Lecture Notes

C^*-Algebras with Aspects of Quantum Physics

Günther Hörmann
Fakultät für Mathematik
Universität Wien
guenther.hoermann@univie.ac.at

Wintersemester 2023/24

(Version of January 23, 2024)

Preface

This course is chronologically a direct follow-up of my course [Hoe23] on advanced functional analysis, although the larger part of its content could be successfully studied already with basic knowledge from bachelor level courses[1] on functional analysis and topology. Overall however, the ideal prerequisites would probably be to be familiar with the key concepts of Chapters I-VII in [Con10].

I am very thankful to the students following the lecture course and to the readers of these notes whose questions and comments are a considerable help in catching flaws and improving the presentation.

Günther Hörmann

[1] Sources on such [in German] are, e.g., http://www.mat.univie.ac.at/~gue/lehre/21fa/Funktionalanalysis.pdf and http://www.mat.univie.ac.at/~gue/lehre/2021gbtop/GBTopologie.pdf

Basic notation

$\mathcal{A}$, $\mathcal{B}$ ... complex Banach algebras or C*-algebras

A, B, C ... elements of a complex Banach algebra or of a C*-algebra

$\mathcal{B}(\mathcal{H})$ ... bounded linear operators on the complex Hilbert space $\mathcal{H}$

$\mathcal{E}^{\#}$ ... dual space of the normed vector space $\mathcal{E}$ (bounded linear functionals on $\mathcal{E}$)

$\mathcal{H}$ ... a complex Hilbert space with inner product $\langle .|. \rangle$, conjugate-linear in the first slot

$\mathcal{M}' = \{B \in \mathcal{B} \mid \forall M \in \mathcal{M}\colon BM = MB\}$ ... the commutant of $\mathcal{M}$ in the complex algebra $\mathcal{B}$

$\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

$\operatorname{res}(A)$ ... resolvent set of A (relative a unital algebra $\mathcal{A}$, $\operatorname{res}(A) = \{\lambda \in \mathbb{C} \mid \exists (A - \lambda)^{-1} \in \mathcal{A}\}$)

$\operatorname{span} Z$ ... linear span (or hull) of the subset Z in a vector space

$\operatorname{sp}(A)$ ... spectrum of A (relative a unital algebra $\mathcal{A}$, $\operatorname{sp}(A) = \mathbb{C} \setminus \operatorname{res}(A)$)

Contents

0. Review of Banach algebras **1**

1. Basic theory of C*-algebras **9**

2. Commutative C*-algebras and functional calculus **19**

3. Representations of C*-algebras **27**

4. A glimpse at von Neumann algebras **39**

5. Canonical commutation relations and the Weyl C*-algebra **53**

6. Fermions and the canonical anticommutation relations **81**

7. A brief outlook on quasi-local algebras **93**

Bibliography **103**

Index **107**

0. Review of Banach algebras

◇ Our main sources for this chapter are [Con10, KRI, Mur90]; see [All11, PI] for more background material.

The default scalar field throughout this course will be $\mathbb{C}$. We will typically speak simply of vector spaces or algebras or linear maps with the understanding that these are complex (linear). If in a particular circumstance the scalar field should be considered to be $\mathbb{R}$ instead, then we will say so.

0.1. Algebraic notions: Recall that a (complex) *algebra* is a $\mathbb{C}$-vector space $\mathcal{A}$ with a multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(A, B) \mapsto AB$ that is associative and bilinear. A *subalgebra* of $\mathcal{A}$ is a vector subspace $\mathcal{B}$ of $\mathcal{A}$ such that $AB \in \mathcal{B}$ for all $A, B \in \mathcal{B}$. In this case, multiplication may be restricted to $\mathcal{B}$ and turns $\mathcal{B}$ itself into an algebra.

An algebra $\mathcal{A}$ is said to be *commutative* or *abelian*, if $AB = BA$ holds for all $A, B \in \mathcal{A}$. Certainly in general, an algebra $\mathcal{A}$ is not abelian and the following concept is nontrivial: If $\mathcal{M}$ is a subset of $\mathcal{A}$, we denote by

$$\mathcal{M}' := \{A \in \mathcal{A} \mid \forall M \in \mathcal{M} \colon AM = MA\}$$

the *commutant* of $\mathcal{M}$ within $\mathcal{A}$ and observe that $\mathcal{M}'$ is a subalgebra of $\mathcal{A}$. We may define the *double commutant* of $\mathcal{M}$ by $\mathcal{M}'' := (\mathcal{M}')'$, and similarly, $\mathcal{M}''' := (\mathcal{M}'')'$. We clearly have $\mathcal{M} \subseteq \mathcal{M}''$ and it is easily seen that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{A}$ always implies $\mathcal{N}' \subseteq \mathcal{M}'$. Hence we obtain $\mathcal{M}''' = \mathcal{M}'$, since $\mathcal{M}' \subseteq (\mathcal{M}')'' = \mathcal{M}'''$ and $\mathcal{M} \subseteq \mathcal{M}''$ implies also $\mathcal{M}''' \subseteq \mathcal{M}'$.

A *left ideal* in an algebra $\mathcal{A}$ is a vector subspace $\mathcal{J} \subseteq \mathcal{A}$ such that $A \in \mathcal{A}$ and $B \in \mathcal{J}$ implies $AB \in \mathcal{J}$. The notion of *right ideal* is defined similarly by requiring $BA \in \mathcal{J}$ in the same situation. If $\mathcal{J}$ is simultaneously a left and a right ideal in $\mathcal{A}$, then we call $\mathcal{J}$ an *ideal* (or two-sided ideal).

A *unit* of $\mathcal{A}$ is an element $I \in \mathcal{A}$, $I \neq 0$, such that $AI = IA = A$ holds for all $A \in \mathcal{A}$. Such element, if it exists, is unique and $\mathcal{A}$ is then said to be a *unital algebra*. Note that by our convention, $\mathcal{A} \neq \{0\}$ for a unital algebra and we always have the one-dimensional subalgebra $\mathbb{C}I := \{\lambda I \mid \lambda \in \mathbb{C}\}$ with maximal commutant $(\mathbb{C}I)' = \mathcal{A}$. If a left or right ideal $\mathcal{J}$ in $\mathcal{A}$ contains the unit I of $\mathcal{A}$, then necessarily $\mathcal{J} = \mathcal{A}$.

A *homomorphism* between complex algebras $\mathcal{A}$ and $\mathcal{B}$ is a $\mathbb{C}$-linear map $\varphi \colon \mathcal{A} \to \mathcal{B}$ that is also multiplicative, i.e., $\varphi(AB) = \varphi(A)\varphi(B)$ for all $A, B \in \mathcal{A}$. Its kernel $\ker \varphi = \{A \in \mathcal{A} \mid \varphi(A) = 0\}$ is an ideal in $\mathcal{A}$ and its image $\operatorname{ran} \varphi = \varphi(\mathcal{A})$ is a subalgebra of $\mathcal{B}$. A bijective homomorphism is called an *isomorphism*. A homomorphism $\varphi \colon \mathcal{A} \to \mathcal{B}$ between unital algebras $\mathcal{A}$ and $\mathcal{B}$ is called *unital*, if[1] $\varphi(I) = I$.

[1] We follow the conventional abuse of notation by having the same symbol I for the units in both algebras. Note that the relation $\varphi(I) = I$ is not implied by the general properties of a homomorphism.

If $\mathcal{A}$ is a unital algebra, then we have a natural injective unital homomorphism $\mathbb{C} \to \mathcal{A}$, $\lambda \mapsto \lambda I$, whose range is $\mathbb{C}I$. In particular, as algebras $\mathbb{C}$ and $\mathbb{C}I$ are isomorphic.

0.2. Banach algebras: An algebra $\mathcal{A}$ that is also a normed vector space with the norm $\|.\|$ satisfying submultiplicativity, i.e.,

$$\forall A, B \in \mathcal{A} \colon \|AB\| \leq \|A\|\|B\|,$$

is called a *normed algebra.* If, in addition, $\mathcal{A}$ is complete with respect to this norm, then $\mathcal{A}$ is said to be a *Banach algebra.* In case $\mathcal{A}$ is a unital algebra, we will also require $\|I\| = 1$ to call it a *unital normed or Banach algebra.*

Remark: (i) Note that $\|I\| \geq 1$ would follow from submultiplicativity in any case for a normed algebra $\mathcal{A}$ with unit (thus $\mathcal{A} \neq \{0\}$) from the existence of some $A \in \mathcal{A}$ with $\|A\| > 0$, since $\|A\| = \|IA\| \leq \|I\|\|A\|$. It is shown in [KRI, Section 3.1] that in any case $\mathcal{A}$ is isomorphic and homeomorphic to a normed unital algebra where the unit has norm 1. (See also [PI, Proposition 1.1.9].)
(ii) As discussed in [KRI, Section 3.1], the above conditions for the norm in a unital Banach algebra are always satisfied by some equivalent norm upon requiring only that multiplication is separately continuous on the Banach space $\mathcal{A}$. (See also [PI, Proposition 1.1.9].)

