $a_n \geq -K$  for all  $n$ ]; hence  $-K - 1$  is a lower bound for  $A$ .

By order completeness  $A$  has an infimum. Let  $a := \inf A$ .

*Claim:*  $a$  is a cluster point of  $(a_n)$ .

Let  $\varepsilon > 0$ .

• Since  $a + \varepsilon > a$  the number  $a + \varepsilon$  is not a lower bound of  $A$  [property of the infimum  $a$ !]. Hence there exists  $y \in A$  such  $a \leq y < a + \varepsilon$ .

By definition of  $A$  we have  $a_n \leq y < a + \varepsilon$  for almost all  $n$ . Therefore we obtain

$$(\star) \quad \exists n_0 \in \mathbb{N} \forall n \geq n_0 : a_n < a + \varepsilon.$$

• Since  $a$  is a lower bound of  $A$  we have that  $a - \varepsilon \notin A$ . Hence  $a_n > a - \varepsilon$  holds for infinitely many  $n$ , that is

$$(\star\star) \quad \forall N \in \mathbb{N} \exists m \geq N : a - \varepsilon < a_m.$$

Combining  $(\star)$  and  $(\star\star)$  gives (3.1), therefore  $a$  is a cluster point of  $(a_n)$ . □

**3.7. Remark:** Note that in the above proof we have found the greatest cluster point of a given bounded sequence  $(a_n)$ . Indeed, if  $b > a$  — using the notation as in the proof — we may choose  $c \in A$  with  $a < c < b$ . If  $\varepsilon := b - c > 0$  then the  $\varepsilon$ -neighborhood  $U_\varepsilon(b)$  contains only finitely many members  $a_n$ . Thus  $b$  is not a cluster point of  $(a_n)$ . Similarly, we can prove that a smallest cluster point exists.

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<sup>1</sup>Bernhard Bolzano (1781–1848) [ˈbeɪnhaʊt bɔlˈtʃaːno], German mathematician  
Karl Weierstraß (1815–1897) [kɑrl ˈvaɪəʃtʁas], German mathematician, was the one who introduced the letter  $\varepsilon$  to mathematical analysis.

**3.8. Definition:** (i) Let  $(a_n)$  be a bounded sequence. The greatest cluster point of the sequence is called the *limit superior* (*Limes superior*) of  $(a_n)$  and is denoted by  $\limsup a_n$  or  $\overline{\lim}a_n$ . The smallest cluster point of  $(a_n)$  is called the *limit inferior* (*Limes inferior*) of  $(a_n)$  and is denoted by  $\liminf a_n$  or  $\underline{\lim}a_n$ .

(ii) If  $(a_n)$  is not bounded above (resp. below), then we set  $\limsup a_n = +\infty$  (resp.  $\liminf a_n = -\infty$ ).

**3.9. Examples:** 1) Let  $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$ , then  $(a_n)$  has cluster points 1 and  $-1$ , thus

$$\limsup(-1)^n \left(1 + \frac{1}{n}\right) = 1, \quad \liminf(-1)^n \left(1 + \frac{1}{n}\right) = -1.$$

2)  $(n)$  has no (real) cluster points and  $\limsup n = +\infty$ .

In deciding whether a certain sequence is convergent or not the defining property of convergence requires to already be in the possession of a good guess for the value of the prospective limit. Unless the sequence is simple enough to be analyzed directly by means of the basic rules for computation of limits [cf. 2.10] (or is seen to be unbounded), it might be difficult or even hopeless to guess the limit with complicated terms or when the sequence members are not defined by an explicit formula or procedure.

To deal with such situations we strive for the development of methods that allows to decide the question of convergence without having to know a candidate for the limit in advance. In some cases, this then also leads to a successful determination of the limit a posteriori. There are also situations where a reasonable candidate for the limit is easily guessed or the only possible value can be determined (under the assumption that the sequence converges), but direct convergence proofs are inaccessible. The principles of Cauchy<sup>2</sup> and that of monotone bounded sequences [cf. 3.10 and 3.12] are among the most powerful alternative methods to prove convergence of real sequences.

### 3.10. Cauchy sequences:

**DEFINITION:** A real sequence  $(a_n)$  is a *Cauchy sequence* if

$$(3.2) \quad \forall \varepsilon > 0 \exists N \in \mathbb{N} \quad \forall n \geq N \forall m \geq N : |a_n - a_m| < \varepsilon.$$

**THEOREM:** Let  $(a_n)$  be a real sequence. The following are equivalent:

- (i)  $(a_n)$  is convergent
- (ii)  $(a_n)$  is a Cauchy sequence.

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<sup>2</sup>Augustin Louis Cauchy (1789–1857) [ogus'tě lwi ko'fi], French mathematician

**REMARK:** Note that condition (ii) can be checked without knowing the limit. Moreover, if (ii) fails to hold we may conclude that the sequence is divergent.

*Proof.* (i)  $\Rightarrow$  (ii): Let  $a := \lim a_n$  and  $\varepsilon > 0$ . Choose  $N$  such that  $|a_n - a| < \varepsilon/2$  holds  $\forall n \geq N$ . Then we have for all  $n, m \geq N$

$$|a_n - a_m| = |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(ii)  $\Rightarrow$  (i): Let  $(a_n)$  be a Cauchy sequence.

*Step 1:*  $(a_n)$  is bounded.

We put  $\varepsilon = 1$  in (3.2) and obtain that there is some  $N$  such that  $n, m \geq N$  implies  $|a_n - a_m| < \varepsilon = 1$ . In particular, if  $m = N$  this means that  $|a_n| - |a_N| \leq |a_n - a_N| < 1$ , hence  $|a_n| \leq 1 + |a_N|$ , holds for all  $n \geq N$ . Since clearly  $|a_n| \leq 1 + \max(|a_0|, \dots, |a_N|) =: K$  for  $n = 1, \dots, N$ , we therefore have  $|a_n| \leq K$  for all  $n \in \mathbb{N}$ .

*Step 2:* By the Theorem of Bolzano-Weierstraß [cf. 3.6]  $(a_n)$  has a cluster point  $a \in \mathbb{R}$ .

*Step 3:*  $a_n \rightarrow a$  ( $n \rightarrow \infty$ )

Let  $\varepsilon > 0$ . Choose  $N$  such that  $|a_n - a_m| < \varepsilon/2$  for all  $n, m \geq N$ . Since  $a$  is a cluster point there exists  $k \geq N$  such that  $|a_k - a| < \varepsilon/2$ . Combining these facts we obtain that for  $n \geq N$

$$|a_n - a| = |(a_n - a_k) + (a_k - a)| \leq |a_n - a_k| + |a_k - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

**3.11. Example:** Let  $(a_k)$  be a sequence and suppose there is some  $\theta \in ]0, 1[$  such that  $|a_k| \leq \theta$  for all  $k$ . Consider the partial sums  $s_n := \sum_{k=0}^n a_k^k$ . If  $m < n$  then

$$\begin{aligned} |s_n - s_m| &\leq \sum_{k=m+1}^n |a_k|^k \leq \sum_{k=m+1}^n \theta^k = \sum_{k=0}^n \theta^k - \sum_{k=0}^m \theta^k = \frac{1 - \theta^{n+1}}{1 - \theta} - \frac{1 - \theta^{m+1}}{1 - \theta} \\ &= \frac{\theta^{m+1} - \theta^{n+1}}{1 - \theta} = \theta^{m+1} \frac{1 - \theta^{n-m}}{1 - \theta} \leq \theta^{m+1} \frac{1}{1 - \theta}. \end{aligned}$$

Let  $\varepsilon > 0$ . Since  $0 \leq \theta < 1$  we can choose  $N$  such that  $0 \leq \theta^{m+1} < \varepsilon(1 - \theta)$  holds for all  $m \geq N$ . Therefore we have for all  $N \leq m < n$  that

$$|s_n - s_m| \leq \theta^{m+1} \frac{1}{1 - \theta} < \varepsilon.$$

If  $n = m \geq N$  we have  $s_n - s_m = 0$  and if  $N \leq n \leq m$  the roles of  $n$  and  $m$  can be interchanged in the above estimates. In summary, we obtain that  $(s_n)$  is a Cauchy sequence.

We conclude that the series  $\sum_{k=0}^{\infty} a_k^k$  is convergent, although we have no clue what its sum might be — and in the general case there is no way to determine the limit.

### 3.12. Monotone sequences:

**DEFINITION:** A real sequence  $(a_n)$  is *(strictly) increasing* (*(streng) monoton wachsend*) if for all  $n \in \mathbb{N}$ :  $a_n \leq a_{n+1}$  (resp.  $a_n < a_{n+1}$ ). A real sequence  $(a_n)$  is *(strictly) decreasing* (*(streng) monoton fallend*) if for all  $n \in \mathbb{N}$ :  $a_n \geq a_{n+1}$  (resp.  $a_n > a_{n+1}$ ). If  $n_0 \in \mathbb{N}$  such that  $(a_n)_{n \geq n_0}$  is (strictly) increasing (or decreasing) we shall occasionally express this by saying that  $(a_n)$  has the corresponding property for  $n \geq n_0$ .

**REMARK:** A decreasing sequence  $(a_n)$  which is bounded below is bounded. Indeed, if  $C$  is a lower bound we have for all  $n \in \mathbb{N}$

$$C \leq a_n \leq a_{n-1} \leq \cdots \leq a_0.$$

Similarly for increasing sequences which are bounded above.

**EXAMPLE:** 1) The Fibonacci sequence  $(f_n)$  [cf. Example 2.2.6] is increasing and strictly increasing for  $n \geq 2$ . We have  $f_0 = 0 < f_1 = 1 = f_2$  and  $f_n > 0$  when  $n \geq 1$ , therefore

$$f_{n+1} = f_n + f_{n-1} > f_n + 0 = f_n \quad \forall n \geq 2.$$

2) Let  $x_0 > 0$  and define the sequence  $(x_n)$  recursively by

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{3}{x_n} \right) \quad (n \in \mathbb{N}).$$

Note that by induction  $x_n > 0$  for all  $n$ , hence the recursion defines a sequence  $(x_n)$ .

*Claim 1:*  $\forall n \geq 1: x_n^2 \geq 3$

This follows from

$$\begin{aligned} x_{n+1}^2 - 3 &= \frac{1}{4} \left( x_n + \frac{3}{x_n} \right)^2 - 3 = \frac{1}{4} \left( x_n^2 + 6 + \frac{9}{x_n^2} \right) - 3 \\ &= \frac{1}{4} \left( x_n^2 - 6 + \frac{9}{x_n^2} \right) = \frac{1}{4} \left( x_n - \frac{3}{x_n} \right)^2 \geq 0. \end{aligned}$$

*Claim 2:*  $(x_n)$  is decreasing for  $n \geq 1$

We have for  $n \geq 1$

$$x_n - x_{n+1} = x_n - \frac{x_n}{2} - \frac{3}{2x_n} = \frac{1}{2x_n}(x_n^2 - 3),$$

which is nonnegative by claim 1, hence  $x_n \geq x_{n+1}$ .

The theorem which we prove below will guarantee that the sequence  $(x_n)$  is convergent, since it is bounded below (by  $\sqrt{3}$ ) and decreasing for  $n \geq 1$ . What is the value of the limit  $x := \lim x_n$ ?

In this case we can make use of the recursion relation. Note first that  $x \geq \sqrt{3} > 0$  [since  $x_n \geq \sqrt{3}$ ] and then take limits for  $n \rightarrow \infty$  on both sides of the recursion relation

$$\begin{array}{rcl} x_{n+1} & = & \frac{1}{2}\left(x_n + \frac{3}{x_n}\right) \\ \downarrow & & \downarrow \\ x & = & \frac{1}{2}\left(x + \frac{3}{x}\right). \end{array}$$

Therefore  $x^2 = 3$ , that is  $x = \sqrt{3}$ .

**THEOREM:** If  $(a_n)$  is increasing (for  $n \geq n_0$ ) and bounded above, then  $(a_n)$  is convergent. A corresponding statement holds for decreasing sequences that are bounded below.

*Proof.* As noted in a remark above  $(a_n)$  is bounded, thus there exists a cluster point  $a$  [Bolzano-Weierstraß!]. We will show that  $a$  is the limit of  $(a_n)$ .

Let  $(a_{n_k})_{k \in \mathbb{N}}$  be a subsequence that converges to  $a$ .

Let  $\varepsilon > 0$ , then there exists  $k_0 \in \mathbb{N}$  such that  $|a_{n_k} - a| < \varepsilon$  holds for all  $k \geq k_0$ . Put  $N := n_{k_0}$  and observe that [by the definition of a subsequence] for given  $n \geq N$  we have

$$\exists k \geq k_0 : n_k \leq n < n_{k+1}.$$

Since  $(a_n)$  is increasing this yields

$$a_{n_k} \leq a_n \leq a_{n_{k+1}} \leq a,$$

and therefore

$$|a - a_n| \underset{[a_n \leq a]}{=} a - a_n \leq a - a_{n_k} = |a - a_{n_k}| < \varepsilon.$$

□

As an application of the preceding theorem we present a useful alternative to describe the limit superior (resp. limit inferior) of an arbitrary bounded sequence as the limit of a specifically constructed decreasing (resp. increasing) sequence.

**3.13. Proposition:** Let  $(a_n)$  be a bounded real sequence, then

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (\sup\{a_k : k \geq n\})$$

and

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (\inf\{a_k : k \geq n\}).$$

A corresponding statement holds for unbounded sequences with improper values  $+\infty$  or  $-\infty$  assigned to sup, inf, lim sup, or lim inf in the appropriate way.

*Proof.* (For lim sup, the case of lim inf being very similar.)

Let  $M_n := \{a_k : k \geq n\}$  and  $x_n := \sup M_n$ . Since  $M_{n+1} \subseteq M_n$  the sequence  $(x_n)$  is decreasing.

Furthermore, if  $C > 0$  is a bound for  $(a_n)$ , that is  $|a_n| \leq C$  for all  $n$ , then it also is one for  $(x_n)$ , hence  $(x_n)$  is bounded too.

By Theorem 3.12  $(x_n)$  is convergent. Let  $x := \lim x_n$ .

*Assertion:*  $x$  is a cluster point of  $(a_n)$ .

Let  $\varepsilon > 0$ .

• We may choose  $N \in \mathbb{N}$  such that for all  $n \geq N$ :  $x \leq x_n < x + \varepsilon$ . By construction  $a_n \leq x_n$ , hence

$$\forall n \geq N : a_n < x + \varepsilon.$$

• Since  $x_n$  is the supremum of the set  $M_n = \{a_k : k \geq n\}$  (and  $x_n - \varepsilon < x_n$ ), we have the following:

$$\forall n \in \mathbb{N} \exists m \geq n : a_m > x_n - \varepsilon \geq x - \varepsilon.$$

Combining the two inequalities above we find that for all  $n$  there is some  $m \geq n$  such that  $a_m \in U_\varepsilon(x)$ . [Proposition 3.3] then implies that  $x$  is a cluster point of  $(a_n)$ .

Finally, if  $(a_{n_k})$  is an arbitrary subsequence of  $(a_n)$ , then clearly  $a_{n_k} \leq x_{n_k}$  holds for all  $k$ . Therefore, if  $(a_{n_k})$  converges we must have

$$\lim_{k \rightarrow \infty} a_{n_k} \leq \lim_{k \rightarrow \infty} x_{n_k} = x.$$

Thus  $x$  is the largest cluster point, i.e.  $x = \limsup a_n$ . □

### 3.14. Adherent points and accumulation points of subsets of $\mathbb{R}$ :

**DEFINITION:** Let  $A \subseteq \mathbb{R}$  and  $a \in \mathbb{R}$ .

(i)  $a$  is an *adherent point* (*Berührungspunkt*) of  $A$  if  $\forall \varepsilon > 0$  we have that  $U_\varepsilon(a) \cap A \neq \emptyset$ .  
(Every  $\varepsilon$ -neighborhood of  $a$  contains at least one point of  $A$ .)

(ii)  $a$  is an *accumulation point* (*Häufungspunkt*) of  $A$  if

$$\forall \varepsilon > 0 : \quad U_\varepsilon(a) \cap A \quad \text{contains infinitely many points.}$$

**BASIC PROPERTIES:** Clearly, every accumulation point of  $A$  and every element of  $A$  is an adherent point of  $A$ .

(a)  $a$  is an adherent point of  $A \iff$  there exists a sequence  $(a_n)$  in  $A$ :  $a_n \rightarrow a$  ( $n \rightarrow \infty$ )  
[( $a_n$ ) in  $A$  means  $a_n \in A$  for all  $n$ ]

$\Rightarrow$  For  $n \in \mathbb{N}$ ,  $n \geq 1$ , choose  $a_n \in U_{1/n}(a) \cap A$ . Then  $(a_n)$  is a sequence in  $A$  with  $a_n \rightarrow a$ .

$\Leftarrow$  By assumption there exists  $(a_n)$  with  $a_n \in A$  for all  $n$  and satisfying for all  $\varepsilon > 0$  that there is some  $N$  such that  $a_n \in U_\varepsilon(a)$  for all  $n \geq N$ . In particular,  $U_\varepsilon(a) \cap A$  is not empty.

(b)  $a$  is an accumulation point of  $A \iff a$  is an adherent point of  $A \setminus \{a\}$

Proof as an exercise.

**EXAMPLES:** 1) 0 is an accumulation point of  $A = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 1\}$  (but 0 does not belong to  $A$ ).

2) Let  $A = \{1\}$  and  $a_n = 1$  for all  $n$ . Then 1 is an adherent point of  $A$  and a cluster point of  $(a_n)$ , but it is not an accumulation point of  $A$ .

Thus we learn that, in general, a cluster point of a sequence  $(a_n)$  need not be an accumulation point of the set  $\{a_n : n \in \mathbb{N}\}$ .

3) Consider  $\mathbb{Q} \subseteq \mathbb{R}$ . Then every point  $x \in \mathbb{R}$  is an accumulation point of  $\mathbb{Q}$ . This follows from the density of  $\mathbb{Q}$  in  $\mathbb{R}$  [cf. 0.7].

4) Let  $a < x_0 < b$  and  $A = ]a, x_0[ \cup ]x_0, b[$ . Every  $x \in [a, b]$  is an accumulation point of  $A$ .

**REMARK:** Let  $A$  be a bounded subset of  $\mathbb{R}$ .

(i) By the Theorem of Bolzano-Weierstraß, every sequence  $(a_n)$  in  $A$  has a cluster point, which is then also an adherent point of  $A$ .

(ii) The supremum (resp. infimum) of  $A$  is an adherent point of  $A$ . Moreover,  $\sup A$  (resp.  $\inf A$ ) is the limit of an increasing (resp. decreasing) sequence in  $A$ : Let  $\alpha := \sup A$ , choose  $\alpha - 2 < a_0 \leq \alpha$ , and construct  $(a_n)$  inductively with the property

$$\forall n \in \mathbb{N}, n \geq 1 : \quad a_n \in A, \quad a_n \geq a_{n-1} \quad \text{and} \quad \alpha - \frac{1}{n} \leq a_n \leq \alpha.$$



### 3.15. Principle of nested intervals *(Intervallschachtelungsprinzip)*:

Let  $a \leq b$  and  $I$  be the closed bounded interval  $[a, b]$ , then

$$(3.3) \quad \text{diam}(I) := b - a$$

is called the *diameter*<sup>3</sup> or *length* of  $I$ . Note that

$$\forall x, y \in I : |x - y| \leq \text{diam}(I).$$

**THEOREM:** Let  $I_n$  ( $n \in \mathbb{N}$ ) be a sequence of closed bounded intervals with the following properties:

$$(i) \quad I_0 \supseteq I_1 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \dots$$

$$(ii) \quad \lim_{n \rightarrow \infty} \text{diam}(I_n) = 0.$$

Then there is a unique point  $x \in \mathbb{R}$  which belongs to every  $I_n$  ( $n \in \mathbb{N}$ ), i.e.

$$\bigcap_{n \in \mathbb{N}} I_n = \{x\}.$$

*Proof.* Let  $I_n = [a_n, b_n]$  ( $n \in \mathbb{N}$ ), where  $a_n \leq b_n$ . By property (i) we have for all  $n \in \mathbb{N}$

$$a_0 \leq a_1 \leq \dots \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq \dots \leq b_1 \leq b_0.$$

Thus  $(a_n)$  is increasing and bounded (above by  $b_0$ ),  $(b_n)$  is decreasing and bounded (below by  $a_0$ ), hence

$$\exists x, y \in \mathbb{R} : x = \lim a_n \text{ and } y = \lim b_n.$$

Since for all  $n \geq k$

$$\begin{array}{ccccccc} a_k & \leq & a_n & \leq & b_n & \leq & b_k \\ & & \downarrow & & \downarrow & & \\ & & x & & y & & \end{array}$$

we have  $x \leq y$  and  $\bigcap_{n \in \mathbb{N}} I_n = [x, y]$ , in particular,  $x, y \in I_k$  for all  $k \in \mathbb{N}$ . Therefore (ii) implies

$$0 \leq y - x \leq \text{diam}(I_k) \rightarrow 0 \quad (k \rightarrow \infty),$$

which in turn gives  $x = y$ .

Hence  $\bigcap_{n \in \mathbb{N}} I_n = \{x\}$ , where  $x$  is uniquely determined as the limit of  $(a_n)$ . □

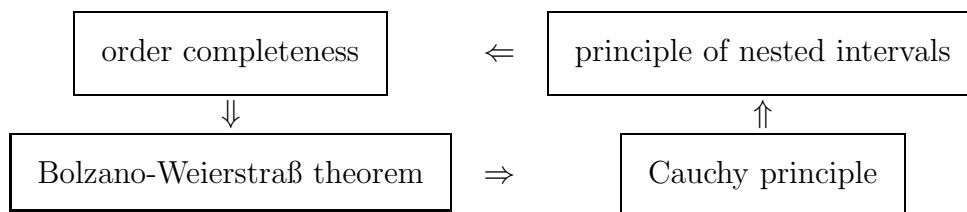
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<sup>3</sup>The word “diameter” was originally derived from the Greek ῥή διάμετρος (literally “measurements”) and is now common in most languages.

**3.16. Completeness revisited:** The current section has been devoted to the derivation of consequences of the Axiom of order completeness 3.1. Along the way, we proved in succession the Theorem of Bolzano-Weierstraß [cf. 3.6], the Cauchy convergence principle [cf. 3.10], and (via the principle of monotone bounded sequences) also the principle of nested intervals [cf. 3.15].

It is an intriguing fact, that in turn the statement of order completeness becomes a provable theorem when any of the other three results is taken as an axiom instead. For example, below we give a proof that the principle of nested intervals implies order completeness.

We summarize this situation in a diagram:



**THEOREM:** The principle of nested intervals implies order completeness.

*Proof.* Let  $A$  be a bounded and nonempty subset of  $\mathbb{R}$ . We will construct a nested sequence of intervals  $I_n = [a_n, b_n]$  ( $n \in \mathbb{N}$ ) with the following properties

- (i) every  $b_n$  is an upper bound for  $A$
- (ii) no  $a_n$  is an upper bound for  $A$ .

Let  $b_0$  be an upper bound for  $A$  [ $A$  is bounded] and choose  $\alpha \in A$  arbitrary [ $A$  is nonempty!]. Then  $\alpha \leq b_0$  and  $a_0 := \alpha - 1 \leq b_0$  cannot be an upper bound for  $A$ .

We proceed by induction: suppose that  $[a_0, b_0] \supseteq \dots \supseteq [a_n, b_n]$  have been constructed satisfying (i) and (ii). Let  $m := (b_n - a_n)/2$  be the midpoint of  $I_n$  and define

$$[a_{n+1}, b_{n+1}] := \begin{cases} [a_n, m] & \text{if } m \text{ is an upper bound of } A \\ [m, b_n] & \text{otherwise.} \end{cases}$$

By the principle of nested intervals we have  $\bigcap_{n \in \mathbb{N}} I_n = \{s\}$  for some  $s \in \mathbb{R}$ .

*Claim 1:*  $s$  is an upper bound for  $A$

Assume the contrary, then there exists  $x \in A$  such that  $x > s$ . Since  $a_n \leq s \leq b_n$  for all  $n$  and  $b_n - a_n \rightarrow 0$  ( $n \rightarrow \infty$ ) there exists  $N$  such that

$$b_N - s \leq b_N - a_N < x - s.$$

Therefore  $b_N < x$ , which is a contradiction to property (i) above.

*Claim 2:*  $s$  is the least upper bound of  $A$

Suppose that  $s' < s$  is also an upper bound of  $A$ . Then there is  $n \in \mathbb{N}$  such that  $\text{diam}(I_n) < s - s'$ . Since  $s \in I_n$  we have

$$s - a_n \leq \text{diam}(I_n) < s - s',$$

which implies that  $a_n > s'$  and thus  $a_n$  is an upper bound of  $A$  — a contradiction  $\nabla$  to (ii) above.  $\square$

## §4. CONVERGENCE OF SERIES

Let  $(a_k)_{k \in \mathbb{N}}$  be a sequence of real numbers. Recall that the series  $\sum_{k=0}^{\infty} a_k$  is defined to be convergent if and only if the corresponding sequence  $(s_m)_{m \in \mathbb{N}}$  of partial sums  $s_m = \sum_{k=0}^m a_k$  is convergent, with  $\lim s_m$  being called the sum of the series.

**4.1. Proposition (Cauchy principle for series):** The series  $\sum_{k=0}^{\infty} a_k$  is convergent if and only if

$$(4.1) \quad \forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n \geq m \geq N : \left| \sum_{k=m}^n a_k \right| < \varepsilon.$$

*Proof.* The sequence of partial sums  $(s_m)$  converges if and only if it is a Cauchy sequence [cf. Theorem 3.10]. The latter is equivalent to the property that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : |s_n - s_{m-1}| < \varepsilon \quad \forall n, m-1 \geq N,$$

where we may assume in addition that  $n \geq m-1$ , which yields  $s_n - s_{m-1} = \sum_{k=m}^n a_k$ .  $\square$

**4.2. Remark:** Note that condition (4.1) is not affected by changing finitely many  $a_k$ 's in the series. Thus the convergence behavior of a series does not depend on alteration of a finite number of terms. (However, the value of the sum might change, of course.)

**4.3. Corollary:** If  $\sum_{k=0}^{\infty} a_k$  is convergent, then  $(a_n)$  is a null sequence.

*Proof.* Let  $\varepsilon > 0$  and put  $m = n \geq N$  in condition (4.1). Thus  $|a_n| < \varepsilon$  holds for all  $n \geq N$ , hence  $a_n \rightarrow 0$  ( $n \rightarrow \infty$ ).  $\square$

**4.4. Proposition:** Let  $\sum a_n$  be a series of nonnegative numbers, i.e.  $a_n \geq 0$  for all  $n \in \mathbb{N}$ , then

$$\sum_{n=0}^{\infty} a_n \text{ is convergent} \iff \text{the sequence of partial sums } (s_m) \text{ is bounded.}$$

*Proof.*  $(\Rightarrow)$   $(s_m)$  is convergent, hence bounded.

$(\Leftarrow)$   $s_{m+1} = s_m + a_{m+1} \geq s_m$  shows that  $(s_m)$  is increasing. Since, by assumption,  $(s_m)$  is also bounded [3.12, Theorem] implies convergence.  $\square$

**4.5. Examples:** 1)  $\sum_{n=0}^{\infty} (-1)^n$  is divergent, since  $(-1)^n \not\rightarrow 0$ .

2) The *harmonic series*  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent, since the partial sums are unbounded:

$$\begin{aligned} s_{2^k} &= \sum_{n=1}^{2^k} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq 2 \cdot \frac{1}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq 4 \cdot \frac{1}{8} = \frac{1}{2}} + \dots + \underbrace{\left(\frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}\right)}_{\geq 2^{k-1} \cdot \frac{1}{2^k} = \frac{1}{2}} \\ &\geq 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{k \text{ terms}} \geq 1 + \frac{k}{2}. \end{aligned}$$

Note that in this example  $a_n := \frac{1}{n} \rightarrow 0$  ( $n \rightarrow \infty$ ) but  $\sum a_n$  is divergent. Thus Corollary 4.3 gives a necessary condition for convergence which is not sufficient ( $a_n \rightarrow 0$  does not imply that  $\sum a_n$  converges)!

3) If  $s \in \mathbb{N}$  with  $s \geq 2$ , then  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  is convergent.

Since all terms are nonnegative it suffices to prove boundedness of the partial sums: Let  $m \in \mathbb{N}$  and choose  $l \in \mathbb{N}$  such that  $m \leq 2^{l+1} - 1$ , then we obtain

$$\begin{aligned} s_m &= \sum_{n=1}^m \frac{1}{n^s} \leq \sum_{n=1}^{2^{l+1}-1} \frac{1}{n^s} = 1 + \underbrace{\left(\frac{1}{2^s} + \frac{1}{3^s}\right)}_{\leq 2 \cdot \frac{1}{2^s}} + \dots + \underbrace{\left(\sum_{n=2^l}^{2^{l+1}-1} \frac{1}{n^s}\right)}_{\leq 2^l \cdot \frac{1}{2^s}} \\ &\leq \sum_{j=0}^l 2^j \frac{1}{2^{js}} = \sum_{j=0}^l \left(\frac{1}{2^{s-1}}\right)^j \leq \sum_{j=0}^{\infty} \left(\frac{1}{2^{s-1}}\right)^j = \frac{1}{1 - \frac{1}{2^{s-1}}}, \end{aligned}$$

which gives an upper bound independent of  $m$ . (The same proof would work for any  $s \in ]1, \infty[$ .)

**4.6. Theorem (Leibniz's<sup>1</sup> criterion for alternating series):** Let  $a_n \geq 0$  for all  $n \in \mathbb{N}$  and consider the series with alternating signs

$$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \dots$$

If in addition  $(a_n)$  satisfies

(i)  $a_n \geq a_{n+1}$  for all  $n$ , i.e.  $(a_n)$  is decreasing,

and

(ii)  $a_n \rightarrow 0$  ( $n \rightarrow \infty$ ),

then the series  $\sum (-1)^n a_n$  is convergent.

Furthermore, for the partial sums  $s_m = \sum_{n=0}^m (-1)^n a_n$  and the sum of the series  $s = \lim_{m \rightarrow \infty} s_m = \sum_{n=0}^{\infty} (-1)^n a_n$  we have the error estimate

$$(4.2) \quad |s - s_m| \leq a_{m+1} \quad \forall m \in \mathbb{N}.$$

(In other words, the error of each partial sum is not larger than the first neglected term.)

*Proof.* Observe that for all  $k$

$$\begin{aligned} s_{2k+1} &= s_{2k} - a_{2k+1} \leq s_{2k} \\ s_{2k+2} &= s_{2k} - \underbrace{(a_{2k+1} - a_{2k+2})}_{\geq 0 \text{ by (i)}} \leq s_{2k} \\ s_{2k+3} &= s_{2k+1} + \underbrace{(a_{2k+2} - a_{2k+3})}_{\geq 0 \text{ by (i)}} \geq s_{2k+1}, \end{aligned}$$

which we summarize by

$$s_1 \leq s_3 \leq s_5 \leq \dots \leq s_{2k+1} \leq s_{2k} \leq \dots \leq s_4 \leq s_2 \leq s_0.$$

In other words, we have a sequence of nested intervals  $I_k := [s_{2k+1}, s_{2k}]$  ( $k \in \mathbb{N}$ ) with  $\text{diam}(I_k) = s_{2k} - s_{2k+1} = a_{2k+1} \rightarrow 0$  as  $k \rightarrow \infty$  [by property (ii)]. Therefore [Theorem 3.15] implies that  $\bigcap_{k \in \mathbb{N}} I_k = \{s\}$ .

Since  $s_{2k+1} \leq s \leq s_{2k}$  for all  $k$  we obtain for every  $m \in \mathbb{N}$  that

$$|s - s_m| \leq |s_{m+1} - s_m| = a_{m+1},$$

which gives (4.2) and also proves convergence of  $(s_n)$ : Let  $\varepsilon > 0$  and choose  $N$  such that  $0 \leq a_n < \varepsilon$  for all  $n \geq N$ . Then for all  $n \geq N$

$$|s_n - s| \leq a_{n+1} < \varepsilon.$$

□

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<sup>1</sup>Gottfried Wilhelm Leibniz (1646–1716) [ˈɡɔtfrɪt ˈvɪlhɛlm ˈlaɪbnɪts], German philosopher and scientist

**4.7. Example:** The alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is convergent, since  $(\frac{1}{n})$  is decreasing and converges to 0.

**4.8. Remark and warning:** The value of the sum — and even the convergence behavior (!) — of a convergent series may depend on the order of summation. To be more precise, if  $\sum_{n=0}^{\infty} a_n$  is a convergent series and  $\tau: \mathbb{N} \rightarrow \mathbb{N}$  is bijective, then the *rearrangement*  $\langle \text{Umordnung} \rangle \sum_{n=0}^{\infty} a_{\tau(n)}$  need not have the same sum or need not converge at all.

For example, consider again the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots,$$

which is convergent as we have shown above. If we rearrange the terms in the series in the following way

$$\begin{aligned} & \underbrace{1 - \frac{1}{2}}_{1/2} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}}_{1/6} - \frac{1}{8} + \underbrace{\frac{1}{5} - \frac{1}{10}}_{1/10} - \frac{1}{12} + \underbrace{\frac{1}{7} - \frac{1}{14}}_{1/14} - \frac{1}{16} + \dots \\ & = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \dots = \frac{1}{2} \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \right), \end{aligned}$$

then we obtain half of the original sum.

We can even find a divergent rearrangement of the (originally convergent) alternating harmonic series, where the negative terms occur with more and more delay as we progress: Let  $n \geq 2$  and observe that

$$\begin{aligned} & \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}_{>0} + \underbrace{\left( \frac{1}{5} + \frac{1}{7} \right) - \frac{1}{6}}_{>2/8=1/4} + \underbrace{\left( \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} \right) - \frac{1}{8}}_{>4/16=1/4} + \dots \\ & \quad + \underbrace{\left( \frac{1}{2^n+1} + \frac{1}{2^n+3} + \dots + \frac{1}{2^{n+1}-1} \right) - \frac{1}{2n+2}}_{>2^{n-1}/2^{n+1}=1/4} \\ & > \frac{n-1}{4} - \left( \frac{1}{6} + \frac{1}{8} + \dots + \frac{1}{2n+2} \right) \geq \frac{n-1}{4} - \frac{n-1}{6} = \frac{n-1}{12}, \end{aligned}$$

thus the partial sums corresponding to such a rearrangement must be unbounded.

The following definition introduces the appropriate convergence notion for series which is stable under rearrangements.

**4.9. Definition:** A series  $\sum a_n$  is *absolutely convergent* (*absolut konvergent*) if  $\sum |a_n|$  is convergent.

**4.10. Remark:** (i) Since  $|a_n| \geq 0$  absolute convergence is thus equivalent to the boundedness of  $s_m = \sum_{n=0}^m |a_n|$ .  
(ii) Convergence does not imply absolute convergence, as can be seen from the example of the alternating harmonic series, where  $a_n = \frac{(-1)^{n-1}}{n}$  ( $n \geq 1$ ): in this case  $\sum |a_n| = \sum \frac{1}{n}$  is the harmonic series, which is divergent.

Absolute convergence is a stronger condition than convergence.

**4.11. Proposition:** Every absolutely convergent series is convergent.

*Proof.* Suppose  $\sum |a_n|$  is convergent. Let  $\varepsilon > 0$ . By the Cauchy principle for series we can find  $N \in \mathbb{N}$  such that

$$\forall n \geq m \geq N : \sum_{k=m}^n |a_k| < \varepsilon.$$

Applying the triangle inequality (for finitely many terms) we obtain

$$\forall n \geq m \geq N : \left| \sum_{k=m}^n a_k \right| \leq \sum_{k=m}^n |a_k| < \varepsilon,$$

which in turn by the Cauchy principle for series yields convergence of  $\sum a_n$ .  $\square$

**4.12. Rearrangement theorem for absolutely convergent series:**

Let  $\sum a_n$  be an absolutely convergent series. Then every rearrangement  $\sum a_{\tau(n)}$ , where  $\tau: \mathbb{N} \rightarrow \mathbb{N}$  is a bijection, is absolutely convergent and has the same limit.

*Proof.* Let  $s := \sum_{n=0}^{\infty} a_n$ .

*Claim 1:*  $\sum_{n=0}^{\infty} a_{\tau(n)}$  is convergent with sum  $s$

Let  $\varepsilon > 0$  and choose  $N$  such that for all  $l \geq N$  we have  $\sum_{k=N}^l |a_k| < \frac{\varepsilon}{2}$  [Cauchy principle for the convergent series  $\sum |a_k|$ ]. Sending  $l \rightarrow \infty$  yields  $\sum_{k=N}^{\infty} |a_k| \leq \frac{\varepsilon}{2}$  and therefore for all



$m \geq N$  also that

$$\left| \sum_{k=0}^m a_k - \sum_{k=0}^{N-1} a_k \right| = \left| \sum_{k=N}^m a_k \right| \leq \sum_{k=N}^m |a_k| \leq \sum_{k=N}^{\infty} |a_k| \leq \frac{\varepsilon}{2}.$$

Upon taking limits as  $m \rightarrow \infty$  we find

$$\left| s - \sum_{k=0}^{N-1} a_k \right| \leq \frac{\varepsilon}{2}.$$

Choose  $M \in \mathbb{N}$  with  $M \geq N$  and large enough to ensure  $\{\tau(0), \tau(1), \dots, \tau(M)\} \supset \{0, 1, \dots, N-1\}$ . [Since  $\tau$  is a bijection of  $\mathbb{N}$  such an  $M$  exists.] Then we have for all  $m \geq M$

$$\left| \sum_{k=0}^m a_{\tau(k)} - s \right| \leq \left| \sum_{k=0}^m a_{\tau(k)} - \sum_{k=0}^{N-1} a_k \right| + \left| \sum_{k=0}^{N-1} a_k - s \right| \leq \sum_{k=N}^{\infty} |a_k| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus  $\sum_{n=0}^{\infty} a_{\tau(n)}$  converges to  $s$ .

*Claim 2:*  $\sum |a_{\tau(n)}|$  is convergent

This follows by application of Claim 1 to the series  $\sum b_n$  with  $b_n := |a_n|$  (and its corresponding sum  $s' := \sum b_n$ ).  $\square$

**4.13. Remark:** One can prove that absolute convergence of a series is, in fact, equivalent to the property that all its rearrangements converge to the same limit. [cf. [BF00]]

Now that we have established the importance of absolute convergence we come to the question of how to determine whether a given series is absolutely convergent.

**4.14. Proposition (Basic comparison test):** (i) Let  $\sum c_n$  be convergent with nonnegative terms  $c_n \geq 0$ . If  $|a_n| \leq c_n$  holds for almost all  $n$ , then  $\sum a_n$  is absolutely convergent. ( $\sum c_n$  ist eine konvergente Majorante für  $\sum a_n$ .)

(ii) Let  $\sum d_n$  be divergent and have nonnegative terms  $d_n \geq 0$ . If  $a_n \geq d_n$  holds for almost all  $n$ , then  $\sum a_n$  is divergent. ( $\sum d_n$  ist eine divergente Minorante für  $\sum a_n$ .)

*Proof.* (i) WLOG (= without loss of generality) we may assume that  $|a_n| \leq c_n$  holds for all  $n$ .  $\sum |a_k|$  is a series with nonnegative terms and for all  $m \in \mathbb{N}$

$$0 \leq \sum_{k=0}^m |a_k| \leq \sum_{k=0}^{\infty} c_n$$

proves that the partial sums are bounded. Thus the convergence of  $\sum |a_n|$  follows from [Proposition 4.4].

(ii) If  $\sum a_n$  were convergent, then by (i)  $\sum d_n$  would converge — a contradiction  $\zeta$ .  $\square$

**4.15. Examples:** 1)  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  is divergent, since

$$\forall n \geq 1 : \frac{1}{\sqrt{n}} \geq \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

2) Let  $(a_n)$  be a sequence with  $|a_n| \leq 1$  for all  $n$  and let  $q \in ]0, 1[$ . Since  $|a_n q^n| \leq q^n$  and the geometric sum  $\sum q^n$  is convergent, we deduce that

$$\sum_{n=0}^{\infty} a_n q^n \quad \text{is absolutely convergent.}$$

**4.16. Proposition (Root test  $\langle$ Wurzeltest $\rangle$ ):** The series  $\sum a_n$  is

(a) absolutely convergent, if there exist  $\theta \in \mathbb{R}$  with  $0 \leq \theta < 1$  and  $n_0 \in \mathbb{N}$  such that

$$\forall n \geq n_0 : |a_n|^{1/n} \leq \theta$$

(b) divergent, if

$$|a_n|^{1/n} \geq 1 \quad \text{for infinitely many } n.$$

*Proof.* (a) Since  $|a_n| \leq \theta^n$  for almost all  $n$  and the geometric series  $\sum \theta^n$  is convergent, the basic comparison test implies convergence of  $\sum |a_n|$ .

