(Fragestunde 27.6.2011)

Reihen:

- $\sum_{k=0}^{\infty} (-1)^k x^k \,, \; |x|<1$ [abs. konv: WT, QT oder geom. R.; Summe $= \frac{1}{1+x}$]

  – d.h. pktw. konv. auf $]-1,1[$

  Bem: für $0<c<1$ auf $[-c,c]$ glm. konv., weil für $f_k(x) = (-1)^k x^k$, $f_k : [-c,c] \to \mathbb{R}$, $\|f_k\|_\infty = c^k$ und $\sum c^k$ konv.

- $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$ [div.: $\frac{k+1}{k^2} = \frac{1}{k} + \frac{1}{k^2} \geq \frac{1}{k}$ div. Min.]

- $\sum_{k=1}^{\infty} \log\left(\frac{k+1}{k}\right)$ [div: $s_N = \sum_{k=1}^{N} (\log(k+1) - \log(k)) = \underbrace{\log(N+1)}_{\to \infty} - \underbrace{\log(1)}_{0}$ $(N \to \infty)$]

  Bem: $\log\left(\frac{k+1}{k}\right) \to \log(1) = 0 \quad (k \to \infty)$

- $\sum_{k=2}^{\infty} \frac{1}{(\log(k))^k}$ [konv: WT]

- $\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^k$ [div: $\left(1-\frac{1}{k}\right)^k \not\to 0$ $\left(\left(1-\frac{1}{k}\right)^k \to e^{-1} = \frac{1}{e}\right)$]

  [[$1 - \frac{1}{k} = \frac{k-1}{k} = \frac{1}{1+\frac{1}{k-1}} \Rightarrow \left(1-\frac{1}{k}\right)^k = \frac{1-\frac{1}{k}}{\left(1+\frac{1}{k-1}\right)^{k-1}} \to \frac{1}{e}$]]

- $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ [div: QT]

- $\displaystyle\sum_{k=1}^{\infty} \frac{\cos((k+1)\pi)}{\sqrt{k}}$ $\quad$ [konv: Leibniz; $\cos((k+1)\pi) = (-1)^{k+1}$]
  nicht abs. konv.

- $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^4-1}}$ $\quad$ [konv: $\displaystyle\frac{1}{\sqrt{k^4-1}} = \frac{1}{k^2\sqrt{1-\frac{1}{k^4}}} \leq \frac{1}{k^2 \cdot \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{k^2}$ konv. Maj.]

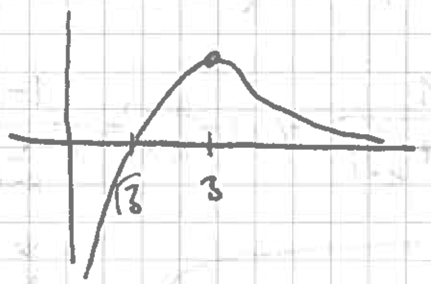
- $\displaystyle\sum_{k=1}^{\infty} \frac{k!\,(2k)!}{(3k)!}$ $\quad$ [konv.: QT; $\displaystyle\frac{\frac{(k+1)!\,(2k+2)!}{(3k+3)!}}{\frac{k!\,(2k)!}{(3k)!}} = \frac{(k+1)\cdot(2k+2)(2k+1)}{(3k+3)(3k+2)(3k+1)}$
  $\downarrow$
  $\displaystyle\frac{1\cdot 2\cdot 2}{3\cdot 3\cdot 3} = \frac{4}{27} < 1$]

Stetigkeit und Differenzierbarkeit

- $f_1 : \mathbb{R} \to \mathbb{R},\ f_1(x) = \begin{cases} 0 & x = 0 \\ x\sin\frac{1}{x} & x \neq 0 \end{cases}$ $\quad$ stetig auf $\mathbb{R}$
  nicht diffbar in $x=0$