We immediately obtain that *multiplication in a normed algebra is jointly continuous*, since the condition $\|A - A_0\| \leq 1$ already implies

$$\begin{aligned}\|AB - A_0B_0\| = \|A(B - B_0) + (A - A_0)B_0\| &\leq \|A\|\|B - B_0\| + \|A - A_0\|\|B_0\| \\ &\leq (\|A_0\| + 1)\|B - B_0\| + \|B_0\|\|A - A_0\|.\end{aligned}$$

We proceed now by assuming that $\mathcal{A}$ is a unital Banach algebra, so that it does make sense to investigate invertibility and related concepts. Recall first that, for any $A \in \mathcal{A}$ with $\|A\| < 1$, the partial sums $\sum_{k=0}^{n} A^k$ of the Neumann series converge[2] as $n \to \infty$ and the limit gives the inverse of $I - A$, i.e.

$$\sum_{k=0}^{\infty} A^k = (I - A)^{-1}.$$

In fact, we may deduce that *the set $\mathcal{G}$ of invertible elements in $\mathcal{A}$ is open and the map $B \mapsto B^{-1}$ is continuous on $\mathcal{G}$*: Suppose $A_0 \in \mathcal{G}$ and $A \in \mathcal{A}$ satisfies $\|A - A_0\| < 1/(2\|A_0^{-1}\|)$, then $\|I - A_0^{-1}A\| = \|A_0^{-1}(A_0 - A)\| \leq \|A_0^{-1}\|\|A_0 - A\| < 1/2$ holds, so that $I - (I - A_0^{-1}A) = A_0^{-1}A$ is invertible by the above, hence A is invertible; moreover, we may now estimate

$$\begin{aligned}\|A^{-1} - A_0^{-1}\| &= \|(A^{-1}A_0 - I)A_0^{-1}\| \leq \|A^{-1}A_0 - I\|\|A_0^{-1}\| = \|(A_0^{-1}A)^{-1} - I\|\|A_0^{-1}\| \\ &\leq \Big(\sum_{k=1}^{\infty} \|I - A_0^{-1}A\|^k\Big)\|A_0^{-1}\| = \|I - A_0^{-1}A\|\Big(\sum_{l=0}^{\infty} \|I - A_0^{-1}A\|^l\Big)\|A_0^{-1}\| \\ &= \frac{\|I - A_0^{-1}A\|\|A_0^{-1}\|}{1 - \|I - A_0^{-1}A\|} \leq 2\|I - A_0^{-1}A\|\|A_0^{-1}\| \leq 2\|A_0^{-1}\|\|A_0 - A\|\|A_0^{-1}\| = 2\|A_0^{-1}\|^2\|A - A_0\|.\end{aligned}$$

[2]Based on the geometric series $\sum_{k=0}^{\infty} \|A\|^k$ and the completeness of $\mathcal{A}$.

0.3. Examples: 1) For any complex or real Banach space $\mathcal{E} \neq \{0\}$, the set $\mathcal{B}(\mathcal{E})$ of bounded linear maps $\mathcal{E} \to \mathcal{E}$ equipped with the operator norm and multiplication defined by composition is a complex or real Banach algebra with the identity map $I := \mathrm{id}_{\mathcal{E}}$ as unit. In the finite dimensional special case $\mathcal{E} = \mathbb{C}^n$ or $\mathcal{E} = \mathbb{R}^n$ and upon choosing a basis in $\mathcal{E}$, we may identify $\mathcal{B}(\mathcal{E})$ with the algebra $M(n, \mathbb{C})$ or $M(n, \mathbb{R})$ of complex or real $(n \times n)$-matrices. These algebras are non-commutative unless $\dim \mathcal{E} = 1$.

2) Any norm closed subalgebra of $\mathcal{B}(\mathcal{E})$ as in 1) is a Banach algebra (possibly without unit). If $\mathcal{E}$ is infinite dimensional, then the set $\mathcal{C}(\mathcal{E})$ of compact operators on $\mathcal{E}$ is a nontrivial closed ideal in $\mathcal{B}(\mathcal{E})$, hence also a closed subalgebra, i.e., $\mathcal{C}(\mathcal{E})$ is a Banach algebra without unit in that case.

3) Let X be a locally compact Hausdorff space and equip the space of continuous functions vanishing at infinity,

$$C_0(X) = \{f \colon X \to \mathbb{C} \text{ continuous} \mid \forall \varepsilon > 0\ \exists K \subseteq X \text{ compact} \colon |f(x)| < \varepsilon \text{ for } x \in X \setminus K\},$$

with the supremum norm $\|.\|_\infty$. Consider the usual pointwise multiplication of functions in $C_0(X)$, then we obtain a commutative Banach algebra. In case X is compact we have $C_0(X) = C(X)$ and the Banach algebra has a unit, namely the constant function 1.

4) Let (Ω, μ) be a measure space and consider the Banach space $L^\infty(\Omega, \mu)$ of (classes of) essentially bounded μ-measurable complex functions on Ω equipped with the essential supremum norm $\|.\|_\infty$. Again with the pointwise defined multiplication of functions we obtain a commutative Banach algebra with unit 1.

5) As a special case of 4) consider $\Omega = \mathbb{N}$ with the counting measure, then we obtain the space l^∞ of bounded complex sequences. The subspace c of convergent sequences is a closed subalgebra and is a Banach algebra with unit 1, while the subspace c_0 of sequences converging to 0 is a closed ideal in l^∞, hence c_0 is another example of a Banach algebra without unit.

6) Consider the Banachspace $l^1(\mathbb{Z})$ (as a special case of $L^1(\Omega, \mu)$ with $\Omega = \mathbb{Z}$ and μ the counting measure) with the *convolution* $f * g$ as multiplication of elements $f, g \in l^1(\mathbb{Z})$. Recall that we have

$$\forall m \in \mathbb{Z} \colon (f * g)(m) := \sum_{k \in \mathbb{Z}} f(k) g(m-k) \quad \text{and} \quad \|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Furthermore, convolution is associative and $g * f = f * g$ holds. We obtain a commutative Banach algebra that possesses the function $e_0(m) := \delta_{0,m}$ as a unit.

7) We also have the *convolution algebra* $L^1(\mathbb{R})$, where

$$(f * g)(s) := \int_{\mathbb{R}} f(t) g(s-t)\, dt \qquad (f, g \in L^1(\mathbb{R}), s \in \mathbb{R})$$

defines an associative and commutative multiplication such that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. Thus we obtain a commutative Banach algebra. It does not have a unit. (Observe that $e * g = g$ implies $\int e(t) g(t)\, dt = g(0)$ for every test function g, which is known to be impossible with $e \in L^1_{\mathrm{loc}}(\mathbb{R})$.)

0.4. Adjoining a unit: In case a normed algebra $\mathcal{A}$ does not possess a unit, one can embed it as a closed subalgebra into a unital normed algebra $\mathcal{A}_1$. Indeed, equip the vector space

$\mathcal{A}_1 := \mathcal{A} \times \mathbb{C}$ with the norm $\|(A, \lambda)\|_1 := \|A\| + |\lambda|$. We define a multiplication on $\mathcal{A}_1$ by $(A, \lambda) \cdot (B, \mu) := (AB + \lambda B + \mu A, \lambda\mu)$ and observe that $I := (0, 1)$ is a unit. It is easily checked that we thereby obtain a normed algebra, which is complete if $\mathcal{A}$ is. If $\mathcal{A}$ is commutative, so is $\mathcal{A}_1$. The map $A \mapsto (A, 0)$ gives an injective and isometric homomorphism $\mathcal{A} \to \mathcal{A}_1$.

One main purpose of the construction is to allow the extension of the notion of spectrum even to non-unital Banach algebras. However, on the one hand, adjoining a unit might occasionally be somewhat unnatural (e.g., in the case of group algebras such as $L^1(\mathbb{R})$, or $L^1(G)$ for any locally compact abelian group G) and one may instead work with so-called approximate units; on the other hand, the relevant C*-algebras in quantum physics are unital from the outset; therefore we will not make a lot of use of the above general construction.

0.5. The spectrum: Let $\mathcal{A}$ be a unital Banach algebra. For a scalar $\lambda \in \mathbb{C}$ we will often write λ as an abbreviation for λI, in particular, in terms like $A - \lambda I$ for any $A \in \mathcal{A}$, which will then read $A - \lambda$.

Let $A \in \mathcal{A}$, then the *spectrum* of A (in $\mathcal{A}$) is defined by

$$\mathrm{sp}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda \text{ is not invertible in } \mathcal{A}\}$$

and its complement in $\mathbb{C}$,

$$\mathrm{res}(A) := \mathbb{C} \setminus \mathrm{sp}(A) = \{\lambda \in \mathbb{C} \mid \exists (A - \lambda)^{-1} \in \mathcal{A}\},$$

is called the *resolvent set* of A (in $\mathcal{A}$).