(b) Since it follows that  $|a_n| \geq 1$  for infinitely many  $n$ , the sequence  $(a_n)$  does not tend to 0, thus  $\sum a_n$  is divergent.  $\square$

**4.17. Remark:** (i) In fact a slightly stronger statement is true [cf. [BF00]]: Let  $\alpha := \limsup |a_n|^{1/n}$ , then  $\sum a_n$  is

(a) absolutely convergent, if  $\alpha < 1$

(b) divergent, if  $\alpha > 1$ .

(ii) An important situation in practice is that  $\beta := \lim a_n$  exists. Then  $\beta < 1$  implies case (a) of the root test, hence absolute convergence, and  $\beta > 1$  implies condition (b), hence divergence. (Note that also  $\alpha = \beta$  in such situations.)

(iii) Note however, that achieving  $\theta = 1$  in condition (a) of the root test (or  $\alpha = 1$  or  $\beta = 1$ ) is not conclusive! For example,

$$\begin{aligned} \sum \frac{1}{n^2} \quad \text{is convergent and} \quad \left(\frac{1}{n^2}\right)^{1/n} < 1, \\ \sum \frac{1}{n} \quad \text{is divergent and} \quad \left(\frac{1}{n}\right)^{1/n} < 1. \end{aligned}$$

**4.18. Proposition (Ratio test <Quotiententest>):** Let  $a_n \neq 0$  for almost all  $n$ .

The series  $\sum a_n$  is

(a) absolutely convergent, if there exist  $\theta \in \mathbb{R}$  with  $0 \leq \theta < 1$  and  $n_0 \in \mathbb{N}$  such that

$$\forall n \geq n_0 : \left| \frac{a_{n+1}}{a_n} \right| \leq \theta$$

(b) divergent, if there exists  $n_0 \in \mathbb{N}$  such that

$$\forall n \geq n_0 : \left| \frac{a_{n+1}}{a_n} \right| \geq 1.$$

*Proof.* (a) We obtain for all  $n \geq n_0$

$$|a_{n+1}| \leq \theta |a_n| \leq \dots \leq \theta^{n-n_0} |a_{n_0}|,$$

where  $\sum \theta^{n-n_0} |a_{n_0}| = |a_{n_0}| \theta^{-n_0} \sum \theta^n$  is convergent. Thus the comparison theorem implies convergence of  $\sum |a_n|$ .

(b) Let  $n_1 \geq n_0$  such that  $a_{n_1} \neq 0$  then the stated condition implies that  $|a_n| \geq |a_{n_1}| > 0$  for all  $n \geq n_1$ . Thus  $(a_n)$  cannot be a null sequence and therefore  $\sum a_n$  diverges.  $\square$

**4.19. Remark:** (i) Again a slightly stronger variant can be shown [cf. [BF00]]: Let  $\alpha := \limsup \left| \frac{a_{n+1}}{a_n} \right|$  and  $\gamma := \liminf \left| \frac{a_{n+1}}{a_n} \right|$ , then  $\sum a_n$  is

(a) absolutely convergent, if  $\alpha < 1$

(b) divergent, if  $\gamma > 1$ .

(ii) Suppose that  $\beta := \lim \frac{a_{n+1}}{a_n}$  exists. Then  $\beta < 1$  implies case (a) of the ratio test, hence absolute convergence, and  $\beta > 1$  implies condition (b), hence divergence. (Note that consequently  $\alpha = \beta = \gamma$ .)

(iii) As with the root test, having  $\theta = 1$  in condition (a) of the ratio test (or  $\alpha = 1$  or  $\gamma = 1$  or  $\beta = 1$ ) is not conclusive! The same examples illustrate this:

$$\begin{aligned} \sum \frac{1}{n^2} \quad \text{is convergent and} \quad \frac{a_{n+1}}{a_n} &= \frac{n^2}{(n+1)^2} < 1, \\ \sum \frac{1}{n} \quad \text{is divergent and} \quad \frac{a_{n+1}}{a_n} &= \frac{n}{n+1} < 1. \end{aligned}$$

(iv) One can prove that condition (a) of the ratio test implies that condition (a) of the root test holds [cf. [BF00, Section 5.3]]. In other words, whenever the ratio test concludes positively the root test is applicable as well. On the other hand, there are absolutely convergent series for which the root test is successful, whereas the ratio test is inconclusive. (We give an example below.)

**4.20. Examples:** 1)  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$  is absolutely convergent, since

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2 2^n}{2^{n+1} n^2} = \frac{1}{2} \left( 1 + \frac{1}{n} \right)^2 \rightarrow \frac{1}{2} \quad (n \rightarrow \infty).$$

2) Consider

$$a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is even} \\ 3^{-n} & \text{if } n \text{ is odd.} \end{cases}$$

Then  $\sum a_n$  is absolutely convergent as can be seen by the root test:

$$|a_n|^{1/n} \leq \max \left( \frac{1}{2}, \frac{1}{3} \right) \leq \frac{1}{2} \quad \forall n$$

showing that we may choose  $\theta = 1/2$ . In this case the quotient test does not provide this, since

$$\frac{a_{2k+1}}{a_{2k}} = \frac{1}{3} \left( \frac{2}{3} \right)^{2k} \rightarrow 0, \quad \frac{a_{2k+2}}{a_{2k+1}} = \frac{1}{2} \left( \frac{3}{2} \right)^{2k+1} \rightarrow \infty \quad (k \rightarrow \infty).$$

**4.21. Decimal and b-adic expansion of real numbers:** In everyday life we are used to seeing rational numbers (e.g. purchase prices of certain products) expressed as decimal numbers like 17,8041 (or 17.8041). The corresponding number  $x \in \mathbb{Q}$  is in this case determined by the expansion with basis 10 and decimal digits from  $\{0, 1, 2, \dots, 9\}$  as

$$\begin{aligned} x &= 1 \cdot 10^1 + 7 \cdot 10^0 + 8 \cdot 10^{-1} + 0 \cdot 10^{-2} + 4 \cdot 10^{-3} + 1 \cdot 10^{-4} \\ &= \frac{10^5 + 7 \cdot 10^4 + 8 \cdot 10^3 + 4 \cdot 10 + 1}{10^4} = \frac{178041}{10000}. \end{aligned}$$

There are two immediate ideas to generalize such representations: The above expansion has finitely many terms. Can this be extended to infinite sums of the same type and be considered then as a representation of the limit? A very similar kind of expansion can be defined with an integer basis other than 10.

**DEFINITION:** Let  $b \in \mathbb{N}$ ,  $b \geq 2$ ,  $N \in \mathbb{Z}$  and  $a_n \in \{0, 1, 2, \dots, b-1\}$  ( $n \in \mathbb{Z}$ ,  $n \geq N$ ). The series

$$\pm \sum_{n=N}^{\infty} a_n b^{-n}$$

is called a *b-adic expansion* (*b-adische Entwicklung*) with *digits* (*Ziffern*)  $a_n$  ( $n \geq -M$ ).

Three questions arise:

1. Is every *b*-adic expansion convergent?
2. Can every  $x \in \mathbb{R}$  be represented as the limit of such an expansion?
3. Are the digits of an expansion uniquely determined by  $x$ ?

The answer to the third one is negative, as can be seen from the following simple example with decimal expansions:

$$0,999\dots = \sum_{n=1}^{\infty} 9 \cdot 10^{-n} = 9 \cdot 10^{-1} \sum_{n=0}^{\infty} (10^{-1})^n = \frac{9}{10} \cdot \frac{1}{1 - 1/10} = \frac{9}{10} \cdot \frac{10}{10 - 1} = 1 = 1,000\dots$$

But questions 2 and 3 can be shown to have positive answers.

**THEOREM:** Let  $b \in \mathbb{N}$ ,  $b \geq 2$ .

- (i) Every *b*-adic expansion is convergent (in  $\mathbb{R}$ ).
- (ii) Every real number  $x$  is the sum of a *b*-adic expansion (with a sequence of digits that can be constructed recursively).

*Proof.* (i) Since  $|a_n b^{-n}| \leq (b-1)b^{-n}$  for all  $n$  and  $(b-1) \sum b^{-n}$  is convergent, this follows from the basic comparison theorem.

- (ii) It suffices to show this for the case  $x \geq 0$ .

By the Archimedian property there is  $m \in \mathbb{N}$  such that  $x < b^{m+1}$ . Hence the minimum  $m_0$  of all  $m \in \mathbb{N}$  such that  $0 \leq x < b^{m+1}$  is well-defined. Put  $N := -m_0$ . We shall construct a sequence  $(a_n)_{n \geq N}$  of integers  $a_n \in \{0, 1, \dots, b-1\}$  inductively such that

$$(\star) \quad \forall n \geq N \quad \exists \xi_n \text{ with } 0 \leq \xi_n < b^{-n} : \quad x = \sum_{k=N}^n a_k b^{-k} + \xi_n.$$

Since by  $(\star)$   $\lim \xi_n = 0$ , hence  $x = \sum_{k=N}^{\infty} a_k b^{-k}$ , the assertion of the theorem will then follow.

Before we proceed, recall that the *floor function*  $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $\lfloor y \rfloor := \max\{l \in \mathbb{Z} : l \leq y\}$ , and that  $0 \leq y - \lfloor y \rfloor < 1$  holds.

Induction base,  $n = N$ : The definition of  $N$  implies  $0 \leq x \cdot b^N < b$ . We define  $a_N \in \{0, 1, \dots, b-1\}$  by

$$a_N := \lfloor x b^N \rfloor \quad \text{and} \quad \xi_N := (x b^N - a_N) \cdot b^{-N}$$

Then we clearly have

$$x = a_N b^{-N} + \xi_N \quad \text{and} \quad 0 \leq \xi_N < b^{-N}.$$

Induction step,  $n \rightarrow n+1$ : Property  $(\star)$  yields  $0 \leq \xi_n \cdot b^{n+1} < b$ . If we define  $a_{n+1} \in \{0, 1, \dots, b-1\}$  by

$$a_{n+1} := \lfloor \xi_n b^{n+1} \rfloor \quad \text{and} \quad \xi_{n+1} := (\xi_n b^{n+1} - a_{n+1}) \cdot b^{-n-1},$$

then  $\xi_n = a_{n+1} b^{-n-1} + \xi_{n+1}$  and therefore

$$x = \sum_{k=N}^n a_k b^{-k} + a_{n+1} b^{-n-1} + \xi_{n+1},$$

where  $0 \leq \xi_{n+1} < b^{-n-1}$ . Thus  $(\star)$  holds with  $n$  replaced by  $n+1$ . □

**COROLLARY:** Every real number is the limit of a sequence of rational numbers. ( $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Compare with the variant of this statement in [0.7].)

*Proof.* Let  $x \in \mathbb{R}$  have decimal expansion  $x = \sum_{n=N}^{\infty} a_n 10^{-n}$ . For every  $m \geq N$  the partial

sum  $s_m := \sum_{n=N}^m a_n 10^{-n}$  is a rational number and  $x = \lim s_m$ . □

**REMARK:** Let  $x = \pm \sum_{n=N}^{\infty} a_n b^{-n}$  be a  $b$ -adic expansion. One can prove (cf. [AE02, II.7]) that  $x$  is a rational number if and only if the sequence of digits is periodic from a certain index  $m \geq N$  onward, that is, with some  $p \in \mathbb{N} \setminus \{0\}$  we have  $a_{n+p} = a_n$  for all  $n \geq m$  (including the case that almost all  $a_n$  vanish, which describes a finite sum).

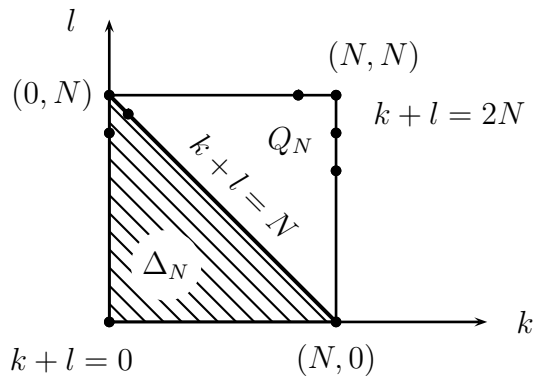
## 4.22. The Cauchy product of absolutely convergent series:

We begin with a simple observation about the product of finite sums.

Let  $A_N := \sum_{n=0}^N a_n$  and  $B_N := \sum_{n=0}^N b_n$ , then

$$A_N \cdot B_N = \left( \sum_{k=0}^N a_k \right) \cdot \left( \sum_{l=0}^N b_l \right) = \sum_{0 \leq k \leq N} \sum_{0 \leq l \leq N} a_k b_l.$$

For the investigation of convergence (as  $N \rightarrow \infty$ ) of such a double sum it is helpful to introduce some notation and to illustrate rearrangements of the terms in a 2-dimensional picture:



Here,  $Q_N := \{(k, l) \in \mathbb{N}^2 : k \leq N, l \leq N\}$  and  $\Delta_N := \{(k, l) \in Q_N : k + l \leq N\}$ . Then the above double sum can be rewritten as

$$\begin{aligned} A_N \cdot B_N &= \sum_{(k,l) \in Q_N} a_k b_l = \sum_{(k,l) \in \Delta_N} a_k b_l + \sum_{(k,l) \in Q_N \setminus \Delta_N} a_k b_l \\ &= \sum_{n=0}^N \sum_{\substack{(k,l) \in \Delta_N \\ k+l=n}} a_k b_l + \sum_{(k,l) \in Q_N \setminus \Delta_N} a_k b_l = \sum_{n=0}^N \sum_{k=0}^n a_k b_{n-k} + \sum_{(k,l) \in Q_N \setminus \Delta_N} a_k b_l. \end{aligned}$$

**PROPOSITION:** Let  $\sum a_n$  and  $\sum b_n$  be absolutely convergent series and define

$$c_n := \sum_{k=0}^n a_k b_{n-k} \quad (n \in \mathbb{N}).$$

Then  $\sum c_n$  is absolutely convergent and

$$\sum_{n=0}^{\infty} c_n = \left( \sum_{k=0}^{\infty} a_k \right) \cdot \left( \sum_{l=0}^{\infty} b_l \right).$$

*Proof.* We use the notation introduced above. Let  $S_N := \sum_{n=0}^N c_n$ , then

$$A_N \cdot B_N - S_N = \sum_{(k,l) \in Q_N \setminus \Delta_N} a_k b_l.$$

*Claim 1:*  $\lim_{N \rightarrow \infty} S_N = \left( \sum_{k=0}^{\infty} a_k \right) \cdot \left( \sum_{l=0}^{\infty} b_l \right)$

Put  $A_N^* := \sum_{n=0}^N |a_n|$  and  $B_N^* := \sum_{n=0}^N |b_n|$ , then  $A_N^* B_N^* = \sum_{(k,l) \in Q_N} |a_k| |b_l|$ . Using  $Q_{\lfloor N/2 \rfloor} \subseteq \Delta_N \subseteq Q_N$  we obtain

$$\begin{aligned} (\star) \quad |A_N B_N - S_N| &\leq \sum_{(k,l) \in Q_N \setminus \Delta_N} |a_k| |b_l| \leq \sum_{(k,l) \in Q_N \setminus Q_{\lfloor N/2 \rfloor}} |a_k| |b_l| \\ &= A_N^* B_N^* - A_{\lfloor N/2 \rfloor}^* B_{\lfloor N/2 \rfloor}^*. \end{aligned}$$

By hypothesis,  $(A_N^* B_N^*)_{N \in \mathbb{N}}$  is convergent, hence a Cauchy sequence, which implies that

$\lim_{N \rightarrow \infty} A_N^* B_N^* - A_{\lfloor N/2 \rfloor}^* B_{\lfloor N/2 \rfloor}^* = 0$ . Since  $\lim_{N \rightarrow \infty} A_N B_N = \left( \sum_{k=0}^{\infty} a_k \right) \cdot \left( \sum_{l=0}^{\infty} b_l \right)$ , the assertion now follows from  $(\star)$ .

*Claim 2:*  $\sum c_n$  is absolutely convergent

Let  $S_N^* := \sum_{n=0}^N \sum_{k=0}^n |a_k| |b_{n-k}|$ , then an application of Claim 1 shows that  $(S_N^*)_{N \in \mathbb{N}}$  is convergent. Since  $\sum |c_n| \leq S_N^*$  the assertion follows.  $\square$



**4.23. The exponential function:** For every  $x \in \mathbb{R}$  the series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is absolutely convergent, as can be verified by the ratio test: The terms  $a_n := x^n/n!$  are all nonzero if  $x \neq 0$ , and in this case

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|^{n+1}n!}{|x|^n(n+1)!} = \frac{|x|}{n+1} \rightarrow 0 \quad (n \rightarrow \infty);$$

in the case  $x = 0$  absolute convergence is clear.

We may thus define a real-valued function on  $\mathbb{R}$  by assigning the value of the sum  $\sum \frac{x^n}{n!}$  to each  $x$ .

**DEFINITION:** The *exponential function* (*Exponentialfunktion*)  $\exp: \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (x \in \mathbb{R}).$$

The value  $e := \exp(1)$  is called *Euler's number* (*Eulersche Zahl*)<sup>2</sup>.

Many important properties of the exponential function are a direct consequence of the so-called functional equation of the exponential function, which we prove in the following theorem. In fact, the exponential function can be characterized as the unique function satisfying this particular equation on  $\mathbb{R}$  and being bounded on some closed finite interval [cf. [BF00, Section 7.5]].

**THEOREM (FUNCTIONAL EQUATION FOR THE EXPONENTIAL FUNCTION):**  
For all  $x, y \in \mathbb{R}$

$$(4.3) \quad \exp(x + y) = \exp(x) \cdot \exp(y).$$

*Proof.* Both series  $\sum \frac{x^n}{n!}$  and  $\sum \frac{y^n}{n!}$  are absolutely convergent, hence the Cauchy product [4.22] gives  $\exp(x) \exp(y) = \sum c_n$ , where

$$c_n = \sum_{k=0}^n \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \stackrel{[\text{Binom. thm.}]}{=} \frac{(x+y)^n}{n!}.$$

Thus we obtain

$$\exp(x) \exp(y) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \exp(x+y).$$

□

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<sup>2</sup>Leonhard Euler (1707–1783) [ˈleɒnhɑːt ˈɔɪlə], Swiss mathematician and physicist, introduced many of the modern mathematic's symbols (e.g.  $e, \pi, i, \sum, f(x)$  etc.).

**COROLLARY:** (i) For all  $x \in \mathbb{R}$ :  $\exp(x) > 0$  and  $\exp(-x) = \frac{1}{\exp(x)}$ .  
(ii) For all  $n \in \mathbb{Z}$ :  $\exp(n) = e^n$ .

*Proof.* (i) The functional equation (4.3) implies

$$1 = \exp(0) = \exp(x - x) = \exp(x) \exp(-x),$$

which proves the second equation in the assertion. If  $x \geq 0$  then

$$\exp(x) = 1 + x + \frac{x^2}{2} + \dots \geq 1 > 0,$$

since all neglected terms are nonnegative. If  $x < 0$  then  $\exp(x) = 1/\exp(-x) > 0$  by what we have just proved.

(ii) Since  $\exp(-n) = 1/\exp(n)$  it suffices to show this for  $n \in \mathbb{N}$ . First,  $\exp(0) = 1 = e^0$  and then inductively

$$\exp(n + 1) = \exp(n) \exp(1) = e^n \cdot e^1 = e^{n+1}.$$

□

**REMARK:** The above Theorem and Corollary (i) show that  $\exp$  is a group homomorphism of the additive group  $(\mathbb{R}, +)$  into the multiplicative group  $(]0, \infty[, \cdot)$ .

Finally, we provide a (crude, but useful!) bound on the error of partial sum approximations of the exponential function. (The error bound shall be improved on later as we learn more about functions of a real variable and Taylor series.)

**PROPOSITION:** Let  $N \in \mathbb{N}$ . For all  $x \in \mathbb{R}$  we have

$$(4.4) \quad \exp(x) = \sum_{n=0}^N \frac{x^n}{n!} + R_{N+1}(x),$$

where the remainder term  $R_{N+1}(x)$  satisfies the following estimate for all  $x \in \mathbb{R}$  with  $|x| \leq 1 + \frac{N}{2}$

$$(4.5) \quad |R_{N+1}(x)| \leq 2 \frac{|x|^{N+1}}{(N+1)!}.$$

*Proof.* The remainder term is

$$R_{N+1}(x) := \exp(x) - \sum_{n=0}^N \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^N \frac{x^n}{n!} = \sum_{n=N+1}^{\infty} \frac{x^n}{n!},$$

which is absolutely convergent. Therefore we have for all  $|x| \leq 1 + N/2$  the following chain of inequalities

$$\begin{aligned} |R_{N+1}(x)| &\leq \sum_{n=N+1}^{\infty} \frac{|x|^n}{n!} = \frac{|x|^{N+1}}{(N+1)!} \left( 1 + \frac{|x|}{N+2} + \underbrace{\frac{|x|^2}{(N+2)(N+3)}}_{\leq \frac{|x|^2}{(N+2)^2}} + \dots \right) \\ &\leq \frac{|x|^{N+1}}{(N+1)!} \sum_{k=0}^{\infty} \left( \frac{|x|}{N+2} \right)^k \stackrel{||x| \leq (N+2)/2}{\leq} \frac{|x|^{N+1}}{(N+1)!} \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k = 2 \cdot \frac{|x|^{N+1}}{(N+1)!}. \end{aligned}$$

□

**EXAMPLE:** Since  $e = \exp(1) = \sum_{n=0}^N \frac{1}{n!} + R_{N+1}(1)$  and  $1 \leq 1 + N/2$  for all  $N \in \mathbb{N}$ , we obtain for  $N = 2$

$$e = 1 + 1 + \frac{1}{2} + R_3(1), \quad \text{where } 0 < R_3(1) \leq 2 \frac{1}{3!} = \frac{2}{6} = \frac{1}{3}.$$

Therefore

$$2 < \frac{5}{2} < e \leq \frac{5}{2} + \frac{1}{3} = \frac{17}{6} < 3.$$

As a matter of fact, the exponential series is numerically efficient. For example, one obtains a value for  $e$  which is accurate up to 100 digits by summation of 73 terms; to give just the first few digits

$$e \approx 2,71828\dots$$

CHAPTER II

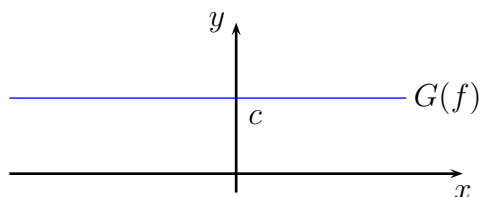
**CONTINUOUS FUNCTIONS  
OF A REAL VARIABLE**

## §5. CONTINUITY

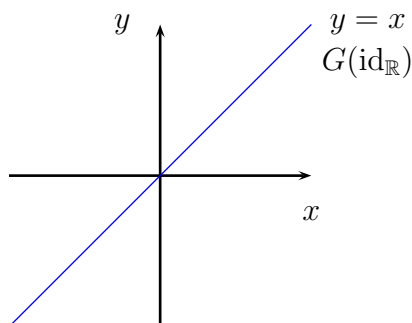
In this section we study real (valued) functions<sup>1</sup> on subsets of  $\mathbb{R}$ , i.e., maps  $f: D \rightarrow \mathbb{R}$ , where  $D \subseteq \mathbb{R}$ . Recall that the *graph* (*Graph*) of  $f$  is defined as the following subset of  $\mathbb{R}^2$ :

$$G(f) := \{(x, f(x)) \in \mathbb{R}^2 : x \in D\}.$$

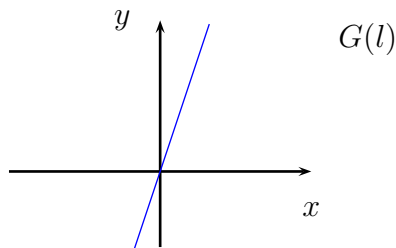
**5.1. Examples:** 1) Let  $c \in \mathbb{R}$  arbitrary, then  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) := c$  for all  $x \in \mathbb{R}$  defines a *constant function*.



2) The *identity map* (*identische Abbildung*) on  $\mathbb{R}$  is given by  $\text{id}_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto x$ .



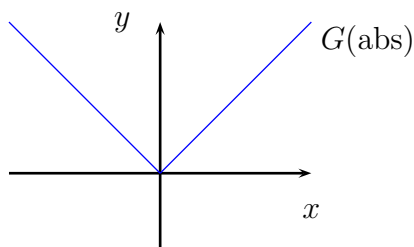
Slightly more general are linear functions  $l: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto a \cdot x$ , where  $a \in \mathbb{R}$  gives the slope of the graph:



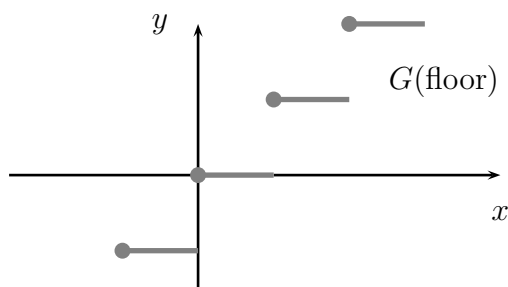
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<sup>1</sup>The mathematical term, “function” (from the Latin *functio*, meaning performance, execution) was first used by Leibniz in 1694 to describe curves.

3) The *absolute value (function)* (Betragsfunktion oder Absolutbetrag) is defined by  $\text{abs}: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|$ .

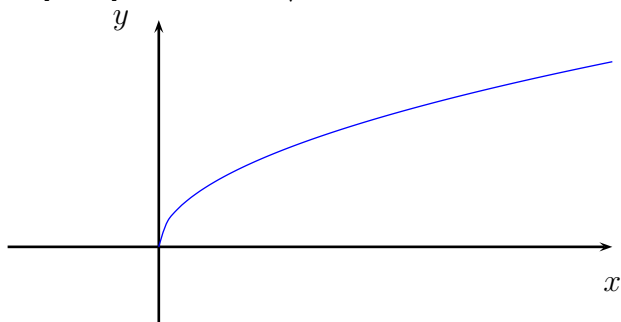


4) floor:  $\mathbb{R} \rightarrow \mathbb{R}, x \mapsto \lfloor x \rfloor$ , where (as on page 44)  $\lfloor x \rfloor = \max\{n \in \mathbb{Z}: n \leq x\}$ .



The floor function is sometimes called *Gauß bracket*<sup>2</sup> (Gaußklammer) and the values are also denoted by  $[x]$  ( $x \in \mathbb{R}$ ).

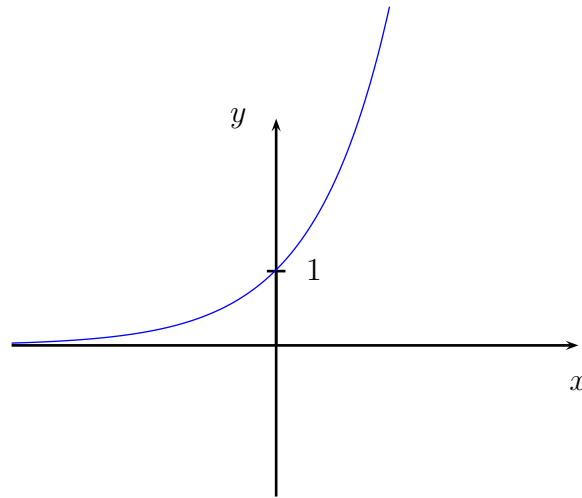
5) The square root  $\text{sqrt}: [0, \infty] \rightarrow \mathbb{R}, x \mapsto \sqrt{x}$




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<sup>2</sup>Carl Friedrich Gauß (1777–1855) [kɑ:l 'fri:trɪç gɑ:ʊs], one of the most outstanding German mathematicians

6) The exponential function  $\exp: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \exp(x)$  as defined in 4.23.

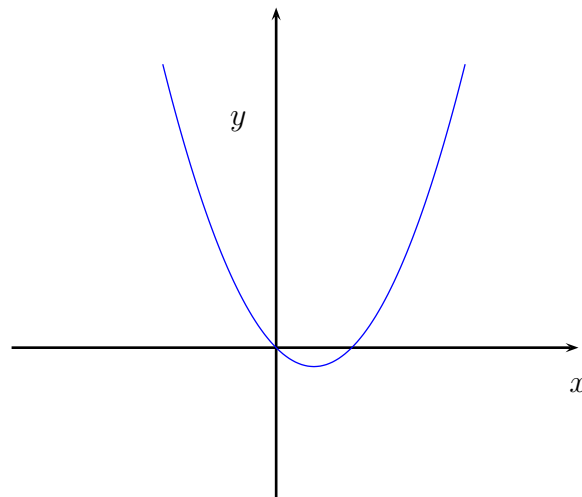


7) *Polynomial functions* (Polynomfunktionen): Let  $m \in \mathbb{N}$  and  $a_0, a_1, \dots, a_m \in \mathbb{R}$ . We define

$$p: \mathbb{R} \rightarrow \mathbb{R} \quad \text{by} \quad p(x) := a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0 \quad \forall x \in \mathbb{R}.$$

The constants  $a_0, \dots, a_m$  are called the *coefficients* (Koeffizienten) of the polynomial function. If  $a_m \neq 0$  then  $p$  is said to be of *degree*  $m$  (vom Grad  $m$ ).

For example, when  $m = 2$  and  $a_0 = 0$ ,  $a_1 = -1$ ,  $a_2 = 1$  we obtain  $p(x) = x^2 - x$



8) *Rational functions* (rationale Funktionen): Let  $p$  and  $q$  be polynomial functions, that is

$$p(x) = a_m x^m + \dots + a_1 x + a_0 \quad \text{and} \quad q(x) = b_n x^n + \dots + b_1 x + b_0$$

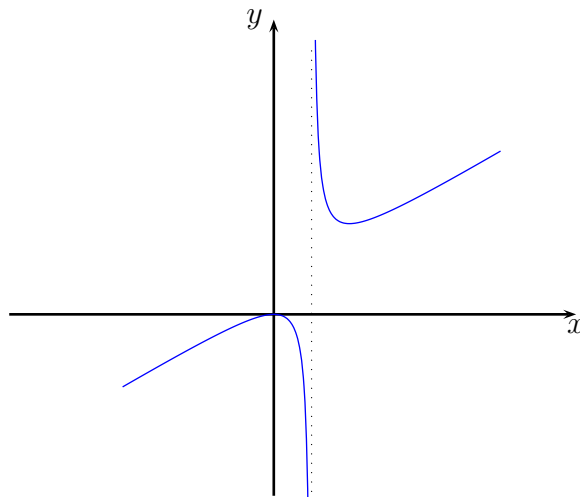
with given coefficients  $a_0, \dots, a_m, b_0, \dots, b_n \in \mathbb{R}$ . Then a rational function is the quotient

function with domain  $D := \{x \in \mathbb{R} : q(x) \neq 0\}$ , defined by

$$r: D \rightarrow \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}.$$

Note that polynomial functions are just rational functions with denominator  $q \equiv 1$ .

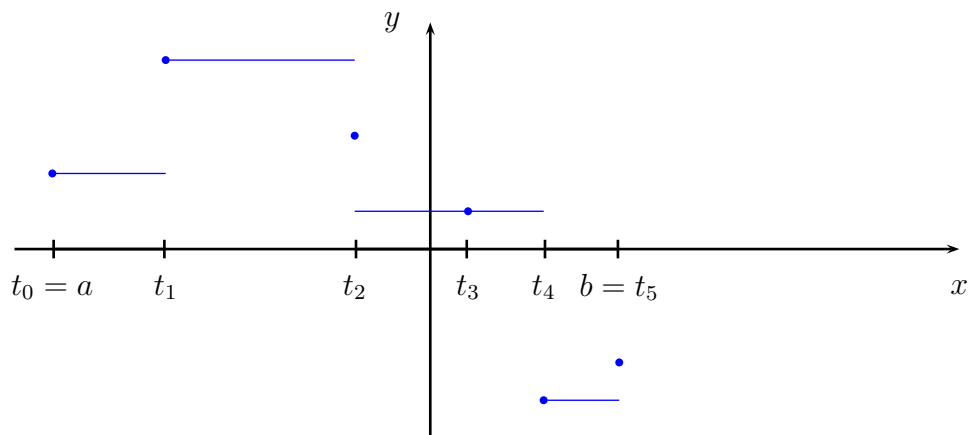
For example, here is the graph of the rational function  $r: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ ,  $r(x) = x^2/(x-1)$



9) *Simple functions (or step functions) (Treppenfunktionen)*: Let  $a, b \in \mathbb{R}$  with  $a < b$ . A function  $\varphi: [a, b] \rightarrow \mathbb{R}$  is called a simple function (or step function), if there is a finite partition  $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$  of the interval  $[a, b]$  and coefficients  $c_1, \dots, c_n \in \mathbb{R}$  such that

$$\varphi(x) = c_k \quad \text{when } x \in ]t_{k-1}, t_k[ \quad (1 \leq k \leq n).$$

Therefore  $\varphi$  is constant on each open subinterval  $]t_{k-1}, t_k[$  ( $1 \leq k \leq n$ ) but the finitely many values  $\varphi(t_k)$  ( $0 \leq k \leq n$ ) are arbitrary.





Note that the restriction floor  $|_{[a,b]}$  of the floor function provides an example of a simple function.

10) The *characteristic function of  $\mathbb{Q}$*  (*charakteristische Funktion von  $\mathbb{Q}$* ) or *Dirichlet function*<sup>3</sup> is given by

$$\mathbf{1}_{\mathbb{Q}}: \mathbb{R} \rightarrow \mathbb{R}, \quad \mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

In this case the graph is

$$G(\mathbf{1}_{\mathbb{Q}}) = \{(q, 1) : q \in \mathbb{Q}\} \cup \{(s, 0) : s \in \mathbb{R} \setminus \mathbb{Q}\},$$

which would be somewhat hard to depict ...

## 5.2. Review of basic operations with functions:

Let  $f, g: D \rightarrow \mathbb{R}$  be functions on  $D \subseteq \mathbb{R}$  and  $\lambda \in \mathbb{R}$ .

- Then the functions

$$f + g: D \rightarrow \mathbb{R}, \quad \lambda f: D \rightarrow \mathbb{R}, \quad f \cdot g: D \rightarrow \mathbb{R}$$

are defined in terms of the corresponding pointwise operations (with real numbers) for all  $x \in D$  by

$$\begin{aligned} (f + g)(x) &:= f(x) + g(x), \\ (\lambda f)(x) &:= \lambda \cdot f(x), \\ (f \cdot g)(x) &:= f(x) \cdot g(x). \end{aligned}$$

Remark: It is easy to check that the set  $\mathcal{F}(D) := \{f: D \rightarrow \mathbb{R}\}$  of all real valued functions on the set  $D$  together with the addition and scalar multiplication as defined by the first two lines above forms a vector space over  $\mathbb{R}$ .

- Let  $D' := \{x \in D : g(x) \neq 0\}$ . The quotient function is defined by

$$\frac{f}{g}: D' \rightarrow \mathbb{R}, \quad x \rightarrow \frac{f(x)}{g(x)}.$$

- Let  $E \subseteq \mathbb{R}$  such that  $f(D) \subseteq E$  and  $h: E \rightarrow \mathbb{R}$ . Recall that the composition of  $f$  and  $h$  is given by

$$h \circ f: D \rightarrow \mathbb{R}, \quad (h \circ f)(x) := h(f(x)) \quad \forall x \in D.$$

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<sup>3</sup>Johann Peter Gustav Lejeune Dirichlet (1805–1859) [ˈjoːhan ˈpɛrtəv ˈɡʊstaf ləˈzɔ̃n dɪrɪˈkle], German mathematician with Belgish origins (the French words Lejeune Dirichlet literally mean “the young chap from Richelet”)

**Examples:** 1) If  $q: \mathbb{R} \rightarrow \mathbb{R}$ ,  $q(x) = x^2$ , then  $q = \text{id} \cdot \text{id}$ .

2) More generally, if  $p$  is a polynomial function, given by

$$p(x) = a_m x^m + \dots a_1 x + a_0,$$

then

$$p = a_m \cdot \underbrace{(\text{id} \cdot \text{id} \cdots \text{id})}_{m \text{ factors}} + \dots a_1 \cdot \text{id} + a_0 \cdot \mathbf{1},$$

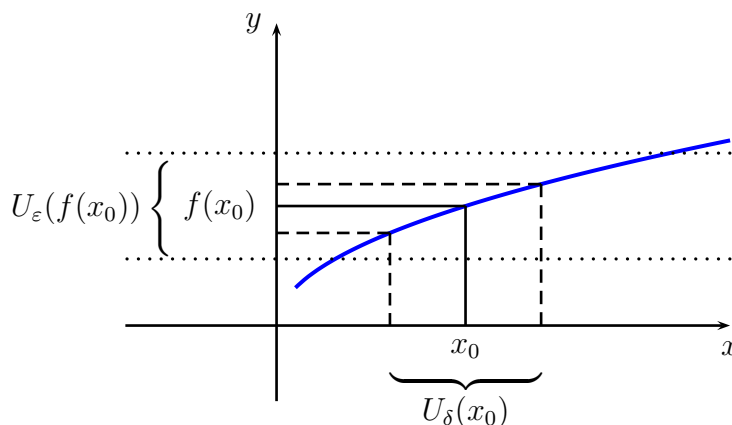
where  $\mathbf{1}$  denotes the constant function  $\mathbf{1}(x) = 1$  for all  $x \in \mathbb{R}$ .

3) With  $q$  as in example 1) we have  $\text{abs} = \text{sqrt} \circ q$ , since for all  $x \in \mathbb{R}$

$$(\text{sqrt} \circ q)(x) = \sqrt{x^2} = |x| = \text{abs}(x).$$

**5.3. Continuity** (*Stetigkeit*): The notion of continuity of a function is a precise way to express an intuitive requirement, which is often implicitly made in model applications: Namely, that small perturbations of a function argument should not result in extreme changes of the function values.

How to specify such a property for a given function  $f$  near a point  $x_0$  of its domain? It might seem practically desirable to first prescribe the acceptable tolerance around the value  $f(x_0)$  and then to look for a safety interval around the argument  $x_0$  on which function values near  $f(x_0)$  within tolerance are guaranteed. If the tolerance is given in terms of an interval  $]f(x_0) - \varepsilon, f(x_0) + \varepsilon[$  with  $\varepsilon > 0$  and the safety interval is sought in the form  $]x_0 - \delta, x_0 + \delta[$  with  $\delta > 0$  we obtain the following picture:



By requiring that for every tolerance  $\varepsilon > 0$  — chosen arbitrarily small — an appropriate safety guard  $\delta > 0$  can (in principle) be found we arrive at the notion of continuity.

**DEFINITION:** Let  $x_0 \in D \subseteq \mathbb{R}$  and  $f: D \rightarrow \mathbb{R}$ . The function  $f$  is *continuous* (*stetig*) at  $x_0$  if

$$(5.1) \quad \forall \varepsilon > 0 \exists \delta > 0 : \quad \forall x \in D : \quad |x - x_0| < \delta \quad \implies \quad |f(x) - f(x_0)| < \varepsilon.$$

Equivalently, upon recalling that  $]x_0 - \delta, x_0 + \delta[ = U_\delta(x_0)$  and  $]f(x_0) - \varepsilon, f(x_0) + \varepsilon[ = U_\varepsilon(f(x_0))$ , we can define the continuity of  $f$  at  $x_0$  in terms of neighborhoods:

$$\forall \varepsilon > 0 \exists \delta > 0 : \quad f(U_\delta(x_0) \cap D) \subseteq U_\varepsilon(f(x_0)).$$

The function  $f$  is said to be *continuous (on  $D$ )* if it is continuous at each point in  $D$ . If  $f$  is not continuous at a point  $b \in D$  then  $f$  is said to be *discontinuous* (*unstetig*) at  $b$ .

**EXAMPLES:** 1) Clearly, a constant function  $f$  is continuous (at every point  $x_0$  in its domain), since  $f(x) - f(x_0) = 0$  and therefore (5.1) is satisfied for all  $\varepsilon > 0$  and  $\delta > 0$  arbitrary.

2) Every linear function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto ax$ , is continuous (at every  $x_0 \in \mathbb{R}$ ): If  $a = 0$  this is clear from Example 1), hence consider  $a \neq 0$ . Let  $\varepsilon > 0$ . From the preparatory observation  $|f(x) - f(x_0)| = |a||x - x_0|$  we learn that we can simply choose  $\delta := \varepsilon/|a|$  to achieve (5.1): Indeed, if  $|x - x_0| < \delta = \varepsilon/|a|$  then

$$|f(x) - f(x_0)| = |a||x - x_0| < |a|\delta = \varepsilon.$$

3) The exponential function  $\exp: \mathbb{R} \rightarrow \mathbb{R}$  is continuous: Let  $x_0 \in \mathbb{R}$  and  $\varepsilon > 0$ . By the properties of the exponential function we have

$$|\exp(x) - \exp(x_0)| = \exp(x_0) |\exp(x - x_0) - 1|,$$

where  $\exp(x_0) > 0$ . From (4.5) we obtain for  $|x - x_0| \leq 1$  that

$$|\exp(x - x_0) - 1| \leq 2|x - x_0|.$$

Thus, putting  $\delta := \min(1, \frac{\varepsilon}{2\exp(x_0)})$  and combining the above inequalities we obtain for all  $x$  with  $|x - x_0| < \delta$  the required estimate

$$|\exp(x) - \exp(x_0)| \leq 2\exp(x_0)|x - x_0| < 2\exp(x_0)\delta \leq \varepsilon.$$

4)  $\text{abs}: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto |x|$ , is continuous: Let  $x_0 \in \mathbb{R}$  and  $\varepsilon > 0$ . Put  $\delta := \varepsilon$  then we have for all  $x \in U_\delta(x_0)$

$$|\text{abs}(x) - \text{abs}(x_0)| = ||x| - |x_0|| \leq |x - x_0| < \delta = \varepsilon.$$

5) The Dirichlet function  $\mathbb{1}_{\mathbb{Q}}$  [Example 5.1, 10)] is discontinuous at every point in  $\mathbb{R}$ : Let  $x_0 \in \mathbb{R}$  and put  $\varepsilon = 1/2$ .