- $f_2 : \mathbb{R} \to \mathbb{R},\ f_2(x) = \begin{cases} 0 & x = 0 \\ x^2\sin\frac{1}{x} & x \neq 0 \end{cases}$ $\quad$ diffbar auf $\mathbb{R}$

- $f :]0,\infty[\to \mathbb{R},\ f(x) = \dfrac{x^2-3}{x^3}$ ..... lokale Maxima, Minima
  globales Max/Min?



lok. Max. bei $x=3$, kein lok. Min.
kein glob. Min; glob. Max. in $x=3$

- $f : \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} 0 & x = 0 \\ x\cdot\log|x| & x \neq 0 \end{cases}$ $\quad$ wo stetig, wo diffb.?
  [stetig auf $\mathbb{R}$; nicht diffb. in 0]

Integration und Differentiation

- $f:[-\pi,\pi]\to\mathbb{R}$, $f(x)=\int_0^x e^{\sin(y)}\,dy$

  Monotonie, Wendepunkte, Max, Min.

  $f'(x)=e^{\sin(x)}>0 \Rightarrow$ str. mon. wachsend

  $\Rightarrow$ Min. in $-\pi$, Max. in π

  $f''(x)=\cos(x)\,e^{\sin(x)}$

  $f''(x)=0 \Leftrightarrow \cos(x)=0 \Leftrightarrow x\in\{-\frac{\pi}{2},\frac{\pi}{2}\}$

  also Wendepkt. in $\pm\frac{\pi}{2}$

- $\int_1^2 \underbrace{x^3}_{f'}\underbrace{\log(x)}_{g}\,dx$ [pert. Int.; $4\log(2)-\frac{15}{16}$]

- $\int_0^\pi x^2\sin(x^3+1)\,dx$ $\left[=\frac{1}{3}\int_0^\pi 3x^2\cdot\sin(x^3+1)\,dx\right]$ [Subst. $y=x^3+1$]

- $\int_0^{1/2}\cos^3(\pi x)\sin(\pi x)\,dx$ $\left[=-\frac{1}{\pi}\underbrace{(-\pi\sin(\pi x))}_{=(\cos(\pi x))'}\cdot\cos^3(\pi x)\right.$ ... Subst: $u=\cos(\pi x)\Big]$

- $(a>0)$ $\int_0^a y\sqrt{a^2-y^2}\,dy=-\frac{1}{2}\int_0^a(-2y)\cdot(a^2-y^2)^{\frac{1}{2}}\,dy=\left[\begin{array}{l} z=a^2-y^2\\ dz=-2y\,dy\end{array}\right]$

  $=-\frac{1}{2}\int_{a^2}^0\sqrt{z}\,dz=\frac{1}{2}\int_0^{a^2}z^{\frac{1}{2}}\,dz=\frac{1}{2}\cdot\frac{2}{3}\cdot z^{\frac{3}{2}}\Big|_0^{a^2}=\frac{1}{3}\cdot a^3$

$\cdot\left(\frac{\pi}{2}\right)$

$$\bullet \int_0^{a} \sqrt{a^2-y^2}\,dy = \left[\begin{array}{l} y = a\sin(t) \\ dy = a\cos(t) \end{array}\right] = \int_0^{\arcsin a} \sqrt{a^2-a^2\sin^2 t}\; a\cos t\; dt =$$

$$= a^2 \cdot \int_0^{\arcsin(a)} \underbrace{\sqrt{1-\sin^2 t}}_{\cos(t)} \cdot \cos t\; dt = a^2 \cdot \underbrace{\int_0^{\arcsin a} \cos^2 t\; dt}_{\text{wie in Ü51}} =$$

$$= a^2 \cdot \left(\frac{t+\cos t\sin t}{2}\right)\Bigg|_0^{\arcsin a} = \frac{a^2}{2} \cdot \left(\arcsin(a) + a\cdot\sqrt{1-a^2}\right)$$