The spectrum of an element $A \in \mathcal{A}$ is a *non-empty compact* subset of the closed disk in $\mathbb{C}$ around 0 with radius $\|A\|$: First, we observe that for $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$ (hence $\lambda \neq 0$), we obtain invertibility of $A - \lambda = \lambda(\frac{1}{\lambda}A - I)$ from the Neumann series since $\|\frac{1}{\lambda}A\| < 1$; hence $\lambda \notin \mathrm{sp}(A)$.
Second, denote by $\mathcal{G} \subset \mathcal{A}$ the open subset of invertible elements; from continuity of $\lambda \mapsto A - \lambda$ and $\mathrm{res}(A) = \{\lambda \in \mathbb{C} \mid A - \lambda \in \mathcal{G}\}$ we directly obtain that $\mathrm{res}(A)$ is open, hence $\mathrm{sp}(A)$ is closed, thus compact due to the first observation.
Third, the proof of non-emptiness of $\mathrm{sp}(A)$ can be based on the weak holomorphy of the resolvent map $R\colon \mathrm{res}(A) \to \mathcal{A}$, $R(\lambda) := (A - \lambda)^{-1}$, which is easily seen to satisfy

$$R(\lambda)R(\mu) = R(\mu)R(\lambda) \quad \text{and} \quad R(\mu) - R(\lambda) = (\mu - \lambda)R(\mu)R(\lambda) \qquad (\lambda, \mu \in \mathrm{res}(A));$$

let ρ be a continuous linear functional on $\mathcal{A}$, i.e., ρ belongs to the dual space $\mathcal{A}^{\#}$ of $\mathcal{A}$ (as a Banach space), then $\rho \circ R$ is holomorphic on $\mathrm{res}(A)$, since we may take the limit $\mu \to \lambda$ in the expression

$$\frac{\rho(R(\mu)) - \rho(R(\lambda))}{\mu - \lambda} = \frac{\rho(R(\mu) - R(\lambda))}{\mu - \lambda} = \frac{(\mu - \lambda)\rho(R(\mu)R(\lambda))}{\mu - \lambda} = \rho(R(\mu)R(\lambda)) \to \rho(R(\lambda)^2);$$

the function $\rho \circ R$ is also bounded, since $|\lambda| \to \infty$ yields $(\lambda^{-1}A - I)^{-1} \to -I$ by continuity of the inverse and this then implies

$$\rho(R(\lambda)) = \lambda^{-1}\rho((\lambda^{-1}A - I)^{-1}) \to 0;$$

if $\mathrm{sp}(A)$ were empty, hence $\mathrm{res}(A) = \mathbb{C}$, the function $\rho \circ R$ would be entire and bounded, hence constant by Liouville's theorem and therefore equal to 0 by the above limit result; in particular, we had $0 \in \mathrm{res}(A)$ and $0 = \rho(R(0)) = \rho(A^{-1})$; since $\rho \in \mathcal{A}^{\#}$ was arbitrary, the Hahn-Banach theorem would imply $A^{-1} = 0$, which is absurd.

We can now directly deduce the following **Corollary (Gelfand-Mazur theorem):** A complex unital Banach division algebra (meaning that every nonzero element is invertible) is isomorphic to $\mathbb{C}$.

Proof: If $\mathcal{A}$ is a complex Banach division algebra and $A \in \mathcal{A}$, then $\mathrm{sp}(A) \neq \emptyset$, hence there is some $\lambda \in \mathrm{sp}(A)$; hence $A - \lambda$ is not invertible and must therefore be 0, i.e., $A = \lambda I$. □

Many more properties of the spectrum in the context of general unital Banach algebras can be derived (see, e.g. [Con10, KRI, Mur90]). For example, upon observing that for $A, B \in \mathcal{A}$, $AB - I$ is invertible if and only if[3] $BA - I$ is invertible, it is elementary to show that

$$\mathrm{sp}(AB) \setminus \{0\} = \mathrm{sp}(BA) \setminus \{0\}.$$

Furthermore, defining $p(A) := a_0 I + a_1 A + \ldots + a_m A^m$ for a complex polynomial $p(z) = a_0 + a_1 z + \ldots + a_m z^m$ and $A \in \mathcal{A}$, it is easily seen that

$$\mathrm{sp}(p(A)) = p(\mathrm{sp}(A)).$$

The second property is the so-called *spectral mapping theorem* and can be extended to the case where p is any holomorphic function defined on some open neighborhood of $\mathrm{sp}(A)$. Invertibility of A implies that of A^{-1}, hence none of these have 0 as a spectral value and the relation $(A - \lambda)^{-1} = \frac{1}{\lambda} A^{-1} (\frac{1}{\lambda} - A^{-1})^{-1}$ proves

$$\mathrm{sp}(A^{-1}) = \Big\{ \frac{1}{\lambda} \mid \lambda \in \mathrm{sp}(A) \Big\}.$$

Analytically more involved is a detailed investigation of the *spectral radius*

$$r(A) := \sup\{|\lambda| \mid \lambda \in \mathrm{sp}(A)\}.$$

We certainly learned above that $r(A) \leq \|A\|$, but it can be shown that, more precisely,

$$r(A) = \inf\{\|A^n\|^{1/n} \mid n \in \mathbb{N}\} = \lim_{n \to \infty} \|A^n\|^{1/n}. \tag{0.1}$$

(A sketch of the proof is as follows: If $\mu \in \mathbb{C}$ with $|\mu| > \|A\| \geq r(A)$, then we certainly have with $z := 1/\mu$ that $(\mu - A)^{-1} = z \sum_{k=0}^{\infty} A^k z^k$ and this power series can easily be shown to have radius of convergence $R = 1/r(A)$; therefore, $r(A) = 1/R = \limsup \|A^k\|^{1/k}$; if $\lambda \in \mathrm{sp}(A)$ then $\lambda^k \in \mathrm{sp}(A^k)$, which implies $|\lambda|^k = |\lambda^k| \leq \|A\|^k$, i.e., $|\lambda| \leq \|A^k\|^{1/k}$ for all $k \in \mathbb{N}$; we conclude that $r(A) \leq \inf\{\|A^k\|^{1/k} \mid k \in \mathbb{N}\} \leq \liminf \|A^k\|^{1/k} \leq \limsup \|A^k\|^{1/k} = r(A)$, hence we have equality throughout and $r(A) = \lim \|A^k\|^{1/k}$.)

0.6. Commutative Banach algebras and the Gelfand transform: We assume here throughout that $\mathcal{A}$ is a unital commutative Banach algebra.

A *multiplicative linear functional* ρ on $\mathcal{A}$ is a homomorphism from $\mathcal{A}$ to $\mathbb{C}$. Note that ρ is non-zero if and only if $\rho(I) = 1$. (Multiplicativity implies $\rho(I) = \rho(I)^2$ in $\mathbb{C}$, hence we must have either $\rho(I) = 0$ or $\rho(I) = 1$; certainly $\rho(I) = 0$ implies $\rho = 0$.)

[3] Note that $(BA - I)^{-1} = B(AB - I)^{-1} A - I$.

Recall that the weak* topology on the dual $\mathcal{E}^{\#}$ of a Banach space $\mathcal{E}$ is the locally convex vector space topology defined by the seminorms p_x ($x \in \mathcal{E}$) with $p_x(\psi) = |\psi(x)|$ for every $\psi \in \mathcal{E}^{\#}$. A typical basis of 0-neighborhoods is given by the following family of subsets (with $\varepsilon > 0$, $m \in \mathbb{N}$, and $x_1, \ldots, x_m \in \mathcal{E}$):

$$U_{\varepsilon; x_1, \ldots, x_m} := \{\psi \in \mathcal{E}^{\#} \mid j = 1, \ldots, m\colon\ p_{x_j}(\psi) < \varepsilon\}.$$

In particular, convergence of nets in the weak* topology of $\mathcal{E}^{\#}$ corresponds exactly to the pointwise convergence of the linear functionals on $\mathcal{E}$. The Hausdorff separation property for the weak* topology follows directly from the elementary fact that for any non-zero $\psi \in \mathcal{E}^{\#}$ we can certainly find some $x \in \mathcal{E}$ such that $p_x(\psi) = |\psi(x)| \neq 0$. By the Banach-Alaoglu theorem, the (norm) closed unit ball $\mathcal{C} := \{\psi \in \mathcal{E}^{\#} \mid \|\psi\| \leq 1\}$ in $\mathcal{E}^{\#}$ is weak* compact.

Proposition: Let X be the set of all non-zero multiplicative linear functionals on the unital commutative Banach algebra $\mathcal{A}$. Then we have

$$\forall A \in \mathcal{A}\colon \quad \operatorname{sp}(A) = \{\rho(A) \mid \rho \in X\}.$$

Furthermore, X is a subset of the unit sphere in $\mathcal{A}^{\#}$ and is a weak* compact Hausdorff space.

Sketch of proof: We first note that $\{\rho(A) \mid \rho \in X\} \subseteq \operatorname{sp}(A)$, since $A - \rho(A)$ belongs to the kernel of ρ, which would contradict the invertibility of $A - \rho(A)$ due to multiplicativity. This already implies that $|\rho(A)| \leq \|A\|$, which shows that $\|\rho\| \leq 1$. In addition with $\rho(I) = 1$ (since $\rho \neq 0$) we obtain $\|\rho\| = 1$, hence X is a subset of the unit sphere in $\mathcal{E}^{\#}$, thus X is also contained in the weak* compact closed unit ball $\mathcal{C}$ of $\mathcal{E}^{\#}$.