If  $x_0 \in \mathbb{R} \setminus \mathbb{Q}$  then  $\mathbf{1}_{\mathbb{Q}}(x_0) = 0$ . By the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , for every  $\delta > 0$  we might choose the interval  $U_\delta(x_0) = ]x_0 - \delta, x_0 + \delta[$  will always contain some (in fact, many) rational number(s)  $r$  [cf. 0.7 or the Corollary in 4.21]. In other words, we can find  $r$  with  $|r - x_0| < \delta$  but

$$|\mathbf{1}_{\mathbb{Q}}(r) - \mathbf{1}_{\mathbb{Q}}(x_0)| = |1 - 0| = 1 \geq \frac{1}{2} = \varepsilon.$$

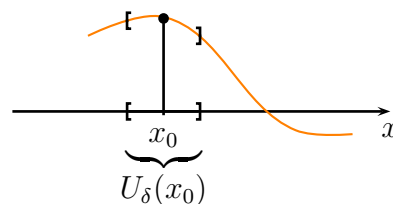
If  $x_0 \in \mathbb{Q}$  then  $\mathbf{1}_{\mathbb{Q}}(x_0) = 1$ . Recall that also  $\mathbb{R} \setminus \mathbb{Q}$  is dense in  $\mathbb{R}$  [cf. 0.7]. Hence for every  $\delta > 0$  we can find  $s \in U_\delta(x_0) \cap (\mathbb{R} \setminus \mathbb{Q})$ , which implies

$$|\mathbf{1}_{\mathbb{Q}}(s) - \mathbf{1}_{\mathbb{Q}}(x_0)| = |0 - 1| = 1 \geq \frac{1}{2} = \varepsilon$$

while  $|s - x_0| < \delta$ .

Knowing that a specific value of a continuous function has positive distance to a certain real number  $c$  already guarantees that the function values will stay away from  $c$  in a whole neighborhood. In the following statement we formulate this for the special case with  $c = 0$ . This can easily be adapted to the case  $c \neq 0$  by a simple translation of the function graph.

**5.4. Lemma:** Let  $f: D \rightarrow \mathbb{R}$  be continuous at  $x_0 \in D$  and assume that  $f(x_0) \neq 0$ . Then there is  $\delta > 0$  such that for all  $x \in U_\delta(x_0) \cap D$  we have  $f(x) \neq 0$ .



*Proof.* Put  $\varepsilon := |f(x_0)|/2$ . Then clearly  $\varepsilon > 0$  and by continuity there exists some  $\delta > 0$  such that for all  $x \in D$  with  $|x - x_0| < \delta$  we have  $|f(x) - f(x_0)| < \varepsilon = |f(x_0)|/2$ . Therefore  $x \in U_\delta(x_0) \cap D$  implies

$$|f(x)| = |f(x_0) + f(x) - f(x_0)| \geq |f(x_0)| - |f(x) - f(x_0)| > |f(x_0)| - \varepsilon = \frac{|f(x_0)|}{2} > 0.$$

□

## 5.5. Continuity test by sequences:

**THEOREM:** Let  $a \in D \subseteq \mathbb{R}$  and  $f: D \rightarrow \mathbb{R}$ . The following are equivalent:

- (i)  $f$  is continuous at  $a$ .
- (ii) For every sequence  $(x_n)$  with  $x_n \in D$  we have: if  $\lim x_n = a$  then  $\lim f(x_n) = f(a)$ .

*Proof.* (i)  $\Rightarrow$  (ii): Let  $x_n \in D$  ( $n \in \mathbb{N}$ ) with  $\lim x_n = a$  and let  $\varepsilon > 0$ . Choose  $\delta > 0$  such that the continuity condition (5.1) is satisfied. There exists  $n_0 \in \mathbb{N}$  such that  $|x_n - a| < \delta$

holds for all  $n \geq n_0$ . Thus (5.1) implies

$$|f(x_n) - f(a)| < \varepsilon \quad \forall n \geq n_0,$$

which proves that  $\lim f(x_n) = f(a)$ .

(ii)  $\Rightarrow$  (i): (proof by contradiction) Assume that (ii) holds but (5.1) is false. That is,

$$\exists \varepsilon > 0 \forall \delta > 0 : \exists x \in U_\delta(a) \cap D : f(x) \notin U_\varepsilon(f(a)).$$

In particular, with this same  $\varepsilon > 0$ , we can choose the  $\delta$ -values to be  $1/n$  ( $n \in \mathbb{N}$ ,  $n \geq 1$ ) successively and obtain:

$$\forall n \in \mathbb{N}, n \geq 1 : \exists x_n \in D : |x_n - a| < \frac{1}{n}, \text{ but } |f(x_n) - f(a)| \geq \varepsilon.$$

Therefore  $\lim x_n = a$  whereas  $f(x_n) \not\rightarrow f(a)$  ( $n \rightarrow \infty$ ) — a contradiction  $\nexists$ . □

**EXAMPLE:** The function floor:  $\mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \lfloor x \rfloor$  is continuous in  $\mathbb{R} \setminus \mathbb{Z}$  and discontinuous in all points  $a \in \mathbb{Z}$ .

If  $a \in \mathbb{Z}$  then  $\lfloor a \rfloor = a$  and the sequence  $x_n := a - \frac{1}{n}$  ( $n \geq 1$ ) has  $\lim x_n = a$  but  $\lim \lfloor x_n \rfloor = \lim(a - 1) = a - 1 \neq \lfloor a \rfloor$ .

If  $a \in \mathbb{R} \setminus \mathbb{Z}$  then  $\lfloor a \rfloor < a < \lfloor a \rfloor + 1$ . Hence for every sequence  $(x_n)$  with  $\lim x_n = a$  there exists some  $n_0$  such that  $\lfloor a \rfloor < x_n < \lfloor a \rfloor + 1$  when  $n \geq n_0$ . Therefore  $\lfloor x_n \rfloor = \lfloor a \rfloor$  for all  $n \geq n_0$ , in particular  $\lim \lfloor x_n \rfloor = \lfloor a \rfloor$ .

**5.6. Basic operations and continuity:** The following results show that we do not leave the class of continuous functions when applying the basic operations summarized in 5.2 to continuous functions. In other words, we can generate many “new” continuous functions from a set of given continuous functions simply by pointwise addition, scalar multiplication, multiplication, division (when the denominator does not vanish), and composition (where the images and domains match appropriately).

**PROPOSITION:** (i) Let  $a \in D \subseteq \mathbb{R}$  and  $\lambda \in \mathbb{R}$ . If  $f, g: D \rightarrow \mathbb{R}$  are continuous at  $a$  then also

$$f + g: D \rightarrow \mathbb{R}, \quad \lambda f: D \rightarrow \mathbb{R}, \quad f \cdot g: D \rightarrow \mathbb{R}$$

are continuous at  $a$ . Furthermore, if  $a \in D' := \{x \in D : g(x) \neq 0\}$  then

$$\frac{f}{g}: D' \rightarrow \mathbb{R}$$

is continuous at  $a$ .

(ii) Let  $D \subseteq \mathbb{R}$ ,  $E \subseteq \mathbb{R}$  and  $f: D \rightarrow \mathbb{R}$ ,  $g: E \rightarrow \mathbb{R}$  such that  $f(D) \subseteq E$ . If  $f$  is continuous at  $a \in D$  and  $g$  is continuous at  $b := f(a) \in E$  then the composition  $g \circ f: D \rightarrow \mathbb{R}$  is continuous at  $a$ .

*Proof.* (i) Let  $(x_n)$  be a sequence in  $D$ , respectively  $D'$ , such that  $x_n \rightarrow a$ . Then by the corresponding properties of basic operations with convergent sequences in 2.10 we obtain that

$$(f + g)(x_n) = f(x_n) + g(x_n) \rightarrow f(a) + g(a) = (f + g)(a) \quad (n \rightarrow \infty)$$

and similarly for the other types of operations. Thus Theorem 5.5 proves continuity at  $a$ .

(ii) Let  $(x_n)$  be a sequence in  $D$  such that  $x_n \rightarrow a$ . Since  $f$  is continuous at  $a$  we have  $y_n := f(x_n) \rightarrow f(a) = b$ . Continuity of  $g$  at  $b$  implies  $g(y_n) \rightarrow g(b)$ . Therefore

$$\lim_{n \rightarrow \infty} (g \circ f)(x_n) = \lim_{n \rightarrow \infty} g(f(x_n)) = \lim_{n \rightarrow \infty} g(y_n) = g(b) = g(f(a)) = (g \circ f)(a)$$

and again by Theorem 5.5 the continuity at  $a$  follows.  $\square$

**COROLLARY:** Polynomial functions and rational functions are continuous (on their respective domains).

*Proof.* By 5.3, Examples 1) and 2), constant functions and the identity map  $\text{id}: \mathbb{R} \rightarrow \mathbb{R}$  are continuous. In 5.2, Example 2), we noted that polynomial functions are just finite linear combinations of products of  $\text{id}$  by itself plus a constant function, thus the above Proposition (i) shows continuity.

Rational functions are quotients of polynomial functions, defined where the denominator does not vanish, and are therefore also continuous by the second part of (i) in the above Proposition.  $\square$

**EXAMPLE:** 1)  $p(x) := -x^2$  defines a continuous function on  $\mathbb{R}$  and  $\exp$  is continuous on  $\mathbb{R}$ . Hence the function  $\exp \circ p: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \exp(-x^2)$  is continuous  $\mathbb{R} \rightarrow \mathbb{R}$ .

2) The *hyperbolic sine and cosine* (*hyperbolischer Sinus und Cosinus*) are defined by

$$\sinh(x) := \frac{\exp(x) - \exp(-x)}{2} \quad \text{and} \quad \cosh(x) := \frac{\exp(x) + \exp(-x)}{2} \quad (x \in \mathbb{R}),$$

hence are continuous functions on  $\mathbb{R}$ .

**5.7. Limit of a function:** Recall that  $a \in \mathbb{R}$  is an adherent point of  $D \subseteq \mathbb{R}$  if and only if there exists a sequence  $(x_n)$  in  $D$  (i.e.,  $x_n \in D$  for all  $n$ ) such that  $x_n \rightarrow a$  ( $n \rightarrow \infty$ ). If  $a$  is an element of  $D$  then the latter condition is clearly satisfied by the constant sequence  $x_n = a$  for all  $n$ . In general, an adherent point of  $D$  need not be a member of the set  $D$ .

**DEFINITION:** Let  $f: D \rightarrow \mathbb{R}$  and  $a$  an adherent point of  $D$ . The function  $f$  has *limit*  $c \in \mathbb{R}$  as  $x$  tends to  $a$ , if every sequence  $(x_n)$  in  $D$  such that  $x_n \rightarrow a$  ( $n \rightarrow \infty$ ) satisfies  $\lim_{n \rightarrow \infty} f(x_n) = c$ . A short-hand notation for this fact is

$$\lim_{x \rightarrow a} f(x) = c \quad \text{or} \quad f(x) \rightarrow c \quad (x \rightarrow a).$$

We also define  $c \in \mathbb{R}$  to be *the limit of  $f$  at  $a$  from the right* (*rechtsseitiger Grenzwert*)

$$\lim_{x \searrow a} f(x) = c, \quad \text{or also} \quad \lim_{x \rightarrow a^+} f(x) = c$$

if  $a$  is an adherent point of  $D \cap [a, \infty[$  and for all sequences  $(x_n)$  with  $x_n \in D$  and  $x_n > a$  such that  $x_n \rightarrow a$  we have  $\lim f(x_n) = c$ .

The notion of *limit from the left* (*linksseitiger Grenzwert*)  $\lim_{x \nearrow a} f(x)$ , also denoted by  $\lim_{x \rightarrow a^-} f(x)$ , is defined analogously using  $] - \infty, a] \cap D$  and  $x_n < a$  instead.

Finally, we define *limits of  $f$  at infinity* as follows:

$$\lim_{x \rightarrow \infty} f(x) = c$$

means that  $D$  is unbounded from above and for every sequence  $(x_n)$  with  $x_n \in D$  and  $x_n \rightarrow \infty$  we have  $\lim f(x_n) = c$ .

We define  $\lim_{x \rightarrow -\infty} f(x)$  similarly when  $D$  is unbounded from below using  $x_n \rightarrow -\infty$ .

Of course, we will often find it convenient to also use the above notions with improper limits  $c = \pm\infty$ . The required adaptations of the definition should be routine and are left to the reader.

**EXAMPLES:** 1) For the rational function  $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ ,  $f(x) = (x^2 - 1)/(x - 1)$ , we have

$$\lim_{x \rightarrow 1} f(x) = 2.$$

Indeed, if  $x_n \rightarrow 1$  with  $x_n \neq 1$  then

$$f(x_n) = \frac{(x_n - 1)(x_n + 1)}{x_n - 1} = x_n + 1 \rightarrow 2 \quad (n \rightarrow \infty).$$

2)  $\lim_{x \searrow 1} [x] = 1$ , since  $[x_n] = 1$  when  $1 < x_n < 2$ . On the other hand,  $\lim_{x \nearrow 1} [x] = 0$  as  $[x_n] = 0$  when  $0 \leq x_n < 1$ .

We conclude that  $\lim_{x \rightarrow 1} [x]$  does not exist, because otherwise the limits from the left and from the right would have to be equal.

3) Let  $m \in \mathbb{N}$ ,  $m \geq 1$ , and  $p: \mathbb{R} \rightarrow \mathbb{R}$  be a polynomial function of the form

$$p(x) = x^m + a_{m-1}x^{m-1} + \dots + a_0.$$

Then we have  $\lim_{x \rightarrow \infty} p(x) = \infty$  and  $\lim_{x \rightarrow \infty} \frac{1}{p(x)} = 0$ .

To see this, we first note that for all  $x > 0$  we have the estimate

$$p(x) = x^m \left( 1 + \frac{a_{m-1}}{x} + \dots + \frac{a_0}{x^m} \right) \geq x^m \left( 1 - \frac{|a_{m-1}|}{|x|} - \dots - \frac{|a_0|}{|x^m|} \right).$$

Let  $x \geq M := 2m \cdot \max(1, |a_{m-1}|, \dots, |a_0|)$ , then the above inequality implies

$$p(x) \geq x^m \left(1 - m \cdot \frac{1}{2m}\right) = \frac{x^m}{2} \quad (\text{in particular, } p(x) \geq \frac{1}{2}).$$

Let  $x_n \rightarrow \infty$  and choose  $n_0 \in \mathbb{N}$  such that  $x_n \geq M$  for all  $n \geq n_0$ . Then we obtain for  $n \geq n_0$

$$p(x_n) \geq \frac{x_n^m}{2} \rightarrow \infty \quad (n \rightarrow \infty),$$

therefore  $\lim p(x_n) = \infty$ , which proves the first assertion above. The second assertion follows immediately from the first, if we note that  $1/p(x)$  is well-defined for  $x \geq M$  (since  $p(x) \geq 1/2$  then, as noted above).

**REMARK:** Note that if  $a \in D$  and  $\lim_{x \rightarrow a} f(x)$  exists then the limit has to be  $f(a)$  (since  $x_n = a$  is a special sequence in  $D$  converging to  $a$ ).

**Warning:** The notion of ‘limit of a function’ is not used in exactly the same way as we do here throughout the literature. Some texts (e.g. [Heu88]) require the admissible sequences  $(x_n)$  in the definition to be in  $D \setminus \{a\}$ , so that the special choice  $x_n = a$  is excluded even in the case where  $a$  belongs to  $D$ . If  $a$  is an adherent point of  $D$  and does not belong to  $D$ , both notions give the same result concerning existence and value of the function limit. However, if  $a \in D$  the conclusions may differ, as can be seen from the following example: Let  $D := \mathbb{R} \setminus \{0\}$  and define  $f: D \rightarrow \mathbb{R}$  by  $f(x) := 1$  if  $x \neq 0$ , and  $f(0) := 0$ . Then in the sense of our definition  $f$  does not have a limit at 0, whereas we obtain for all sequences  $(x_n)$  with  $x_n \neq 0$  and  $x_n \rightarrow 0$  that  $\lim f(x_n) = 1$  (note that this value differs from  $f(0)$ ), which would give existence of the limit of  $f$  at 0 in the alternative definition.

Since the notion of ‘limit of a function’ is essentially a short-hand notation to describe the way how a function translates converging sequences into sequences of corresponding function values, we can rephrase the sequence test of continuity 5.5 in these terms.

**PROPOSITION:** A function  $f: D \rightarrow \mathbb{R}$  is continuous at a point  $a \in D$  if and only if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

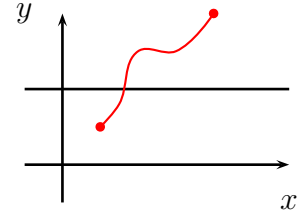
*Proof.* This is immediate from Theorem 5.5 and the remark made above. □



## 5.8. The intermediate value property (Zwischenwertsatz):

**THEOREM:** Suppose  $f: [a, b] \rightarrow \mathbb{R}$  is continuous and  $c \in \mathbb{R}$  lies between  $f(a)$  and  $f(b)$ , that is  $f(a) \leq c \leq f(b)$  or  $f(b) \leq c \leq f(a)$ . Then there exists  $x_0 \in [a, b]$  such that  $f(x_0) = c$ .

In other words, a continuous function on  $[a, b]$  attains every value between  $f(a)$  and  $f(b)$  at least once — there are no gaps in  $f([a, b])$ .



An important **special case** of the Theorem is the following: If  $f: [a, b] \rightarrow \mathbb{R}$  is continuous and  $f(a) < 0$  and  $f(b) > 0$  (resp.  $f(a) > 0$  and  $f(b) < 0$ ), then  $f$  has a *zero* (Nullstelle) in  $[a, b]$ , i.e.,  $\exists x_0 \in [a, b]: f(x_0) = 0$ .

**EXAMPLE OF AN APPLICATION:** Let  $p: \mathbb{R} \rightarrow \mathbb{R}$  be a polynomial function of odd degree  $m = 2n + 1$  (with  $n \in \mathbb{N}$ ), say,

$$p(x) = b_{2n+1}x^{2n+1} + b_{2n}x^{2n} + \dots + b_0 \quad (x \in \mathbb{R}),$$

where  $b_{2n+1} \neq 0$ . Then  $p$  has at least one real zero.

To show this, we first write

$$p(x) = b_{2n+1} \cdot \left( x^{2n+1} + \frac{b_{2n}}{b_{2n+1}}x^{2n} + \dots + \frac{b_0}{b_{2n+1}} \right) = b_{2n+1} \cdot q(x),$$

where the polynomial function  $q$  is of the form  $q(x) = x^{2n+1} + a_{2n}x^{2n} + \dots + a_0$  (with  $a_j = b_j/b_{2n+1}$  for  $j = 0, \dots, 2n$ ).

By 5.7, Example 3), we have  $\lim_{x \rightarrow \infty} q(x) = \infty$ , hence there exists  $x_+ > 0$  such that  $q(x_+) > 0$ . Similarly, upon observing that

$$q(-x) = -x^{2n+1} + a_{2n}x^{2n} - \dots + a_0 = -(x^{2n+1} - a_{2n}x^{2n} + \dots - a_0)$$

we obtain that  $\lim_{x \rightarrow -\infty} q(x) = -\infty$ , hence there exists  $x_- < 0$  such that  $q(x_-) < 0$ .

Since  $q|_{[x_-, x_+]}: [x_-, x_+] \rightarrow \mathbb{R}$  is continuous and  $q(x_-) < 0 < q(x_+)$ , the above Theorem implies that there exists  $x_0 \in [x_-, x_+]$  such that  $q(x_0) = 0$  (in fact,  $x_- < x_0 < x_+$ , because the values of  $q$  at  $x_{\pm}$  are known to be nonzero). Therefore also  $p(x_0) = b_{2n+1}q(x_0) = 0$ .

*Proof of the Theorem.*

WLOG (:= without loss of generality) (OBdA (:= ohne Beschränkung der Allgemeinheit)) we may assume that  $f(a) < c < f(b)$ . [otherwise we just have to consider  $-f$  instead]

If  $c \neq 0$  we can reduce the statement to that of the special case of a zero by putting  $f_1(x) := f(x) - c$ . Then  $f_1(a) = f(a) - c < 0$  and  $f_1(b) = f(b) - c > 0$  and the assertion of the Theorem is equivalent to the existence of a zero  $x_0 \in [a, b]$  of the function  $f_1$ .

So, again WLOG we may assume that  $c = 0$  and  $f(a) < 0 < f(b)$ . We have to show that there is some  $x_0 \in [a, b]$  such that  $f(x_0) = 0$ .

We will find  $x_0$  by constructing a sequence of nested intervals in the fashion of a so-called *bisection method*. To be more precise, we claim that we can define  $[a_n, b_n] \subseteq [a, b]$  for all  $n \in \mathbb{N}$  with the following properties:

1.  $\forall n \in \mathbb{N}, n \geq 1: [a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$
2.  $b_n - a_n = \frac{b - a}{2^n}$
3.  $f(a_n) < 0$  and  $f(b_n) \geq 0$ .

Put  $a_0 := a$  and  $b_0 := b$ , then properties 2 and 3 are satisfied. We proceed inductively and assume that  $[a_0, b_0], \dots, [a_n, b_n]$  have been defined satisfying properties 1-3. Let  $m := (a_n + b_n)/2$  (this is the midpoint of  $[a_n, b_n]$ ) and distinguish two cases:

If  $f(m) \geq 0$  put  $a_{n+1} := a_n$  and  $b_{n+1} := m$

If  $f(m) < 0$  put  $a_{n+1} := m$  and  $b_{n+1} := b_n$ .

Then properties 1-3 are valid for  $[a_{n+1}, b_{n+1}]$  as well. By the principle of nested intervals we obtain that  $(a_n)$  and  $(b_n)$  converge to the same limit  $x_0 \in [a, b]$ , that is  $\lim a_n = \lim b_n = x_0$ . Since  $f$  is continuous we have that

$$f(x_0) = \lim f(a_n) = \lim f(b_n).$$

By property 3 we obtain in addition

$$f(x_0) = \lim f(a_n) \leq 0 \leq \lim f(b_n) = f(x_0),$$

which proves that  $f(x_0) = 0$ . □

**COROLLARY:** Let  $I \subseteq \mathbb{R}$  be a nonempty interval and  $f: I \rightarrow \mathbb{R}$  be continuous. Then  $f(I) \subseteq \mathbb{R}$  is an interval as well.

*Proof.* Let  $A := \inf f(I)$  and  $B := \sup f(I)$ , where we allow for the improper values  $A = -\infty$  (unbounded below) and  $B = \infty$  (unbounded above). If  $A = B$  then  $f(I)$  contains just a single point, in which case the statement is true. So we henceforth assume that  $A < B$ .

We assert that  $]A, B[ \subseteq f(I)$ : Let  $y \in ]A, B[$ , then there exist  $r, s \in I$  such that  $f(r) < y < f(s)$ . By the above Theorem we have some  $x_0 \in I$  such  $f(x_0) = y$ . Therefore  $y \in f(I)$ .

To summarize,  $]A, B[ \subseteq f(I) \subseteq [A, B]$ , hence  $f(I)$  equals one of the intervals  $]A, B[$  or  $]A, B]$  or  $[A, B[$  or  $[A, B]$ . □

## 5.9. Continuous functions on bounded closed intervals:

**DEFINITION:** A function  $f: D \rightarrow \mathbb{R}$  is called *bounded* (*beschränkt*) if the image set  $f(D) \subseteq \mathbb{R}$  is bounded, i.e.,

$$\exists M > 0 \quad \forall x \in D : |f(x)| \leq M.$$

**THEOREM:** Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f$  is bounded and attains maximum and minimum values, i.e., there exist  $x_1, x_2 \in [a, b]$  such that

$$\begin{aligned} f(x_1) &= \min f([a, b]) = \min \{f(x) : x \in [a, b]\} && (= \inf f([a, b])) \\ f(x_2) &= \max f([a, b]) = \max \{f(x) : x \in [a, b]\} && (= \sup f([a, b])). \end{aligned}$$

**REMARK:** (i) In the hypothesis of this theorem it is essential that the interval  $[a, b]$ , where  $f$  is defined and continuous, is bounded (i.e.,  $-\infty < a \leq b < \infty$ ) and closed (i.e., the boundary points  $a$  and  $b$  belong to the interval). Otherwise the statement is not true in general as can be seen from the following examples: Consider the continuous functions

$$f_1: ]0, 1[ \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}, \quad f_2: ]0, 1[ \rightarrow \mathbb{R}, x \mapsto x, \quad f_3: [0, \infty[ \rightarrow \mathbb{R}, x \mapsto x.$$

Then  $f_1$  and  $f_3$  are unbounded and do not attain a maximum,  $f_2$  does neither attain a maximum nor a minimum.

(ii) As is shown by the simple example of a constant function, the locations  $x_1$  and  $x_2$  of a minimum or a maximum need not be unique.

*Proof of the Theorem.* It suffices to give the proof for boundedness from above and concerning the maximum, the case of minimum and boundedness from below can be reduced to the latter by switching to  $-f$ .

Let  $A := \sup f([a, b]) \in \mathbb{R} \cup \{\infty\}$ , then there exists a sequence  $(a_n)$  in  $[a, b]$  such that  $f(a_n) \rightarrow A$  ( $n \rightarrow \infty$ ).

Since  $[a, b]$  is a bounded subset of  $\mathbb{R}$  the Theorem of Bolzano-Weierstraß implies that there is a convergent subsequence  $(a_{n_k})_{k \in \mathbb{N}}$ . Let  $x_2 := \lim_{k \rightarrow \infty} a_{n_k} \in [a, b]$ .

Since  $f$  is continuous we obtain that

$$\mathbb{R} \ni f(x_2) = \lim_{k \rightarrow \infty} f(a_{n_k}) = A = \sup f([a, b]).$$

In particular,  $f([a, b])$  is bounded above and the supremum is a maximum, which is attained by  $f$  at  $x_2 \in [a, b]$ .  $\square$

**5.10. Uniform continuity:** If we are to check continuity of a real valued function  $f$  at a certain point  $x_0$  in its domain  $D$ , then for given  $\varepsilon > 0$  we have to find  $\delta > 0$  such that the condition  $|f(x) - f(x_0)| < \varepsilon$  is met whenever  $x \in D$  satisfies  $|x - x_0| < \delta$ . We observe that, in general,  $\delta$  will depend on  $\varepsilon > 0$  as well as on the point  $x_0$ . Consider the following example, where the range of possible values for  $\delta$  is strictly shrinking as  $\varepsilon$  gets smaller or  $x_0$  varies: Let  $D = ]0, 1]$  and  $f: D \rightarrow \mathbb{R}$  with  $f(x) = 1/x$ , which is continuous in every point  $x_0 \in D$ .

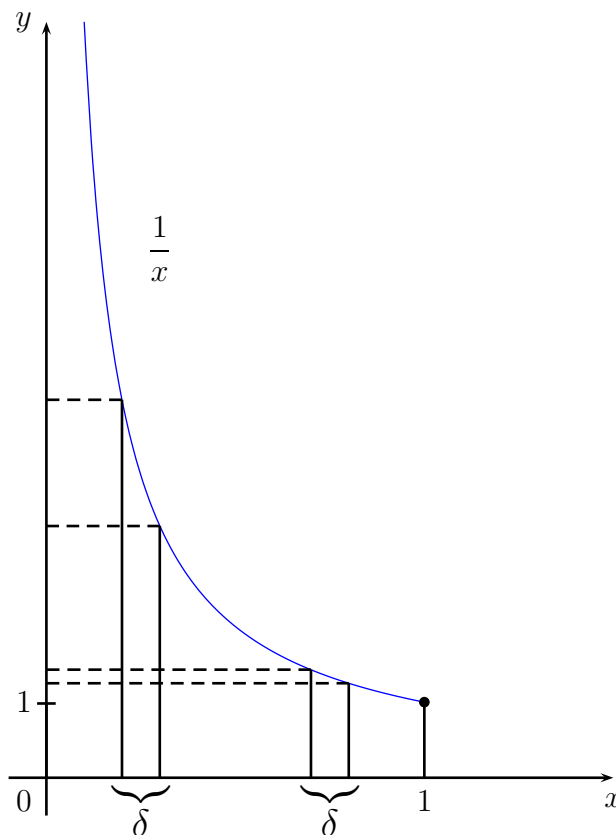
Fix some  $x_0 \in D$  and  $\varepsilon > 0$  arbitrarily and let us test the allowed tolerance in varying the argument in  $0 < x \leq x_0$  while maintaining  $|f(x) - f(x_0)| < \varepsilon$ . For every  $0 < \delta < x_0$  let  $x_\delta := x_0 - \delta$ . Then

$$\begin{aligned} |f(x_\delta) - f(x_0)| &= \frac{1}{x_\delta} - \frac{1}{x_0} \\ &= \frac{x_0 - x_\delta}{x_0 x_\delta} = \frac{\delta}{x_0(x_0 - \delta)}. \end{aligned}$$

Thus requiring  $|f(x) - f(x_0)| < \varepsilon$  for all  $x \in ]0, 1]$  with  $|x - x_0| < \delta$  implies  $\varepsilon > \frac{\delta}{x_0^2 - x_0\delta}$ . Equivalently,  $\varepsilon x_0^2 - \varepsilon x_0\delta > \delta$  and hence

$$\delta < \frac{\varepsilon x_0^2}{1 + \varepsilon x_0} < \varepsilon x_0^2.$$

This shows that the smaller  $x_0 > 0$  is the smaller we have to choose  $\delta > 0$  (even at fixed value of  $\varepsilon > 0$ ).



We thus obtain a stronger form of continuity notion, if we require that for each  $\varepsilon > 0$  a suitable  $\delta > 0$  can be found which guarantees the typical  $\varepsilon$ - $\delta$ -estimate to hold for all pairs of points in the domain of relative distance less than  $\delta$ .

**DEFINITION:** Let  $D \subseteq \mathbb{R}$ . A function  $f: D \rightarrow \mathbb{R}$  is *uniformly continuous* (*gleichmäßig stetig*) (in  $D$ ), if the following holds:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, x' \in D : |x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon.$$

**REMARK:** It is immediate from the definition that every uniformly continuous function is continuous (at every point in the domain). The converse is not true as is illustrated by the example above with  $D = ]0, 1]$ ,  $f(x) = 1/x$ : If  $x_n = 1/n$  and  $x'_n = 1/(2n)$  ( $n \in \mathbb{N}$ ,  $n \geq 1$ ) then

$$|x_n - x'_n| = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$$

is arbitrarily small when  $n$  is sufficiently large, but

$$|f(x_n) - f(x'_n)| = 2n - n = n$$

is unbounded, hence will never stay below a given  $\varepsilon$ -tolerance.

However, as the following theorem will show, there is no distinction between continuity and uniform continuity if  $D$  is a bounded closed interval.

**THEOREM:** If  $f: [a, b] \rightarrow \mathbb{R}$  is continuous then  $f$  is uniformly continuous (on  $[a, b]$ ).

*Proof.* (by contradiction) If  $f$  is not uniformly continuous then

$$\exists \varepsilon > 0 \forall n \in \mathbb{N}, n > 0 \exists x_n, x'_n \in [a, b] : |x_n - x'_n| < \frac{1}{n} \text{ and } |f(x_n) - f(x'_n)| \geq \varepsilon.$$

The sequence  $(x_n)$  is bounded, thus by the Theorem of Bolzano-Weierstraß possesses a convergent subsequence  $(x_{n_k})_{k \in \mathbb{N}}$ . Let  $x_0 := \lim x_{n_k} \in [a, b]$ .

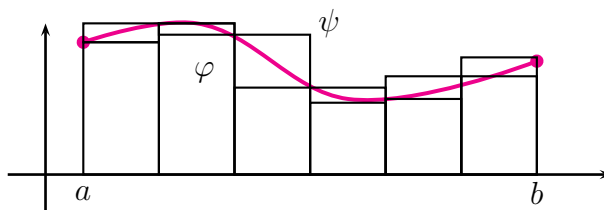
Since  $|x_{n_k} - x'_{n_k}| < 1/n_{n_k} \rightarrow 0$  ( $k \rightarrow \infty$ ) we have that  $\lim x'_{n_k} = \lim x_{n_k} = x_0$ . Then the continuity of  $f$  at  $x_0$  yields

$$0 < \varepsilon \leq |f(x_{n_k}) - f(x'_{n_k})| \rightarrow 0 \quad (k \rightarrow \infty),$$

— a contradiction  $\zeta$ .

□

**5.11. Approximation by step (or simple) functions:** As an application of the above Theorem 5.10 we show that “the area under the graph of a continuous function” can be approximated by sums over areas of small vertical rectangles. This will be used later in the chapter on integration theory.



The rectangles can be represented as graphs of step or simple functions and the approximation result is stated in terms of these as a uniform approximation from above and from below.

**PROPOSITION:** Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous. For every  $\varepsilon > 0$  there exist simple functions  $\varphi, \psi: [a, b] \rightarrow \mathbb{R}$  with the following properties valid for all  $x \in [a, b]$ :

- (a)  $\varphi(x) \leq f(x) \leq \psi(x)$
- (b)  $|\psi(x) - \varphi(x)| = \psi(x) - \varphi(x) \leq \varepsilon$ .

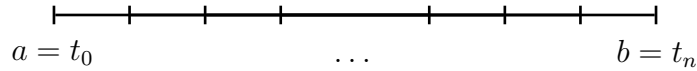
*Proof.* By Theorem 5.10  $f$  is uniformly continuous on  $[a, b]$ . Therefore we can find  $\delta > 0$  such that

$$\forall x, x' \in [a, b] : |x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon.$$

Choose  $n \in \mathbb{N}$  large enough to ensure  $(b - a)/n < \delta$  and define partition points

$$t_k := a + k \cdot \frac{b - a}{n} \quad (k = 0, \dots, n).$$

In this way we obtain an equidistant partition of  $[a, b]$



by  $t_0 = a < t_1 < \dots < t_n = b$  with

$$t_k - t_{k-1} = \frac{b - a}{n} < \delta.$$

As heights of the approximating rectangles we choose the maximum or minimum values of  $f$  on the corresponding subintervals  $[t_{k-1}, t_k]$  ( $k = 1, \dots, n$ ) of the partition, that is we define

$$c_k := \max\{f(x) : t_{k-1} \leq x \leq t_k\}, \quad c'_k := \min\{f(x) : t_{k-1} \leq x \leq t_k\}.$$

By Theorem 5.9 there exist  $\xi_k, \xi'_k \in [t_{k-1}, t_k]$  such that  $f(\xi_k) = c_k$  and  $f(\xi'_k) = c'_k$  ( $k = 1, \dots, n$ ). Since  $|\xi_k - \xi'_k| < \delta$  we have  $|c_k - c'_k| < \varepsilon$  from the uniform continuity property noted in the beginning.

Finally, we define the simple functions  $\varphi, \psi: [a, b] \rightarrow \mathbb{R}$  as follows:

Let  $\varphi(a) := f(a)$  and  $\psi(a) := f(a)$ ,

for  $t_{k-1} < x \leq t_k$  we set  $\psi(x) := c_k$  and  $\varphi(x) := c'_k$  ( $k = 1, \dots, n$ ).

Then the conditions (a) and (b) follow by construction of  $\varphi$  and  $\psi$ . □

**5.12. Continuous inverse function:** Let  $A, B \subseteq \mathbb{R}$ . Assume that the function  $f: A \rightarrow B$  is bijective, then the inverse function  $f^{-1}: B \rightarrow A$  exists. If we know that  $f$  is continuous, does this imply that  $f^{-1}$  is also continuous? In general, the answer is ‘no’ (see the exercises for an example).

It turns out that there is a positive answer to the above question under the two additional hypotheses of strict monotonicity on  $f$  and that  $A$  is an interval.

Recall that  $f$  is strictly increasing (resp. decreasing) if  $x_1 < x_2$  implies  $f(x_1) < f(x_2)$  (resp.  $f(x_1) > f(x_2)$ ) and that a strictly monotone function necessarily is injective.

**THEOREM:** Let  $I \subseteq \mathbb{R}$  be an interval and  $f: I \rightarrow \mathbb{R}$  be continuous and strictly increasing (resp. decreasing). Then  $f$  maps the interval  $I$  bijectively onto the interval  $J := f(I)$  and the corresponding inverse function  $J \rightarrow I$  is also continuous and strictly increasing (resp. decreasing).

(Strictly speaking, we deal here with the inverse of the map  $\tilde{f}: I \rightarrow J, x \mapsto f(x)$ ; but we will follow the common abuse of language and denote  $\tilde{f}$  again by  $f$  and its inverse by  $f^{-1}: J \rightarrow I$ .)

*Proof.* We present the proof for the case that  $f$  is strictly increasing, the case where  $f$  is strictly decreasing is reduced to this by considering  $-f$  instead.

Corollary 5.8 implies that  $J = f(I)$  is an interval. Since  $f$  is strictly increasing it is injective, hence  $f$  is bijective as a map  $I \rightarrow J$ . Let  $f^{-1}: J \rightarrow I$  denote the inverse of this map.

Note that for  $x_1, x_2 \in I$  the inequality  $f(x_1) < f(x_2)$  in turn implies  $x_1 < x_2$  (since then  $x_1 = x_2$  is impossible with different function values and  $x_1 > x_2$  contradicts the fact that  $f$  increases), therefore we have

$$\forall x_1, x_2 \in I : \quad x_1 < x_2 \iff f(x_1) < f(x_2),$$

which shows that  $f^{-1}$  is strictly increasing as well.

It remains to prove that  $f^{-1}$  is continuous at every point  $b \in J$ .

*Case 1,* if  $b \in J$  is not a boundary point of  $J$ : Let  $a := f^{-1}(b) \in I$ . Then  $a$  is not a boundary point of  $I$  (for otherwise by monotonicity  $b$  would have to be boundary point of  $J$ ). Choose  $\varepsilon > 0$  so small that both  $a - \varepsilon$  and  $a + \varepsilon$  belong to  $I$ . Since  $f$  is strictly increasing we have

$$f(a - \varepsilon) < f(a) = b < f(a + \varepsilon).$$

Thus we can find  $\delta > 0$  such that

$$f(a - \varepsilon) < b - \delta < b + \delta < f(a + \varepsilon),$$

which simply means that

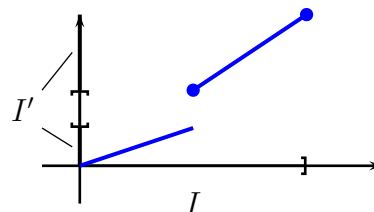
$$f^{-1}(U_\delta(b)) \subseteq U_\varepsilon(f^{-1}(b))$$

and therefore proves the continuity of  $f^{-1}$  at  $b$ .

*Case 2*, if  $b \in J$  is the left boundary point of  $J$ : Then  $a := f^{-1}(b)$  has to be the left boundary point of  $I$  (since  $f$  is strictly increasing). We can copy the proof of case 1 with the only changes that we use  $U_\delta \cap J$  and  $U_\varepsilon(f^{-1}(b)) \cap I$  as neighborhoods and the chain of inequalities reads  $f(a) = b < b + \delta < f(a + \varepsilon)$ .