Since weak* convergence means pointwise convergence, it is clear that a weak* limit of multiplicative linear functionals is also multiplicative. Therefore, X is a weak* closed subset of $\mathcal{C}$, hence itself compact. (The Hausdorff property is certainly inherited from the weak* topology.)

It remains to show $\operatorname{sp}(A) \subseteq \{\rho(A) \mid \rho \in X\}$. Let $\lambda \in \operatorname{sp}(A)$, then $B := A - \lambda$ is not invertible in $\mathcal{A}$. It suffices to show that for any non-invertible element $B \in \mathcal{A}$ one can find some $\rho \in X$ such that $\rho(B) = 0$, because then we would deduce from $\rho(A - \lambda) = 0$ that $\lambda = \rho(A)$ belongs to the right-hand side of the claimed inclusion relation.

Here, we will quickly borrow some results from basic algebra upon noting that $\mathcal{A}$ is a commutative ring with unit. Recall that an ideal $\mathcal{J} \neq \mathcal{A}$ in $\mathcal{A}$ is called maximal, if $\mathcal{J}$ and $\mathcal{A}$ are the only ideals containing $\mathcal{J}$; any ideal $\mathcal{I} \neq \mathcal{A}$ is contained in some maximal ideal $\mathcal{J} \supseteq \mathcal{I}$. If $\mathcal{J}$ is a maximal ideal then the quotient ring $\mathcal{A}/\mathcal{J}$ is a field. Since in our context of a Banach algebra, the maximal ideal $\mathcal{J}$ is closed ([KRI, Proposition 3.1.8] or [Mur90, Theorem 1.3.1]) and the quotient $\mathcal{A}/\mathcal{J}$ is a Banach algebra ([KRI, Proposition 3.1.8] or [Mur90, Theorem 1.1.1]), the field $\mathcal{A}/\mathcal{J}$ can only be $\mathbb{C}$ due to the Gelfand-Mazur theorem (see the corollary in 0.5).

We finally show: For any non-invertible element $B \in \mathcal{A}$ there is some $\rho \in X$ such that $\rho(B) = 0$.

The set $\mathcal{I} := \{BC \mid C \in \mathcal{A}\}$ is an ideal in $\mathcal{A}$ with $I \notin \mathcal{I}$. Let $\mathcal{J}$ be a maximal ideal containing $\mathcal{I}$, then $\mathcal{A}/\mathcal{J} \cong \mathbb{C}$. The canonical surjection $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{J}$ is a ring homomorphism and linear, hence induces a non-zero multiplicative linear functional ρ on $\mathcal{A}$ with $\ker \rho = \mathcal{J}$, in particular, $\rho(B) = 0$. $\square$

We recall that $\mathcal{A}$ is canonically embedded into the bidual $\mathcal{A}^{\#\#}$ by assigning to $A \in \mathcal{A}$ the linear functional $\iota(A)\colon \mathcal{A}^{\#} \to \mathbb{C}$, $\psi \mapsto \psi(A)$, which is obviously weak* continuous. We may thus define $\hat{A}$ as the restriction of $\iota(A)$ to $X \subseteq \mathcal{A}^{\#}$ and obtain a continuous function $\hat{A}\colon X \to \mathbb{C}$ on the compact Hausdorff space X. We have thus established the *Gelfand transform*

$$\mathcal{A} \to C(X), \quad A \mapsto \hat{A} \quad \text{with} \quad \hat{A}(\rho) = \rho(A) \quad (\rho \in X). \tag{0.2}$$

Theorem: The Gelfand transform is a continuous unital homomorphism of Banach algebras with the following properties:

$$\forall A \in \mathcal{A}\colon \quad \|\hat{A}\|_\infty = r(A) \leq \|A\| \quad \text{and} \quad \mathrm{sp}(A) = \hat{A}(X) = \mathrm{sp}(\hat{A}).$$

Proof: Linearity of the Gelfand transform is clear and the multiplicativity follows immediately from the relation $\widehat{AB}(\rho) = \rho(AB) = \rho(A)\rho(B) = \hat{A}(\rho)\hat{B}(\rho)$; furthermore, $\hat{I}(\rho) = \rho(I) = 1$. By the above proposition, $\mathrm{sp}(A) = \hat{A}(X)$ and therefore also $\|\hat{A}\|_\infty = r(A) \leq \|A\|$. The equality $\hat{A}(X) = \mathrm{sp}(\hat{A})$ follows from the elementary fact that $\mathrm{sp}(f) = f(X)$ for any $f \in C(X)$. $\square$

Remark: (i) Let Y be a compact Hausdorff space and $\mathcal{A} = C(Y)$. For any $y \in Y$ we have the multiplicative linear functional ρ_y on $\mathcal{A}$, given by $\rho_y(f) := f(y)$ for all $f \in C(Y)$. (We see that ρ_y corresponds to the Dirac measure concentrated at y.) It can be shown (see, e.g., [Con10, Chapter VII, Theorem 8.7] or [KRI, Section 3.4]) that these comprise all elements in X, the set of non-zero multiplicative linear functionals on $C(Y)$, and that $y \mapsto \rho_y$ gives a homeomorphism $Y \to X$. In summary, we have in this special case $\hat{f}(\rho_y) = \rho_y(f) = f(y)$ for all $y \in Y$, $f \in C(Y)$ and $C(Y) \cong C(X)$.

(ii) One can show ([KRI, Theorem 3.4.3]) that for compact Hausdorff spaces X and Y, the existence of an algebraic isomorphism between $C(X)$ and $C(Y)$ is equivalent to the condition that X and Y are homeomorphic.

A C*-algebraic variant of the following corollary to the above theorem will re-occur later. It can be viewed as the abstract background for the continuous functional calculus for a self-adjoint bounded operator on a Hilbert space.

Corollary: Suppose that $\mathcal{A}$ is generated from an element $A \in \mathcal{A}$ in the sense that the subalgebra $\{p(A) \mid p \text{ a polynomial}\}$ is dense in $\mathcal{A}$. Then X is homeomorphic to $\mathrm{sp}(A)$. In particular, we may interpret the Gelfand transform as a homomorphism $\mathcal{A} \to C(\mathrm{sp}(A))$.

Proof: Let the map $h\colon X \to \mathrm{sp}(A)$ be defined by $h(\rho) := \rho(A)$. Then h is weak* continuous on X and by the above theorem, $h(X) = \hat{A}(X) = \mathrm{sp}(A)$, hence h is surjective. We claim that h is also injective: $h(\rho_1) = h(\rho_2)$ means $\rho_1(A) = \rho_2(A)$ and multiplicativity and linearity then yields that $\rho_1(p(A)) = \rho_2(p(A))$ for every polynomial p. The continuity of ρ_1 and ρ_2 then implies $\rho_1 = \rho_2$. We obtained that h is a continuous bijective map between the compact Hausdorff spaces X and $\mathrm{sp}(A)$. Therefore, h is a homeomorphism. $\square$

As we will show later in the course, for a unital commutative C*-algebra $\mathcal{A}$ the Gelfand transform is always an isometric isomorphism onto $C(X)$. This is not true in general for unital commutative Banach algebras. For example, let D denote the closed unit disk in $\mathbb{C}$ and $\mathcal{A}(D)$ be the closed subalgebra of $C(D)$ consisting of all functions that are holomorphic in the interior of D. Then this unital commutative *disk algebra* $\mathcal{A}(D)$ is generated by the function id_D. It can be shown (e.g., [Mur90, Section 1.3]) that in this case, the Gelfand transform is (modulo the homeomorphism $X \to \mathrm{sp}(\mathrm{id}_D)$) the identity map $\mathcal{A}(D) \to C(D)$, hence is injective but not surjective. An example with a non-injective Gelfand transform can be found in [Con10, Chapter VII, Example 8.12]. As we can see in the theorem above, the kernel of the Gelfand transform consists of the elements in $\mathcal{A}$ with vanishing spectral radius.

A remark on the non-unital case: If $\mathcal{A}$ is a Banach algebra without unit, then we may still define a notion of spectrum by referring to the Banach algebra $\mathcal{A}_1$ constructed in 0.4 by adjoining a unit. Let $\gamma\colon \mathcal{A} \to \mathcal{A}_1$ denote the isometric embedding $A \mapsto (A, 0)$, then we define

$$\mathrm{sp}(A) := \mathrm{sp}(\gamma(A)) = \{\lambda \in \mathbb{C} \mid \gamma(A) - \lambda \text{ is not invertible in } \mathcal{A}_1\}.$$

We automatically have $0 \in \mathrm{sp}(A)$, since $\gamma(A)\cdot(B, \mu) = (A, 0)\cdot(B, \mu) = (AB+\mu A, 0) \neq (0, 1) = I$ for all $B \in \mathcal{A}$ and $\mu \in \mathbb{C}$.

The set X of non-zero multiplicative linear functionals on $\mathcal{A}$ is then a locally compact Hausdorff space with respect to the weak* topology as a subset of $\mathcal{A}^\#$ ([Mur90, Theorem 1.3.5]) and

$$\mathrm{sp}(A) = \hat{A}(X) \cup \{0\}$$

holds ([Mur90, Theorem 1.3.4]). If $X \neq \emptyset$, then the Gelfand transform is a continuous homomorphism $\mathcal{A} \to C_0(X)$ satisfying $\|\hat{A}\|_\infty = r(A) \leq \|A\|$ for all $A \in \mathcal{A}$ ([Mur90, Theorem 1.3.6]).

1. Basic theory of C*-algebras

◇ Our main sources for this chapter are [KRI, Con00, Mur90, BR1]; see [TakI, PII] for more on the background.

1.1. Involution and *-homomorphisms: An *involution* (or sometimes called $*$-structure) on a complex algebra $\mathcal{A}$ is a map $A \mapsto A^*$ from $\mathcal{A}$ into $\mathcal{A}$ such that for all $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$ the following hold:

(i) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$ (i.e., $A \mapsto A^*$ is conjugate linear),

(ii) $(AB)^* = B^* A^*$,

(iii) $(A^*)^* = A$.

An algebra $\mathcal{A}$ with an involution is called a $*$-*algebra*. A subalgebra $\mathcal{B} \subseteq \mathcal{A}$ that is invariant under the involution, i.e., $\mathcal{B}^* \subseteq \mathcal{B}$, is said to be a $*$-*subalgebra*.

In case a $*$-algebra $\mathcal{A}$ has a unit I, then $I^* A = (I^* A)^{**} = (A^* I)^* = (A^*)^* = A$ and similarly $AI^* = A$; hence by the uniqueness of the unit we conclude that

$$I^* = I.$$

It follows that for any $\alpha \in \mathbb{C}$, writing again α to mean αI, we have $\alpha^* = (\alpha I)^* = \overline{\alpha} I = \overline{\alpha}$.

A $*$-*homomorphism* between $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is an algebra homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ with the additional property $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathcal{A}$.

1.2. Definition: A C*-*algebra* is a complex Banach algebra $\mathcal{A}$ with an involution such that

$$\forall A \in \mathcal{A}\colon \quad \|A^* A\| = \|A\|^2.$$

A C*-*subalgebra* of $\mathcal{A}$ is a closed $*$-subalgebra and is itself a C*-algebra.

As a first implication of the above C*-property we observe in the following statement that the involution preserves norm, hence is a continuous conjugate linear map.

1.3. Lemma: If $\mathcal{A}$ is a C*-algebra, then the involution is an isometry,

$$\forall A \in \mathcal{A}\colon \quad \|A^*\| = \|A\|.$$

Furthermore, also $\|AA^*\| = \|A\|^2$ holds in general.

Proof: It suffices to consider $A \neq 0$. Then $A^* \neq 0$, for otherwise $\|A\|^2 = \|A^* A\| = \|0\| = 0$. We have $\|A\|^2 = \|A^* A\| \leq \|A^*\| \|A\|$, hence $\|A\| \leq \|A^*\|$; using $A^{**} = A$, we also have $\|A^*\|^2 = \|A^{**} A^*\| = \|AA^*\| \leq \|A\| \|A^*\|$, hence also $\|A^*\| \leq \|A\|$. In summary, $\|A^*\| = \|A\|$.

Finally, $\|AA^*\| = \|A^{**} A^*\| = \|A^*\|^2 = \|A\|^2$. □

1.4. Examples: 1) Let $\mathcal{H} \neq \{0\}$ be a complex Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a Banach algebra as a special case of Example 0.3, 1). The standard involution on $\mathcal{B}(\mathcal{H})$ is given by the definition of the adjoint operator, where we recall that the C*-property with respect to the operator norm can be verified in the following way: For any $x \in \mathcal{H}$,

$$\|Ax\|^2 = \langle Ax|Ax\rangle = \langle A^*Ax|x\rangle \leq \|A^*A\|\|x\|^2,$$

which implies $\|A\|^2 \leq \|A^*A\| \leq \|A^*\|\|A\|$ and $\|A\| \leq \|A^*\|$. Replacing A by A^* and using $A^{**} = A$ we obtain also $\|A^*\| \leq \|A\|$. This now yields $\|A^*\| = \|A\|$ and then $\|A\|^2 \leq \|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2$, so that also $\|A\|^2 = \|A^*A\|$ follows.

We conclude that $\mathcal{B}(\mathcal{H})$ is a unital C*-algebra. Since the adjoint of a compact operator is compact, we obtain $\mathcal{C}(\mathcal{H})$ as a (non-unital) C*-subalgebra of $\mathcal{B}(\mathcal{H})$.

2) For any compact Hausdorff space X, we have the unital Banach algebra $C(X)$ with the supremum norm $\|.\|_\infty$ from Example 0.3, 3). In this case, pointwise complex conjugation, $\bar{f}(x) := \overline{f(x)}$ ($f \in C(X)$, $x \in X$), defines an involution $f \mapsto \bar{f}$ on $C(X)$ and we obtain an example of a unital commutative C*-algebra.

3) Similarly, $L^\infty(\Omega, \mu)$ from 0.3, 4), and l^∞ from 0.3, 5), become unital commutative C*-algebras.

4) The commutative convolution Banach algebras $L^1(\mathbb{R})$ and $l^1(\mathbb{Z})$ in the Examples 0.3, 6) and 7), are not C*-algebras, although $f^*(s) := \overline{f(-s)}$ defines an involution in both cases that even satisfies $\|f^*\|_1 = \|f\|_1$. Banach algebras with an isometric involution are sometimes called *Banach $*$-algebras*. (For a concrete example violating the C*-property in $l^1(\mathbb{Z})$ consider $f(m) := \delta_{-1,m} + \delta_{0,m} - \delta_{1,m}$; then $\|f\|_1 = 3$ and $\|f^* * f\|_1 = 5$. For a general reason why $L^1(G)$, G a locally compact abelian group, is almost never a C*-algebra see [Dix82, 13.3.6].)

1.5. Remark (adjoining a unit to a C*-algebra): If $\mathcal{A}$ is a C*-algebra without unit, we may algebraically adjoin a unit exactly as in 0.4 by considering $\mathcal{A}_1 := \mathcal{A} \times \mathbb{C}$ and extend the involution by $(A, \lambda)^* := (A^*, \overline{\lambda})$. Although the norm given in 0.4 provides us then with the structure of a Banach $*$-algebra, it fails to produce a C*-algebra. The remedy is to use instead the norm

$$\|(A, \lambda)\|_0 := \sup\{\|\lambda B + AB\| \mid B \in \mathcal{A}, \|B\| = 1\},$$

which can be shown to have all the required properties ([BR1, Proposition 2.1.5]). This norm is derived from the operator norm of $\lambda I + L_A \in \mathcal{B}(\mathcal{A})$, where L_A denotes left multiplication by A, i.e., $L_A B := AB$, and is the unique C*-norm on $\mathcal{A}_1$ extending the norm on $\mathcal{A}$ ([Mur90, Theorem 2.1.6]).

Starting from here we will be using the following two "default specifications":

1) A C*-algebra $\mathcal{A}$ has a unit I,

2) a $*$-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ between C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is unital, i.e., $\varphi(I) = I$.

We will now extend a few well-known notions from the special case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space to the abstract C*-algebra setting.

1.6. Definition: Let $\mathcal{A}$ be a C*-algebra and $A \in \mathcal{A}$. We refer to $A^* \in \mathcal{A}$ as the *adjoint* of A.

(i) A is *self-adjoint* (or *hermitian*), if $A^* = A$.

(ii) A is *normal*, if A commutes with A^*, i.e., $A^*A = AA^*$.

(iii) A is *unitary*, if A^* is the inverse of A, i.e., $A^*A = AA^* = I$.

Clearly, a self-adjoint or unitary element is normal. The unit is unitary and self-adjoint.

The unitary elements of $\mathcal{A}$ form a multiplicative group. The set of self-adjoint elements in $\mathcal{A}$ is a real vector space. A subset $\mathcal{B}$ of $\mathcal{A}$ is said to be *self-adjoint*, if it is invariant under the involution. Thus, a self-adjoint subalgebra is a *-subalgebra. By continuity of the involution, the closure of a $*$-subalgebra is a $*$-subalgebra, hence a C*-subalgebra.

Every element $A \in \mathcal{A}$ has a unique representation in the form $A = H_1 + iH_2$ with self-adjoint elements $H_1, H_2 \in \mathcal{A}$, namely

$$A = \frac{1}{2}(A + A^*) + i\frac{1}{2i}(A - A^*).$$

Thus, we may call $H_1 = (A + A^*)/2$ the *real part* of A and $H_2 = (A - A^*)/2i$ the *imaginary part* of A. We see that A is normal if and only if H_1 and H_2 commute.

An element A is invertible if and only if A^* is invertible; in this case, $(A^*)^{-1} = (A^{-1})^*$. Noting that $(A - \lambda)^* = A^* - \overline{\lambda}$ we obtain

$$\mathrm{sp}(A^*) = \{\overline{\lambda} \mid \lambda \in \mathrm{sp}(A)\} \quad \text{and} \quad r(A^*) = r(A).$$

1.7. Proposition: Let $\mathcal{A}$ be a C*-algebra and $U, A \in \mathcal{A}$.

(i) If U is unitary, then $\|U\| = 1$ and $\mathrm{sp}(U) \subseteq S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$.