*Case 3*, if  $b \in J$  is the right boundary point of  $J$ : Similarly to case 2.  $\square$

**REMARK:** The second part of the above proof shows, in fact, the following result: If  $I \subseteq \mathbb{R}$  is an interval and  $f: I \rightarrow \mathbb{R}$  is strictly increasing (not necessarily continuous!), then  $f^{-1}: f(I) \rightarrow I$  is continuous. But  $f(I)$  need not be an interval, if  $f$  is discontinuous:



**ROOT FUNCTIONS:** As an application of the above Theorem we consider for any  $k \in \mathbb{N}$ ,  $k \geq 1$ , the functions<sup>4</sup>

$$f_{2k}: [0, \infty[ \rightarrow [0, \infty[, x \mapsto x^{2k}, \quad \text{and} \quad f_{2k+1}: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^{2k+1}.$$

All these functions are continuous, strictly increasing, and bijective, therefore the corresponding inverse functions

$$f_{2k}^{-1}: [0, \infty[ \rightarrow [0, \infty[ \quad \text{and} \quad f_{2k+1}^{-1}: \mathbb{R} \rightarrow \mathbb{R}$$

are continuous and strictly increasing. We use the the following notation for their function values (for  $x$  in the appropriate domain)

$$n \geq 2: \quad \sqrt[n]{x} = x^{\frac{1}{n}} := f_n^{-1}(x).$$

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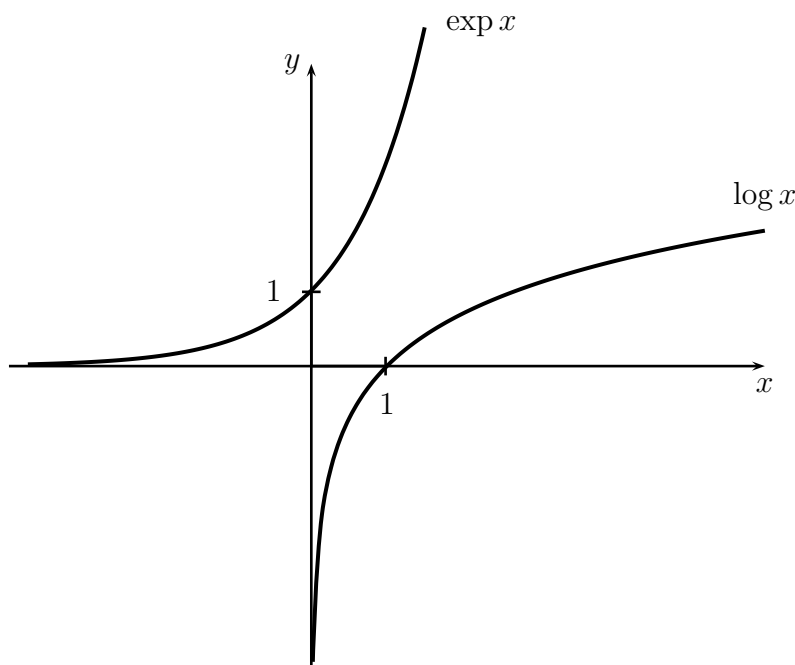
<sup>4</sup>Although the origin of the radical symbol  $\sqrt{\phantom{x}}$  is rather unclear, many believe that it is an abbreviation of the Latin word *radix* (root). The symbol was first used in Germany in the 16th century without the *winkulum* (i.e. the term  $\sqrt{a+b}$  was originally denoted by  $\sqrt{(a+b)}$ )



## §6. ELEMENTARY TRANSCENDENTAL FUNCTIONS

**6.1. Proposition:** The exponential function  $\exp: \mathbb{R} \rightarrow \mathbb{R}$  is continuous, strictly increasing, and  $\exp(\mathbb{R}) = ]0, \infty[$ . Its inverse function  $\log: ]0, \infty[ \rightarrow \mathbb{R}$  is continuous, strictly increasing and is called the *natural logarithm* (*natürlicher Logarithmus*).<sup>1</sup> Furthermore, the following functional equation holds for all  $x, y \in ]0, \infty[$ :

$$(6.1) \quad \log(x \cdot y) = \log(x) + \log(y).$$



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<sup>1</sup>Logarithms have been introduced by the Scottish mathematician John Napier in 1614, the term logarithm being derived from the Greek words λόγος (“proportion”) and ἀριθμός (“number”).

Nowadays, there are different notations used for logarithmic functions: While mathematicians often write  $\log(x)$  for the natural logarithm and  $\log_b(x)$  for the base- $b$  logarithm, in many calculus textbooks a notations such as  $\ln(x)$  can be found for the natural logarithm,  $\lg(x)$  for the base-10 logarithm etc. In these lecture notes we will always use  $\log(x)$  to denote the natural logarithm and  $\log_b(x)$  for the base- $b$  logarithm.

*Proof.* The continuity of  $\exp$  has already been established in the previous section.

*Step 1:* We show that  $\exp$  is strictly increasing.

For every  $\xi > 0$  we have

$$\exp(\xi) = 1 + \xi + \sum_{k=2}^{\infty} \frac{\xi^k}{k!} > 1 + \xi > 1.$$

Let  $x_1 < x_2$  then  $\xi := x_2 - x_1 > 0$  and

$$\exp(x_2) = \exp(x_1 + \xi) = \exp(x_1) \cdot \exp(\xi) > \exp(x_1).$$

*Step 2:* We show that  $\exp(\mathbb{R}) = ]0, \infty[$ .

Since  $\exp(x) > 0$  for all  $x \in \mathbb{R}$  [4.23] we have  $\exp(\mathbb{R}) \subseteq ]0, \infty[$ . To show the reverse inclusion relation, it suffices to show that

$$\lim_{n \rightarrow \infty} \exp(n) = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \exp(-n) = 0,$$

since then by the intermediate value theorem all values in  $]0, \infty[$  are indeed attained.

For  $n \in \mathbb{N}$  we had shown  $\exp(n) = e^n$ . Since  $e > 2$  we therefore have  $e^n \rightarrow \infty$  ( $n \rightarrow \infty$ ), which implies that

$$\exp(-n) = \frac{1}{\exp(n)} = \frac{1}{e^n} \rightarrow 0 \quad (n \rightarrow \infty).$$

We may thus define the function  $f: \mathbb{R} \rightarrow ]0, \infty[$ ,  $f(x) := \exp(x)$ , which is again continuous and strictly increasing. Due to Theorem 5.12 the inverse function  $\log := f^{-1}: ]0, \infty[ \rightarrow \mathbb{R}$  is also continuous and strictly increasing.

*Step 3:* We prove the functional equation (6.1).

Let  $x, y \in ]0, \infty[$  and put  $\xi := \log(x)$ ,  $\eta := \log(y)$ . Then  $\exp(\xi + \eta) = \exp(\xi) \cdot \exp(\eta) = x \cdot y$  and therefore

$$\log(x \cdot y) = \xi + \eta = \log(x) + \log(y).$$

□

As a simple consequence we obtain

$$\log(x^k) = k \log(x) \quad \text{for all } x > 0 \text{ and } k \in \mathbb{N}.$$

**6.2. Real powers and general exponentials:** We can use the exponential function and the logarithm to give a simple definition of expressions of the form  $r^s$  when  $r > 0$  and  $s \in \mathbb{R}$ . Observe that if  $s$  is a natural number then  $r^s = \exp(\log(r^s)) = \exp(s \log(r))$ .

**DEFINITION:** (i) Let  $r > 0$  and  $s \in \mathbb{R}$  then

$$r^s := \exp(s \log(r)) \in ]0, \infty[.$$

As an immediate consequence of this definition we thus obtain the formula

$$\log(r^s) = s \log(r) \quad (r > 0, s \in \mathbb{R}).$$

(ii) For any  $\alpha \in \mathbb{R}$  we define general power or root functions  $w_\alpha: ]0, \infty[ \rightarrow \mathbb{R}$  by

$$x \mapsto x^\alpha = \exp(\alpha \log(x)).$$

(iii) The Exponential function with base  $a \in ]0, \infty[$  is given by

$$\exp_a: \mathbb{R} \rightarrow \mathbb{R}, \quad \exp_a(x) := a^x = \exp(x \log(a)).$$

Note that  $\exp(x) = \exp_e(x) = e^x$  for all  $x \in \mathbb{R}$ .

We list the basic properties of exponential function with base  $a > 0$ , which are immediate consequences of those for the exponential function and the logarithm.

**PROPOSITION:**  $\exp_a$  is continuous on  $\mathbb{R}$  and we have the following:

(i) If  $a > 1$  then  $\exp_a$  is strictly increasing, if  $0 < a < 1$  then  $\exp_a$  is strictly decreasing.

(ii) The functional equation:  $a^{x+y} = a^x \cdot a^y$  for all  $x, y \in \mathbb{R}$ .

(iii) Let  $a > 0$ . For all  $m \in \mathbb{Z}$ :  $\exp_a(m) = a^m = a \cdot a \cdots a$  ( $m$  factors).

(In other words, the notation  $a^m$  is consistent with the algebraically defined integer powers.)

(iv) Let  $a > 0$ . If  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$ ,  $q \geq 1$ , then  $a^{\frac{p}{q}} = \sqrt[q]{a^p} = (a^p)^{\frac{1}{q}}$ .

(Consistency with the root functions as defined in 5.12.)

(v) Let  $a > 0$ . For all  $x, y \in \mathbb{R}$ :  $(a^x)^y = a^{xy} = (a^y)^x$ .

(vi) For all  $a > 0$ ,  $b > 0$ , and  $x \in \mathbb{R}$ :  $a^x \cdot b^x = (a \cdot b)^x$ .

(vii) Let  $a > 0$ . For all  $x \in \mathbb{R}$ :  $(\frac{1}{a})^x = a^{-x}$ .

*Proof.* Immediate from the definition. □

### 6.3. A collection of useful limits:

1) For all  $k \in \mathbb{N}$ :  $\lim_{x \rightarrow \infty} \frac{e^x}{x^k} = \infty$ .

We may assume that  $x > 0$ , which yields  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > \frac{x^{k+1}}{(k+1)!}$ . Therefore

$$\frac{e^x}{x^k} > \frac{x}{(k+1)!} \rightarrow \infty \quad \text{as } x \rightarrow \infty.$$

2) For all  $k \in \mathbb{N}$ :  $\lim_{x \rightarrow \infty} \frac{x^k}{e^x} = 0$ . (Follows directly from 1).)

3) For all  $k \in \mathbb{N}$ :  $\lim_{x \searrow 0} x^k e^{1/x} = \infty$ .

Writing  $y = 1/x$  gives  $\lim_{x \searrow 0} x^k e^{1/x} = \lim_{y \rightarrow \infty} \frac{e^y}{y^k} = \infty$  (by 1)).

4)  $\lim_{x \rightarrow \infty} \log(x) = \infty$  and  $\lim_{x \searrow 0} \log(x) = -\infty$ .

Both assertions follows from Proposition 6.1, which implies that  $\log: ]0, \infty[ \rightarrow \mathbb{R}$  is strictly increasing and bijective.

5) For all  $\alpha > 0$ :  $\lim_{x \searrow 0} x^\alpha = 0$  and  $\lim_{x \searrow 0} x^{-\alpha} = \infty$ .

The second assertion follows from the first. To prove the first we write  $x = e^{-y/\alpha}$  (equivalently,  $y = -\alpha \log(x)$ ) and compute

$$\lim_{x \searrow 0} x^\alpha = \lim_{y \rightarrow \infty} e^{-y} = 0.$$

6) For all  $\alpha > 0$ :  $\lim_{x \rightarrow \infty} \frac{\log(x)}{x^\alpha} = 0$ .

We may assume that  $x > 0$  and write  $x^\alpha = e^y$  (equivalently,  $y = \alpha \log(x)$ ) to obtain

$$\lim_{x \rightarrow \infty} \frac{\log(x)}{x^\alpha} = \frac{1}{\alpha} \cdot \lim_{y \rightarrow \infty} \frac{y}{e^y} = 0.$$

7) For all  $\alpha > 0$ :  $\lim_{x \searrow 0} x^\alpha \log(x) = 0$ .

Upon writing  $x = 1/y$  (so that  $y \rightarrow \infty$ ) we have  $x^\alpha \log(x) = -\log(y)/y^\alpha$ , then use 6).

8)  $\lim_{\substack{x \rightarrow 0 \\ x \neq 0}} \frac{e^x - 1}{x} = 1$

The remainder term estimate (4.4), (4.5) for the exponential sum gives for all  $x$  with  $|x| \leq 3/2$  that

$$|e^x - (1 + x)| \leq 2 \frac{|x|^2}{2!} = |x|^2.$$

In other words, if  $0 < |x| \leq 3/2$  then

$$\left| \frac{e^x - 1}{x} - 1 \right| = \frac{|e^x - 1 - x|}{|x|} \leq |x| \rightarrow 0 \quad \text{as } x \rightarrow 0.$$

## 6.4. The Landau-symbols<sup>2</sup> — comparison of asymptotic growth:

**DEFINITION:** (i) Let  $a \in \mathbb{R}$  and  $f, g: ]a, \infty[ \rightarrow \mathbb{R}$ . We write

$$f(x) = o(g(x)) \quad (x \rightarrow \infty),$$

to mean that  $\forall \varepsilon \exists R > a: |f(x)| \leq \varepsilon |g(x)|$  holds for all  $x \geq R$ . (“ $f(x)$  is a little-oh of  $g(x)$  as  $x$  tends to infinity”.)

We write

$$f(x) = O(g(x)) \quad (x \rightarrow \infty),$$

to mean that  $\exists K > 0 \exists R > a: |f(x)| \leq K |g(x)|$  holds for all  $x \geq R$ . (“ $f(x)$  is a big-oh of  $g(x)$  as  $x$  tends to infinity”.)

(ii) Let  $D \subseteq \mathbb{R}$  and  $x_0$  be an adherent point of  $D$ . We write

$$f(x) = o(g(x)) \quad (x \rightarrow x_0, x \in D),$$

to mean that  $\forall \varepsilon > 0 \exists \delta > 0: |f(x)| \leq \varepsilon |g(x)|$  holds  $\forall x \in U_\delta(x_0) \cap D$ .

We write

$$f(x) = O(g(x)) \quad (x \rightarrow x_0, x \in D),$$

to mean that  $\exists K > 0 \exists \delta > 0: |f(x)| \leq K |g(x)|$  holds  $\forall x \in U_\delta(x_0) \cap D$ .

**REMARK:** (i) If, for example,  $g(x) \neq 0$  for all  $x$  near  $x_0$  and  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$ , then  $f(x) = o(g(x))$  ( $x \rightarrow x_0$ ).

(ii) We will occasionally make use of a notation like

$$f(x) = h(x) + O(g(x))$$

to mean that  $f(x) - h(x) = O(g(x))$ .

**EXAMPLES:** 1) If  $\alpha > 0$  then  $\log(x) = o(x^\alpha)$  ( $x \rightarrow \infty$ ) [cf. 6.3.6].

2)  $e^x = 1 + x + O(x^2)$  ( $x \rightarrow 0$ ), since  $|e^x - 1 - x| \leq |x|^2$  when  $|x| \leq 3/2$  [as seen in 6.3.8].

3)  $f(x) = f(x_0) + o(1)$  ( $x \rightarrow x_0$ )  $\iff \lim_{x \rightarrow x_0} f(x) = f(x_0)$   $\iff f$  is continuous at  $x_0$ .

4) If  $p$  is a polynomial function of degree  $m$ , then  $p(x) = O(x^m)$  ( $x \rightarrow \infty$ ).

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<sup>2</sup>Edmund Georg Hermann Landau (1877–1938) [ˈɛdmʊnt ˈgɛrəɛk ˈhɛrman ˈlandəʊ], German mathematician

## 6.5. A digression into basic analysis on $\mathbb{C}$ :

Let  $z = x + iy \in \mathbb{C}$ , so that  $x = \operatorname{Re}(z)$ ,  $y = \operatorname{Im}(z)$  and  $\bar{z} = x - iy \in \mathbb{C}$ . Then the product  $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$  always gives a non-negative real number and we may define the *absolute value* of  $z$  by

$$|z| := \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}.$$

Note that, since real numbers  $x$  are embedded as complex numbers<sup>3</sup> of the form  $x + i0$ , the absolute value of  $x$  as a real number is the same as its absolute value as a complex number.

**LEMMA:** The absolute value as a map  $|\cdot|: \mathbb{C} \rightarrow \mathbb{R}$  has the following properties, valid for all  $z, z_1, z_2 \in \mathbb{C}$ :

- (i)  $|z| \geq 0$  and  $|z| = 0 \Leftrightarrow z = 0$
- (ii)  $|\bar{z}| = |z|$
- (iii)  $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- (iv)  $|\operatorname{Re}(z)| \leq |z|$  and  $|\operatorname{Im}(z)| \leq |z|$
- (v)  $|z_1 + z_2| \leq |z_1| + |z_2|$  (triangle inequality).

*Proof.* Let  $z = x + iy$ ,  $z_k = x_k + iy_k$  ( $k = 1, 2$ ).

(i):  $|z| \geq 0$  and  $|0| = 0$  is immediate. If  $|z| = 0$ , then  $0 \leq x^2 \leq x^2 + y^2 = 0$  as well as  $0 \leq y^2 \leq x^2 + y^2 = 0$ , hence  $x = 0$  and  $y = 0$ .

(ii): Clear from the definition.

(iii):  $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2$ .

(iv):  $|x|^2 = x^2 \leq x^2 + y^2$  and  $|y|^2 = y^2 \leq x^2 + y^2$ .

(v):  $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + 2 \operatorname{Re}(z_1 \bar{z}_2) + |z_2|^2 \leq$  [by (iv)]  
 $|z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$ . □

### (a) Convergence in $\mathbb{C}$ :

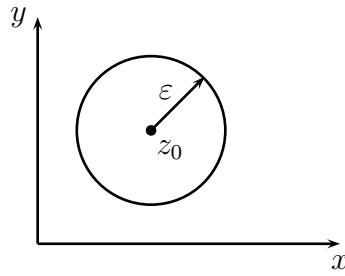
**DEFINITION:** (i) Let  $z_0 \in \mathbb{C}$  and  $\varepsilon > 0$ , then the  $\varepsilon$ -neighborhood  $U_\varepsilon(z_0)$  of  $z_0$  is defined by

$$U_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}.$$

In a planar representation of the complex numbers,  $U_\varepsilon(z_0)$  is an open disk with radius  $\varepsilon$  around  $z_0$ :

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<sup>3</sup>Square roots of negative numbers were “invented” by the Italian mathematicians Gerolamo Cardano and Raffaele Bombelli. In modern mathematics, complex numbers are generally denoted by  $z = a + bi$  or sometimes  $z = a + bj$ .



(ii) A *sequence of complex numbers* is a map  $c: \mathbb{N} \rightarrow \mathbb{C}$ . We use the notation  $(c_n)_{n \in \mathbb{N}}$  with  $c_n := c(n)$ .

(iii) The complex sequence  $(c_n)$  *converges to*  $z_0 \in \mathbb{C}$ , denoted by  $c_n \rightarrow z_0$  ( $n \rightarrow \infty$ ) or  $\lim_{n \rightarrow \infty} c_n = z_0$ , if

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 : |c_n - z_0| < \varepsilon.$$

Equivalently, we may require that

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 : c_n \in U_\varepsilon(z_0).$$

**PROPOSITION:** Let  $(c_n)$  be a complex sequence. Then the following are equivalent:

- (i)  $(c_n)$  is convergent (in  $\mathbb{C}$ ).
- (ii) Both sequences  $(\operatorname{Re} c_n)$  and  $(\operatorname{Im} c_n)$  converge (in  $\mathbb{R}$ ).

In this case we have  $\lim c_n = \lim \operatorname{Re} c_n + i \lim \operatorname{Im} c_n$ .

*Proof.* Let  $a_n := \operatorname{Re} c_n$ ,  $b_n := \operatorname{Im} c_n$  ( $n \in \mathbb{N}$ ).

(i)  $\Rightarrow$  (ii): Let  $c := \lim c_n$ ,  $a := \operatorname{Re} c$  and  $b := \operatorname{Im} c$ .

If  $\varepsilon > 0$  is given arbitrarily, we can find  $n_0 \in \mathbb{N}$  such that  $|c_n - c| < \varepsilon$  holds for all  $n \geq n_0$ . Therefore we have for all  $n \geq n_0$

$$|a_n - a| = |\operatorname{Re}(c_n - c)| \leq |c_n - c| < \varepsilon \quad \text{as well as} \quad |b_n - b| = |\operatorname{Im}(c_n - c)| \leq |c_n - c| < \varepsilon,$$

which proves that  $a_n \rightarrow a$  and  $b_n \rightarrow b$ .

(ii)  $\Rightarrow$  (i): Let  $\varepsilon > 0$ . Put  $a := \lim a_n$ ,  $b := \lim b_n$ , and  $c := a + ib$ . Choose  $n_0 \in \mathbb{N}$  such that  $|a_n - a| < \varepsilon/2$  and  $|b_n - b| < \varepsilon/2$  holds for all  $n \geq n_0$ . Then we have for  $n \geq n_0$

$$|(a_n + ib_n) - (a + ib)| = |(a_n - a) + i(b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus  $c_n \rightarrow c$  ( $n \rightarrow \infty$ ).

□

**COROLLARY:** If  $(c_n)$  is a convergent complex sequence, then  $\lim \overline{c_n} = \overline{\lim c_n}$ .

*Proof.*  $\overline{\lim c_n} = \lim \operatorname{Re} c_n - i \lim \operatorname{Im} c_n = \lim \operatorname{Re} \overline{c_n} + i \lim \operatorname{Im} \overline{c_n} = \lim \overline{c_n}$ . □

Precisely as in the case of real sequences one proves the following rules for basic operations with convergent sequences:

If  $(c_n), (d_n)$  are convergent complex sequences and  $\lambda \in \mathbb{C}$ , then

$$\begin{aligned}\lim(c_n + d_n) &= \lim c_n + \lim d_n \\ \lim(\lambda c_n) &= \lambda \lim c_n \\ \lim(c_n d_n) &= (\lim c_n)(\lim d_n) \\ \lim \frac{c_n}{d_n} &= \frac{\lim c_n}{\lim d_n} \quad (\text{if } d_n \neq 0 \text{ for almost all } n).\end{aligned}$$

**THEOREM (COMPLETENESS OF  $\mathbb{C}$ ):** A sequence  $(c_n)$  of complex numbers converges if and only if it is a *Cauchy sequence*, i.e.,

$$(6.2) \quad \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \forall n, m \geq n_0 : |c_n - c_m| < \varepsilon.$$

*Proof.* (6.2)  $\Leftrightarrow$  both  $(\operatorname{Re} c_n)$  and  $(\operatorname{Im} c_n)$  are Cauchy sequences in  $\mathbb{R} \Leftrightarrow$  both  $(\operatorname{Re} c_n)$  and  $(\operatorname{Im} c_n)$  are convergent in  $\mathbb{R} \stackrel{[\text{Prop.}]}{\Leftrightarrow} (c_n)$  is convergent in  $\mathbb{C}$ . □

## (b) Complex series:

**DEFINITION:** Let  $(c_n)$  be a sequence of complex numbers. The series  $\sum_{k=0}^{\infty} c_k$  is *convergent*,

if the corresponding sequence  $(s_n)$  of partial sums  $s_n := \sum_{k=0}^n c_k$  is convergent (in  $\mathbb{C}$ ).

The series  $\sum_{k=0}^{\infty} c_k$  is *absolutely convergent*, if the (real) series  $\sum_{k=0}^{\infty} |c_k|$  converges (in  $\mathbb{R}$ ).

**PROPOSITION:** (i) Basic comparison test: Let  $(a_n)$  be a sequence with  $a_n \geq 0$  (thus, real!) and such that  $\sum a_n$  is convergent. If  $(c_n)$  is a complex sequence with the property

$$\exists n_0 \in \mathbb{N} \forall n \geq n_0 : |c_n| \leq a_n,$$

then the series  $\sum_{n=0}^{\infty} c_n$  is absolutely convergent.

(ii) The root test and the ratio test both are valid for complex sequences literally as stated in Section 4. In particular, if a complex sequence  $(c_n)$  with  $c_n \neq 0$  (for almost all  $n$ ) satisfies

$$\exists \theta \in [0, 1[: \left| \frac{c_{n+1}}{c_n} \right| \leq \theta,$$

then the series  $\sum_{n=0}^{\infty} c_n$  is absolutely convergent.



(iii) The Proposition concerning the Cauchy product for absolutely convergent series holds literally as stated in Section 4.

*Proof.* Can be literally copied from those of the corresponding statements about real series.  $\square$

### (c) Continuity of functions of a complex variable:

**DEFINITION:** Let  $D \subseteq \mathbb{C}$ ,  $z_0 \in D$ . A function  $f: D \rightarrow \mathbb{C}$  is *continuous* at  $z_0$ , if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in D: |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon,$$

or equivalently

$$\forall \varepsilon > 0 \exists \delta > 0: f(U_\delta(z_0) \cap D) \subseteq U_\varepsilon(f(z_0)).$$

$f$  is said to be continuous on  $D$ , if  $f$  is continuous at all points in  $D$ .

**REMARK:** As in the case of functions on  $\mathbb{R}$ , continuity can be tested by sequences (the proof is also a literal translation of that in the real case):  $f: D \rightarrow \mathbb{C}$  is continuous at  $w \in D$  if and only if for all sequences  $(z_n)$  with  $z_n \in D$  and  $z_n \rightarrow w$  ( $n \rightarrow \infty$ ) we have that  $\lim f(z_n) = f(w)$ . We also express the latter fact by  $\lim_{z \rightarrow w} f(z) = f(w)$ .

## 6.6. The complex exponential function:

**THEOREM:** (i) For all  $z \in \mathbb{C}$  the series  $\sum_{k=0}^{\infty} \frac{z^k}{k!}$  is absolutely convergent. We thus define the complex exponential function  $\exp: \mathbb{C} \rightarrow \mathbb{C}$  by

$$\exp(z) = e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad (z \in \mathbb{C}).$$

When restricted to  $\mathbb{R}$  it coincides with the exponential function defined in Section 4. We will continue to use the same notation for both functions.

(ii) For all  $N \in \mathbb{N}$

$$\exp(z) = \sum_{k=0}^N \frac{z^k}{k!} + R_{N+1}(z),$$

where

$$|R_{N+1}(z)| \leq 2 \frac{|z|^{N+1}}{(N+1)!} \quad (z \in \mathbb{C}, |z| \leq 1 + \frac{N}{2}).$$

(iii) We have the functional equation

$$\forall z_1, z_2 \in \mathbb{C}: \exp(z_1 + z_2) = \exp(z_1) \cdot \exp(z_2).$$

- (iv) For all  $z \in \mathbb{C}$ :  $\exp(\bar{z}) = \overline{\exp(z)}$ .  
 (v) For all  $z \in \mathbb{C}$ :  $\exp(z) \neq 0$ .  
 (vi)  $\lim_{z \neq 0, z \rightarrow 0} \frac{e^z - 1}{z} = 1$ .  
 (vii)  $\exp: \mathbb{C} \rightarrow \mathbb{C}$  is continuous (at all points of  $\mathbb{C}$ ).

*Proof.* (i): If  $z = 0$  the assertion is trivial. If  $z \neq 0$  we apply the ratio test with  $c_n = z^k/k!$ . For all  $n \geq 2|z|$  we have

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{z^{n+1}n!}{z^n(n+1)!} \right| = \frac{|z|}{n+1} \leq \frac{1}{2} < 1,$$

which proves absolute convergence.

If we temporarily use the notation  $\exp_{\mathbb{R}}$  for the (real) exponential function defined in Section 4, then for  $x \in \mathbb{R}$  we have  $\exp(x + i0) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp_{\mathbb{R}}(x)$ .

(ii), (iii): Literally as in the corresponding proofs in Section 4.

(iv): Let  $s_n(z) := \sum_{k=0}^n z^k/k!$  and use 6.5(a):  $\exp(\bar{z}) = \lim s_n(\bar{z}) = \overline{\lim s_n(z)} = \overline{\exp(z)}$ .

(v): The functional equation gives  $\exp(z)\exp(-z) = \exp(z-z) = \exp(0) = 1$ , hence  $\exp(z) \neq 0$ .

(vi): By (ii) we have  $|e^z - 1 - z| \leq 2\frac{|z|^2}{2} = |z|^2$  for all  $|z| \leq 3/2$ , hence

$$\left| \frac{e^z - 1}{z} - 1 \right| \leq |z| \rightarrow 0 \quad (z \rightarrow 0).$$

(vii): By (vi) we have  $e^z - 1 = O(z)$  ( $z \rightarrow 0$ ). Therefore  $\lim_{z \rightarrow 0} \exp(z) = 1 = \exp(0)$ , which shows continuity of  $\exp$  at 0.

Let  $w \in \mathbb{C}$  arbitrary and assume that  $(z_n)$  is a sequence in  $\mathbb{C}$  such that  $z_n \rightarrow w$  ( $n \rightarrow \infty$ ). Then  $z_n - w \rightarrow 0$ , thus

$$1 = \exp(0) = \lim_{n \rightarrow \infty} \exp(z_n - w) = \lim_{n \rightarrow \infty} \exp(z_n) \exp(-w),$$

which implies that  $\lim \exp(z_n) = \exp(w)$ , hence the continuity of  $\exp$  at  $w$ . □

## 6.7. Trigonometric functions (trigonometrische Funktionen oder Winkelfunktionen)<sup>4</sup>:

DEFINITION: We define the *cosine (function)* (Cosinus (Funktion)) by

$$\cos: \mathbb{R} \rightarrow \mathbb{R}, \quad \cos(x) := \operatorname{Re}(\exp(ix)) = \operatorname{Re}(e^{ix}),$$

and the *sine (function)* (Sinus (Funktion)) by

$$\sin: \mathbb{R} \rightarrow \mathbb{R}, \quad \sin(x) := \operatorname{Im}(\exp(ix)) = \operatorname{Im}(e^{ix}).$$

BASIC PROPERTIES: (i) Since  $e^{ix} = \operatorname{Re}(e^{ix}) + i \operatorname{Im}(e^{ix})$  we obtain *Euler's formula*

$$(6.3) \quad \forall x \in \mathbb{R} : \quad e^{ix} = \cos(x) + i \sin(x).$$

Furthermore,  $\cos$  and  $\sin$  are continuous  $\mathbb{R} \rightarrow \mathbb{R}$ , since  $\exp(ix_n) \rightarrow \exp(ia)$  if and only if  $\operatorname{Re}(\exp(ix_n)) \rightarrow \operatorname{Re}(\exp(ia))$  and  $\operatorname{Im}(\exp(ix_n)) \rightarrow \operatorname{Im}(\exp(ia))$ .

(ii) Geometric interpretation: Since any real  $t$  gives  $|e^{it}|^2 = e^{it} \cdot \overline{(e^{it})} = e^{it} e^{-it} = e^0 = 1$ , we obtain

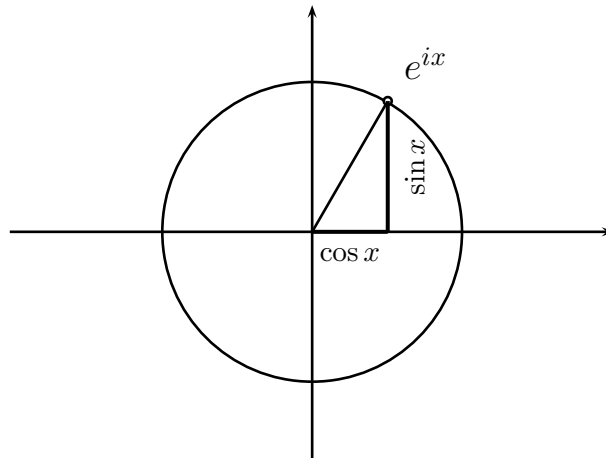
$$|e^{it}| = 1 \quad \forall t \in \mathbb{R}.$$

Therefore every number of the form  $e^{it}$  lies on the unit circle

$$S^1 := \{z \in \mathbb{C} : |z| = 1\} \cong \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$$

and  $(\cos(t), \sin(t))$  represents the (Cartesian) coordinates in the plane. In particular, we have the relation

$$(6.4) \quad \cos^2(x) + \sin^2(x) = 1 \quad \forall x \in \mathbb{R}.$$




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<sup>4</sup>These so-called trigonometric functions have a very long history: They were first used by the Babylonians in around 1900 BC and later in the Hellenistic world, in medieval India, in the Islamic Persia and in the medieval Europe. The terms sine and cosine (from the Latin sinus, i.e. “arch”) were introduced by the German mathematician Georg von Peurbach.

**Remark:** Note that we avoided any reference to notions like arc length (*Bogenlänge*) or angle (*Winkel*) in defining the trigonometric functions for reasons of a deductive presentation. Arc length will be introduced rigorously, and in more generality, later during the course (based on the notion of integrals along curves), but it certainly is useful to have the intuitive meaning at hand already as suggested by the above geometric interpretation.

(iii) Recall that for any complex number  $w$  the real and imaginary part can be obtained from  $\operatorname{Re}(w) = (w + \bar{w})/2$  and  $\operatorname{Im}(w) = (w - \bar{w})/2i$ . Therefore we have

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i},$$

which in turn implies

$$\cos(-x) = \cos(x) \quad \text{and} \quad \sin(-x) = -\sin(x),$$

telling that  $\cos$  is an even (*gerade*) function (the graph is symmetric with respect to the vertical axis) and  $\sin$  is an odd (*ungerade*) function (the graph is reflected by lines through the origin  $(0, 0)$ ).

(iv) The fundamental relations for the addition of arguments (“angles”) (*Additionstheoreme*) are the following: For all  $x, y \in \mathbb{R}$

$$\begin{aligned} \cos(x + y) &= \cos(x) \cos(y) - \sin(x) \sin(y) \\ \sin(x + y) &= \cos(x) \sin(y) + \sin(x) \cos(y) \end{aligned}$$

and

$$\begin{aligned} \cos(x) - \cos(y) &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2} \\ \sin(x) - \sin(y) &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}. \end{aligned}$$

*Proof.* The first two equations are obtained by taking real and imaginary parts in the relation

$$e^{i(x+y)} = e^{ix} \cdot e^{iy}.$$

The third (resp. fourth) equation follows from the first (resp. second) equation upon setting  $u = (x + y)/2$  and  $v = (x - y)/2$  ( $\Leftrightarrow x = u + v, y = u - v$ ):

$$\begin{aligned} \cos(x) - \cos(y) &= \cos(u + v) - \cos(u - v) \\ &= \cos(u) \cos(v) - \sin(u) \sin(v) - (\cos(u) \cos(v) + \sin(u) \sin(v)) \\ &= -2 \sin(u) \sin(v) = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}, \end{aligned}$$

and similarly for the last equation. □

(v) The natural integer powers of  $i$  show a simple repetitive pattern: Since  $i^2 = -1$ ,  $i^3 = i^2i = -i$ ,  $i^4 = i^3i = -i^2 = 1$ , we have for  $n \in \mathbb{N}$  that

$$i^n = \begin{cases} 1 & \text{if } n = 4m \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 0 \pmod{4}) \\ i & \text{if } n = 4m + 1 \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 1 \pmod{4}) \\ -1 & \text{if } n = 4m + 2 \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 2 \pmod{4}) \\ -i & \text{if } n = 4m + 3 \text{ for some } m \in \mathbb{N} \quad (\Leftrightarrow n \equiv 3 \pmod{4}). \end{cases}$$

Therefore we obtain for all  $x \in \mathbb{R}$

$$\begin{aligned} \cos(x) + i \sin(x) &= e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} \\ &= \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}_{\operatorname{Re}(e^{ix})} + i \cdot \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{\operatorname{Im}(e^{ix})}, \end{aligned}$$

which proves the following series expansions for cosine and sine:

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

(vi) Suppose we are to use the above series expansions to approximate cosine and sine for small  $x$  by simply dropping all terms that contain  $x$  to quadratic or higher order. If justified, this would give the following simple heuristic relations when  $|x|$  is small:

$$\cos(x) \approx 1 \quad \text{and} \quad \sin(x) \approx x.$$

As a matter of fact, we have the limit equations

$$\lim_{x \neq 0, x \rightarrow 0} \frac{\cos(x) - 1}{x} = 0, \quad \lim_{x \neq 0, x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$

*Proof.* By Theorem 6.6(vi) we have that

$$1 + i \cdot 0 = \lim_{x \rightarrow 0} \frac{e^{ix} - 1}{ix} = \lim_{x \rightarrow 0} \operatorname{Re}\left(\frac{e^{ix} - 1}{ix}\right) + i \cdot \lim_{x \rightarrow 0} \operatorname{Im}\left(\frac{e^{ix} - 1}{ix}\right).$$

Therefore we have for  $x \in \mathbb{R}$  as  $x \rightarrow 0$

$$\begin{aligned} \frac{\cos(x) - 1}{x} &= -\operatorname{Im}\left(\frac{\cos(x) - 1 + i \sin(x)}{ix}\right) = -\operatorname{Im}\left(\frac{e^{ix} - 1}{ix}\right) \rightarrow 0 \\ \frac{\sin(x)}{x} &= \operatorname{Re}\left(\frac{\cos(x) - 1 + i \sin(x)}{ix}\right) = \operatorname{Re}\left(\frac{e^{ix} - 1}{ix}\right) \rightarrow 1. \end{aligned}$$

□

**6.8. Definition of  $\pi$ :** We will show that  $\cos$  is strictly decreasing on the interval  $[0, 2]$  and possesses a unique zero  $x_0$  in that interval. We will define  $\pi$  as the value of  $2x_0$ . We postpone the precise identification of  $\pi$  with half the (length of the) circumference of the unit circle until integration theory allows us to provide a simple calculation.<sup>5</sup>

LEMMA: (i)  $\cos(0) = 1$  and  $\cos(2) \leq -1/3$ .

(ii) If  $0 < x \leq 2$  then  $\sin(x) > 0$ .

(iii)  $\cos$  is strictly decreasing on  $[0, 2]$ .

*Proof.* (i): We clearly have  $\cos(0) = \operatorname{Re}(e^{i0}) = 1$ .

The series expansion for the cosine function gives the alternating sum

$$\cos(2) = 1 - \frac{2^2}{2!} + \sum_{k=2}^{\infty} (-1)^k \frac{2^{2k}}{(2k)!} = -1 + r,$$

where  $r$  represents the error when approximating  $\cos(2)$  by the partial sum  $s_1 = -1$ . Thus the error estimate (4.2) from the Leibniz criterion tells that  $|r|$  is bounded by the absolute value of the first neglected term. Therefore we have

$$\cos(2) \leq -1 + |r| \leq -1 + \frac{2^4}{4!} = -1 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}.$$

(ii): Let  $0 < x \leq 2$ . We have the alternating sum for the sine function

$$\sin(x) = x + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x + r(x),$$

where  $r(x)$  now denotes the error when approximating  $\sin(x)$  by the partial sum  $s_1(x) = x$ . We apply again the estimate (4.2), which now reads

$$|r(x)| \leq \frac{x^3}{3!} = x \cdot \frac{x^2}{6} \leq x \cdot \frac{4}{6} = \frac{2x}{3},$$

and therefore

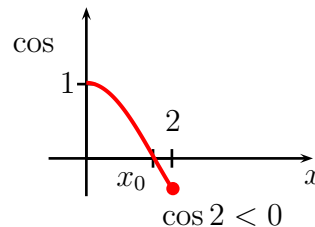
$$\sin(x) \geq x - |r(x)| \geq x - \frac{2x}{3} = \frac{x}{3} > 0.$$

(iii): Let  $0 \leq x_1 < x_2 \leq 2$ , then  $0 < (x_1 + x_2)/2 \leq 2$  as well as  $0 < (x_2 - x_1)/2 \leq 2$ . By 6.7(iv) and property (ii) proved above we obtain

$$\cos(x_2) - \cos(x_1) = -2 \cdot \underbrace{\sin \frac{x_2 + x_1}{2}}_{>0} \cdot \underbrace{\sin \frac{x_2 - x_1}{2}}_{>0} < 0,$$

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<sup>5</sup>This constant was first named “ $\pi$ ” by the Welsh scientist William Jones in 1706 because it is the first letter of the Greek words περιφέρεια (“periphery”) and περίμετρος (“circumference”). This notation was later adopted by Euler.



hence  $\cos(x_2) < \cos(x_1)$ .  $\square$

**PROPOSITION:** There exists a unique  $x_0 \in [0, 2]$  such that  $\cos(x_0) = 0$ .

*Proof.* By the above lemma,  $\cos$  is strictly decreasing on  $[0, 2]$ , hence  $\cos|_{[0,2]}$  is injective. Furthermore, the same lemma gives that  $\cos(0) > 0$  and  $\cos(2) < 0$ . Since  $\cos$  is continuous, the intermediate value theorem implies the existence of a zero  $x_0 \in [0, 2]$ . This zero must be unique, since  $\cos$  is injective on that interval.  $\square$

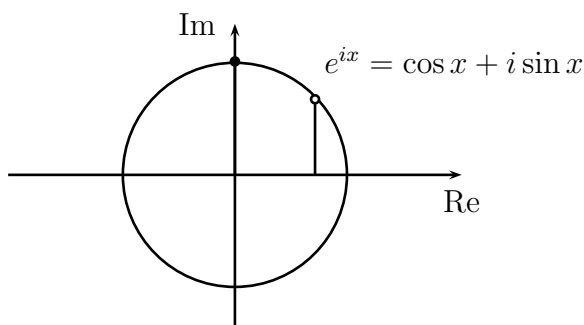
**DEFINITION:** Let  $x_0$  denote the unique zero of  $\cos$  in the interval  $[0, 2]$  (according to the above proposition). Then the real number  $\pi$  is defined by  $\pi := 2x_0$ .