(ii) If A is self-adjoint, then $r(A) = \|A\|$ and we have $\mathrm{sp}(A) \subseteq \mathbb{R}$. In particular, $-\|A\|$ or $\|A\|$ belongs to $\mathrm{sp}(A)$.

(iii) If A is normal, then $r(A) = \|A\|$.

Proof: (i) From the C*-property, $\|U\|^2 = \|U^*U\| = \|I\| = 1$, hence $\|U\| = 1$ and therefore $r(U) \leq 1$. Note that $0 \notin \mathrm{sp}(U)$, since U is invertible.

Let $\mu \in \mathrm{sp}(U)$. Recall that $\overline{\mu} \in \mathrm{sp}(U^*) = \mathrm{sp}(U^{-1})$, hence also $\frac{1}{\mu} \in \mathrm{sp}(U)$ (thanks to what we observed in 0.5) and therefore,

$$|\mu| \leq r(U) \leq 1 \quad \text{and} \quad \frac{1}{|\mu|} \leq r(U) \leq 1,$$

which can only hold if $|\mu| = 1$.

(ii) Let $n \in \mathbb{N}$, then $\|A^{2n}\| = \|(A^n)^* A^n\| = \|A^n\|^2$ and inductively, $\|A^{2^m}\| = \|A\|^{2^m}$ for every $m \in \mathbb{N}$, so that (0.1) yields

$$r(A) = \lim_{k\to\infty} \|A^k\|^{1/k} = \lim_{m\to\infty} \|A^{2^m}\|^{1/2^m} = \|A\|.$$

Since $\mathrm{sp}(A)$ is compact it hence follows from the definition of the spectral radius that there is some $\lambda \in \mathrm{sp}(A)$ with $|\lambda| = r(A) = \|A\|$. It remains to show that the spectrum is real.

Define $U := \exp(iA) := \sum_{k=0}^{\infty} (iA)^k/k!$ (using the absolute convergence of the series) and note[1] that $U^* = \exp(-iA) = U^{-1}$, hence U is unitary. Let $\lambda \in \mathbb{C}$, then direct calculation shows $U - e^{i\lambda} = e^{i\lambda}(A-\lambda)\sum_{n=1}^{\infty} i^n (A-\lambda)^{n-1}/n!$, where all factors commute. If $\lambda \in \mathrm{sp}(A)$, the non-invertibility of $A - \lambda$ thus implies that of $U - e^{i\lambda}$ and we obtain $e^{i\lambda} \in S^1$ from (i), which forces $\lambda \in \mathbb{R}$.

(iii) We have $\|A^2\|^2 = \|(A^2)^*(A^2)\| = \|A^*A^*AA\| = \|A^*AA^*A\| = \|(A^*A)^2\| = \|A^*A\|^2 = (\|A\|^2)^2$, where in the next to last equality we used that A^*A is self-adjoint. Hence $\|A^2\| = \|A\|^2$ and we may proceed exactly as in the first part of the proof of (ii), since A^n is normal. □

1.8. Positive elements and partial ordering: We define an element A of the C*-algebra to be *positive* and write $A \geq 0$, if there is some self-adjoint $C \in \mathcal{A}$ such that $A = C^2$. Note that a positive element is automatically self-adjoint. We define a relation on the real vector subspace of self-adjoint elements in $\mathcal{A}$ by writing $A \leq B$, if $B - A \geq 0$, i.e., $B - A$ is positive. It will follow from the results we will establish in 2.11 that this is a partial ordering.

Examples: 1) Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and A be positive in the above sense. There is some self-adjoint $C \in \mathcal{B}(\mathcal{H})$ such that $A = C^2$. Let $x \in \mathcal{H}$ be arbitrary, then

$$\langle x|Ax\rangle = \langle x|C^2x\rangle = \langle Cx|Cx\rangle = \|Cx\|^2 \geq 0$$

and therefore, A is positive in the sense of operators. Using the construction of a square root as given in [Hoe23, Corollary 1.4 and Example 1.20] (and being independent of the C*-algebraic arguments to follow below), it is easy to see that positivity in the operator sense implies positivity according to the C*-algebraic definition (cf. also [KRI, Theorem 4.2.6(iv)]).

2) Let $\mathcal{A} = C(X)$, where X is a compact Hausdorff space. Note that self-adjointness of $f \in C(X)$ means that f is real-valued. Suppose $f = g^2$ with some real-valued $g \in C(X)$, then $f(x) = g(x)^2 \geq 0$ for every $x \in X$. Conversely, if f is non-negative in the pointwise sense of continuous functions, then we may define $g(x) := \sqrt{f(x)}$ ($x \in X$) to obtain a continuous real-valued function on X satisfying $f = g^2$. We conclude that in $C(X)$ positivity in the C*-algebraic sense means pointwise non-negativity as a function.

From Proposition 1.7(ii) and the elementary spectral mapping theorem in 0.5 for the polynomial $p(z) = z^2$, we obtain that the spectrum of a positive element A is contained in the non-negative real numbers: Let C be self-adjoint such that $A = C^2$, then

$$\mathrm{sp}(A) = \mathrm{sp}(C^2) = \{\lambda^2 \mid \lambda \in \mathrm{sp}(C)\} \subseteq \{\lambda^2 \mid -\|C\| \leq \lambda \leq \|C\|^2\} \subseteq [0, \|C\|^2].$$

[1] reasoning like $U^* = (\lim_{m\to\infty} \sum_{k=0}^{m} (iA)^k/k!)^* = \lim_{m\to\infty} (\sum_{k=0}^{m} (iA)^k/k!)^* = \lim_{m\to\infty} \sum_{k=0}^{m} ((iA)^*)^k/k!$ and $\exp(iA)\exp(-iA) = \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{l=0}^{n} \binom{n}{l} (iA)^l (-iA)^{n-l} = \sum_{n=0}^{\infty} (iA - iA)^n = I$ etc

This observation suggests condition (ii) in the following theorem (and shows (i) $\Rightarrow$ (ii)), while (iii) is a trivial consequence of (i). A complete proof will be given in the next chapter (see 2.11) once the continuous functional calculus is available.

Theorem: If A is self-adjoint[2] in the C*-algebra $\mathcal{A}$, then following conditions are equivalent:

(i) $A \geq 0$, i.e., there is some self-adjoint $C \in \mathcal{A}$ such that $A = C^2$,

(ii) $\mathrm{sp}(A) \subseteq [0, \infty[$,

(iii) there is some $C \in \mathcal{A}$ such that $A = C^*C$.

Corollary: Let $A, B \in \mathcal{A}$ be self-adjoint and $C \in \mathcal{A}$ be arbitrary.

(i) $A \leq B$ implies $C^*AC \leq C^*BC$,

(ii) $-\|A\| \leq A \leq \|A\|$,

(iii) $0 \leq C^*C \leq \|C\|^2$,

(iv) if A is invertible and positive, then A^{-1} is positive.

Proof: (i) Suppose $B - A = D^*D$ with $D \in \mathcal{A}$, then $C^*BC - C^*AC = C^*(B - A)C = C^*D^*DC = (DC)^*(DC)$, so that (iii) in the above theorem shows that $C^*BC - C^*AC$ is positive.

(ii) Considering the polynomials $p_\pm(z) = \|A\| \pm z$ and recalling $\mathrm{sp}(A) \subseteq [-\|A\|, \|A\|]$ we obtain $\mathrm{sp}(\|A\| \pm A) = \{\|A\| \pm \lambda \mid \lambda \in \mathrm{sp}(A)\} \subseteq [0, \infty[$ and may appeal to (ii) in the above theorem.

(iii) By the above theorem, $C^*C \geq 0$. Since C^*C is self-adjoint, we deduce from (ii) also that $C^*C \leq \|C^*C\| = \|C\|^2$.

(iv) Clearly, A^{-1} is self-adjoint and $\mathrm{sp}(A^{-1}) = \{1/\lambda \mid \lambda \in \mathrm{sp}(A)\} \subseteq]0, \infty[$, so the claim follows again by (ii) in the above theorem. $\square$

Remark: While it is clear that the positive multiple of a positive element in an abstract C*-algebra is positive, two other statements about positivity are less obvious and will be proved only later in 2.11: First, for positive A, B we always have that $A + B$ is positive. Second, $-C^*C$ positive implies $C = 0$. Both are deceivingly easy to show in case A, B, C are operators on a Hilbert space $\mathcal{H}$, since

$$\langle x|(A+B)x\rangle = \langle x|Ax\rangle + \langle x|Bx\rangle \geq 0 \quad \text{and} \quad 0 \leq -\langle x|C^*Cx\rangle = -\langle Cx|Cx\rangle = -\|Cx\|^2.$$

We will now show automatic continuity of $*$-homomorphims between C*-algebras.

1.9. Theorem: Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism of C*-algebras, then $\mathrm{sp}(\varphi(A)) \subseteq \mathrm{sp}(A)$ and $\|\varphi(A)\| \leq \|A\|$ for every $A \in \mathcal{A}$. If φ is a $*$-isomorphism[3], then φ is isometric and $\mathrm{sp}(\varphi(A)) = \mathrm{sp}(A)$.

[2] Without the assumption self-adjointness, (ii) is weaker than (i) or (iii): Consider $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in M(2, \mathbb{C})$, then $\mathrm{sp}(A) = \{0\} \subseteq [0, \infty[$ and $\langle x|Ax\rangle = -1 < 0$ for $x = (1, -1)$.

[3] Unfortunately, in some of the literature (e.g., [KRI, KRII]), the term $*$-isomorphism requires only injectivity and does not refer to bijections in general.

Proof: If $\lambda \in \mathbb{C}$ and $A - \lambda$ is invertible (in $\mathcal{A}$) then clearly $\varphi(A) - \lambda = \varphi(A - \lambda)$ is invertible (in $\mathcal{B}$), hence $\mathrm{sp}(\varphi(A)) \subseteq \mathrm{sp}(A)$. Since A^*A (and therefore also $\varphi(A^*A)$) is self-adjoint, we have from Proposition 1.7(ii), $\|\varphi(A)\|^2 = \|\varphi(A)^*\varphi(A)\| = \|\varphi(A^*A)\| = r(\varphi(A^*A)) \leq r(A^*A) = \|A^*A\| = \|A\|^2$.

In case φ is a $*$-isomorphism, then φ^{-1} is one too, hence also $\|A\| = \|\varphi^{-1}(\varphi(A))\| \leq \|\varphi(A)\|$ and $\mathrm{sp}(A) = \mathrm{sp}(\varphi^{-1}(\varphi(A))) \subseteq \mathrm{sp}(\varphi(A))$. $\square$

1.10. Positive linear functionals: Let $\mathcal{A}$ be a C*-algebra. A linear functional ρ on $\mathcal{A}$ is said to be *hermitian* if $\rho(A^*) = \overline{\rho(A)}$ holds for every $A \in \mathcal{A}$. Upon writing $A = H_1 + iH_2$ with real part H_1 and imaginary part H_2 we see that ρ is hermitian if and only if $\rho(A)$ is real for each self-adjoint $A \in \mathcal{A}$.

Lemma: If ρ is a bounded hermitian linear functional on $\mathcal{A}$, then

$$\|\rho\| = \sup\{\rho(H) \mid H \in \mathcal{A} \text{ self-adjoint and } \|H\| \leq 1\}.$$

Proof: It is evident that the right-hand side is a lower bound for $\|\rho\|$, thus it remains to show that for every $\varepsilon > 0$ we can find some self-adjoint $H_0 \in \mathcal{A}$ with $\|H_0\| \leq 1$ such that $\rho(H_0) > \|\rho\| - \varepsilon$.

There is some $A \in \mathcal{A}$ with $\|A\| \leq 1$ such that $|\rho(A)| > \|\rho\| - \varepsilon$. Choose $\alpha \in \mathbb{C}$, $|\alpha| = 1$, such that $|\rho(A)| = \alpha\rho(A) = \rho(\alpha A)$ and let H_0 be the real part of αA, i.e., $H_0 = (\alpha A + (\alpha A)^*)/2$. We have $\|H_0\| \leq \|\alpha A\| = \|A\| \leq 1$ and

$$\rho(H_0) = \frac{1}{2}(\rho(\alpha A) + \rho((\alpha A)^*)) = \frac{1}{2}(\rho(\alpha A) + \overline{\rho(\alpha A)}) = |\rho(A)| > \|\rho\| - \varepsilon.$$

$\square$

We saw that for a bounded hermitian linear functional, the norm is the same as that for the restriction to the real vector subspace of self-adjoint elements.

Definition: A linear functional ρ on $\mathcal{A}$ is said to be *positive* if $\rho(A) \geq 0$ for all positive elements $A \in \mathcal{A}$. If, in addition, $\rho(I) = 1$, then ρ is called a *state*.

A positive linear functional is hermitian: Let A be self-adjoint, then $\|A\| \pm A \geq 0$ so that $\rho(\|A\| \pm A) \geq 0$ and further $\rho(A) = (\rho(\|A\| + A - (\|A\| - A)))/2 = (\rho(\|A\| + A) - \rho(\|A\| - A))/2 \in \mathbb{R}$.

Clearly, for a positive linear functional ρ and $A \leq B$ we always have $\rho(A) \leq \rho(B)$.

Examples: (i) Let $\mathcal{A}$ be a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$. The linear functional $\omega_x(A) := \langle x|Ax\rangle$ $(A \in \mathcal{A})$ is positive and $\omega_x(I) = \|x\|^2$. Therefore, ω_x is a state if and only if $\|x\| = 1$. The states given in this way on a C*-algebra of operators by unit vectors in $\mathcal{H}$ are called *vector states*.

(ii) Let $\mathcal{A} = C(X)$ with X a compact Hausdorff space, then any (positive) regular Borel measure μ on X defines a positive linear functional by $f \mapsto \int_X f\, d\mu$. In fact, according to the Riesz representation theorem, every positive linear functional on $C(X)$ arises in this way. The states on $C(X)$ correspond to the probability measures on X.

Proposition (Cauchy-Schwarz-inequality for a positive linear functional): If ρ is a positive linear functional on a C*-algebra $\mathcal{A}$, then we have for any $A, B \in \mathcal{A}$,

$$|\rho(A^*B)|^2 \leq \rho(A^*A)\rho(B^*B).$$

Proof: The map $\gamma\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$, $(A, B) \mapsto \rho(A^*B)$ is a hermitian sesquilinear form (conjugate-linear in the first slot) and positive-semidefinite, since $\gamma(A, A) = \rho(A^*A) \geq 0$. Thus, the standard Cauchy-Schwarz inequality for γ ([Con10, Chapter I, 1.4] or [KRI, Proposition 2.1.1]) implies

$$|\rho(A^*B)|^2 = |\gamma(A, B)|^2 \leq \gamma(A, A)\gamma(B, B) = \rho(A^*A)\rho(B^*B).$$

□

Corollary: Let $\mathcal{A}$ be a C*-algebra.

(i) A linear functional ρ on $\mathcal{A}$ is positive if and only if ρ is bounded and $\|\rho\| = \rho(I)$.

(ii) The set $\mathcal{S}(\mathcal{A})$ of states on $\mathcal{A}$ is a convex weak* compact subset of the (norm closed) unit ball of the dual space $\mathcal{A}^\#$.

Proof: (i) If ρ is positive and $A \in \mathcal{A}$, then recalling $0 \leq A^*A \leq \|A\|^2$ from Corollary 1.8(iii) the Cauchy-Schwarz inequality gives

$$|\rho(A)|^2 = |\rho(I^*A)|^2 \leq \rho(I)\rho(A^*A) \leq \rho(I)\|A\|^2\rho(I).$$

Therefore $|\rho(A)| \leq \rho(I)\|A\|$, i.e., ρ is bounded and $\|\rho\| \leq \rho(I)$. Since $\rho(I) \leq \|\rho\|$ is obvious, we have in summary $\|\rho\| = \rho(I)$.

Conversely, suppose ρ is a bounded linear functional with $\|\rho\| = \rho(I)$. Since the assertion is trivial for $\rho = 0$, we may focus on the case $\|\rho\| \neq 0$ and suppose, upon scaling by a positive number, that $\|\rho\| = \rho(I) = 1$.

Let $A \in \mathcal{A}$ be positive and $\rho(A) = a + ib$ with $a, b \in \mathbb{R}$. We will show that $a \geq 0$ and $b = 0$.

Since $\mathrm{sp}(A) \subseteq [0, \infty[$, there is some $\varepsilon > 0$ such that for all $0 < s < \varepsilon$ we have

$$\mathrm{sp}(I - sA) = \{1 - s\lambda \mid \lambda \in \mathrm{sp}(A)\} \subseteq [0, 1].$$

By self-adjointness of $I - sA$, $\|I - sA\| = r(I - sA) \leq 1$ and therefore,

$$1 - sa \leq |1 - s(a + ib)| = |\rho(I - sA)| \leq \|\rho\|\|I - sA\| \leq 1,$$

which already implies $a \geq 0$. Let $A_t := A - a + itb$ ($t > 0$), then

$$\|A_t\|^2 = \|A_t^*A_t\| = \|(A - a - itb)(A - a + itb)\| = \|(A - a)^2 + t^2b^2\| \leq \|(A - a)^2\| + t^2b^2$$

and $\rho(A_t) = \rho(A) - a + itb = a + ib - a + itb = i(t + 1)b$. Combining these we find

$$\forall t > 0\colon \quad (t + 1)^2b^2 = |\rho(A_t)|^2 \leq \|\rho\|^2\|A_t\|^2 \leq \|(A - a)^2\| + t^2b^2,$$

thus $(2t + 1)b^2 \leq \|(A - a)^2\|$, which can only hold for all $t > 0$, if $b = 0$.

(ii) We have from (i) that

$$\mathcal{S}(\mathcal{A}) = \{\rho \in \mathcal{A}^\# \mid \rho(I) = 1 \text{ and } \rho(A) \geq 0 \text{ for all positive } A \in \mathcal{A}\} \subseteq \{\mu \in \mathcal{A}^\# \mid \|\mu\| \leq 1\}.$$

From this representation of $\mathcal{S}(\mathcal{A})$ convexity is obvious. Furthermore, $\mathcal{S}(\mathcal{A})$ is weak* closed, because we recall that weak* convergence means pointwise convergence which preserves both conditions in the set specification. Since $\mathcal{S}(\mathcal{A})$ is a weak* closed subset of the weak* compact closed unit ball of $\mathcal{A}^{\#}$ (Banach-Alaoglu theorem), we obtain also weak* compactness of $\mathcal{S}(\mathcal{A})$. □

Recall that the weak* topology is always Hausdorff, hence we obtain the *state space* $\mathcal{S}(\mathcal{A})$ as a compact Hausdorff space in a natural way.