The properties of  $\cos$  and  $\sin$  established above can now be reformulated in more familiar terms: For example, we obtain that

$$\cos(x) > 0 \quad \text{for } 0 \leq x < \frac{\pi}{2}, \quad \cos\left(\frac{\pi}{2}\right) = 0, \quad \cos(x) < 0 \quad \text{for } \frac{\pi}{2} < x \leq 2.$$

Since  $\sin^2(\frac{\pi}{2}) = 1 - \cos^2(\frac{\pi}{2}) = 1$  and  $\sin(\frac{\pi}{2}) > 0$  (by the above lemma), we have

$$\sin\left(\frac{\pi}{2}\right) = 1 \quad \text{and} \quad e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = i.$$



Taking integer powers for all  $k \in \mathbb{Z}$  we obtain  $e^{ik\frac{\pi}{2}} = (e^{i\frac{\pi}{2}})^k = i^k$ . In particular,

$$e^{i0} = 1 = \cos(0) + i \sin(0), \quad e^{i\frac{\pi}{2}} = i = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right), \quad e^{i\pi} = -1 = \cos(\pi) + i \sin(\pi),$$

$$e^{i\frac{3\pi}{2}} = -i = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right), \quad e^{i2\pi} = 1 = \cos(2\pi) + i \sin(2\pi),$$

which is summarized in the table

$x$	0	$\frac{\pi}{2}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$
$\sin(x)$	0	1	0	-1	0
$\cos(x)$	1	0	-1	0	1

**6.9. Further properties of the trigonometric functions:** For all  $x \in \mathbb{R}$  we have the following properties:

(a)  $\cos$  and  $\sin$  are *periodic* (*periodisch*) with *period* (*Periode*) of  $2\pi$ , i.e.,

$$\cos(x + 2\pi) = \cos(x), \quad \sin(x + 2\pi) = \sin(x).$$

This follows from 6.7(iv) and the fact that  $\cos(2\pi) = 1$ ,  $\sin(2\pi) = 0$ :

$$\cos(x + 2\pi) = \cos(x) \cos(2\pi) - \sin(x) \sin(2\pi) = \cos(x).$$

(b) Since  $\cos(x + \pi) = \cos(x) \cos(\pi) - \sin(x) \sin(\pi) = -\cos(x)$ , and a similar calculation for  $\sin$ , we have

$$\cos(x + \pi) = -\cos(x), \quad \sin(x + \pi) = -\sin(x).$$

(c) By  $\sin(\frac{\pi}{2} - x) = \sin(\frac{\pi}{2}) \cos(-x) + \cos(\frac{\pi}{2}) \sin(-x) = \cos(x)$ , and a similar calculation for  $\cos$ , we obtain

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x), \quad \cos\left(\frac{\pi}{2} - x\right) = \sin(x).$$

(d)  $\sin(x) = 0 \iff x \in \pi\mathbb{Z} := \{k\pi : k \in \mathbb{Z}\}$

*Proof.* By  $2\pi$ -periodicity it suffices to show the assertion for  $x \in [0, 2\pi[$ .

Let  $0 < x < \pi$  arbitrary, then  $\frac{\pi}{2} - x \in ] -\frac{\pi}{2}, \frac{\pi}{2}[$  and therefore  $\sin(x) = \cos(\frac{\pi}{2} - x) > 0$ .

Furthermore, note that  $]\pi, 2\pi[ = \{r + \pi : 0 < r < \pi\}$  and  $\sin(x + \pi) = -\sin(x) < 0$ .

Thus, 0 and  $\pi$  are the only zeros of  $\sin$  in the interval  $[0, 2\pi[$ , which proves the assertion.  $\square$

(e)  $\cos(x) = 0 \iff x \in \{\frac{\pi}{2}\} + \pi\mathbb{Z} := \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$

*Proof.* Use  $\cos(x) = -\sin(x - \frac{\pi}{2})$  and apply (d).  $\square$

(f)  $e^{ix} = 1 \iff x \in 2\pi\mathbb{Z} := \{2k\pi : k \in \mathbb{Z}\}$

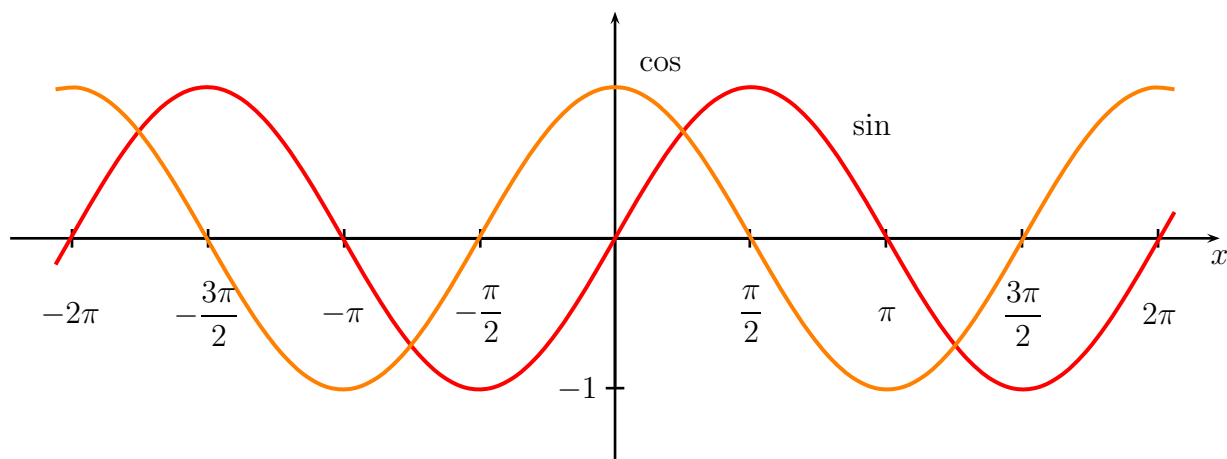
*Proof.* We have  $e^{ix} - 1 = e^{i\frac{x}{2}} \cdot (e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}) = 2i e^{i\frac{x}{2}} \sin \frac{x}{2}$  and  $2i e^{i\frac{x}{2}} \neq 0$ . Therefore

$$e^{ix} = 1 \iff \sin\left(\frac{x}{2}\right) = 0 \iff \frac{x}{2} \in \pi\mathbb{Z} \iff x \in 2\pi\mathbb{Z}.$$

$\square$

Using the above list of basic properties of  $\cos$  and  $\sin$  we can get a good qualitative picture of their graphs. Note in particular the following features: a shift of the graph of  $\cos$  by  $\frac{\pi}{2}$  along the horizontal axes gives the graph of  $\sin$ ;  $\cos$  is even and strictly decreasing on  $[0, \pi]$  (thus increasing on  $[-\pi, 0]$ ),  $\sin$  is odd and strictly increasing on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ; besides the zeros we can read off locations of maximum and minimum values, where each functions changes monotonicity type from increasing to decreasing or vice versa.





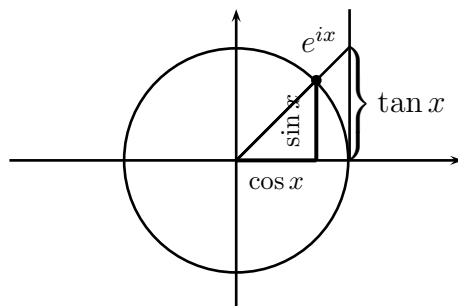
**DEFINITION:** (i) The *tangent (function)*<sup>6</sup> *(Tangens (-Funktion))*  $\tan: \mathbb{R} \setminus (\frac{\pi}{2} + \pi\mathbb{Z}) \rightarrow \mathbb{R}$  is given by

$$\tan(x) := \frac{\sin(x)}{\cos(x)}.$$

(ii) The *cotangent (function)* *(Cotangens (-Funktion))*  $\cot: \mathbb{R} \setminus \pi\mathbb{Z} \rightarrow \mathbb{R}$  is given by

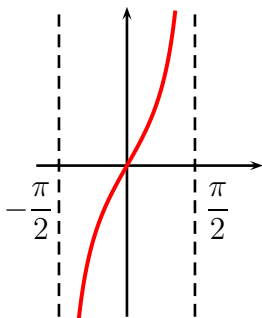
$$\cot(x) := \frac{\cos(x)}{\sin(x)}.$$

A geometric interpretation of  $\tan(x)$ , when  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ , is easy by comparing the right-angled triangles in the following illustration:



<sup>6</sup>The term tangent was first used by the Danish mathematician Thomas Fincke in 1583.

Here is the part of the graph of  $\tan$  above the interval  $]-\frac{\pi}{2}, \frac{\pi}{2}[$ , whose basic qualitative features can be derived from the properties of  $\cos$  and  $\sin$ :



Note that  $\tan(x + \pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin(x)}{-\cos(x)} = \tan(x)$ , so that the complete graph of  $\tan$  can be obtained from shifts of the basic part on  $]-\frac{\pi}{2}, \frac{\pi}{2}[$  by integer multiples of  $\pi$ .

## 6.10. Inverse trigonometric functions *(Arcusfunktionen)*:

**Arc cosine:** We assert that  $\cos$  is strictly decreasing on  $[0, \pi]$  and  $\cos([0, \pi]) = [-1, 1]$ .

Indeed, that  $\cos$  is strictly decreasing on  $[0, \frac{\pi}{2}]$  follows from the Lemma in 6.8; since  $\cos(\pi - x) = -\cos(x)$  the same follows for the interval  $[\frac{\pi}{2}, \pi]$ ; by continuity and injectivity,  $\cos([0, \pi]) = [\cos(\pi), \cos(0)] = [-1, 1]$ .

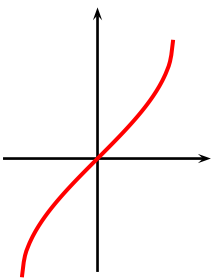
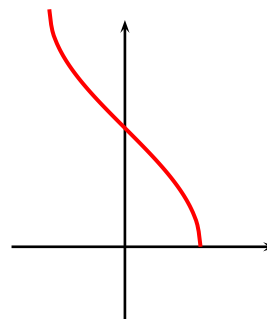
Thus  $\cos$  is continuous, strictly decreasing, and bijective as a map  $[0, \pi] \rightarrow [-1, 1]$ , hence possesses a strictly decreasing continuous inverse function

$$\arccos: [-1, 1] \rightarrow [0, \pi],$$

called the *arc cosine (function)* *(Arcus Cosinus)*.

We have for all  $x \in [0, \pi]$  that  $\arccos(\cos(x)) = x$  and  $\cos(\arccos(y)) = y$  for all  $y \in [-1, 1]$ .

Of course we could have constructed similar inverses on any interval of strict monotonicity for  $\cos$ . Unless stated otherwise we will usually refer to the one constructed above as  $\arccos$ .



**Arc sine:** Using  $\sin(x) = \cos(\frac{\pi}{2} - x)$  and (i) we deduce the following:  $\sin$  is strictly increasing on  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  and  $\sin([-\frac{\pi}{2}, \frac{\pi}{2}]) = [-1, 1]$ . The corresponding inverse function

$$\arcsin: [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}],$$

called the *arc sine (function)* *(Arcus Sinus)*, is continuous and strictly increasing.

**Arc tangent:** We claim that  $\tan$  is strictly increasing on  $] -\frac{\pi}{2}, \frac{\pi}{2}[$  and  $\tan(] -\frac{\pi}{2}, \frac{\pi}{2}[) = \mathbb{R}$ .

*Proof.* Since

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x),$$

it suffices to consider the subinterval  $[0, \frac{\pi}{2}[$ . If  $0 \leq x < x' < \frac{\pi}{2}$  then  $\sin(x) < \sin(x')$  and  $\cos(x) > \cos(x') > 0$ , hence

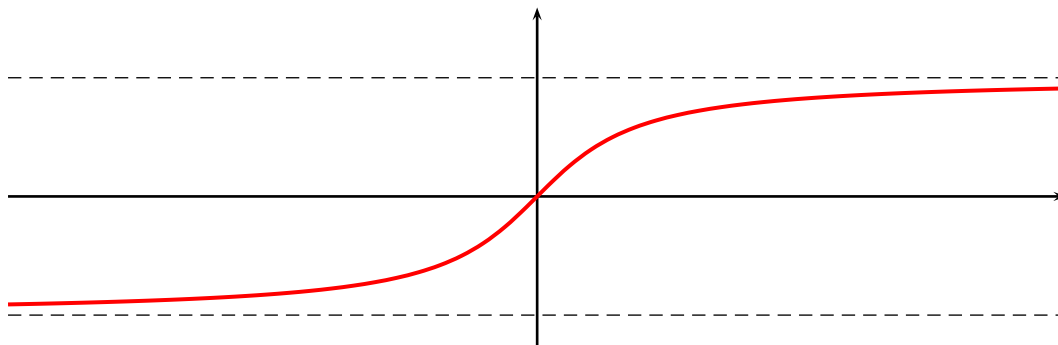
$$\tan(x) = \frac{\sin(x)}{\cos(x)} < \frac{\sin(x')}{\cos(x')} = \tan(x').$$

Note that  $\frac{\cos(x)}{\sin(x)} > 0$  for all  $x \in ]0, \frac{\pi}{2}[$  and that

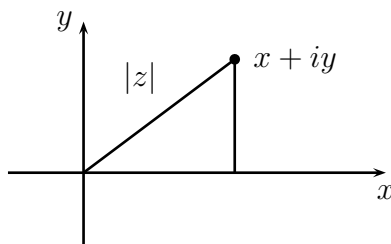
$$\lim_{x \nearrow \frac{\pi}{2}} \frac{\cos(x)}{\sin(x)} = \frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} = 0.$$

Therefore we obtain that  $\tan(x) \rightarrow \infty$  as  $x \nearrow \frac{\pi}{2}$  and by the intermediate value theorem ( $\tan$  is continuous!) that  $\tan([0, \frac{\pi}{2}[) = [0, \infty[$ . By symmetry of  $\tan$  we obtain that  $\tan(] -\frac{\pi}{2}, \frac{\pi}{2}[) = ] -\infty, \infty[$ .  $\square$

We conclude that the restriction of  $\tan$  to  $] -\frac{\pi}{2}, \frac{\pi}{2}[$  has a continuous and strictly increasing inverse function  $\arctan: \mathbb{R} \rightarrow ] -\frac{\pi}{2}, \frac{\pi}{2}[$ , called *arc tangent (function)* (*Arcus Tangens*).



**6.11. Polar coordinates<sup>7</sup> for complex numbers:** If  $z = x + iy \in \mathbb{C}$  we may interpret the absolute value  $|z| = \sqrt{x^2 + y^2}$  as the distance of  $z$  to the origin in the plane:

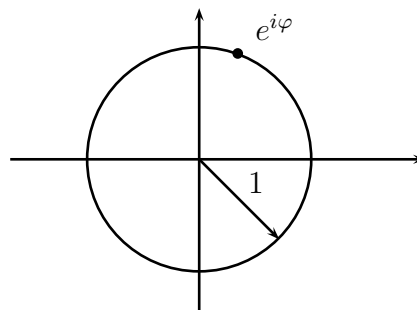


How do we obtain information on the direction towards  $z$  as seen from the origin with respect to the positive real axis (the  $x$ -axis)?

Recall that for any  $\varphi \in \mathbb{R}$  we have  $|e^{i\varphi}| = 1$  and  $e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$ .

Furthermore, the points where the unit circle  $S^1$  intersects the Cartesian axes are given by  $e^{i0} = 1$ ,  $e^{i\frac{\pi}{2}} = i$ ,  $e^{i\pi} = -1$ ,  $e^{i\frac{3\pi}{2}} = -i$  and we have  $2\pi$ -periodicity

$$e^{i(\varphi+2\pi)} = e^{i\varphi}.$$



Let  $z \neq 0$  and set  $w := \frac{z}{|z|}$ . Then  $w$  lies on the unit circle and can be written in the form

$$w = \xi + i\eta, \quad \text{where } \xi, \eta \in \mathbb{R} \text{ such that } 1 = |w|^2 = \xi^2 + \eta^2.$$

Therefore  $\xi \in [-1, 1]$  and  $\alpha := \arccos(\xi) \in [0, \pi]$  and we have

$$\sin^2(\alpha) = 1 - \cos^2(\alpha) = 1 - \xi^2 = \eta^2,$$

hence  $\sin(\alpha) = \eta$  or  $\sin(\alpha) = -\eta$ . If we define

$$\varphi := \begin{cases} \alpha & \text{if } \sin(\alpha) = \eta \\ -\alpha & \text{if } \sin(\alpha) = -\eta, \end{cases}$$

then we obtain  $w = \cos(\varphi) + i \sin(\varphi) = e^{i\varphi}$ , which in turn yields the *polar representation*  $\langle \text{Polardarstellung} \rangle$  of  $z$  in the form

$$z = |z| \cdot e^{i\varphi}.$$

In this representation  $\varphi$ , the so-called the *argument*  $\langle \text{Argument} \rangle$  of  $z$ ,  $\varphi = \arg(z)$ , is unique up to an addition of integer multiples of  $2\pi$ .

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<sup>7</sup>The history of polar coordinates is about as long as the one of trigonometry. The term polar coordinates was introduced by 18th-century Italian mathematicians.

CHAPTER III

# DIFFERENTIATION

## §7. DIFFERENTIABILITY AND DERIVATIVE

**7.1. Definition:** Consider  $V \subseteq \mathbb{R}$ ,  $f: V \rightarrow \mathbb{R}$ , and let  $x \in V$  be an accumulation point of  $V$ . Then  $f$  is said to be *differentiable at  $x$*  (*differenzierbar in  $x$* ) if the limit

$$(7.1) \quad f'(x) := \lim_{\substack{\xi \rightarrow x \\ \xi \in V \setminus \{x\}}} \frac{f(\xi) - f(x)}{\xi - x}$$

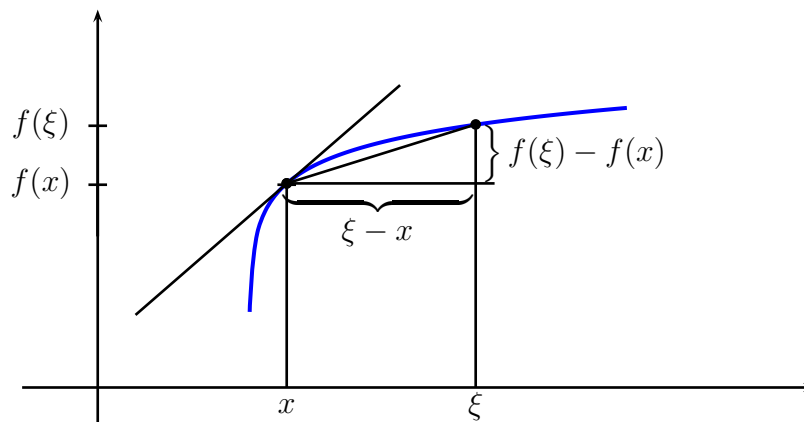
exists. The value  $f'(x)$  is the *derivative of  $f$  at  $x$*  (*Ableitung von  $f$  an der Stelle  $x$* ). The function  $f$  is called *differentiable in  $V$* , if  $f$  is differentiable at all points  $x \in V$ .

**7.2. Remark:** (i) An alternative expression to (7.1) is

$$(7.2) \quad f'(x) := \lim_{\substack{h \rightarrow 0 \\ x+h \in V, h \neq 0}} \frac{f(x+h) - f(x)}{h}$$

(We will often suppress the additional conditions on  $h$  in the specification of the limit.)

(ii) **Geometric interpretation:** Let  $V$  be an interval and consider the graph of  $f$  of a differentiable function  $f: V \rightarrow \mathbb{R}$ .



The *difference quotient* (Differenzenquotient)

$$\frac{f(\xi) - f(x)}{\xi - x}$$

gives the slope of the straight line, the so-called secant (Sekante), through the points  $(x, f(x))$  and  $(\xi, f(\xi))$ . As  $\xi$  is getting closer and closer to  $x$ , the slopes of the corresponding secant lines approach the limit  $f'(x)$ , which therefore can be thought of being the slope of the tangent (Tangente) to the graph of  $f$  at the point  $(x, f(x))$ .<sup>1</sup>

(iii) A common notation for  $f'(x)$  is the *differential quotient* (Differenzialquotient)  $\frac{df(x)}{dx}$ , which however is not a quotient but merely a reminder of the fact that  $f'(x)$  is the limit of the difference quotients  $\frac{\Delta f(x)}{\Delta x} = \frac{f(\xi) - f(x)}{\xi - x}$ .

(iv) If  $f$  is differentiable in  $V$  then the derivative of  $f$  defines a function  $f': V \rightarrow \mathbb{R}$ ,  $x \mapsto f'(x)$ .

**7.3. Examples:** 1) Let  $c \in \mathbb{R}$ . The constant function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = c$ , is differentiable in  $\mathbb{R}$  and has derivative

$$f'(x) = \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{f(x+h) - f(x)}{h} = \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{c - c}{h} = 0.$$

2) Let  $c \in \mathbb{R}$  and  $n \in \mathbb{N}$  with  $n \geq 1$ . The function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto cx^n$ , is differentiable in  $\mathbb{R}$  and has derivative

$$\begin{aligned} f'(x) &= \lim_{0 \neq h \rightarrow 0} \frac{c(x+h)^n - cx^n}{h} = c \lim_{0 \neq h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= c \lim_{0 \neq h \rightarrow 0} \frac{x^n + \sum_{k=1}^n \binom{n}{k} h^k x^{n-k} - x^n}{h} = c \lim_{0 \neq h \rightarrow 0} \frac{h \cdot \sum_{k=1}^n \binom{n}{k} h^{k-1} x^{n-k}}{h} \\ &= c \lim_{0 \neq h \rightarrow 0} \left( \binom{n}{1} x^{n-1} + \sum_{k=2}^n \binom{n}{k} h^{k-1} x^{n-k} \right) = cnx^{n-1}. \end{aligned}$$

In particular, we have  $(cx)' = cx^0 = c$  and  $(x^2)' = 2x$ .

3)  $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ ,  $x \mapsto \frac{1}{x}$ , is differentiable at every  $x \neq 0$  and

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{x+h} - \frac{1}{x} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{x(x+h)} = \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

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<sup>1</sup>This solution of the so-called “tangent problem” was developed independently by Isaac Newton and Gottfried Wilhelm Leibnitz in the 17th century, thereby inventing modern calculus.

4) The exponential function is differentiable and

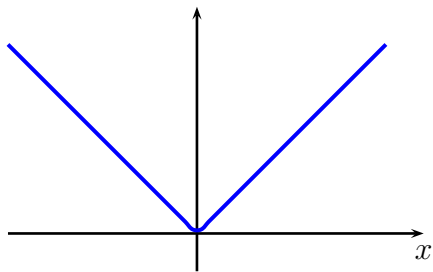
$$\exp'(x) = \lim_{h \rightarrow 0} \frac{\exp(x+h) - \exp(x)}{h} = \exp(x) \lim_{h \rightarrow 0} \frac{\exp(h) - 1}{h} \stackrel{[6.6(vi)]}{=} \exp(x) \cdot 1 = \exp(x).$$

5)  $\sin: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable and

$$\begin{aligned} \sin'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \stackrel{[6.7(iv)]}{=} \lim_{h \rightarrow 0} \frac{2 \cos(x + \frac{h}{2}) \sin(\frac{h}{2})}{h} \\ &= \lim_{h \rightarrow 0} \underbrace{\cos(x + \frac{h}{2})}_{[\text{cos continuous}]} \underbrace{\frac{\sin(h/2)}{h/2}}_{[\rightarrow 1 \text{ by } 6.7(vi)]} = \lim_{h \rightarrow 0} \cos(x + \frac{h}{2}) \cdot \lim_{h \rightarrow 0} \frac{\sin(h/2)}{h/2} = \cos(x). \end{aligned}$$

Similarly, one can show that  $\cos$  is differentiable and  $\cos'(x) = -\sin(x)$  (left as an exercise).

6)  $\text{abs}: \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto |x|$ , is differentiable for all  $x \neq 0$ . This follows easily from the fact that  $\text{abs}|_{]0, \infty[} = \text{id}$  and  $\text{abs}|_{]-\infty, 0[} = -\text{id}$ .



But  $\text{abs}'(0)$  does not exist: Let  $h_n := (-1)^n/n$ , then  $\lim h_n = 0$  and the difference quotients do not converge

$$\frac{\text{abs}(0 + h_n) - \text{abs}(0)}{h_n} = \frac{1/n}{(-1)^n/n} = (-1)^n.$$

Thus  $\text{abs}$  is a continuous function on  $\mathbb{R}$  but not differentiable at the point 0.

**7.4. One-sided derivatives:** Let  $x$  be an accumulation point of  $V \subseteq \mathbb{R}$  such that  $x \in V$ .  $f: V \rightarrow \mathbb{R}$  is *differentiable from the right* (*rechtsseitig differenzierbar*) at  $x \in V$ , if

$$f'_+(x) := \lim_{\substack{\xi \searrow x \\ \xi \in V \setminus \{x\}}} \frac{f(\xi) - f(x)}{\xi - x}$$

exists.  $f$  is *differentiable from the left* (*linksseitig differenzierbar*) at  $x \in V$ , if

$$f'_-(x) := \lim_{\substack{\xi \nearrow x \\ \xi \in V \setminus \{x\}}} \frac{f(\xi) - f(x)}{\xi - x}$$

exists.

For example,  $\text{abs}'_-(0) = -1$  and  $\text{abs}'_+(0) = 1$ . (Compare with the graph!)



**7.5. Proposition:** If  $f: V \rightarrow \mathbb{R}$  is differentiable at  $x \in V$ , then  $f$  is continuous at  $x$ .

*Proof.* Let  $V \ni \xi \neq x$ , then we have

$$f(\xi) - f(x) = \frac{f(\xi) - f(x)}{\xi - x} \cdot (\xi - x) \rightarrow f'(x) \cdot 0 = 0 \quad (\xi \rightarrow x).$$

□

**7.6. Proposition:** Let  $f, g: V \rightarrow \mathbb{R}$  be differentiable in  $x \in V$ .

(i) If  $\lambda, \mu \in \mathbb{R}$  then  $\lambda f + \mu g$  is differentiable at  $x$  and

$$(\lambda f + \mu g)'(x) = \lambda f'(x) + \mu g'(x).$$

(ii) *Leibniz or product rule:*  $f \cdot g$  is differentiable at  $x$  and

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x).$$

(iii) *Quotient rule:* If  $g(\xi) \neq 0$  for all  $\xi \in V$  then  $\frac{f}{g}: V \rightarrow \mathbb{R}$  is differentiable at  $x$  and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

*Proof.* (i) Follows from the basic rules for limits.

(ii) Let  $h \neq 0$  such that  $x + h \in V$ . Then we have

$$\begin{aligned} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{1}{h} \left( f(x+h)(g(x+h) - g(x)) + (f(x+h) - f(x))g(x) \right) \\ &= f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} \cdot g(x) \\ &\xrightarrow{(h \rightarrow 0)} f(x) \cdot g'(x) + f'(x) \cdot g(x) \end{aligned}$$

since  $f$  is continuous at  $x$  by Proposition 7.5.

(iii) Consider first the case that  $f(\xi) = 1$  for all  $\xi \in V$ . Then we have with  $h \neq 0$  such that  $x + h \in V$

$$\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \frac{g(x) - g(x+h)}{hg(x)g(x+h)} \rightarrow -\frac{g'(x)}{g(x)^2} \quad (h \rightarrow 0).$$

Thus  $(1/g)'(x) = -g'(x)/g(x)^2$  and the general case now follows from the product rule:

$$\left(\frac{f}{g}\right)'(x) = \left(f \cdot \frac{1}{g}\right)'(x) = f'(x)\frac{1}{g(x)} - f(x)\frac{g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

□

**7.7. Examples:** 1) Let  $n \in \mathbb{N}$ ,  $n \geq 1$  and  $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ ,  $x \mapsto 1/x^n$ :

$$f'(x) = \left( \frac{1}{x^n} \right)' = \frac{-nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}},$$

in other words,  $(x^{-n})' = -nx^{-n-1}$ .

2)  $\tan: \mathbb{R} \setminus (\frac{\pi}{2} + \pi\mathbb{Z}) \rightarrow \mathbb{R}$  is differentiable and

$$\begin{aligned} (\tan x)' &= \left( \frac{\sin(x)}{\cos(x)} \right)' = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = 1 + \tan^2(x). \end{aligned}$$

**7.8. The derivative as linear approximation:** Let  $V \subseteq \mathbb{R}$  and  $x_0 \in V$  be an accumulation point of  $V$ . Suppose that  $f: V \rightarrow \mathbb{R}$  is differentiable at  $x_0$ . We have seen that in a geometric interpretation the tangent to the graph of  $f$  at the point  $(x_0, f(x_0))$  has slope  $f'(x_0)$ . Thus we define the *tangent* (*Tangente*) to the graph of  $f$  at  $(x_0, f(x_0))$  as the straight line given by the following linear equation for  $(x, y) \in \mathbb{R}^2$

$$(7.3) \quad y = f(x_0) + f'(x_0)(x - x_0).$$

Intuitively, if  $f$  is sufficiently “well-behaved” then the tangent, considered as the affine linear function  $x \mapsto f(x_0) + f'(x_0)(x - x_0)$ , should give a reasonable approximation of the function values when  $x$  is close to  $x_0$ , i.e.,  $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$  as  $x \rightarrow x_0$ . The following statement makes this more precise.

**THEOREM:** The following statements are equivalent:

(i)  $f$  is differentiable at  $x_0$ .

(ii) There exist a constant  $c \in \mathbb{R}$  and a function  $r: V \rightarrow \mathbb{R}$  with  $\lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{r(x)}{x - x_0} = 0$  such that

$$(7.4) \quad f(x) = f(x_0) + c \cdot (x - x_0) + r(x) \quad \forall x \in V.$$

In this case  $c = f'(x_0)$ .

Note that, in particular,  $r(x) = o(|x - x_0|)$  ( $x \rightarrow x_0$ ).

*Proof.* (i)  $\Rightarrow$  (ii): Put  $r(x) := f(x) - f(x_0) - f'(x_0)(x - x_0)$  and let  $x \in V$  with  $x \neq x_0$ . Then

$$\frac{r(x)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \longrightarrow f'(x_0) - f'(x_0) = 0 \quad (x \rightarrow x_0).$$

(ii)  $\Rightarrow$  (i): Let  $x \in V$  with  $x \neq x_0$ . Then (7.4) implies

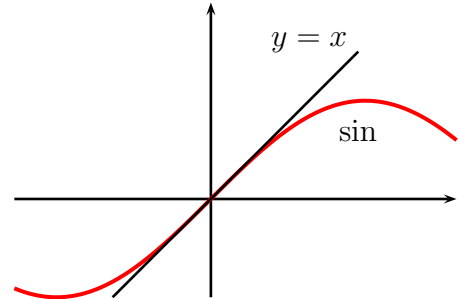
$$\frac{f(x) - f(x_0)}{x - x_0} = c + \frac{r(x)}{x - x_0} \longrightarrow c + 0 \quad (x \rightarrow x_0).$$

Thus  $f$  is differentiable at  $x_0$  and  $f'(x_0) = c$ .  $\square$

**EXAMPLE:** Consider  $\sin: \mathbb{R} \rightarrow \mathbb{R}$  with  $\sin'(0) = \cos(0) = 1$ . We obtain the relation

$$\begin{aligned} \sin(x) &= \sin(0) + \sin'(0)(x - 0) + r(x) \\ &= 0 + 1 \cdot x + r(x) = x + r(x), \end{aligned}$$

where  $r(x) = \sin(x) - x = o(x)$  ( $x \rightarrow 0$ ).



## 7.9. The chain rule:

**THEOREM:** Let  $V, W \subseteq \mathbb{R}$  and  $f: V \rightarrow \mathbb{R}$ ,  $g: W \rightarrow \mathbb{R}$  such that  $f(V) \subseteq W$ .

If  $f$  is differentiable at  $x \in V$  and  $g$  is differentiable at  $y := f(x) \in W$ , then the composition  $g \circ f: V \rightarrow \mathbb{R}$  is differentiable at  $x$  and

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

*Idea of the proof:* We will put in a little extra effort to make the following operations well-defined for all  $\xi \in V$  with  $\xi \neq x$

$$\frac{g(f(\xi)) - g(f(x))}{\xi - x} = \frac{g(f(\xi)) - g(f(x))}{f(\xi) - f(x)} \cdot \frac{f(\xi) - f(x)}{\xi - x} \approx g'(f(x)) \cdot f'(x);$$

note that we cannot guarantee that  $f(\xi) - f(x) \neq 0$ .

*Proof.* Define the modified difference quotient  $g^*: W \rightarrow \mathbb{R}$  of  $g$  at  $y$  by

$$g^*(\eta) := \begin{cases} \frac{g(\eta) - g(y)}{\eta - y} & \text{if } \eta \neq y \\ g'(y) & \text{if } \eta = y. \end{cases}$$

Since  $\lim_{\eta \rightarrow y} g^*(\eta) = g'(y) = g^*(y)$  we deduce that  $g^*$  is continuous. Furthermore, by construction we have for all  $\eta \in W$

$$g(\eta) - g(y) = g^*(\eta) (\eta - y).$$

Therefore we obtain for  $\xi \in V$  with  $\xi \neq x$

$$\frac{g(f(\xi)) - g(f(x))}{\xi - x} = \frac{g^*(f(\xi))(f(\xi) - f(x))}{\xi - x} = g^*(f(\xi)) \cdot \frac{f(\xi) - f(x)}{\xi - x}$$

$$\xrightarrow{(\xi \rightarrow x)} g^*(y) \cdot f'(x) = g'(f(x)) \cdot f'(x).$$

□

**APPLICATION:** Let  $D, W \subseteq \mathbb{R}$  and  $f: D \rightarrow W$  be a bijective map. Suppose that  $f$  is differentiable at  $x \in D$  and  $f^{-1}$  is differentiable at  $y := f(x) \in W$ . Then the chain rule applied to  $f^{-1} \circ f = \text{id}_D$  yields

$$(f^{-1})'(f(x)) \cdot f'(x) = \text{id}'(x) = 1.$$

In particular, this implies  $f'(x) \neq 0$  and thus we may rewrite the above relation with  $y = f(x)$  in the form

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

**7.10. Differentiability and derivative of the inverse function:** Let  $I, J \subseteq \mathbb{R}$  be intervals and  $f: I \rightarrow J$  be continuous and strictly increasing with (continuous) inverse function  $f^{-1}: J \rightarrow I$ .

**THEOREM:** Under the assumptions stated above: If  $f$  is differentiable at  $x \in I$  with  $f'(x) \neq 0$ , then  $f^{-1}$  is differentiable at  $y := f(x) \in J$  and

$$(7.5) \quad (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

*Proof.* Let  $(\eta_n)$  be a sequence with  $\eta_n \in J \setminus \{y\}$  (for all  $n \in \mathbb{N}$ ) such that  $\eta_n \rightarrow y$  ( $n \rightarrow \infty$ ).

If we put  $\xi_n := f^{-1}(\eta_n)$  then  $\xi_n \in I \setminus \{x\}$  and  $\xi_n \rightarrow x$  as  $n \rightarrow \infty$  (by continuity of  $f^{-1}$  [cf. 5.12]). Therefore

$$\frac{f^{-1}(\eta_n) - f^{-1}(y)}{\eta_n - y} = \frac{\xi_n - x}{f(\xi_n) - f(x)} \longrightarrow \frac{1}{f'(x)} \quad (n \rightarrow \infty),$$

which proves that  $f^{-1}$  is differentiable at  $y$  and equals  $1/f'(f^{-1}(y))$ . □

**7.11. Examples:** 1) Let  $x > 0$  then

$$\log'(x) = \frac{1}{\exp'(\log(x))} = \frac{1}{\exp(\log(x))} = \frac{1}{x}.$$

A simple application of this result proves a famous limit relation: Since  $\log'(1) = 1$  we have (using the continuity of  $\exp$ )

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{\log(1 + \frac{1}{n}) - \log(1)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} n \cdot \log(1 + \frac{1}{n}) = \lim_{n \rightarrow \infty} \log(1 + \frac{1}{n})^n \\ &= \log \exp \lim_{n \rightarrow \infty} \log(1 + \frac{1}{n})^n = \log(\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n) \end{aligned}$$

and upon exponentiation

$$(7.6) \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \exp(1) = e.$$

2) If  $x \in ]-1, 1[$  then

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))} \stackrel{[\cos = \sqrt{1 - \sin^2}]}{=} \frac{1}{\sqrt{1 - x^2}}.$$

$$3) \arctan'(x) = \frac{1}{\tan'(\arctan(x))} = \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + x^2}.$$

4) If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable and  $a, b \in \mathbb{R}$  then  $g(x) := f(ax + b)$  is differentiable and by the chain rule

$$g'(x) = f'(ax + b) (ax + b)' = a f'(ax + b).$$

5) For  $\alpha \in \mathbb{R}$  the function  $x \mapsto x^\alpha, ]0, \infty[ \rightarrow \mathbb{R}$  is differentiable and

$$\begin{aligned} (x^\alpha)' &= (\exp(\alpha \log(x)))' = \exp'(\alpha \log(x)) (\alpha \log(x))' \\ &= \exp(\alpha \log(x)) \alpha \log'(x) = x^\alpha \alpha \frac{1}{x} = \alpha x^{\alpha-1}. \end{aligned}$$

**7.12. Higher derivatives:** Let  $f: V \rightarrow \mathbb{R}$  be differentiable and  $f': V \rightarrow \mathbb{R}$  be its derivative. If  $f'$  is differentiable in  $x \in V$ , then

$$f''(x) := (f')'(x)$$

is called the *second derivative* (*zweite Ableitung*) of  $f$  at  $x$ . A classical notation for  $f''(x)$  is also  $\frac{d^2 f(x)}{dx^2}$ .

Suppose that  $f'$  is differentiable in  $U_\varepsilon(x) \cap V$ . Then we can consider the function  $f'': U_\varepsilon(x) \cap V \rightarrow \mathbb{R}$  and investigate its differentiability properties in turn. In this way, we can define

the property of a function  $f$  to be  $k$  times differentiable at  $x$  ( $k \in \mathbb{N}$ ,  $k \geq 1$ ) inductively, if there exists  $\varepsilon > 0$  such that  $f|_{U_\varepsilon(x) \cap V}$  is  $(k-1)$  times differentiable (in all points of  $U_\varepsilon(x) \cap V$ ) and the  $(k-1)$ st derivative  $f^{(k-1)}$  is differentiable at  $x$ . In this case we set

$$f^{(k)}(x) := (f^{(k-1)})'(x).$$

It is common to put  $f^{(0)}(x) = f(x)$ . A function  $f: V \rightarrow \mathbb{R}$  is called *infinitely (often) differentiable* (*unendlich oft differenzierbar*) at  $x$  if it is  $k$  times differentiable for all  $k \in \mathbb{N}$ .

**EXAMPLE:**  $\sin$  is infinitely (often) differentiable on all of  $\mathbb{R}$ . We have  $\sin' = \cos$ ,  $\sin'' = -\sin$ ,  $\sin^{(3)} = -\cos$ , and  $\sin^{(4)} = \sin$  etc.

Let  $\omega \in \mathbb{R}$  be a constant (frequency). Then  $u(x) := \sin(\omega x)$  is infinitely differentiable on  $\mathbb{R}$ . We have  $u'(x) = \omega \cos(\omega x)$  and  $u''(x) = -\omega^2 \sin(\omega x) = -\omega^2 u(x)$ . Therefore  $u$  satisfies the differential equation of the isochronous pendulum

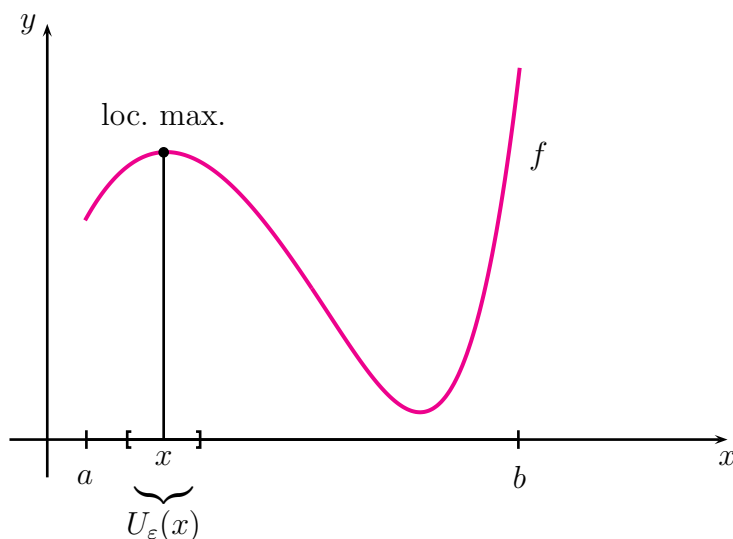
$$u'' + \omega^2 u = 0.$$

## §8. PROPERTIES OF DIFFERENTIABLE FUNCTIONS

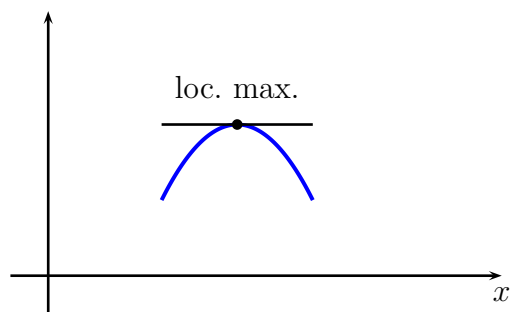
**8.1. Definition:** Let  $f: ]a, b[ \rightarrow \mathbb{R}$ . A point  $x \in ]a, b[$  is a *local maximum* (*lokales Maximum*) (resp. *local minimum* (*lokales Minimum*)) of  $f$ , if

$$\exists \varepsilon > 0 \forall \xi \in U_\varepsilon(x) : f(x) \geq f(\xi) \quad (\text{resp. } f(x) \leq f(\xi)).$$

If  $f(x) > f(\xi)$  (resp.  $f(x) < f(\xi)$ ) for all  $\xi \in U_\varepsilon(x)$  with  $\xi \neq x$ , then  $x$  is a *strict local maximum* (resp. *minimum*) (*striktes lokales Maximum* (bzw. *Minimum*)). In either case  $x$  is called a *local extreme* (*lokales Extremum*) of  $f$ .



Geometrically, we expect that the graph of  $f$  has a horizontal tangent in a local extreme, which means that the slope  $f'(x)$  must be 0.



**8.2. Proposition:** Suppose that  $f: ]a, b[ \rightarrow \mathbb{R}$  has a local extreme in  $x$  and is differentiable at  $x$ . Then  $f'(x) = 0$ .

*Proof.* Let  $x$  be a local maximum of  $f$  (the case of a local minimum is analogous). There is  $\varepsilon > 0$  such that

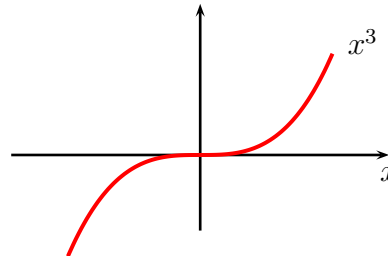
$$\forall \xi \in U_\varepsilon(x) : f(x) \geq f(\xi).$$

Since  $f$  is differentiable at  $x$  we deduce that

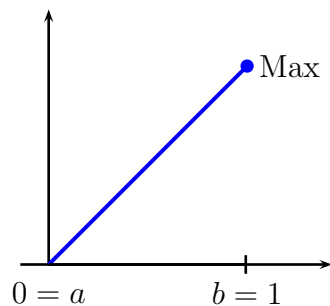
$$\lim_{\xi \searrow x} \underbrace{\frac{f(\xi) - f(x)}{\xi - x}}_{\leq 0} = f'(x) = \lim_{\xi \nearrow x} \underbrace{\frac{f(\xi) - f(x)}{\xi - x}}_{\geq 0}$$

and therefore  $0 \leq f'(x) \leq 0$ , hence  $f'(x) = 0$ . □

**8.3. Remark:** (i) The above proposition gives a necessary condition, namely  $f'(x) = 0$ , for a local extreme at a point  $x$  where  $f$  is differentiable. This condition is, in general, *not sufficient* as the following simple example shows: Let  $f: ]-1, 1[ \rightarrow \mathbb{R}$ ,  $x \mapsto x^3$ ; then  $f'(0) = 0$ , but  $x = 0$  is neither a local maximum nor a local minimum.



Thus a solution of the equation  $f'(x) = 0$  merely provides a candidate for a local extremum.

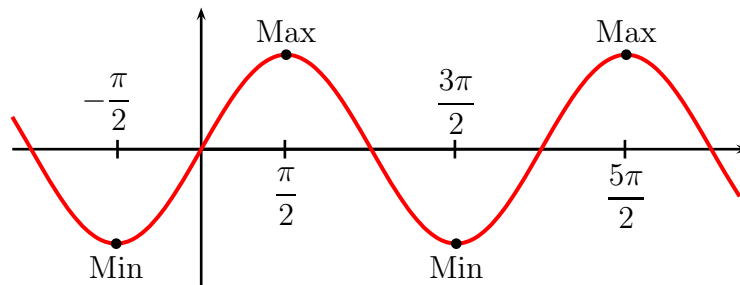


(ii) Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous. By the results in Section 5 we know that  $f$  possesses a (global) maximum and a minimum in the closed interval  $[a, b]$ . Such an extreme point can lie at the boundary, i.e.,  $x = a$  or  $x = b$ , in which case the derivative need not vanish. For example, the function  $f: [0, 1] \rightarrow \mathbb{R}$ ,  $f(x) = x$ , has a maximum in  $x = 1$ , but  $f'(1) = 1 \neq 0$ .



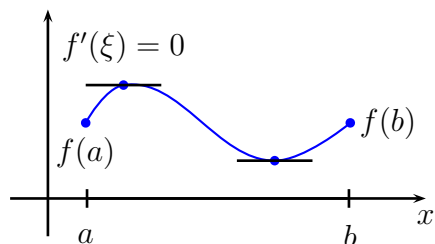
**8.4. Example:** Consider the function  $\sin: \mathbb{R} \rightarrow \mathbb{R}$ .

We already deduced from its strict monotonicity properties on the intervals  $[k\frac{\pi}{2}, (k+1)\frac{\pi}{2}]$  that we have strict maxima at the points  $\frac{\pi}{2} + 2k\pi$  and strict minima at the points  $\frac{\pi}{2} + (2k+1)\pi$  ( $k \in \mathbb{Z}$ ). And indeed we have  $\sin'(\frac{\pi}{2} + k\pi) = \cos(\frac{\pi}{2} + k\pi) = 0$ .



**8.5. The mean value theorem of differential calculus:** In the following statements we suppose that  $f: [a, b] \rightarrow \mathbb{R}$  is continuous and differentiable in  $]a, b[$ .<sup>1</sup>

**PROPOSITION (ROLLE'S THEOREM):** If  $f(a) = f(b)$  then there exists  $\xi \in ]a, b[$  such that  $f'(\xi) = 0$ .<sup>2</sup>



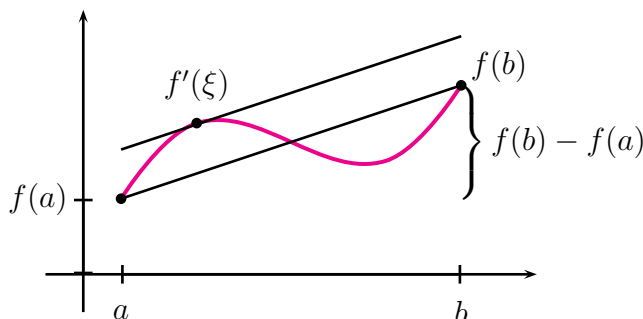
*Proof.* If  $f$  is a constant function the assertion is trivial, hence we may assume that  $f$  is not constant. Then there exists  $x_0 \in ]a, b[$  such that  $f(x_0) > f(a) = f(b)$  or  $f(x_0) < f(a) = f(b)$ . Thus the maximum or minimum of  $f$  must be attained at some  $\xi \in ]a, b[$ . Since  $\xi$  has to be a local extreme as well we obtain that  $f'(\xi) = 0$ .  $\square$

**MEAN VALUE THEOREM:** There exists  $\xi \in ]a, b[$  such that

$$(8.1) \quad \frac{f(b) - f(a)}{b - a} = f'(\xi)$$

<sup>1</sup>In case you are wondering about the relation between the two conditions 'continuity on the closed interval  $[a, b]$ ' and 'differentiability in the open interval  $]a, b[$ ' or their implying 'differentiability on the closed interval  $[a, b]$ ': First, recall that continuity at a point  $x$  does not imply differentiability at  $x$ ; second, consider the examples  $f_1: [0, 1] \rightarrow \mathbb{R}$ ,  $f_1(x) = \sqrt{x}$  and  $f_2: ]0, 1] \rightarrow \mathbb{R}$ ,  $f_2(x) = 1/x$ ; then  $f_1$  is continuous on  $[0, 1]$ , differentiable in  $]0, 1[$ , but not differentiable at 0 and  $f_2$  is differentiable in  $]0, 1[$  but has no continuous extension to  $[0, 1]$ .

<sup>2</sup>Michel Rolle (1652–1719) [mi'ʃɛl ʁɔʎ], French mathematician



*Proof.* Define  $g: [a, b] \rightarrow \mathbb{R}$  by  $g(x) := f(x) - \frac{f(b) - f(a)}{b - a} \cdot (x - a)$ .

Then  $g$  is continuous on  $[a, b]$ , differentiable in  $]a, b[$ , and  $g(a) = f(a) = g(b)$ . Therefore Rolle's theorem implies that there is some  $\xi \in ]a, b[$  such that

$$0 = g'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a},$$

which proves the assertion.  $\square$

**COROLLARY:** (i) If  $f'$  is bounded, i.e., there are  $m, M \in \mathbb{R}$  such that  $m \leq f'(\xi) \leq M$  holds for all  $\xi \in ]a, b[$ , then we have for all  $x_1, x_2 \in [a, b]$  with  $x_1 \leq x_2$

$$m(x_2 - x_1) \leq f(x_2) - f(x_1) \leq M(x_2 - x_1).$$

In particular,  $f$  is *Lipschitz continuous* (*Lipschitz-stetig*) with *Lipschitz constant*  $L := \max(|m|, |M|)$ , that is

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1| \quad \forall x_1, x_2 \in [a, b].$$

(ii) If  $f'(x) = 0$  for all  $x \in ]a, b[$  then  $f$  is constant.

*Proof.* (i) Apply (8.1) to each interval  $[x_1, x_2]$ .

(ii) follows from (i) with  $m = M = 0$ .  $\square$

**APPLICATION:** Let  $c \in \mathbb{R}$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  be differentiable. If  $f$  satisfies the differential equation

$$f'(x) = c \cdot f(x) \quad \forall x \in \mathbb{R},$$

then  $f$  has to be of the form  $f(x) = f(0) \exp(cx)$ .

To prove this we put  $g(x) := f(x) \exp(-cx)$  and observe that

$$g'(x) = -ce^{-cx} f(x) + e^{-cx} f'(x) = -ce^{-cx} f(x) + e^{-cx} c f(x) = 0.$$

Thus  $g(x) = g(0) = f(0)$  for all  $x$ , which implies  $f(x) = f(0) \exp(cx)$ .

**8.6. Proposition:** Suppose  $f: [a, b] \rightarrow \mathbb{R}$  is continuous and differentiable in  $]a, b[$ .

(i) If  $f'(x) \geq 0$  (resp.  $f'(x) > 0$ ) for all  $x \in ]a, b[$  then  $f$  is increasing (resp. strictly increasing) on  $[a, b]$ .

(ii) If  $f$  is increasing on  $[a, b]$  then  $f'(x) \geq 0$  for all  $x \in [a, b]$ .

Corresponding statements hold for decreasing functions.

*Proof.* (i) Proof by contradiction: If  $f$  is not increasing (resp. strictly increasing) there are  $x_1, x_2 \in [a, b]$ ,  $x_1 < x_2$  such that  $f(x_1) > f(x_2)$  (resp.  $f(x_1) \geq f(x_2)$ ). By the mean value theorem there is a  $\xi \in ]x_1, x_2[$  such that

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

which implies that  $f'(\xi) < 0$  (resp.  $f'(\xi) \leq 0$ ) — a contradiction  $\nabla$ .

(ii) Since  $f$  is increasing we have for all  $x, \xi \in ]a, b[$  with  $x \neq \xi$  that

$$\frac{f(\xi) - f(x)}{\xi - x} \geq 0.$$

Thus  $f'(x) \geq 0$ . □

**REMARK:** Note that strict monotonicity does not imply that the derivative is strictly greater or smaller than 0: For example,  $f: [-1, 1] \rightarrow \mathbb{R}$ ,  $f(x) = x^3$ , is strictly increasing whereas  $f'(0) = 0$ .

**8.7. Corollary (Sufficient condition for a local extreme):** Let  $f: ]a, b[ \rightarrow \mathbb{R}$  be differentiable. Suppose that  $x \in ]a, b[$  and that  $f$  is twice differentiable at  $x$ . If

$$f'(x) = 0 \quad \text{and} \quad f''(x) > 0 \quad (\text{resp. } f''(x) < 0)$$

then  $x$  is a strict local minimum (resp. maximum) of  $f$ .

*Proof.* Assume that  $f'(x) = 0$  and  $f''(x) > 0$  (the case  $f''(x) < 0$  is analogous.)

We obtain that

$$0 < f''(x) = \lim_{x \neq \xi \rightarrow x} \frac{f'(\xi) - f'(x)}{\xi - x}.$$

Hence there exists  $\varepsilon > 0$  such that for all  $\xi$  with  $0 < |\xi - x| < \varepsilon$

$$\frac{f'(\xi)}{\xi - x} = \frac{f'(\xi) - f'(x)}{\xi - x} > 0.$$

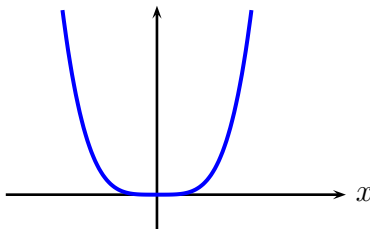
From this we deduce the following: If  $\xi \in ]x - \varepsilon, x[$  then  $f'(\xi) < 0$  and if  $\xi \in ]x, x + \varepsilon[$  then  $f'(\xi) > 0$ . Therefore  $f$  is strictly decreasing in  $[x - \varepsilon, x]$  and strictly increasing in  $[x, x + \varepsilon]$ , which implies that  $x$  is a strict local minimum for  $f$ . □

**8.8. Examples:** 1) We already know that  $\sin: \mathbb{R} \rightarrow \mathbb{R}$  has strict local maxima at the points  $\frac{\pi}{2} + 2k\pi$  and strict local minima at  $\frac{\pi}{2} + (2k+1)\pi$ , where  $k \in \mathbb{Z}$ . We can now confirm this earlier observation by observing  $\sin'(\frac{\pi}{2} + l\pi) = \cos(\frac{\pi}{2} + l\pi) = 0$  for all  $l \in \mathbb{Z}$

and that  $\sin''(\frac{\pi}{2} + 2k\pi) = -\sin(\frac{\pi}{2} + 2k\pi) = -1 < 0$ ,

$\sin''(\frac{\pi}{2} + (2k+1)\pi) = -\sin(\frac{\pi}{2} + (2k+1)\pi) = 1 > 0$ .

2) The function  $f: ]-1, 1[ \rightarrow \mathbb{R}$ ,  $f(x) = x^4$ , has a strict (global) minimum at 0 since  $f(x) = x^4 > 0$  for all  $x \neq 0$  and  $f(0) = 0$ . But we also have  $f''(0) = 0$ .



This example illustrates that the condition  $f''(x) > 0$  is *sufficient* but *not necessary* for a strict local extreme.

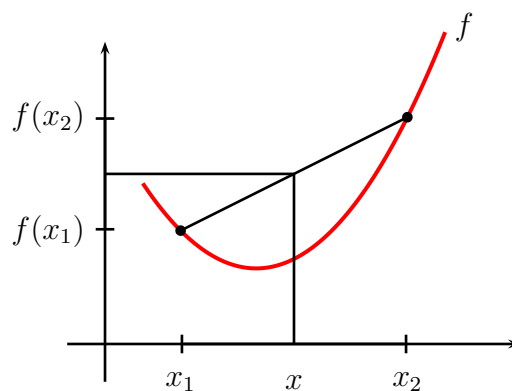
## 8.9. Convexity:

**DEFINITION:** Let  $I \subseteq \mathbb{R}$  be an interval and  $f: I \rightarrow \mathbb{R}$ . The function  $f$  is *convex* (*konvex*) if for all  $x_1, x_2 \in I$  and for all  $\lambda \in [0, 1]$

$$(8.2) \quad f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

$f$  is said to be *concave* (*konkav*) if  $-f$  is convex.

When the parameter  $\lambda$  runs through  $[0, 1]$  the corresponding point  $x = \lambda x_1 + (1 - \lambda)x_2 \in I$  runs through the closed interval with boundary points  $x_1$  and  $x_2$ . The inequality (8.2) means that the points  $(x, f(x))$  of the graph of  $f$  do not lie above the secant line through  $(x_1, f(x_1))$  and  $(x_2, f(x_2))$ .



**Remark:** A subset  $B$  of a real vector space  $W$  is *convex* (*konvex*) if for all  $u, v \in B$  the entire line segment  $\{\lambda u + (1 - \lambda)v : 0 \leq \lambda \leq 1\}$  connecting  $u$  and  $v$  belongs to  $B$ . In the definition given above the convexity of a function  $f: I \rightarrow \mathbb{R}$  corresponds to convexity of the subset  $B := \{(x, y) \in \mathbb{R}^2 : x \in I, y \geq f(x)\} \subseteq \mathbb{R}^2$  that lies above the graph of  $f$ .

**PROPOSITION:** Let  $I \subseteq \mathbb{R}$  be an open interval and  $f: I \rightarrow \mathbb{R}$  twice differentiable. Then

$$f''(x) \geq 0 \quad \forall x \in I \quad \iff \quad f \text{ is convex.}$$

*Proof.*  $(\Rightarrow)$  When checking (8.2) we may assume that  $x_1 < x_2$  and  $0 < \lambda < 1$ . If  $x := \lambda x_1 + (1 - \lambda)x_2$  then  $x_1 < x < x_2$ .

By the Proposition in 8.5  $f'$  is increasing and the mean value theorem provides us with  $\xi_1 \in ]x_1, x[$  and  $\xi_2 \in ]x, x_2[$  such that

$$(\star) \quad \frac{f(x) - f(x_1)}{x - x_1} = f'(\xi_1) \leq f'(\xi_2) = \frac{f(x_2) - f(x)}{x_2 - x}.$$

Since  $x - x_1 = (1 - \lambda)(x_2 - x_1) > 0$  and  $x_2 - x = \lambda(x_2 - x_1) > 0$  we obtain from  $(\star)$  that

$$\frac{f(x) - f(x_1)}{1 - \lambda} \leq \frac{f(x_2) - f(x)}{\lambda},$$

which implies

$$f(x) \leq \lambda f(x_1) + (1 - \lambda)f(x_2),$$

thus the convexity of  $f$ .

$(\Leftarrow)$  (Proof by contradiction) Suppose there is an  $x_0 \in I$  such that  $f''(x_0) < 0$ .

Let  $\varphi(x) := f(x) - f'(x_0)(x - x_0)$  ( $x \in I$ ). Then  $\varphi: I \rightarrow \mathbb{R}$  is twice differentiable and  $\varphi'(x_0) = f'(x_0) - f'(x_0) = 0$ . Since  $\varphi''(x_0) = f''(x_0) < 0$  we deduce that  $\varphi$  possesses a strict local maximum at  $x_0$ .

Therefore we have  $\varepsilon_0 > 0$  (sufficiently small) such that  $\varphi(x) < \varphi(x_0)$  holds for all  $x \in U_\varepsilon(x_0) \subseteq I$ . In particular, we obtain for  $0 < \varepsilon < \varepsilon_0$  that  $\varphi(x_0 - \varepsilon) < \varphi(x_0)$  and  $\varphi(x_0 + \varepsilon) < \varphi(x_0)$ , which in turn yields

$$f(x_0) = \varphi(x_0) > \frac{1}{2}(\varphi(x_0 - \varepsilon) + \varphi(x_0 + \varepsilon)) = \frac{1}{2}(f(x_0 - \varepsilon) + f(x_0 + \varepsilon)).$$

If we put  $\lambda := 1/2$ ,  $x_1 := x_0 - \varepsilon$ , and  $x_2 := x_0 + \varepsilon$ , the latter inequality means  $f(\lambda x_1 + (1 - \lambda)x_2) > \lambda f(x_1) + (1 - \lambda)f(x_2)$  — a contradiction  $\zeta$  to the convexity of  $f$ .  $\square$

**EXAMPLES:** 1) Consider the general quadratic function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = ax^2 + bx + c$ , where  $a, b, c \in \mathbb{R}$  are constants. Since  $f''(x) = 2a$  we have:

$f$  is convex if and only if  $a > 0$



$f$  is concave if and only if  $a < 0$



2)  $\exp'' = (\exp')' = \exp' = \exp > 0$ , hence  $\exp$  is convex.

3)  $\log'(x) = 1/x$  ( $x > 0$ ) and  $\log''(x) = -1/x^2 < 0$ , hence  $\log: ]0, \infty[ \rightarrow \mathbb{R}$  is concave.

An interesting application of the concavity of  $\log$  is the following: Let  $p, q \in ]1, \infty[$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . For all  $x, y > 0$  we have that  $x/p + y/q = x/p + (1 - 1/p)y$  is a convex combination and thus

$$\log\left(\frac{1}{p}x + \frac{1}{q}y\right) \geq \frac{1}{p}\log(x) + \frac{1}{q}\log(y) = \log(x^{1/p}) + \log(y^{1/q}) = \log(x^{1/p} \cdot y^{1/q}).$$

Upon applying the exponential function we obtain the following inequality

$$(8.3) \quad \frac{x}{p} + \frac{y}{q} \geq x^{1/p} \cdot y^{1/q} \quad (x, y > 0; p, q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1).$$

As a special case with  $p = q = 2$  we have proved the inequality between the *arithmetic* and the *geometric mean* (*Ungleichung zwischen arithmetischem und geometrischem Mittel*)

$$(8.4) \quad \frac{x + y}{2} \geq \sqrt{xy} \quad (x, y \geq 0).$$

## 8.10. Generalized mean value theorem and de l'Hospital's rules:

**LEMMA:** Let  $f, g: [a, b] \rightarrow \mathbb{R}$  continuous and differentiable in  $]a, b[$ . Then there exists  $\xi \in ]a, b[$  such that

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

(Generalized mean value theorem.)

*Proof.* Define  $h: [a, b] \rightarrow \mathbb{R}$  by

$$h(x) := (f(b) - f(a))g(x) - (g(b) - g(a))f(x).$$

Then  $h$  is continuous on  $[a, b]$  and differentiable in  $]a, b[$ . Furthermore,  $h(a) = f(b)g(a) - g(b)f(a) = h(b)$  implies that there is some  $\xi \in ]a, b[$  such that  $0 = h'(\xi) = (f(b) - f(a))g'(\xi) - (g(b) - g(a))f'(\xi)$ , which proves the assertion.  $\square$

**PROPOSITION (RULES OF DE L'HOSPITAL<sup>3</sup>):** Let  $-\infty \leq a < b \leq \infty$  and  $f, g: ]a, b[ \rightarrow \mathbb{R}$  be differentiable functions such that  $g'(x) \neq 0$  for all  $a < x < b$ . Suppose that the following limit exists

$$(8.5) \quad \eta := \lim_{x \nearrow b} \frac{f'(x)}{g'(x)}.$$

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<sup>3</sup>Guillaume François Antoine, Marquis de l'Hospital (1661–1704) [gijom frãswa mar'ki də lopi'tal] was a French mathematician. He is the author of the first known textbook on differential calculus, *l'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes*, published in 1696. It includes the lectures of his teacher, Johann Bernoulli, who was paid 300 Francs a year to tell de L'Hospital about his discoveries which, including the above rule, were then published under the Marquis' name.

(i) If in addition  $\lim_{x \nearrow b} f(x) = \lim_{x \nearrow b} g(x) = 0$ , then  $g(x) \neq 0$  for all  $a < x < b$  and

$$\lim_{x \nearrow b} \frac{f(x)}{g(x)} = \eta.$$

(ii) If in addition  $\lim_{x \nearrow b} g(x) = \pm\infty$ , then there is some  $x_0 \in ]a, b[$  such that  $g(x) \neq 0$  for all  $x_0 < x < b$  and

$$\lim_{x \nearrow b} \frac{f(x)}{g(x)} = \eta.$$

Analogous statements hold for the limits  $x \searrow a$ .

**REMARK:** The exact same statement is true if the limit in (8.5) is improper (cf. [Heu88, Abschnitt 50]).

*Proof.* (i) Since  $g'$  never vanishes  $g$  is strictly increasing or decreasing, hence  $g$  is injective. Therefore  $\lim_{x \nearrow b} g(x) = 0$  implies that  $g(x) \neq 0$  for all  $x$  ( $g(x_0) = 0$  with  $x_0 < b$  would contradict the strict monotonicity).

Let  $\varepsilon > 0$ . By (8.5) we may choose  $\beta \in ]a, b[$  such that  $f'(r)/g'(r) \in U_\varepsilon(\eta)$  for all  $\beta < r < b$ .

Let  $x, y \in ]\beta, b[$ ,  $x \neq y$ . By the above Lemma there is some  $\xi$  between  $x$  and  $y$  such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)} \in U_\varepsilon(\eta).$$

In other words,  $\left| \frac{f(x) - f(y)}{g(x) - g(y)} - \eta \right| < \varepsilon$ . Since  $g(x) \neq 0$  we may send  $y \rightarrow b$  while keeping  $x$  fixed and obtain

$$\left| \frac{f(x)}{g(x)} - \eta \right| \leq \varepsilon,$$

which in turn shows that  $f(x)/g(x) \rightarrow \eta$  as  $x \rightarrow b$ .

(ii) There is some  $x_0 \in ]a, b[$  such that  $|g(x)| \geq 1$  for all  $x \geq x_0$ . In particular,  $g(x) \neq 0$  for  $x \geq x_0$ .

Let  $\varepsilon > 0$  and choose  $\beta \in ]x_0, b[$  such that

$$\forall r \geq \beta : \quad \frac{f'(r)}{g'(r)} \in U_\varepsilon(\eta) \quad \text{and} \quad |g(r)| \geq \frac{1}{\varepsilon}.$$

Let  $(x_n)$  be a sequence in  $]\beta, b[$  such that  $\lim x_n = b$ . For each  $n \in \mathbb{N}$  the above Lemma provides some  $y_n$  between  $\beta$  and  $x_n$  such that

$$\frac{f(x_n) - f(\beta)}{g(x_n) - g(\beta)} = \frac{f'(y_n)}{g'(y_n)} \in U_\varepsilon(\eta).$$

Thus we have

$$\frac{f(x_n)}{g(x_n)} = \frac{g(x_n) - g(\beta)}{g(x_n)} \cdot \frac{f'(y_n)}{g'(y_n)} + \frac{f(\beta)}{g(x_n)}$$

and

$$\begin{aligned} \left| \frac{f(x_n)}{g(x_n)} - \eta \right| &\leq \left| \frac{g(x_n) - g(\beta)}{g(x_n)} \cdot \frac{f'(y_n)}{g'(y_n)} - \eta \right| + \left| \frac{f(\beta)}{g(x_n)} \right| \\ &= \left| \frac{g(x_n) - g(\beta)}{g(x_n)} \left( \frac{f'(y_n)}{g'(y_n)} - \eta \right) + \left( \frac{g(x_n) - g(\beta)}{g(x_n)} - 1 \right) \eta \right| + \left| \frac{f(\beta)}{g(x_n)} \right| \\ &\leq \left| \frac{g(x_n) - g(\beta)}{g(x_n)} \right| \left| \frac{f'(y_n)}{g'(y_n)} - \eta \right| + \left| \frac{g(x_n) - g(\beta)}{g(x_n)} - 1 \right| |\eta| + \left| \frac{f(\beta)}{g(x_n)} \right| \\ &= \left| 1 - \frac{g(\beta)}{g(x_n)} \right| \left| \frac{f'(y_n)}{g'(y_n)} - \eta \right| + \left| -\frac{g(\beta)}{g(x_n)} \right| |\eta| + \left| \frac{f(\beta)}{g(x_n)} \right| \\ &\leq (1 + \varepsilon |g(\beta)|) \varepsilon + \varepsilon |g(\beta)| |\eta| + \varepsilon |f(\beta)| \\ &= (1 + \varepsilon |g(\beta)| + |g(\beta)| |\eta| + |f(\beta)|) \cdot \varepsilon \leq K \varepsilon, \end{aligned}$$

for some  $K > 0$  if  $\varepsilon$  stays less than 1, say. Therefore  $f(x_n)/g(x_n) \rightarrow \eta$  as  $n \rightarrow \infty$ .  $\square$

**EXAMPLES:** 1) We can give an alternative proof for 6.3,6):  $\lim_{x \rightarrow \infty} \frac{\log(x)}{x^\alpha} = 0$  ( $\alpha > 0$ ).

With  $f(x) := \log(x) \rightarrow \infty$ ,  $g(x) := x^\alpha \rightarrow \infty$ ,  $f'(x) = 1/x$ , and  $g'(x) = \alpha x^{\alpha-1}$  we obtain that

$$\eta := \frac{f'(x)}{g'(x)} = \frac{1}{\alpha x^\alpha} \rightarrow 0 \quad (x \rightarrow \infty),$$

hence Proposition (i) proves the above limit assertion.

$$2) \lim_{x \searrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \boxed{?}$$

Note that  $\frac{1}{\sin(x)} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$ , hence we attempt to apply the Proposition with  $f(x) := x - \sin x \rightarrow 0$  and  $g(x) := x \sin x \rightarrow 0$  as  $x \rightarrow 0$ .

We have

$$f'(x) = 1 - \cos x \rightarrow 0 \text{ and } g'(x) = \sin x + x \cos x \rightarrow 0 \text{ as } x \rightarrow 0$$

$$f''(x) = \sin x \rightarrow 0 \text{ and } g''(x) = \cos x + \cos x - x \sin x \rightarrow 2 \text{ as } x \rightarrow 0.$$

At this point we may observe that de l'Hospital's rule is applicable with  $f$  replaced by  $f'$  and  $g$  replaced by  $g'$ . Therefore we may summarize as follows

$$\lim_{x \searrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \searrow 0} \frac{f(x)}{g(x)} = \lim_{x \searrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \searrow 0} \frac{f''(x)}{g''(x)} = \frac{0}{2} = 0.$$



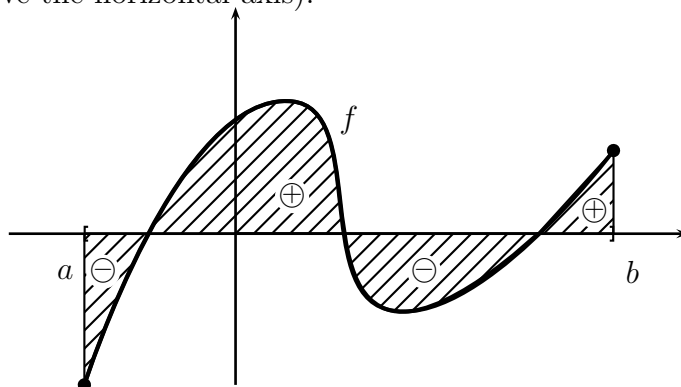
CHAPTER IV

# INTEGRATION

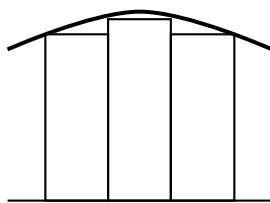
## §9. THE RIEMANN INTEGRAL

### 9.0. Two paths leading to the idea of integration:

(A) **Geometric motivation:** Suppose we are to determine whether more area of the subset of the plane between the horizontal axis and the graph of a function  $f: [a, b] \rightarrow \mathbb{R}$  is below or above the axis. Or, in case  $f(t) \geq 0$  for all  $t$ , we want to determine the area underneath the graph (and above the horizontal axis).



In the small, if  $f$  is reasonably flat, we could hope to approximate the area quite well by using narrow vertical rectangles:



Such a rectangular approximation can be thought of being the area underneath the graph of a simple function  $\varphi: [a, b] \rightarrow \mathbb{R}$  corresponding to a partition  $a = t_0 < t_1 \dots < t_n = b$  and satisfying

$$\varphi(t) = c_j \quad t \in [t_j, t_{j+1}[ \quad (j = 0, \dots, n-1)$$

for suitably chosen constants  $c_0, \dots, c_{n-1}$ . The approximate value of the area  $A$  is then given by

$$A \approx \sum_{j=0}^{n-1} c_j (t_{j+1} - t_j) = \sum_{j=0}^{n-1} \varphi(t_j) (t_{j+1} - t_j).$$

**(B) A basic problem from physics:** If  $x: [0, T] \rightarrow \mathbb{R}$  represents the position of a one-dimensional particle at all times  $t \in [0, T]$ , then the speed  $v(t)$  at time  $t$  is given by the derivative (assuming that it exists)

$$v(t) = \dot{x}(t) = \frac{dx(t)}{dt} \approx \frac{\Delta \text{ distance}}{\Delta \text{ time}}.$$

Suppose that we only know the initial position  $x_0 := x(0)$  of the particle and its speed at all times, that is,  $v: [0, T] \rightarrow \mathbb{R}$ . How can we recover the position (function)  $x: [0, T] \rightarrow \mathbb{R}$ ?

Within any short time interval  $[t_1, t_2] \subseteq [0, T]$  we might try to approximate the position  $x(t)$  as the gain in distance relative to the position  $x(t_1)$  according to the law ‘distance = speed  $\times$  time’ for a constant speed motion:

$$x(t) \approx x(t_1) + v(t_1)(t - t_1) \quad (t \in [t_1, t_2]).$$

We may successively carry out such approximations in time intervals  $[0, t_1], [t_1, t_2], \dots, [t_{n-1}, T]$ . Finally, we sum over the respective intervals (putting  $t_0 = 0$  and  $t_n = T$ ) and obtain

$$x(T) \approx x_0 + \sum_{j=0}^{n-1} v(t_j)(t_{j+1} - t_j).$$

We observe that the sum can be interpreted as the area between the horizontal axis and the graph of the simple function  $\varphi: [0, T] \rightarrow \mathbb{R}$  with values  $\varphi(t) := v(t_j)$  when  $t \in [t_j, t_{j+1}[$  ( $j = 0, \dots, n-1$ ).

We saw that in both cases 1) and 2) we could employ simple functions as the basic building blocks for an approximate solution of the posed problems. Thus we introduce a notation for the set of all simple functions on a given interval.

**DEFINITION:** Let  $a < b$ . We define  $\mathcal{T}[a, b] := \{\varphi: [a, b] \rightarrow \mathbb{R} : \varphi \text{ is a simple function}\}$ .

**9.1. Proposition:** (i) If  $\varphi, \psi \in \mathcal{T}[a, b]$  then  $\varphi + \psi \in \mathcal{T}[a, b]$ .

(ii) If  $\lambda \in \mathbb{R}$  and  $\varphi \in \mathcal{T}[a, b]$  then  $\lambda \cdot \varphi \in \mathcal{T}[a, b]$ .

In other words,  $\mathcal{T}[a, b]$  is a linear subspace of the real vector space  $\mathbb{R}^{[a, b]} = \{g: [a, b] \rightarrow \mathbb{R}\}$  of all real-valued functions on  $[a, b]$ .

*Proof.* (ii) is clear from the definition.

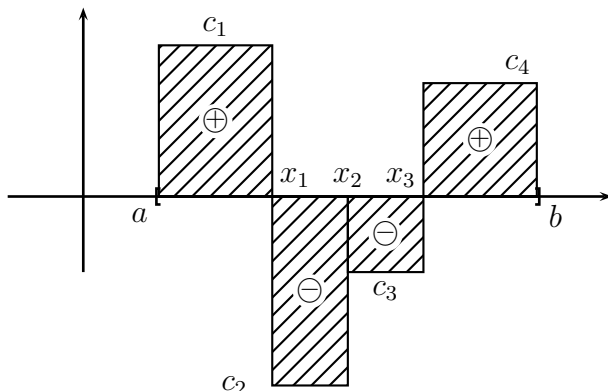
(i) Let  $\varphi$  be defined via the partition  $\mathcal{Z} = \{x_0, x_1, \dots, x_n\}$  with  $a = x_0 < x_1 < \dots < x_n = b$  and  $\psi$  be defined via the partition  $\mathcal{Z}' = \{x'_0, x'_1, \dots, x'_m\}$  with  $a = x'_0 < x'_1 < \dots < x'_m = b$ . Then we define the partition  $\tilde{\mathcal{Z}} := \mathcal{Z} \cup \mathcal{Z}'$ , which may be written in the form  $a = t_0 < t_1 < \dots < t_k = b$ , where each  $t_j$  ( $0 \leq j \leq k$ ) equals some  $x_l$  ( $0 \leq l \leq n$ ) or  $x'_p$  ( $0 \leq p \leq m$ ) and  $\max(n, m) \leq k \leq n + m$ .

Since each interval  $]t_{j-1}, t_j[$  is contained in some subinterval of the partition  $\mathcal{Z}$  and  $\mathcal{Z}'$  respectively, we have that  $\varphi$  and  $\psi$  is constant on each interval  $]t_{j-1}, t_j[$  ( $j = 1, \dots, k$ ). Hence also  $\varphi + \psi$  is constant on these intervals, which proves that  $\varphi + \psi$  is a simple function.  $\square$

## 9.2. Integral of simple functions:

**DEFINITION:** Let  $\varphi \in \mathcal{T}[a, b]$  with partition  $a = x_0 < \dots < x_n = b$  and values  $\varphi(x) = c_j$  when  $x \in ]x_{j-1}, x_j[$  ( $j = 1, \dots, n$ ). Then the integral<sup>1</sup> of  $\varphi$  over  $[a, b]$  is defined by

$$(9.1) \quad \int_a^b \varphi(x) dx := \sum_{j=1}^n c_j (x_j - x_{j-1}).$$



**LEMMA:**  $\varphi \mapsto \int_a^b \varphi(x) dx$  defines a map  $\mathcal{T}[a, b] \rightarrow \mathbb{R}$ , that is, the value of the integral does not depend on the choice of representation (and partition) of  $\varphi$ .

*Proof.* Denote by  $\mathcal{Z} : a = x_0 < \dots < x_n = b$  and  $\mathcal{Z}' : a = t_0 < \dots < t_m = b$  two partitions for  $\varphi$  with respective values

$$\varphi(x) = c_j \quad (x \in ]x_{j-1}, x_j[, j = 1, \dots, n) \quad \text{and} \quad \varphi(x) = c'_k \quad (x \in ]t_{k-1}, t_k[, k = 1, \dots, m)$$

and put

$$\int_{\mathcal{Z}} \varphi := \sum_{j=1}^n c_j (x_j - x_{j-1}), \quad \int_{\mathcal{Z}'} \varphi := \sum_{k=1}^m c'_k (t_k - t_{k-1}).$$

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<sup>1</sup>The term “integral” is derived from the Latin integer, meaning whole, untouched. The modern integral sign was introduced by Leibnitz in 1675. It was originally an abbreviation of the Latin word *summa*, written *fumma* in Leibnitz’s handwriting.

We have to show that  $\int_{\mathcal{Z}} \varphi = \int_{\mathcal{Z}'} \varphi$ .

*Case 1:*  $\mathcal{Z} \subseteq \mathcal{Z}'$

For all  $j \in \{0, \dots, n\}$  we can find  $k_j$  such that  $x_j = t_{k_j}$ . Hence we have

$$x_{j-1} = t_{k_{j-1}} < t_{k_{j-1}+1} < \dots < t_{k_j} = x_j \quad (1 \leq j \leq n)$$

and  $c_j = c'_l$  when  $k_{j-1} < l \leq k_j$ . Therefore

$$\begin{aligned} \int_{\mathcal{Z}'} \varphi &= \sum_{k=1}^m c'_k (t_k - t_{k-1}) = \sum_{j=1}^n \sum_{l=k_{j-1}+1}^{k_j} c'_l (t_l - t_{l-1}) \\ &= \sum_{j=1}^n c_j \underbrace{\sum_{l=k_{j-1}+1}^{k_j} (t_l - t_{l-1})}_{=t_{k_j} - t_{k_{j-1}} = x_j - x_{j-1}} = \sum_{j=1}^n c_j (x_j - x_{j-1}) = \int_{\mathcal{Z}} \varphi. \end{aligned}$$

*Case 2:*  $\mathcal{Z}' \subseteq \mathcal{Z}$  is treated as in case 1 with the roles of  $\mathcal{Z}$  and  $\mathcal{Z}'$  interchanged.

*Case 3:* Neither  $\mathcal{Z} \subseteq \mathcal{Z}'$  nor  $\mathcal{Z}' \subseteq \mathcal{Z}$ .

Define the common refinement of the partitions by  $\mathcal{Z}^* := \mathcal{Z} \cup \mathcal{Z}'$ . Then  $\mathcal{Z} \subseteq \mathcal{Z}^*$  and  $\mathcal{Z}' \subseteq \mathcal{Z}^*$ , thus we have

$$\int_{\mathcal{Z}} \varphi = \int_{\mathcal{Z}^*} \varphi = \int_{\mathcal{Z}'} \varphi$$

by cases 1 and 2. □

**9.3. Proposition (Linearity and monotonicity of the integral):** Let  $\varphi, \psi \in \mathcal{T}[a, b]$  and  $\lambda \in \mathbb{R}$ . We have the following:

$$(i) \int_a^b (\varphi + \psi)(x) dx = \int_a^b \varphi(x) dx + \int_a^b \psi(x) dx$$

$$(ii) \int_a^b (\lambda \varphi)(x) dx = \lambda \int_a^b \varphi(x) dx$$

$$(iii) \text{ If } \varphi \leq \psi, \text{ i.e., } \varphi(x) \leq \psi(x) \text{ for all } x, \text{ then } \int_a^b \varphi(x) dx \leq \int_a^b \psi(x) dx.$$

In other words, the map  $\varphi \mapsto \int_a^b \varphi(x) dx, \mathcal{T}[a, b] \rightarrow \mathbb{R}$  is a linear monotone functional.

*Proof.* Use a common partition for both functions  $\varphi$  and  $\psi$  (as in the previous proofs), then the assertions follow from corresponding properties of the defining finite sums. □

**9.4. Definition (Riemann-integral):** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a bounded function.

(i) We define the *upper integral* of  $f$  by

$$(9.2) \quad \int_a^b f(x) dx := \inf \left\{ \int_a^b \psi(x) dx : \psi \in \mathcal{T}[a, b], f \leq \psi \right\}$$

and the *lower integral* of  $f$  by

$$(9.3) \quad \int_a^b f(x) dx := \sup \left\{ \int_a^b \varphi(x) dx : \varphi \in \mathcal{T}[a, b], \varphi \leq f \right\}.$$

Note that we always have  $\int_a^b f(x) dx \geq \int_a^b f(x) dx$ .

(ii) The function  $f$  is *Riemann<sup>2</sup>-integrable* (*Riemann-integrierbar*) if

$$\int_a^b f(x) dx = \int_a^b f(x) dx.$$

In this case we define the *R-integral* of  $f$  from  $a$  to  $b$  by

$$(9.4) \quad \int_a^b f(x) dx := \int_a^b f(x) dx.$$

Occasionally we will also employ the short-hand notation  $\int_a^b f$  instead of  $\int_a^b f(x) dx$ .

**9.5. Examples:** 1) If  $\varphi \in \mathcal{T}[a, b]$ , then clearly the lower integral equals the upper integral, thus  $\varphi$  is R-integrable and the R-integral coincides with the integral of simple functions as defined above.

2) The Dirichlet function  $\mathbf{1}_{\mathbb{Q}}: \mathbb{R} \rightarrow \mathbb{R}$  is bounded, thus its restriction  $f := \mathbf{1}_{\mathbb{Q}}|_{[0,1]}: [0, 1] \rightarrow \mathbb{R}$  is bounded. But  $f$  is not R-integrable: Indeed, since every open interval contains rational as well irrational numbers, we have that

$$\psi \in \mathcal{T}[0, 1], \psi \geq f \implies \psi \geq 1 \text{ on every open subinterval of any partition}$$

and

$$\varphi \in \mathcal{T}[0, 1], \varphi \leq f \implies \varphi \leq 0 \text{ on every open subinterval of any partition.}$$

$$\text{Therefore } \int_0^1 f(x) dx = 1 \neq 0 = \int_0^1 f(x) dx.$$

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<sup>2</sup>Georg Friedrich Bernhard Riemann (1826–1866) [*ˈɡeːɔk ˈfʁiːtʁɪç ˈbɛənhɑːt ˈʁiːmən*], German mathematician

For practical purposes as well as for a further theoretical investigation it is essential to have a list of basic properties of the integral and alternative characterizations of R-integrability at hand. To develop such is our task in the remainder of this section.

**9.6. Theorem:** Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded. The following are equivalent:

- (i)  $f$  is R-integrable
- (ii) For all  $\varepsilon > 0$  we can find  $\varphi, \psi \in \mathcal{T}[a, b]$  with  $\varphi \leq f \leq \psi$  such that

$$0 \leq \int_a^b \psi(x) dx - \int_a^b \varphi(x) dx \leq \varepsilon.$$

*Proof.* This follows directly from the definition of the R-integral and the general properties of the infimum and the supremum.  $\square$

**9.7. Corollary:** Every continuous function  $f: [a, b] \rightarrow \mathbb{R}$  is R-integrable.

*Proof.* Let  $\varepsilon > 0$ . By Proposition 5.11 we can find  $\varphi, \psi \in \mathcal{T}[a, b]$  with  $\varphi \leq f \leq \psi$  such that

$$\sup_{x \in [a, b]} (\psi(x) - \varphi(x)) \leq \frac{\varepsilon}{b - a}.$$

Then monotonicity and additivity of the integral gives

$$0 \leq \int_a^b \psi(x) dx - \int_a^b \varphi(x) dx = \int_a^b (\psi - \varphi)(x) dx \leq \frac{\varepsilon}{b - a} \int_a^b 1 dx = \varepsilon,$$

which implies R-integrability by the above theorem.  $\square$

**9.8. Corollary:** Every monotone function  $f: [a, b] \rightarrow \mathbb{R}$  is R-integrable.

**REMARK:** Note that a monotone function on a bounded closed interval is necessarily bounded, since  $f([a, b]) \subseteq [\min(f(a), f(b)), \max(f(a), f(b))]$ .

*Proof.* We may assume that  $f$  is increasing, since the case  $f$  being decreasing is analogous.

For any  $n \in \mathbb{N}$ ,  $n \geq 1$ , we define an equidistant partition of  $[a, b]$  by  $x_k := a + k(b - a)/n$  ( $k = 0, \dots, n$ ).

Associated with this partition we define simple functions  $\varphi, \psi \in \mathcal{T}[a, b]$  as follows:

$$\begin{aligned} \varphi(x) = f(x_{k-1}) \text{ and } \psi(x) = f(x_k) \text{ when } x \in [x_{k-1}, x_k[ \quad (k = 0, \dots, n), \\ \text{and } \varphi(b) = \psi(b) = f(b). \end{aligned}$$

Since  $f$  is increasing we have  $\varphi \leq f \leq \psi$ . Furthermore, we obtain

$$\begin{aligned} 0 \leq \int_a^b \psi(x) dx - \int_a^b \varphi(x) dx &= \sum_{k=1}^n f(x_k)(x_k - x_{k-1}) - \sum_{k=1}^n f(x_{k-1})(x_k - x_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=1}^n (f(x_k) - f(x_{k-1})) = \frac{b-a}{n} (f(x_n) - f(x_0)) \\ &= \frac{(b-a)(f(b) - f(a))}{n} \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Hence  $0 \leq \int_a^b \psi - \int_a^b \varphi \leq \varepsilon$  for sufficiently large  $n$ , which proves R-integrability due to the above theorem.  $\square$

**9.9. Proposition (Linearity and monotonicity again):** Let  $f, g: [a, b] \rightarrow \mathbb{R}$  be R-integrable and  $\lambda \in \mathbb{R}$ . Then the following statements hold:

(i)  $f + g$  is R-integrable and

$$\int_a^b (f + g)(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

(ii)  $\lambda f$  is R-integrable and

$$\int_a^b (\lambda f)(x) dx = \lambda \int_a^b f(x) dx.$$

(iii) If  $f \leq g$  then  $\int_a^b f(x) dx \leq \int_a^b g(x) dx$ .

*Proof.* (iii) is immediate from the definition.

(i): Let  $\varepsilon > 0$ . We may choose  $\varphi_j, \psi_j \in \mathcal{T}[a, b]$  ( $j = 1, 2$ ) such that  $\varphi_1 \leq f \leq \psi_1$ ,  $\varphi_2 \leq g \leq \psi_2$  and  $\int_a^b \psi_j - \int_a^b \varphi_j \leq \varepsilon/2$  ( $j = 1, 2$ ). Then clearly

$$\varphi_1 + \varphi_2 \leq f + g \leq \psi_1 + \psi_2 \quad \text{and} \quad \int_a^b (\psi_1 + \psi_2)(x) dx - \int_a^b (\varphi_1 + \varphi_2)(x) dx \leq \varepsilon,$$

thus  $f + g$  is R-integrable.



Furthermore, since  $\int_a^b(\psi_1 + \psi_2) = \int_a^b \psi_1 + \int_a^b \psi_2$  and  $\int_a^b(\varphi_1 + \varphi_2) = \int_a^b \varphi_1 + \int_a^b \varphi_2$  we obtain

$$\int_a^b f + \int_a^b g = \int_a^b f + \int_a^b g \leq \int_a^b (f + g) \leq \int_a^* (f + g) \leq \int_a^* f + \int_a^* g = \int_a^b f + \int_a^b g,$$

therefore equality throughout and thus the asserted equation.

The proof of (ii) is left as an exercise (hint: consider the cases  $\lambda = 0$ ,  $\lambda > 0$ ,  $\lambda = -1$ , and  $\lambda < 0$  separately).  $\square$

**9.10. Positive and negative part of a function:** Let  $D$  be an arbitrary set and  $f: D \rightarrow \mathbb{R}$  be a real-valued function. Then the positive part  $f_+: D \rightarrow \mathbb{R}$  and the negative part  $f_-: D \rightarrow \mathbb{R}$  of  $f$  are defined as follows:

$$f_+(x) := \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_-(x) := \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have  $f_+(x) = \max(f(x), 0)$ ,  $f_-(x) = -\min(f(x), 0)$  (for all  $x \in D$ ) and

$$(9.5) \quad f = f_+ - f_- \quad |f| = f_+ + f_-.$$

**9.11. Proposition:** Let  $f, g: [a, b] \rightarrow \mathbb{R}$  be R-integrable, then the following statements hold:

(i)  $f_+$ ,  $f_-$ , and  $|f|$  are R-integrable as well and we have the triangle inequality for integrals

$$(9.6) \quad \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

(ii) For all  $p \in [1, \infty[$ :  $|f|^p$  is R-integrable.

(iii)  $f \cdot g$  is R-integrable.

*Proof.* (i): Let  $\varepsilon > 0$ . Choose  $\varphi, \psi \in \mathcal{T}[a, b]$  with  $\varphi \leq f \leq \psi$  such that  $0 \leq \int_a^b \psi - \int_a^b \varphi \leq \varepsilon$ . Then  $\varphi_+, \psi_+$  (as well as  $\varphi_-, \psi_-$ ) also belong to  $\mathcal{T}[a, b]$  and we have  $\varphi_+ \leq f_+ \leq \psi_+$  (as well as  $\varphi_- \leq f_- \leq \psi_-$ ). Since  $\psi_+ - \psi_- = -\psi_- \leq -\varphi_- = \varphi_+ - \varphi_-$  we obtain

$$0 \leq \int_a^b \psi_+ - \int_a^b \varphi_+ = \int_a^b (\psi_+ - \varphi_+) \leq \int_a^b (\psi - \varphi) \leq \varepsilon,$$

hence the R-integrability of  $f_+$ . Similarly, we can prove that  $f_-$  is R-integrable.

Using the relation  $|f| = f_+ + f_-$  we deduce that  $|f|$  is R-integrable. Since  $-f \leq |f|$  and  $f \leq |f|$  Equation (9.6) follows from the monotonicity and linearity of the integral.

(ii): Assume that  $f$  is bounded by  $K > 0$ , i.e.,  $0 \leq |f(x)| \leq K$  for all  $x \in [a, b]$ .

Let  $\varepsilon > 0$ . Since by (i)  $|f|$  is R-integrable and  $0 \leq |f| \leq K$  we can find simple functions  $\varphi, \psi \in \mathcal{T}[a, b]$  with  $0 \leq \varphi \leq |f| \leq \psi \leq K$  such that  $0 \leq \int_a^b \psi - \int_a^b \varphi \leq \varepsilon / (pK^{p-1})$ .

Consider  $h: [0, K] \rightarrow \mathbb{R}$  given by  $h(x) := x^p$ . By the mean value theorem of differential calculus we have for all  $y, z \in [0, K]$

$$|h(z) - h(y)| \leq \sup_{\xi \in [0, K]} |h'(\xi)| \cdot |z - y| = pK^{p-1}|z - y|.$$

Inserting  $z = \psi(x)$  and  $y = \varphi(x)$  we obtain for all  $x \in [a, b]$

$$|\psi^p(x) - \varphi^p(x)| \leq pK^{p-1}|\psi(x) - \varphi(x)|.$$

Note that  $\varphi^p$  and  $\psi^p$  are simple functions satisfying  $0 \leq \varphi^p \leq |f|^p \leq \psi^p$ . Moreover, by the previous estimate we obtain

$$0 \leq \int_a^b \psi^p - \int_a^b \varphi^p = \int_a^b (\psi^p - \varphi^p) \leq pK^{p-1} \int_a^b (\psi - \varphi) \leq \varepsilon.$$

Therefore we conclude that  $|f|^p$  is R-integrable.

(iii): Since  $f \pm g$  is R-integrable the assertion follows from (ii) using the decomposition  $fg = ((f + g)^2 - (f - g)^2)/4$ .  $\square$

**9.12. Proposition (Mean value theorem of integral calculus):** Let  $f, \varphi: [a, b] \rightarrow \mathbb{R}$  be continuous and assume that  $\varphi \geq 0$ . Then there exists  $\xi \in [a, b]$  such that

$$(9.7) \quad \int_a^b f(x)\varphi(x) dx = f(\xi) \int_a^b \varphi(x) dx.$$

In particular, if  $\varphi(x) := 1$  for all  $x$  we obtain

$$(9.8) \quad \int_a^b f(x) dx = f(\xi) (b - a).$$

**REMARK:** The number

$$M(f) := \frac{1}{b-a} \int_a^b f(x) dx$$

is called the *mean (value)* of  $f$  on  $[a, b]$ .

*Proof.*  $f$  is bounded, hence the values of

$$m := \inf\{f(x) : x \in [a, b]\}, \text{ and } M := \sup\{f(x) : x \in [a, b]\}$$

are finite. Since  $\varphi \geq 0$  we have  $m\varphi \leq f\varphi \leq M\varphi$  and therefore

$$m \int_a^b \varphi \leq \int_a^b (f\varphi) \leq M \int_a^b \varphi.$$

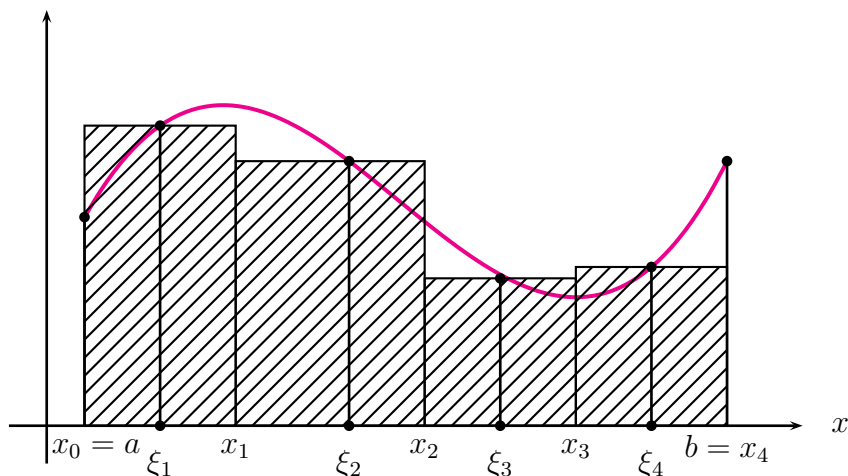
Thus there must be a number  $\mu \in [m, M]$  such that  $\int_a^b (f\varphi) = \mu \int_a^b \varphi$ . Since  $f$  is continuous the intermediate value theorem guarantees the existence of some  $\xi \in [a, b]$  such that  $\mu = f(\xi)$ . Thus the assertion follows.  $\square$

**9.13. Riemann sums:** Here we discuss an important alternative to compute (and define) R-integrals. The following approach is, in fact, in the spirit of Riemann's original treatment.

**DEFINITION:** Let  $f: [a, b] \rightarrow \mathbb{R}$  and  $\mathcal{Z} : a = x_0 < x_1 < \dots < x_n = b$  be a partition of  $[a, b]$ . For all  $k \in \{1, \dots, n\}$  let  $\xi_k$  be an arbitrary point (*Stützstelle*) in  $[x_{k-1}, x_k]$ . We summarize all division points  $x_0, \dots, x_n$  and intermediate points  $\xi_1, \dots, \xi_n$  in the notation  $\mathfrak{Z} := ((x_k)_{k=0}^n; (\xi_j)_{j=1}^n)$  and call

$$(9.9) \quad S(\mathfrak{Z}, f) := \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1})$$

the *Riemann sum* (*Riemann-Summe*) of  $f$  with respect to  $\mathfrak{Z}$ . We also introduce the value  $\mu(\mathfrak{Z}) := \max_{1 \leq k \leq n} (x_k - x_{k-1})$  as a measure of the (maximum size of) subintervals of the partition  $\mathcal{Z}$  (*Feinheit der Zerlegung*). (Note that  $\mu(\mathfrak{Z})$  depends only on the division points in  $\mathcal{Z}$ , and not on the  $\xi_k$ . But it is common practice to associate it with  $\mathfrak{Z}$ , thus the notation.)



We see that the Riemann sum can be interpreted as the integral of a simple function  $\varphi$  with values  $\varphi(x) = f(\xi_k)$  when  $x \in ]x_{k-1}, x_k[$ , i.e.,

$$S(\mathfrak{Z}, f) = \int_a^b \varphi(x) dx,$$

where  $\varphi$  interpolates  $f$  at the points  $\xi_1, \dots, \xi_n$ . Riemann's idea was to consider the limit behavior of  $S(\mathfrak{Z}, f)$  as  $\mu(\mathfrak{Z}) \rightarrow 0$ , and, if the limit exists, take this to be the integral of  $f$ .

**THEOREM:** Let  $f: [a, b] \rightarrow \mathbb{R}$  be bounded. Then the following are equivalent:

- (i)  $f$  is R-integrable
- (ii) There exists  $s \in \mathbb{R}$  with the following property:  $\forall \varepsilon > 0 \exists \delta > 0$  such that for any choice of  $\mathfrak{Z}$  with  $\mu(\mathfrak{Z}) < \delta$  we have

$$|S(\mathfrak{Z}, f) - s| \leq \varepsilon.$$

Moreover, in this case  $s = \int_a^b f(x) dx$ . (Hence the integral can then be obtained as the limit of Riemann sums.)

**REMARK:** One can show that property (ii) in the above theorem implies that  $f$  is bounded on  $[a, b]$  (cf. [Heu88, Satz 79.7]).

*Proof.* (i)  $\Rightarrow$  (ii): For arbitrary  $\mathfrak{Z}$  and any choice  $\varphi, \psi \in \mathcal{T}[a, b]$  with  $\varphi \leq f \leq \psi$  we clearly have

$$S(\mathfrak{Z}, \varphi) \leq S(\mathfrak{Z}, f) \leq S(\mathfrak{Z}, \psi).$$

Therefore it suffices to prove that  $S(\mathfrak{Z}, \varphi) \rightarrow \int_a^b \varphi(x) dx$ , respectively  $S(\mathfrak{Z}, \psi) \rightarrow \int_a^b \psi(x) dx$ , as  $\mu(\mathfrak{Z}) \rightarrow 0$ , since due to the Sandwich property this will imply that  $S(\mathfrak{Z}, f) \rightarrow \int_a^b f(x) dx$  upon letting the integrals over the simple functions approach the integral over  $f$ .

Thus it remains to prove the following *assertion*: If  $h \in \mathcal{T}[a, b]$  then property (ii) holds for  $h$  (with  $s = \int_a^b h$  instead).

Let  $\mathcal{Z} : a = t_0 < t_1 < \dots < t_m = b$  be a partition for  $h$ . Let  $\mathfrak{Z} = ((x_k)_{k=0}^n; (\xi_j)_{j=1}^n)$  and define the simple function  $\tilde{h} \in \mathcal{T}[a, b]$  as follows:

$$\tilde{h}(a) := h(a), \text{ and } \tilde{h}(x) := h(\xi_k) \text{ for } x \in ]x_{k-1}, x_k] \text{ (} 1 \leq k \leq n \text{)}.$$

Then we have

$$S(\mathfrak{Z}, h) = \int_a^b \tilde{h}(x) dx$$

and therefore

$$\left| \int_a^b h(x) dx - S(\mathfrak{Z}, h) \right| = \left| \int_a^b (h - \tilde{h})(x) dx \right| \leq \int_a^b |h(x) - \tilde{h}(x)| dx.$$

Whenever  $]x_{k-1}, x_k] \cap \{t_1, \dots, t_m\} = \emptyset$  we have  $(h - \tilde{h})(x) = 0$  for all  $x \in ]x_{k-1}, x_k]$ .

Since every  $t_j$  ( $1 \leq j \leq m$ ) can lie in at most one of the subintervals  $]x_{k-1}, x_k]$  we can have  $(h - \tilde{h}) \neq 0$  in at most  $m$  intervals of length at most  $\mu(\mathfrak{Z})$  each. This gives a total length of at most  $m \mu(\mathfrak{Z})$  of the union of subintervals where the above integrand  $|h - \tilde{h}|$  can be non-zero.

The simple functions  $h$  and  $\tilde{h}$  are bounded, hence there is some  $M > 0$  such that  $|h(x) - \tilde{h}(x)| \leq M$  for all  $x \in [a, b]$ .

In summary, we obtain

$$\left| \int_a^b h(x) dx - S(\mathfrak{Z}, h) \right| \leq \int_a^b |h(x) - \tilde{h}(x)| dx \leq m \mu(\mathfrak{Z}) M,$$

where the upper bound becomes arbitrarily small as  $\mu(\mathfrak{Z}) \rightarrow 0$ .

(ii)  $\Rightarrow$  (i): Let  $s$  be the limit of the Riemann sums according to (ii). Let  $\varepsilon > 0$  arbitrary.

We may choose  $\delta > 0$  such that the statement in (ii) holds with this  $\delta$ . Let  $\mathfrak{Z} = ((x_k), (\xi_k))$  with  $\mu(\mathfrak{Z}) < \delta$ .

We define the simple functions  $\varphi, \psi$  as follows: Put  $\varphi(a) = \psi(a) = f(a)$  and for  $x \in ]x_{k-1}, x_k]$

$$\varphi(x) = m_k := \inf \{f(x) : x_{k-1} < y \leq x_k\}, \quad \psi(x) = M_k := \sup \{f(x) : x_{k-1} < y \leq x_k\}.$$

Then clearly  $\varphi \leq f \leq \psi$ .

For each  $k \geq 1$  choose  $\alpha_k, \beta_k \in ]x_{k-1}, x_k]$  such that

$$m_k \leq f(\alpha_k) < m_k + \varepsilon, \quad M_k - \varepsilon < f(\beta_k) \leq M_k.$$

Put  $\mathfrak{Z}' := ((x_k), (\alpha_k))$  and  $\mathfrak{Z}'' := ((x_k), (\beta_k))$ , then  $\mu(\mathfrak{Z}') < \delta$  and  $\mu(\mathfrak{Z}'') < \delta$  holds. Therefore, by assumption, we have

$$(\star) \quad |s - S(\mathfrak{Z}', f)| \leq \varepsilon \quad \text{and} \quad |s - S(\mathfrak{Z}'', f)| \leq \varepsilon.$$

By construction,  $m_k = \varphi(\alpha_k) \leq f(\alpha_k) < \varphi(\alpha_k) + \varepsilon$  and therefore

$$S(\mathfrak{Z}', \varphi) \leq S(\mathfrak{Z}', f) \leq S(\mathfrak{Z}', \varphi) + \varepsilon(b - a),$$

which combined with  $(\star)$  gives

$$(\star\star) \quad S(\mathfrak{Z}', \varphi) - \varepsilon \leq s \leq S(\mathfrak{Z}', \varphi) + \varepsilon + \varepsilon(b - a).$$

Taking the supremum in  $(\star\star)$  over all  $\varphi \in \mathcal{T}[a, b]$  with  $\varphi \leq f$  yields

$$\int_a^b f - \varepsilon \leq s \leq \int_a^b f + \varepsilon(1 + b - a).$$

Since  $\varepsilon > 0$  was arbitrary we have, in fact, shown that  $\int_a^b f(x) dx = s$ .

Similarly it follows that also  $\int_a^b f(x) dx = s$ , which in summary proves that  $f$  is R-integrable and  $\int_a^b f(x) dx = s$ .  $\square$

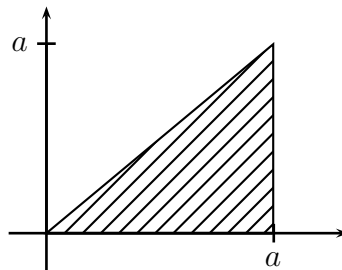
**EXAMPLES:** We illustrate the technique of Riemann sums in the computation of two integrals. Both could, of course, be determined quickly by using *primitive functions* (*Stammfunktionen*) as discussed in the following section.

1) Let  $a > 0$ . We design particular Riemann sums to determine the value of  $\int_0^a x dx$  (note that the integrand is continuous, hence R-integrable): For  $n \in \mathbb{N}$ ,  $n \geq 1$ , consider the division points  $x_k := ka/n$  ( $k = 0, \dots, n$ ) and intermediate points  $\xi_k := x_k$  ( $k = 1, \dots, n$ ). Let  $\mathfrak{Z} := ((x_k), (\xi_l))$ , then  $\mu(\mathfrak{Z}) = a/n \rightarrow 0$  as  $n \rightarrow \infty$ . The corresponding Riemann sum for  $f := \text{id}|_{[0, a]}$  gives an approximating sequence

$$S_n := S(\mathfrak{Z}, f) = \sum_{k=1}^n \frac{ka}{n} \cdot \frac{a}{n} = \frac{a^2}{n^2} \sum_{k=1}^n k = \frac{a^2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{a^2}{2} \left(1 + \frac{1}{n}\right).$$

Since  $S_n \rightarrow a^2/2$  as  $n \rightarrow \infty$  we obtain that  $\int_0^a x dx = \frac{a^2}{2}$ .

This is also clear by looking at the triangular area underneath the graph of  $f$ :



2) Let  $a > 0$ . We determine  $\int_0^a \cos(x) dx$  via Riemann sums:

Define  $\mathfrak{Z}$  for  $n \in \mathbb{N}$ ,  $n \geq 1$  as above by  $x_k := ka/n$  and  $\xi_k = x_k$ . Then

$$S_n := S(\mathfrak{Z}, \cos) = \sum_{k=1}^n \cos\left(\frac{ka}{n}\right) \cdot \frac{a}{n} = \frac{a}{n} \cdot \sum_{k=1}^n \cos\left(\frac{ka}{n}\right).$$

**MINI-LEMMA:** Let  $n \in \mathbb{N}$ ,  $n \geq 1$ , and  $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ . Then we have

$$(9.10) \quad \frac{1}{2} + \sum_{k=1}^n \cos(kt) = \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{t}{2}\right)}.$$

*Mini-Proof.* Since  $\cos(kt) = (e^{ikt} + e^{-ikt})/2$  and  $e^{it} \neq 1$  we have

$$\begin{aligned} \frac{1}{2} + \sum_{k=1}^n \cos(kt) &= \frac{1}{2} \sum_{k=-n}^n e^{ikt} = \frac{e^{-int}}{2} \sum_{k=0}^{2n} (e^{it})^k \\ &= \frac{e^{-int}}{2} \cdot \frac{e^{i(2n+1)t} - 1}{e^{it} - 1} = \frac{1}{2} \cdot \frac{e^{it} e^{int} - e^{-int}}{e^{it} - 1} \\ &= \frac{1}{2} \cdot \frac{e^{i(n+1/2)t} - e^{-i(n+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{1}{2} \cdot \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{\sin\left(\frac{t}{2}\right)}. \end{aligned}$$

□

Applying (9.10) to the above Riemann sum we obtain for  $n$  large enough such that  $a/n \notin 2\pi\mathbb{Z}$

$$S_n = \frac{a}{2n} \left( \frac{\sin\left(\left(n + \frac{1}{2}\right)\frac{a}{n}\right)}{\sin\left(\frac{a}{2n}\right)} - 1 \right) = \underbrace{\frac{\frac{a}{2n}}{\sin\left(\frac{a}{2n}\right)}}_{\rightarrow 1} \cdot \underbrace{\sin\left(\left(1 + \frac{1}{2n}\right)a\right)}_{\rightarrow \sin(a)} - \underbrace{\frac{a}{2n}}_{\rightarrow 0} \rightarrow \sin(a) \quad (n \rightarrow \infty).$$

Therefore we have shown that  $\int_0^a \cos(x) dx = \sin(a)$ .

**9.14. Integration over subintervals and orientation:** Let  $a < b < c$  and  $f: [a, c] \rightarrow \mathbb{R}$  be bounded. By restricting and patching together corresponding simple functions one easily proves that

$$f \text{ is R-integrable} \iff \text{both } f|_{[a,b]} \text{ and } f|_{[b,c]} \text{ are R-integrable.}$$

In this case we also have that  $\int_a^c f = \int_a^b f + \int_b^c f$ .

We will use the following convention in the sequel: If  $a < b$  we put

$$\int_b^a f(x) dx := - \int_a^b f(x) dx,$$

which reflects the idea that the  $x$ -axis is oriented by considering the direction from smaller to larger  $x$ -values positive, and negative otherwise.

Finally, we also put  $\int_a^a f(x) dx := 0$ .

(This is also the value of the limit of  $\int_{a-\delta}^{a+\delta} f(x) dx$  as  $\delta \rightarrow 0$ , if  $f$  is defined and integrable on some interval  $[a - c, a + c]$  for some  $c > 0$ .)



## §10. INTEGRAL AND DERIVATIVE; TAYLOR'S THEOREM

Throughout this section let  $I \subseteq \mathbb{R}$  denote an interval.

### 10.1. The fundamental theorem of calculus *(Der Hauptsatz der Differential- und Integralrechnung)*:

**DEFINITION:** (i) Let  $f: I \rightarrow \mathbb{R}$ . A differentiable function  $F: I \rightarrow \mathbb{R}$  is called a *primitive (function) of  $f$  (on  $I$ )* *(Stammfunktion von  $f$  (auf  $I$ ))* if  $F' = f$  (on  $I$ ).

(ii) A function  $g: I \rightarrow \mathbb{R}$  is said to be *continuously differentiable* *(stetig differenzierbar)* if  $g$  is differentiable (on  $I$ ) and  $g': I \rightarrow \mathbb{R}$  is continuous.

Let  $k \in \mathbb{N}$ ,  $k \geq 1$ . The function  $g$  is  *$k$ -times continuously differentiable* if  $g$  is  $k$ -times differentiable (in all points of  $I$ ) and  $g^{(k)}: I \rightarrow \mathbb{R}$  is continuous. We denote the set of  $k$ -times continuously differentiable functions  $I \rightarrow \mathbb{R}$  by  $\mathcal{C}^k(I, \mathbb{R})$ . By convention, we also write  $\mathcal{C}^0(I, \mathbb{R}) := \mathcal{C}(I, \mathbb{R}) = \{f: I \rightarrow \mathbb{R} : f \text{ is continuous}\}$ .

Two questions naturally arise about primitive functions:

How many primitives does a given function possess?

How can we systematically determine primitive functions?

**PROPOSITION:** Let  $F: I \rightarrow \mathbb{R}$  be a primitive of  $f: I \rightarrow \mathbb{R}$ . For a function  $G: I \rightarrow \mathbb{R}$  we have:

$$G \text{ is a primitive of } f \iff F - G \text{ is a constant function.}$$

As a consequence, in order to determine all primitive functions of  $f$  it suffices to find a particular primitive  $F$ . Then any primitive of  $f$  has to be of the form  $x \mapsto F(x) + c$ , where  $c \in \mathbb{R}$ .

*Proof.*  $(\Rightarrow)$  If  $G$  is also a primitive of  $f$ , then  $G' = f = F'$ . Hence  $(F - G)'(x) = 0$  for all  $x \in I$ , which implies that  $F - G$  is constant.

$(\Leftarrow)$  Let  $c \in \mathbb{R}$  and  $G(x) = F(x) + c$  ( $x \in I$ ). Then  $G$  is differentiable as well and  $G' = (F + c)' = F' = f$ .  $\square$

It remains to address the question how to determine a primitive. For continuous functions this is settled by the following result.

**THEOREM (FUNDAMENTAL THEOREM OF CALCULUS):** Let  $f: I \rightarrow \mathbb{R}$  be continuous and  $a, b \in I$  arbitrary.

(i) If we define  $F: I \rightarrow \mathbb{R}$  by

$$(10.1) \quad F(x) := \int_a^x f(y) dy \quad (x \in I),$$

then  $F$  is continuously differentiable (in short:  $f$  is  $\mathcal{C}^1$ ) and  $F' = f$ . In particular,  $F$  is a primitive of  $f$ .

(ii) If  $F: I \rightarrow \mathbb{R}$  is an arbitrary primitive of  $f$ , then

$$(10.2) \quad \int_a^b f(x) dx = F(b) - F(a) =: F(x) \Big|_a^b$$

holds.

*Proof.* (i): Let  $x \in I$  and  $h \neq 0$  such that  $x + h \in I$ . Then we have

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \left( \int_a^{x+h} f - \int_a^x f \right) = \frac{1}{h} \int_x^{x+h} f(y) dy.$$

By the mean value theorem for integrals there is some  $\xi_h$  between  $x$  and  $x+h$  such that

$$\int_x^{x+h} f(y) dy = h f(\xi_h).$$

Note that  $\xi_h \rightarrow x$  as  $h \rightarrow 0$ , since  $|x - \xi_h| \leq |x - (x+h)| = |h|$ . Thus by continuity of  $f$  we obtain

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(y) dy = \lim_{h \rightarrow 0} f(\xi_h) = f(x),$$

which implies that  $F'(x) = f(x)$ .

(ii): Define the primitive  $G: I \rightarrow \mathbb{R}$  of  $f$  according to formula (10.1), i.e.,  $G(x) := \int_a^x f(y) dy$ . By the above proposition there exists a constant  $c \in \mathbb{R}$  such that  $F(x) - G(x) = c$  for all  $x \in I$ . Therefore we have

$$F(b) - F(a) = G(b) - G(a) = \int_a^b f - \int_a^a f = \int_a^b f(y) dy.$$

□

**REMARKS:** (i) One often finds a short-hand notation like  $F(x) = \int f(x) dx$  to denote a (or sometimes all) primitive function(s) of  $f$ . The latter is also referred to as *indefinite integral* (*unbestimmtes Integral*). Within this set of lecture courses we will adopt the following **convention**: “determine  $\int f(x) dx$ ” shall mean “find a primitive of  $f$ ”.

(ii) The fundamental theorem of calculus clarifies that “the processes of differentiation and integration are essentially the inverses of one another”. To be more precise, we have the following scheme:

Fix  $a \in I$ . For a continuous function  $g: I \rightarrow \mathbb{R}$  denote by  $R(g)$  the function  $R(g)(x) := \int_a^x g(y) dy$ . Then  $R(g): I \rightarrow \mathbb{R}$  is continuously differentiable and  $R: g \mapsto R(g)$  defines a linear map  $\mathcal{C}(I, \mathbb{R}) \rightarrow \mathcal{C}^1(I, \mathbb{R})$ . Denote by  $D: \mathcal{C}^1(I, \mathbb{R}) \rightarrow \mathcal{C}(I, \mathbb{R})$  the map given by  $h \mapsto h'$ , which is also linear. Then the statement of the above theorem can be rephrased as follows:

- $D \circ R$  is the identity map on  $\mathcal{C}(I, \mathbb{R})$ , in other words, for  $f: I \rightarrow \mathbb{R}$  continuous the following holds

$$\frac{d}{dx} \left( \int_a^x f(y) dy \right) = f(x)$$

- $(R \circ D)(f) = f - f(a)$  for all  $f \in \mathcal{C}^1(I, \mathbb{R})$ , i.e., since  $f$  is a primitive of  $f'$ ,

$$\int_a^x f'(y) dy = f(x) - f(a)$$

**EXAMPLES:** 1) Let  $s \in \mathbb{R}$ ,  $s \neq -1$ , and  $a, b > 0$ . Since  $(x^{s+1})' = (s+1)x^s$  we have

$$\int_a^b x^s dx = \frac{x^{s+1}}{s+1} \Big|_a^b$$

In case  $s \in \mathbb{Z}$ ,  $s \neq -1$ , the same holds with  $a, b \in \mathbb{R}$  arbitrary.

2) The case  $s = -1$  in 1) leads to  $\int_a^b \frac{dx}{x}$ .

Since  $\log'(x) = 1/x$  we obtain  $\int_a^b \frac{dx}{x} = \log(x) \Big|_a^b = \log(b) - \log(a)$  (if  $a, b > 0$ ).

Observe that for  $x < 0$  we obtain  $(\log(-x))' = -1/(-x) = 1/x$ , so that we may state

$$\int_a^b \frac{dx}{x} = \log(-x) \Big|_a^b, \quad \text{if } a, b < 0.$$

The two cases can be summarized by

$$\int_a^b \frac{dx}{x} = (\log |x|) \Big|_a^b, \quad \text{if } a, b < 0 \text{ or } a, b > 0.$$

3)  $\int \sin x \, dx = -\cos x$ ,  $\int \cos x \, dx = \sin x$

4)  $\int e^x \, dx = e^x$

5) In the interval  $I = ]-1, 1[$  we have  $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$

6)  $\int \frac{dx}{1+x^2} = \arctan x$

7) In any interval  $I$  such that  $I \cap (\{\frac{\pi}{2}\} + \pi\mathbb{Z}) = \emptyset$  we have  $\int \frac{dx}{\cos^2 x} = \tan x$

## 10.2. Change of variable in an integral *(Substitutionsregel)*:

**PROPOSITION:** Let  $f: I \rightarrow \mathbb{R}$  be continuous and  $\varphi: [a, b] \rightarrow \mathbb{R}$  be  $\mathcal{C}^1$  with  $\varphi([a, b]) \subseteq I$ . Then the following substitution formula holds

$$(10.3) \quad \int_a^b f(\varphi(t)) \varphi'(t) \, dt = \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx.$$

*Proof.* Let  $F$  be a primitive of  $f$ , then  $F \circ \varphi: [a, b] \rightarrow \mathbb{R}$  is differentiable and by the chain rule

$$(F \circ \varphi)'(t) = F'(\varphi(t)) \varphi'(t) = f(\varphi(t)) \varphi'(t).$$

Therefore the fundamental theorem of calculus gives

$$\int_a^b f(\varphi(t)) \varphi'(t) \, dt = (F \circ \varphi)(t) \Big|_a^b = F(\varphi(a)) - F(\varphi(b)) = \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx.$$

□

**REMARK:** (i) In practice, the following symbolic notation is useful and intuitively helps to remember the above formula: Think of  $x$  representing the new variable via the transformation  $x = \varphi(t)$ , then  $dx = d(\varphi(t)) = \varphi'(t) dt$  gives the correct transformation on the infinitesimal level (it simply corresponds to the equation  $\frac{dx}{dt} = \varphi'(t)$ ) and, finally, the integration limits are transformed via  $\varphi$  accordingly, i.e.,  $a \mapsto \varphi(a)$  and  $b \mapsto \varphi(b)$ .

(ii) Keep in mind that formula (10.3) works both ways:

(a) from right to left: if we encounter a complicated integrand  $f(x)$ , we might be lucky to find  $\varphi$  such that  $(f \circ \varphi)\varphi'$  becomes simple

(b) from left to right: an at first sight nasty looking integrand might turn out to be of the structure  $f(\varphi(t))\varphi'(t)$  with appropriate  $\varphi$  such that  $f$  has a known primitive.

**10.3. Examples:** 1)  $\int_a^b f(t+c) dt \stackrel{[t+c=\varphi(t), \varphi'(t)=1]}{=} \int_{a+c}^{b+c} f(x) dx$

2) Let  $c \neq 0$ , then

$$\int_a^b f(ct) dt = \frac{1}{c} \int_a^b f(ct)c dt \stackrel{[\varphi(t)=ct, \varphi'(t)=c]}{=} \frac{1}{c} \int_{ac}^{bc} f(x) dx$$

3)  $\int_a^b tf(t^2) dt \stackrel{[\varphi(t)=t^2, \varphi'(t)=2t]}{=} \frac{1}{2} \int_{a^2}^{b^2} f(x) dx$

4) Let  $\varphi: [a, b] \rightarrow \mathbb{R}$  be  $\mathcal{C}^1$  and such that  $\varphi(t) \neq 0$  for all  $t$ . Then we can interpret an expression of the form  $\varphi'(t)/\varphi(t)$  as  $f(\varphi(t))\varphi'(t)$  with  $f(x) = 1/x$ . Therefore we obtain

$$\int_a^b \frac{\varphi'(t)}{\varphi(t)} dt = \int_a^b \varphi'(t)f(\varphi(t)) dt = \int_{\varphi(a)}^{\varphi(b)} f(x) dx = (\log |x|) \Big|_{\varphi(a)}^{\varphi(b)} = (\log |\varphi(t)|) \Big|_a^b$$

5) Exercise: Apply 4) to find  $\int_a^b \tan(t) dt$  (where  $[a, b]$  does not contain a zero of  $\cos$ ).

6) Assume that  $-1, 1 \notin [a, b]$ . We determine  $\int_a^b \frac{dx}{1-x^2}$  in two steps:

First, we decompose the rational function  $1/(1-x^2)$  into *partial fractions* (*Partialbruchzerlegung*). To do so, we use the factorization

$$1-x^2 = (1-x)(1+x)$$

of the denominator and make the ansatz

$$\frac{1}{1-x^2} = \frac{\alpha}{1-x} + \frac{\beta}{1+x}.$$

Upon multiplication with  $1-x^2$  the constants  $\alpha$  and  $\beta$  can be determined from the equation

$$1 = \alpha(1+x) + \beta(1-x) = (\alpha + \beta) + (\alpha - \beta)x$$

by comparing coefficients of the resulting polynomials. We obtain  $1 = \alpha + \beta$  and  $0 = \alpha - \beta$ , thus  $\alpha = \beta = 1/2$  and the integrand can be written in the form

$$\frac{1}{1-x^2} = \frac{1}{2} \cdot \frac{1}{1-x} + \frac{1}{2} \cdot \frac{1}{1+x}.$$

Hence we have

$$\int_a^b \frac{dx}{1-x^2} = \frac{1}{2} \left( \int_a^b \frac{dx}{1-x} + \int_a^b \frac{dx}{1+x} \right),$$

where each integral can be computed by a simple application of the rule in 4) and gives

$$\begin{aligned} \int_a^b \frac{dx}{1-x^2} &= \frac{1}{2} \left( - \int_a^b \frac{-1}{1-x} dx + \int_a^b \frac{dx}{1+x} \right) \\ &= \frac{1}{2} ( - \log |1-x| + \log |1+x| ) \Big|_a^b = \frac{1}{2} \cdot \left( \log \left| \frac{1+x}{1-x} \right| \right) \Big|_a^b. \end{aligned}$$

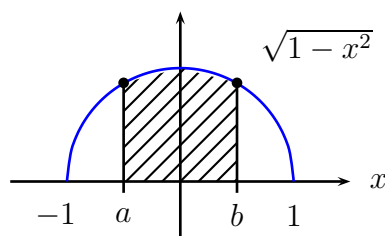
**REMARK:** The procedure of *decomposition into partial fractions* (*Partialbruchzerlegung*) can (in principle) be applied to arbitrary rational functions  $p(x)/q(x)$ . By carrying out a polynomial division first, one may assume that  $q$  is of higher degree than  $p$ . Then starting from a factorization of  $q$  into linear factors (when allowing for complex roots) or quadratic factors (when insisting on real roots only) one can make an ansatz similar as above and write the rational function as a sum of simpler rational functions, which can be integrated by standard means. (See, e.g., [Heu88, §69] or [HW96, Section II.5] for details.)

7) Let  $-1 \leq a < b \leq 1$  and consider  $\int_a^b \sqrt{1-x^2} dx$ .

The idea is to use the relation  $\cos t = \sqrt{1-\sin^2 t}$ , which suggests that we substitute  $x = \sin t = \varphi(t)$  and hope for the best ...

We have  $\varphi'(t) = \cos t$ , in other words,  $dx = d\varphi(t) = \cos t dt$ , thus

$$\int_a^b \sqrt{1-x^2} dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} \cos t \cdot \cos t dt = \int_u^v \cos^2 t dt,$$



where we have put  $u := \varphi^{-1}(a) = \arcsin a$  and  $v := \varphi^{-1}(b) = \arcsin b$ .

Since  $\cos(t+t) = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$  we may write  $\cos^2 t = (1 + \cos(2t))/2$  and therefore compute

$$\begin{aligned} \int_u^v \cos^2 t \, dt &= \frac{1}{2} \int_u^v \cos(2t) \, dt + \frac{1}{2} \int_u^v 1 \, dt = \frac{1}{4} \int_{2u}^{2v} \cos r \, dr + \frac{1}{2}(t) \Big|_u^v \\ &= \frac{1}{4} (\sin r) \Big|_{2u}^{2v} + \frac{\arcsin x}{2} \Big|_a^b = \left( \frac{\sin(2 \arcsin x)}{4} + \frac{\arcsin x}{2} \right) \Big|_a^b. \end{aligned}$$

Using the relation  $\sin(t+t) = 2\cos t \sin t = 2\sin t \sqrt{1 - \sin^2 t}$  we can simplify the first term on the right-hand side and finally obtain

$$\int_a^b \sqrt{1-x^2} \, dx = \frac{1}{2} \left( x\sqrt{1-x^2} + \arcsin x \right) \Big|_a^b.$$

Note that as a special case we deduce that the area of the upper half of the unit disc is

$$\int_{-1}^1 \sqrt{1-x^2} \, dx = \frac{1}{2} \left( (1\sqrt{0} + \arcsin 1) - (-1\sqrt{0} + \arcsin(-1)) \right) = \frac{1}{2} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{2}.$$

#### 10.4. Integration by parts (*Partielle Integration*):

**PROPOSITION:** Let  $f, g: [a, b] \rightarrow \mathbb{R}$  be continuously differentiable, then we have

$$(10.4) \quad \int_a^b f(x)g'(x) \, dx = \left( f(x)g(x) \right) \Big|_a^b - \int_a^b f'(x)g(x) \, dx.$$

*Proof.* If we set  $F := fg$  then  $F' = f'g + fg'$ . Therefore the fundamental theorem of calculus gives

$$\left( f(x)g(x) \right) \Big|_a^b = F(x) \Big|_a^b = \int_a^b F'(x) \, dx = \int_a^b f'(x)g(x) \, dx + \int_a^b f(x)g'(x) \, dx.$$

□

EXAMPLES: 1) Let  $a, b > 0$ , then

$$\int_a^b \log x \, dx = \int_a^b \log x \cdot 1 \, dx \stackrel{\substack{[f(x)=\log x, \\ g(x)=x]}}{=}}{(x \log x) \Big|_a^b - \int_a^b \frac{1}{x} \cdot x \, dx = (x \log x - x) \Big|_a^b$$

$$2) \int \arctan x \, dx = \int \arctan x \cdot 1 \, dx = x \arctan x - \int x \arctan'(x) \, dx$$

Using  $\arctan'(x) = 1/(1+x^2)$  and the substitution  $t = \varphi(x) = x^2$ ,  $dt = 2x \, dx$ , this becomes

$$x \arctan x - \int x \cdot \frac{1}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \int \frac{dt}{1+t}.$$

Upon observing that  $\int \frac{dt}{1+t} = \log(1+t) = \log(1+x^2)$  (note that  $t = x^2 \geq 0$ ) we arrive at

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2} \log(1+x^2).$$

$$3) \text{ Let } m \in \mathbb{N} \text{ and consider } I_m(y) := \int_0^y \sin^m(x) \, dx \quad (y \in \mathbb{R}).$$

We derive a recursion relation for  $m \geq 2$ :

$$\begin{aligned} I_m(y) &= \int_0^y \sin^{m-1}(x) \underbrace{\sin(x)}_{-\cos'(x)} \, dx = -\cos(y) \sin^{m-1}(y) + (m-1) \int_0^y \sin^{m-2}(x) \underbrace{\cos^2(x)}_{1-\sin^2(x)} \, dx \\ &= -\cos(y) \sin^{m-1}(y) + (m-1) I_{m-2}(y) - (m-1) I_m(y), \end{aligned}$$

hence

$$m I_m(y) = (m-1) I_{m-2}(y) - \cos(y) \sin^{m-1}(y).$$

To summarize, we have

$$\begin{aligned} I_0(y) &= \int_0^y 1 \, dx = y, & I_1(y) &= \int_0^y \sin(x) \, dx = 1 - \cos(y) \\ I_m(y) &= \frac{m-1}{m} \cdot I_{m-2}(y) - \frac{\cos(y) \sin^{m-1}(y)}{m} \quad (m \geq 2). \end{aligned}$$

As an application we consider the sequence defined by  $A_m := I_m(\frac{\pi}{2})$  ( $m \in \mathbb{N}$ ). We have

$$A_0 = \frac{\pi}{2}, \quad A_1 = 1$$



and for  $m \geq 2$

$$A_m = \frac{m-1}{m} A_{m-2} - \frac{\cos(\frac{\pi}{2}) \sin^{m-1}(\frac{\pi}{2})}{m} = \frac{m-1}{m} A_{m-2}.$$

Therefore we obtain for even  $m = 2n$

$$A_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot A_0 = \frac{(2n-1)(2n-3) \cdots 3 \cdot 1}{2n(2n-2) \cdots 4 \cdot 2} \cdot \frac{\pi}{2}$$

and for odd  $m = 2n+1$

$$A_{2n+1} = \frac{2n(2n-2) \cdots 4 \cdot 2}{(2n+1)(2n-1) \cdots 5 \cdot 3} \cdot \underbrace{A_1}_1.$$

When  $x \in [0, \frac{\pi}{2}]$  we have  $\sin^{2n+2}(x) \leq \sin^{2n+1}(x) \leq \sin^{2n}(x)$ , thus by the monotonicity of the integral  $A_{2n+2} \leq A_{2n+1} \leq A_{2n}$ , which implies

$$\frac{A_{2n+2}}{A_{2n}} \leq \frac{A_{2n+1}}{A_{2n}} \leq 1.$$

Since  $\frac{A_{2n+2}}{A_{2n}} = \frac{2n+1}{2n+2} \rightarrow 1$  ( $n \rightarrow \infty$ ) we obtain

$$\lim_{n \rightarrow \infty} \frac{A_{2n+1}}{A_{2n}} = 1.$$

On the other hand,

$$\begin{aligned} \frac{A_{2n+1}}{A_{2n}} &= \frac{(2n)^2(2n-2)^2 \cdots 4^2 \cdot 2^2}{(2n+1)(2n-1)^2(2n-3)^2 \cdots 5^2 \cdot 3^2 \cdot 1} \cdot \frac{2}{\pi} \\ &= \frac{2}{\pi} \cdot \prod_{k=1}^n \frac{4k^2}{(2k-1)(2k+1)} = \frac{2}{\pi} \cdot \prod_{k=1}^n \frac{4k^2}{4k^2-1}. \end{aligned}$$

Thus we obtain *Wallis'*<sup>1</sup> *product representation of  $\pi$*

$$(10.5) \quad \pi = 2 \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{4k^2}{4k^2-1} = 2 \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1}.$$

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<sup>1</sup>John Wallis (1616–1703), English mathematician

## 10.5. Riemann's lemma and a series representation of $\pi$ :

LEMMA (RIEMANN): Let  $f: [a, b] \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$  function and define  $F: \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(\nu) := \int_a^b f(x) \sin(\nu x) dx.$$

Then we have

$$\lim_{|\nu| \rightarrow \infty} F(\nu) = 0.$$

*Proof.* We may assume that  $\nu \neq 0$ . Integration by parts gives

$$F(\nu) = \int_a^b f(x) \underbrace{\sin(\nu x)}_{\left(-\frac{\cos(\nu x)}{\nu}\right)'} dx = - \left( f(x) \frac{\cos(\nu x)}{\nu} \right) \Big|_a^b + \frac{1}{\nu} \int_a^b f'(x) \cos(\nu x) dx.$$

Since  $f$  and  $f'$  are continuous on  $[a, b]$  they are bounded, hence there is some  $M > 0$  such that  $|f(x)| \leq M$  and  $|f'(x)| \leq M$  for all  $x \in [a, b]$ . Therefore we obtain from the above equation that

$$|F(\nu)| \leq \frac{2M}{|\nu|} + \frac{1}{|\nu|} \int_a^b M \cdot 1 dx = \frac{2M + (b-a)M}{|\nu|} \rightarrow 0 \quad (|\nu| \rightarrow \infty).$$

□

APPLICATION (THE FOURIER<sup>2</sup> SERIES OF  $\frac{\pi-x}{2}$ ): Recall the relation (9.10), which gives

$$\sum_{k=1}^n \cos(kt) = \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{t}{2}\right)} - \frac{1}{2}.$$

If we integrate both sides in this relation from  $\pi$  to  $x \in ]0, 2\pi[$  with respect to  $t$  we obtain on the left-hand side

$$\int_{\pi}^x \sum_{k=1}^n \cos(kt) dt = \sum_{k=1}^n \frac{\sin(kt)}{k} \Big|_{\pi}^x = \sum_{k=1}^n \frac{\sin(kx)}{k},$$

and on the right-hand side

$$\underbrace{\int_{\pi}^x \sin\left(\left(n + \frac{1}{2}\right)t\right) \cdot \frac{1}{2 \sin\left(\frac{t}{2}\right)} dt}_{=F\left(n + \frac{1}{2}\right), \text{ with } f(t) = 1/(2 \sin\left(\frac{t}{2}\right))} - \frac{x - \pi}{2}$$

---

<sup>2</sup>Jean Baptiste Joseph Fourier (1768–1830) [ʒɑ̃ bab'tist ʒo'sef fur'je], French mathematician and physicist

Sending  $n \rightarrow \infty$  the above equality must remain and gives

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2} \quad (0 < x < 2\pi).$$

In particular, if we set  $x = \pi/2$  and observe that  $\sin(k\pi/2) = 0$  for  $k$  even, and  $\sin(k\pi/2) = (-1)^l$  for  $k = 2l + 1$  odd, we obtain the following sum

$$\frac{\pi}{4} = \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

**10.6. Taylor's<sup>3</sup> formula:** Suppose  $f: I \rightarrow \mathbb{R}$  is  $\mathcal{C}^2$  and  $x_0 \in I$ . By the fundamental theorem of calculus we have for all  $x \in I$  the identity

$$(10.6) \quad f(x) = f(x_0) + \int_{x_0}^x f'(t) dt.$$

Note that  $\frac{d}{dt}(x-t) = -1$  and  $f'$  is  $\mathcal{C}^1$ , hence we may apply integration by parts to  $\int_{x_0}^x f'(t) dt = -\int_{x_0}^x \frac{d}{dt}(x-t) \cdot f'(t) dt$  and obtain

$$(10.7) \quad f(x) = f(x_0) - \left. ((x-t)f'(t)) \right|_{t=x_0}^{t=x} + \int_{x_0}^x (x-t)f''(t) dt$$

$$= f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^x (x-t)f''(t) dt.$$

Observe that the first two terms on the right-hand side represent the (affine) linear approximation to  $f$  near  $x_0$ . If  $f$  is  $\mathcal{C}^3$  we can carry out one more integration by parts and obtain a quadratic approximation plus an integral remainder term. Proceeding inductively we arrive at the following

**TAYLOR'S THEOREM:** Let  $f: I \rightarrow \mathbb{R}$  be in  $\mathcal{C}^{n+1}(I, \mathbb{R})$  (i.e. an  $(n+1)$ -times continuously differentiable function) and let  $x_0 \in I$  arbitrary. Then we have for all  $x \in I$  *Taylor's formula*

$$(10.8) \quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2} \cdot (x-x_0)^2 + \dots$$

$$+ \frac{f^{(n)}(x_0)}{n!} \cdot (x-x_0)^n + R_{n+1}(x),$$

---

<sup>3</sup>Brook Taylor (1685–1731), English mathematician

where

$$(10.9) \quad R_{n+1}(x) = \frac{1}{n!} \int_{x_0}^x (x-t)^n f^{(n+1)}(t) dt.$$

*Proof.* We introduce the notation  $T_n[f, x_0](x) := \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$ , then the asserted equation reads

$$f(x) = T_n[f, x_0](x) + R_{n+1}(x).$$

If  $n = 0$  or  $n = 1$  this is precisely (10.6) or (10.7). We prove (10.8) by induction, assuming that

$$f(x) = T_{n-1}[f, x_0](x) + R_n(x)$$

already holds. We apply integration by parts to the remainder term

$$\begin{aligned} R_n(x) &= \frac{1}{(n-1)!} \int_{x_0}^x \underbrace{(x-t)^{n-1}}_{-\frac{d}{dt}[\frac{(x-t)^n}{n}]} f^{(n)}(t) dt = -\frac{(x-t)^n}{n!} \cdot f^{(n)}(t) \Big|_{t=x_0}^{t=x} + \frac{1}{n!} \int_{x_0}^x (x-t)^n f^{(n+1)}(t) dt \\ &= \frac{(x-x_0)^n}{n!} \cdot f^{(n)}(x_0) + R_{n+1}(x), \end{aligned}$$

which proves the assertion.  $\square$

**10.7. Definition:** Let  $f: I \rightarrow \mathbb{R}$  be  $n$ -times differentiable and  $x_0 \in I$ . For  $m \in \mathbb{N}$ ,  $m \leq n$  the *Taylor polynomial* of order  $m$  of  $f$  at  $x_0$  is defined by

$$T_m[f, x_0](x) := \sum_{k=0}^m \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k.$$

If  $f$  is differentiable the *Taylor series* of  $f$  about  $x_0$  is defined by

$$T[f, x_0](x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

(regardless of its convergence or not).

**10.8. Corollary (Alternative formulae for the remainder term):**

(i) Let  $f: I \rightarrow \mathbb{R}$  be  $\mathcal{C}^{n+1}$  and  $x_0, x \in I$ . Then there exists some  $\xi$  between  $x_0$  and  $x$  such that

$$f(x) = T_n[f, x_0](x) + R_{n+1}(x)$$

with the *Lagrange formula*

$$(10.10) \quad R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.$$

(ii) Let  $f: I \rightarrow \mathbb{R}$  be  $\mathcal{C}^n$  and  $x_0 \in I$ . Then there exists a function  $\varphi: I \rightarrow \mathbb{R}$  with  $\lim_{x \rightarrow x_0} \varphi(x) = 0$  such that for all  $x \in I$

$$f(x) = T_n[f, x_0](x) + \varphi(x) (x - x_0)^n.$$

We may rephrase this as

$$(10.11) \quad f(x) = T_n[f, x_0](x) + o(|x - x_0|^n) \quad (x \rightarrow x_0).$$

*Proof.* (i): Applying the mean value theorem of integral calculus to the integral formula for the remainder term (10.9) we obtain with some  $\xi$  between  $x_0$  and  $x$  that

$$\begin{aligned} R_{n+1}(x) &= \frac{1}{n!} \int_{x_0}^x (x-t)^n f^{(n+1)}(t) dt = \frac{f^{(n+1)}(\xi)}{n!} \int_{x_0}^x (x-t)^n dt \\ &= \frac{f^{(n+1)}(\xi)}{n!} \cdot \frac{-(x-t)^{n+1}}{n+1} \Big|_{t=x_0}^{t=x} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-x_0)^{n+1}. \end{aligned}$$

(ii): We may apply (i) with  $n$  in place of  $n+1$ . Hence for all  $x \in I$  there exists  $\xi(x) \in I$  with  $|x_0 - \xi(x)| \leq |x_0 - x|$  such that

$$\begin{aligned} f(x) - T_{n-1}[f, x_0](x) &= \frac{f^{(n)}(\xi(x))}{n!} (x - x_0)^n \\ &= \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \underbrace{\frac{f^{(n)}(\xi(x)) - f^{(n)}(x_0)}{n!}}_{=: \varphi(x)} (x - x_0)^n. \end{aligned}$$

We have  $\lim_{x \rightarrow x_0} \xi(x) = x_0$ , thus by continuity of  $f^{(n)}$  that  $\lim_{x \rightarrow x_0} f^{(n)}(\xi(x)) = f^{(n)}(x_0)$ , which in turn implies that  $\lim_{x \rightarrow x_0} \varphi(x) = 0$ .  $\square$

**10.9. Corollary:** Let  $f: I \rightarrow \mathbb{R}$  be infinitely differentiable and  $x_0 \in I$ . Then we have for all  $n \in \mathbb{N}$  and for all  $x \in I$

$$f(x) = T_n[f, x_0](x) + R_{n+1}(x),$$

where  $R_{n+1}(x)$  is given by (10.9) or (10.10).

Moreover, for  $x \in I$  the Taylor series  $T[f, x_0](x)$  converges with sum  $f(x)$ , that is

$$f(x) = T[f, x_0](x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

if and only if  $\lim_{n \rightarrow \infty} R_n(x) = 0$ .

*Proof.* Follows immediately from Theorem 10.6 and Corollary 10.8.  $\square$

**10.10. Corollary:** Let  $f: I \rightarrow \mathbb{R}$  be  $(n+1)$ -times differentiable. If  $f^{(n+1)}(x) = 0$  for all  $x \in I$ , then  $f$  is a polynomial function of degree at most  $n$ .

*Proof.* Since  $f^{(n+1)} = 0$  it is continuous and thus  $f$  is in  $\mathcal{C}^{n+1}(I, \mathbb{R})$ . Therefore we may apply Theorem 10.6. Since the remainder term  $R_{n+1}(x)$  in (10.8) now vanishes, the assertion follows directly.  $\square$

**10.11. Examples:** 1) For all  $x \in \mathbb{R}$  and  $k \in \mathbb{N}$  we have  $\exp^{(k)}(x) = \exp(x)$ , hence  $\exp^{(k)}(0) = 1$ . Thus we obtain the Taylor series  $T[\exp, 0](x) = \sum_{k=0}^{\infty} x^k/k!$  and the remainder term with some  $\xi$  between 0 and  $x$  satisfies

$$R_n(x) = \exp(\xi) \frac{x^n}{n!} \rightarrow 0 \quad (n \rightarrow \infty),$$

since we have already shown in Section 4 that the above series is absolutely convergent. Therefore we have that

$$\exp(x) = T[\exp, 0](x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

holds for all  $x \in \mathbb{R}$ . To summarize, the Taylor series of the exponential function is convergent and its sum gives the function value for all  $x$ .

Similarly, we can determine the Taylor series of  $\sin$  and  $\cos$  (e.g., simply compute  $\sin^{(2k)}(0) = 0$  and  $\sin^{(2k+1)}(0) = (-1)^k$  for all  $k$ ) and find that these agree with the already given series expansion in Section 6. (Work out all details as an exercise!)

2) **Series expansion of the logarithm:** Consider  $f(x) := \log(1+x)$ ,  $f: [0, 1] \rightarrow \mathbb{R}$ . Then we obtain

$$f'(x) = \frac{1}{1+x}$$

and, inductively,

$$f^{(k)}(x) = (-1)^{k-1} \frac{(k-1)!}{(1+x)^k} \quad (k \geq 1).$$

Hence  $f(0)/0! = 0$  and  $f^{(k)}(0)/k! = (-1)^{k-1}/k$ , which gives the Taylor series

$$T[f, 0](x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

and the remainder term (with  $0 \leq \xi \leq x \leq 1$ )

$$|R_n(x)| = \left| \frac{(-1)^{n-1}}{n(1+\xi)^n} x^n \right| \leq \frac{|x|^n}{n} \rightarrow 0 \quad (n \rightarrow \infty).$$

Therefore we obtain the Taylor series expansion

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \quad (0 \leq x \leq 1).$$

In particular, we have  $\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

The above expansion is also valid when  $-1 < x < 0$ . We do not prove this here, since it will be a simple consequence of results in Section 12.

By the following trick we can also find a series expansion for  $\log(y)$  when  $y \geq 2$ : Let  $0 < \lambda < 1$  and choose  $r > 1$  such that  $y = (1 + \lambda)^r$ . Then  $\log(y) = r \log(1 + \lambda)$  and the problem is reduced to the series expansion for  $\log(1 + \lambda)$ .

**3) The binomial series:** Let  $\alpha \in \mathbb{R} \setminus \mathbb{N}$  and consider  $f: ]-1, 1[ \rightarrow \mathbb{R}$ ,  $f(x) := (1+x)^\alpha$ . We have

$$f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-k+1)(1+x)^{\alpha-k} \quad (k \in \mathbb{N}),$$

hence  $f^{(k)}(0)/k! = \binom{\alpha}{k}$ , yielding the Taylor series

$$T[f, 0](x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

The ratio test implies absolute convergence for all  $x$  with  $|x| < 1$ :

$$\left| \frac{\binom{\alpha}{k+1} x^{k+1}}{\binom{\alpha}{k} x^k} \right| = \left| \frac{\alpha - k}{k + 1} \right| |x| \rightarrow |x| \quad (k \rightarrow \infty).$$

But note that this still leaves open the question whether we have

$$T[f, 0](x) \stackrel{?}{=} f(x) \quad (|x| < 1).$$

We will answer this in the positive by estimating the remainder term for two cases separately.

$0 \leq x < 1$ : We have with some  $\xi \in [0, x]$

$$R_n(x) = \frac{f^{(n)}(\xi)}{n!} x^n = (1+\xi)^{\alpha-n} \binom{\alpha}{n} x^n.$$

Since  $(1 + \xi)^{\alpha-n} \leq 1$  when  $n \geq \alpha$  and the series  $T[f, 0](x)$  is absolutely convergent we deduce that

$$|R_n(x)| \leq \left| \binom{\alpha}{n} x^n \right| \rightarrow 0 \quad (n \rightarrow \infty).$$

$-1 < x < 0$ : Using the integral form of the remainder term we obtain, after substitution of  $-t$  in place of  $t$ ,

$$R_n(x) = n \binom{\alpha}{n} \int_0^{|x|} (x+t)^{n-1} (1-t)^{\alpha-n} dt.$$

Note that for  $x < 0$  and  $0 \leq t \leq |x| < 1$  we have

$$|x+t| = |x| - t \leq |x| - |x|t = |x|(1-t)$$

and therefore

$$\begin{aligned} |R_n(x)| &\leq \left| n \binom{\alpha}{n} \right| |x|^{n-1} \int_0^{|x|} (1-t)^{n-1} (1-t)^{\alpha-n} dt \\ &= \left| \alpha \binom{\alpha-1}{n-1} \right| |x|^{n-1} \int_0^{|x|} (1-t)^{\alpha-1} dt = \underbrace{|\alpha| \int_0^{|x|} (1-t)^{\alpha-1} dt}_{=: C(\alpha, |x|)} \cdot \underbrace{\left| \binom{\alpha-1}{n-1} x^{n-1} \right|}_{=: a_n}, \end{aligned}$$

where  $C(\alpha, |x|)$  is independent of  $n$  and  $a_n \rightarrow 0$  ( $n \rightarrow \infty$ ) by absolute convergence of  $T[f, 0](x)$ .

Observe that for  $\alpha = m \in \mathbb{N}$  the series  $T[f, 0](x)$  simply reduces to the finite sum according to the binomial theorem applied to  $(1+x)^m$ . In summary, we have shown that

$$(10.12) \quad (1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad (|x| < 1, \alpha \in \mathbb{R}).$$

**10.12. Remark and warning:** Even when a Taylor series converges (for all  $x$ ) its sum need not give the corresponding function values. Consider, for example, the function  $f: \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$f(x) := \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

We will prove below that  $f$  is infinitely differentiable and that  $f^{(k)}(0) = 0$  for all  $k$ . Therefore  $T[f, 0](x) = 0$  for all  $x \in \mathbb{R}$ , which is clearly convergent. But note that  $f(x) > 0$  for all  $x \neq 0$ , therefore

$$0 = T[f, 0](x) \neq f(x), \quad \text{if } x \in \mathbb{R} \setminus \{0\}.$$



It remains to prove the following

*Claim:*  $f$  is infinitely differentiable and that  $f^{(k)}(0) = 0$  for all  $k$ .

We will show inductively that  $f$  is  $n$ -times differentiable for all  $n \in \mathbb{N}$  and that the following assertion holds

$$(\star) \quad \exists \text{ polynomial function } p_n : \quad f^{(n)}(x) = \begin{cases} p_n\left(\frac{1}{x}\right) e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

The claim then follows immediately from  $(\star)$ .

By definition of  $f$ , the assertion holds for  $n = 0$ . Assume that it holds for  $1, \dots, n$  as well.

Let  $x \neq 0$ , then by the induction hypothesis

$$\begin{aligned} f^{(n+1)}(x) &= \left( p_n\left(\frac{1}{x}\right) e^{-1/x^2} \right)' = p_n'\left(\frac{1}{x}\right) \left(\frac{-1}{x^2}\right) e^{-1/x^2} + \frac{2}{x^3} p_n\left(\frac{1}{x}\right) e^{-1/x^2} \\ &= \left( p_n'\left(\frac{1}{x}\right) \left(\frac{-1}{x^2}\right) + \frac{2}{x^3} p_n\left(\frac{1}{x}\right) \right) e^{-1/x^2} =: p_{n+1}\left(\frac{1}{x}\right) e^{-1/x^2}. \end{aligned}$$

To show differentiability at 0 we have for  $h \neq 0$

$$\frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \frac{p_n\left(\frac{1}{h}\right) e^{-1/h^2}}{h} = y p_n(y) e^{-y^2},$$

where we have put  $y = 1/h$ . Thus,  $h \rightarrow 0$  is equivalent to sending  $|y| \rightarrow \infty$ , which shows that  $f^{(n+1)}(0) = 0$ .

## §11. IMPROPER INTEGRALS

### 11.1. Unbounded integration interval:

DEFINITION: Let  $a \in \mathbb{R}$  and  $f: [a, \infty[ \rightarrow \mathbb{R}$  be such for all  $R > a$  the function  $f|_{[a, R]}$  is R-integrable. If the limit  $\lim_{R \rightarrow \infty} \int_a^R f(x) dx$  exists, then the *improper integral* (das *uneigentliche Integral*)  $\int_a^\infty f(x) dx$  is said to be *convergent* and we set

$$\int_a^\infty f(x) dx := \lim_{R \rightarrow \infty} \int_a^R f(x) dx.$$

Similarly for the case  $\int_{-\infty}^a f(x) dx$ .

EXAMPLE:  $\int_1^\infty \frac{dx}{x^s}$  is convergent  $\iff s > 1$

Indeed, if  $s \neq 1$  and  $R > 1$  we have

$$\int_1^R \frac{dx}{x^s} = \frac{x^{1-s}}{1-s} \Big|_1^R = \frac{R^{1-s} - 1}{1-s},$$

which converges as  $R \rightarrow \infty$  if and only if  $s > 1$ . If  $s = 1$  we have for all  $R > 1$

$$\int_1^R \frac{dx}{x} = \log(x) \Big|_1^R = \log(R) \rightarrow \infty \quad (R \rightarrow \infty).$$

### 11.2. Unbounded or undefined integrand at the boundary:

DEFINITION: Let  $f: ]a, b] \rightarrow \mathbb{R}$  be such that for all  $\varepsilon > 0$  (with  $0 < \varepsilon < b - a$ ) the function  $f|_{[a+\varepsilon, b]}$  is R-integrable. The *improper integral*  $\int_a^b f(x) dx$  is said to be *convergent* if the limit  $\lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^b f(x) dx$  exists. In this case we set

$$\int_a^b f(x) dx := \lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^b f(x) dx.$$

EXAMPLE:  $\int_0^1 \frac{dx}{x^s}$  is convergent  $\iff s < 1$

Indeed, if  $s \neq 1$  and  $1 > \varepsilon > 0$  we have

$$\int_{\varepsilon}^1 \frac{dx}{x^s} = \frac{1 - \varepsilon^{1-s}}{1-s},$$

which converges as  $\varepsilon \rightarrow 0$  if and only if  $s < 1$ . If  $s = 1$  and  $\varepsilon > 0$  we have

$$\int_{\varepsilon}^1 \frac{dx}{x} = -\log(\varepsilon) \rightarrow \infty \quad (\varepsilon \rightarrow 0).$$

### 11.3. Combined cases:

DEFINITION: Let  $-\infty \leq a < b \leq \infty$  and  $f: ]a, b[ \rightarrow \mathbb{R}$  be such that  $f$  is R-integrable on  $[\alpha, \beta]$  for all  $\alpha, \beta \in ]a, b[$  with  $\alpha < \beta$ . Let  $c \in ]a, b[$  arbitrary. If both improper integrals  $\int_a^c f(x) dx$  and  $\int_c^b f(x) dx$  are convergent, then the *improper integral*  $\int_a^b f(x) dx$  is said to be *convergent* and we set

$$\int_a^b f(x) dx := \int_a^c f(x) dx + \int_c^b f(x) dx.$$

EXAMPLES: 1) For all  $s \in \mathbb{R}$ :  $\int_0^{\infty} \frac{dx}{x^s}$  is divergent.

Indeed for all  $c > 0$  we have that  $\int_0^c \frac{dx}{x^s}$  diverges if  $s \geq 1$  and  $\int_c^{\infty} \frac{dx}{x^s}$  diverges if  $s \leq 1$ .

2)  $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$  is convergent:

We have

$$\int_{-1+\varepsilon}^0 \frac{dx}{\sqrt{1-x^2}} = -\arcsin(-1+\varepsilon) \rightarrow -\arcsin(-1) = \frac{\pi}{2} \quad (\varepsilon \rightarrow 0)$$

and

$$\int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} = \arcsin(1-\varepsilon) \rightarrow \arcsin(1) = \frac{\pi}{2} \quad (\varepsilon \rightarrow 0).$$

Therefore

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^0 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

3)  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$  is convergent:

Let  $R > 0$ . We have

$$\int_{-R}^0 \frac{dx}{1+x^2} = -\arctan(-R) \rightarrow \frac{\pi}{2}, \quad \int_0^R \frac{dx}{1+x^2} = \arctan(R) \rightarrow \frac{\pi}{2} \quad (R \rightarrow \infty),$$

hence

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

**11.4. Basic convergence principles:** To keep notation simple we will formulate everything for a one-sided infinite interval  $[a, \infty[$  of integration and a function  $f: [a, \infty[ \rightarrow \mathbb{R}$  that is R-integrable on all finite subintervals  $[a, R]$  ( $R > a$ ). The required adaptation to cases involving functions with critical behavior or undefined at the boundary should be routine.

**(A) Cauchy principle:**  $\int_a^{\infty} f(x) dx$  is convergent if and only if

$$(11.1) \quad \forall \varepsilon > 0 \exists R > a : \forall r, s \geq R \quad \left| \int_r^s f(x) dx \right| < \varepsilon.$$

*Proof.* ( $\Rightarrow$ ) Let  $(R_n)$  be a sequence with  $R_n \rightarrow \infty$  ( $n \rightarrow \infty$ ). Let  $\varepsilon > 0$  and choose  $R > a$  such (11.1) holds. There exists  $n_0 \in \mathbb{N}$  such that  $R_n \geq R$  for all  $n \geq n_0$ . Therefore

$$\left| \int_{R_m}^{R_n} f(x) dx \right| < \varepsilon$$

holds for all  $m, n \geq n_0$ , which shows that  $\left( \int_a^{R_n} f \right)_{n \in \mathbb{N}}$  is a Cauchy sequence, thus convergent.

( $\Rightarrow$ ) (By contradiction.) Assume that there is some  $\varepsilon > 0$  such that (11.1) does not hold. Then for every  $n \in \mathbb{N}$  we can find  $r_n, s_n \geq n$  such that

$$(\star) \quad \left| \int_{r_n}^{s_n} f(x) dx \right| \geq \varepsilon.$$

We have that  $r_n \rightarrow \infty$  and  $s_n \rightarrow \infty$  as  $n \rightarrow \infty$ . We define a new sequence  $(R_m)$  by mixing:  $R_{2k} := r_k$  and  $R_{2k+1} := s_k$  ( $k \in \mathbb{N}$ ). Then  $R_m \rightarrow \infty$  ( $m \rightarrow \infty$ ), hence  $\left( \int_a^{R_m} f \right)_m$  is convergent, thus it is a Cauchy sequence — a contradiction  $\nabla$  to  $(\star)$ .  $\square$

**(B) Comparison tests** (*Majoranten- und Minorantenkriterium*): (i) Let  $h: [a, \infty[ \rightarrow \mathbb{R}$  with  $h \geq 0$  and such that  $\int_a^\infty h(x) dx$  is convergent. If  $|f(x)| \leq h(x)$  for all  $x$ , then  $\int_a^\infty f(x) dx$  is convergent.

Indeed, since

$$\left| \int_r^s f(x) dx \right| \leq \int_r^s |f(x)| dx \leq \int_r^s h(x) dx$$

the assertion follows from the Cauchy principle.

(ii) Let  $g: [a, \infty[ \rightarrow \mathbb{R}$  with  $g \geq 0$  and such that  $\int_a^\infty g(x) dx$  is divergent. If  $f(x) \geq g(x)$  for all  $x$ , then  $\int_a^\infty f(x) dx$  is divergent.

This follows from (i), since otherwise  $\int_a^\infty g$  would have to converge.

**EXAMPLES:** 1)  $\int_1^\infty \frac{\cos(x)}{x^2} dx$  converges due to (B)(i):  $\left| \frac{\cos(x)}{x^2} \right| \leq \frac{1}{x^2}$  and  $\int_1^\infty \frac{dx}{x^2}$  is convergent.

2)  $\int_1^\infty \frac{\sin(x)}{x} dx$  is convergent: Let  $1 < r < s$ , then integration by parts with  $f(x) = 1/x$  and  $g'(x) = \sin(x)$  and applying the Cauchy principle to the convergent integral  $\int_1^\infty dx/x^2$  we obtain

$$\begin{aligned} \left| \int_r^s \frac{\sin(x)}{x} dx \right| &= \left| \frac{-\cos(x)}{x} \Big|_r^s - \int_r^s \frac{\cos(x)}{x^2} dx \right| \leq \left| \frac{\cos(r)}{r} - \frac{\cos(s)}{s} \right| + \left| \int_r^s \frac{\cos(x)}{x^2} dx \right| \\ &\leq \left( \frac{1}{r} + \frac{1}{s} \right) + \int_r^s \frac{dx}{x^2} < \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

if  $r$  and  $s$  are sufficiently large. Thus the Cauchy principle (A) proves convergence.

*Remark:* However, in contrast to Example 2), one can show that the improper integral  $\int_1^\infty \frac{|\sin(x)|}{x} dx$  is divergent.

(Hint: Write  $\int_1^{n\pi}$  as a sum of  $n$  integrals; upon a suitable change of variable, each of these can be reduced to an integration over  $[0, \pi]$ , which can be estimated from below such that the harmonic series occurs in a lower bound.)

## 11.5. Integral test for series:

**PROPOSITION:** Let  $f: [1, \infty[ \rightarrow [0, \infty[$  be a decreasing function. Then the following holds:

$$\sum_1^\infty f(n) \text{ is convergent} \iff \int_1^\infty f(x) dx \text{ is convergent.}$$

*Proof.* Define  $\varphi, \psi: [1, \infty[ \rightarrow \mathbb{R}$  as follows:  $\psi(x) := f(n)$ ,  $\varphi(x) := f(n+1)$  when  $x \in [n, n+1[$ . Then the restrictions of  $\varphi$  and  $\psi$  to each finite subinterval are simple functions and  $0 \leq \varphi \leq f \leq \psi$ . Therefore we have for all  $N \in \mathbb{N}$ ,  $N \geq 1$ ,

$$\underbrace{\int_1^N \varphi(x) dx}_{\sum_{n=2}^N f(n)} \leq \int_1^N f(x) dx \leq \underbrace{\int_1^N \psi(x) dx}_{\sum_{n=1}^{N-1} f(n)}.$$

If  $\int_1^\infty f(x) dx$  is convergent, then we obtain that  $\sum_{n=2}^N f(n)$  is bounded, hence convergent.

On the other hand, if  $\sum_{n=1}^\infty f(n)$  is convergent, then for all increasing sequences  $R_l \rightarrow \infty$  ( $l \rightarrow \infty$ ) the sequence  $\left(\int_1^{R_l} f(x) dx\right)_{l \in \mathbb{N}}$  is increasing and bounded, hence convergent.

Therefore  $\int_1^\infty f(x) dx$  is convergent.  $\square$

**EXAMPLE:** Let  $s > 0$ , then we have

$$\sum_1^\infty \frac{1}{n^s} \text{ is convergent} \iff \int_1^\infty \frac{dx}{x^s} \text{ is convergent} \iff s > 1.$$

( $\sum_1^\infty \frac{1}{n^s}$  is clearly divergent when  $s \leq 0$ .)

## 11.6. The Gamma function:

LEMMA: If  $x > 0$  then  $\int_0^{\infty} t^{x-1} e^{-t} dt$  is convergent.

*Proof.* For all  $t > 0$  we have that

$$0 \leq t^{x-1} e^{-t} \leq \frac{1}{t^{1-x}} = \frac{1}{t^s},$$

where  $s := 1 - x < 1$ . Therefore, the comparison test shows that for arbitrary  $R > 0$  the improper integral  $\int_0^R t^{x-1} e^{-t} dt$  is convergent.

To obtain also convergence of  $\int_R^{\infty} t^{x-1} e^{-t} dt$  (with some fixed  $R > 1$ ) we proceed as follows:

Since  $\lim_{t \rightarrow \infty} t^{x+1} e^{-t} = 0$  there is some  $R > 1$  such that

$$\forall t \geq R: \quad t^{x-1} e^{-t} = t^{-2} t^{x+1} e^{-t} \leq t^{-2} \cdot 1 \leq \frac{1}{t^2}.$$

Hence  $\int_R^{\infty} t^{x-1} e^{-t} dt$  is convergent by the comparison test.  $\square$

DEFINITION: The *Gamma function*  $\Gamma: ]0, \infty[ \rightarrow \mathbb{R}$  is given by

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt.$$

PROPOSITION: (i) (Functional equation) For all  $x \in ]0, \infty[$ :  $\Gamma(x+1) = x \cdot \Gamma(x)$ .

(ii) For all  $n \in \mathbb{N}$ :  $\Gamma(n+1) = n!$

In other words, the Gamma function is an interpolation of the factorial function.

*Proof.* (i): Let  $0 < \varepsilon < R$ . Integration by parts gives

$$\int_{\varepsilon}^R t^x e^{-t} dt = -t^x e^{-t} \Big|_{t=\varepsilon}^{t=R} + \int_{\varepsilon}^R x t^{x-1} e^{-t} dt = \frac{\varepsilon^x}{e^{\varepsilon}} - \frac{R^x}{e^R} + x \int_{\varepsilon}^R t^{x-1} e^{-t} dt.$$

Hence  $\Gamma(x+1) = \lim_{\varepsilon \rightarrow 0} \lim_{R \rightarrow \infty} \int_{\varepsilon}^R t^x e^{-t} dt = 0 + x \Gamma(x)$ .

(ii):  $\Gamma(1) = \int_0^{\infty} e^{-t} dt = \lim_{R \rightarrow \infty} (-e^{-t}) \Big|_0^R = 1$  and inductively by (i)

$$\Gamma(n+1) = n \Gamma(n) = n(n-1)! = n!$$

$\square$

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