1.11. Proposition: Let $\mathcal{A}$ be a C*-algebra, $A \in \mathcal{A}$ and $a \in \mathrm{sp}(A)$. Then there is a state $\rho \in \mathcal{S}(\mathcal{A})$ such that $\rho(A) = a$. If A is normal, there is a state $\rho \in \mathcal{S}(\mathcal{A})$ such that $|\rho(A)| = \|A\|$.

Proof: The assertion is trivial if $A \in \mathbb{C}I$ or $A = 0$, since $\mathrm{sp}(\lambda I) = \{\lambda\}$ and $\mathrm{sp}(0) = \{0\}$, so it remains to consider the case where $A \notin \mathbb{C}I$ and $A \neq 0$. We define a linear functional ρ_0 on the subspace $\mathcal{M} := \mathrm{span}\{A, I\}$ by $\rho_0(\alpha A + \beta I) := a\alpha + \beta$. We have $\mathrm{sp}(\alpha A + \beta) = \{\alpha\lambda + \beta \mid \lambda \in \mathrm{sp}(A)\}$, hence $\rho_0(\alpha A + \beta) = a\alpha + \beta \in \mathrm{sp}(\alpha A + \beta)$ implies $|\rho_0(\alpha A + \beta)| \leq \|\alpha A + \beta\|$. Since $\rho_0(I) = 1$ we obtain $\|\rho_0\| = 1$. By the Hahn-Banach theorem, there is an extension $\rho \in \mathcal{A}^{\#}$ of ρ_0 with $\|\rho\| = \|\rho_0\| = 1 = \rho_0(I) = \rho(I)$. By Corollary 1.10, ρ is positive, hence a state with $\rho(A) = \rho_0(A) = a$.

If A is normal, then Proposition 1.7(iii) guarantees that there is some $a \in \mathrm{sp}(A)$ such that $|a| = \|A\|$. The above construction yields a state ρ with $|\rho(A)| = |a| = \|A\|$. □

1.12. Pure states: We have seen above that the state space $\mathcal{S}(\mathcal{A})$ of a C*-algebra $\mathcal{A}$ is a convex and weak* compact subset of the dual space $\mathcal{A}^{\#}$. Since $\mathcal{A} \neq \{0\}$ (for unital $\mathcal{A}$) we certainly have that $\mathcal{S}(\mathcal{A}) \neq \emptyset$, hence the Krein-Milman theorem ([Con10, 7.4 in Chapter V] or [KRI, Theorem 1.4.3] or [Hoe23, 5.11]) asserts that $\mathcal{S}(\mathcal{A})$ is the closed convex hull $\overline{\mathrm{co}}\,(\mathcal{P}(\mathcal{A}))$ of the (non-empty) subset $\mathcal{P}(\mathcal{A}) := \mathrm{ex}\,\mathcal{S}(\mathcal{A})$ of extreme points in $\mathcal{S}(\mathcal{A})$.

The elements of $\mathcal{P}(\mathcal{A})$ are called the *pure states* on $\mathcal{A}$. Recall that $\rho \in \mathcal{P}(\mathcal{A})$, if any representation $\rho = c\rho_1 + (1-c)\rho_2$ with $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{A})$ and $0 < c < 1$ forces $\rho_1 = \rho_2 = \rho$.

Lemma: Let ρ be a state on the C*-algebra $\mathcal{A}$. Then ρ is a pure state, i.e., $\rho \in \mathcal{P}(\mathcal{A})$, if and only if for every positive linear functional μ on $\mathcal{A}$ with $\mu \leq \rho$, meaning $\mu(A) \leq \rho(A)$ whenever $A \geq 0$, there is some number $t \in [0, 1]$ such that $\mu = t\rho$.

Proof: : Let ρ be pure and $0 \leq \mu \leq \rho$. We have $0 \leq \mu(I) \leq \rho(I) = 1$.

If $\mu(I) = 0$, then $\mu = 0$ (hence $t := 0$ works) since for every self-adjoint $A \in \mathcal{A}$, $-\|A\| \leq A \leq \|A\|$ implies $0 = -\|A\|\mu(I) \leq \mu(A) \leq \|A\|\mu(I) = 0$, hence $\mu(A) = 0$ and $\mu(B) = 0$ for any $B \in \mathcal{A}$ follows upon decomposition into real and imaginary parts.

If $\mu(I) = 1$, then similarly, $\rho - \mu = 0$ and we may put $t := 1$.

Suppose then that $0 < \mu(I) < 1$. Put $t := \mu(I)$ and define states $\rho_1 := (\rho - \mu)/(1-t)$ and $\rho_2 := \mu/t$. Then $(1-t)\rho_1 + t\rho_2 = \rho$. Since ρ is pure, we must have $\rho_2 = \rho$, i.e., $\mu = t\rho$.

Conversely, suppose ρ satisfies the second condition in the hypothesis and let $0 < c < 1$ and $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{A})$ be such that $\rho = c\rho_1 + (1-c)\rho_2$. We have $c\rho_1 \leq \rho$, so that $c\rho_1 = t\rho$ for some $t \in [0, 1]$. Since $\rho_1(I) = 1 = \rho(I)$, we must have $c = t$ and thus $\rho_1 = \rho$. In the same way we obtain $\rho_2 = \rho$. Therefore, ρ is pure. □

Examples: 1) On $\mathcal{A} = C(X)$, X a compact Hausdorff space, every Dirac measure δ_{x_0} ($x_0 \in X$) defines a pure state ρ_{x_0} with $\rho_{x_0}(f) := f(x_0)$ ($f \in C(X)$). Indeed, suppose μ is a regular positive Borel measure on X such that $\int f\,d\mu \leq \rho_{x_0}(f) = f(x_0)$ holds for every $f \in C(X)$ with $f \geq 0$. Then μ must be concentrated on the singleton set $\{x_0\}$, since it follows that $\int f\,d\mu = 0$ whenever $f(x_0) = 0$. Therefore, $\int f\,d\mu = f(x_0)\mu(\{x_0\}) = \mu(\{x_0\})\rho_{x_0}(f)$, i.e., the positive functional corresponding to μ equals $t\rho_{x_0}$ with $t := \mu(\{x_0\})$. The previous lemma implies that ρ_{x_0} is pure.

We note that ρ_{x_0} is a $*$-homomorphism $C(X) \to \mathbb{C}$, since $\rho_{x_0}(fg) = f(x_0)g(x_0) = \rho_{x_0}(f)\rho_{x_0}(g)$ and $\rho_{x_0}(\bar{f}) = \overline{f(x_0)} = \overline{\rho_{x_0}(f)}$ for all $f, g \in C(X)$.

It is true that *every pure state on $C(X)$ is given by some Dirac measure* concentrated on a point $x_0 \in X$. This can be shown along the standard paths upon proving that the pure states are exactly the non-zero multiplicative linear functionals (as we will do in Proposition 2.4), identifying the kernels of the latter with the maximal ideals in $C(X)$, and characterizing the closed ideals in $C(X)$ by the zero sets of families of functions (combine Corollary 3.4.2 and Theorem 3.4.7 in [KRI]). But there is also a more direct measure theoretic argument that we may sketch here: First note that one can show that a regular Borel probability measure μ on X that takes on only the values 0 and 1 is of the form δ_{x_0} with some $x_0 \in X$. (A proof of this [in German] can be found in [Wer18, Beispiel (f) in VIII.4].) Let μ be a probability measure on X that is different from every δ_{x_0} ($x_0 \in X$). We may choose a Borel set $D \subseteq X$ such that $0 < \mu(D) < 1$ and put $c := \mu(D)$. Then $\mu_1(E) := \mu(E \cap D)/c$ and $\mu_2(E) := \mu(E \cap (X \setminus D))/(1-c)$ define two probability measures with $\mu_1 \neq \mu_2$ and $\mu = c\mu_1 + (1-c)\mu_2$ is a nontrivial convex combination. Hence μ is not pure.

2) We will show that every vector state ω_x ($x \in \mathcal{H}$, $\|x\| = 1$) on $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is a pure state. Recall that $\omega_x(A) = \langle x|Ax\rangle$ for all $A \in \mathcal{B}(\mathcal{H})$. Denote by P the (orthogonal) projection onto the one-dimensional subspace defined by x, i.e., $Py = \langle x|y\rangle x$ for all $y \in \mathcal{H}$. Note that as (orthogonal) projections both P as well as $I - P$ are positive idempotent operators and that $\omega_x(P) = 1$ while $\omega_x(I-P) = 0$. Furthermore, a direct calculation shows that $PBP = \langle x|Bx\rangle P$ holds for all $B \in \mathcal{B}(\mathcal{H})$.

We will again apply the above lemma to show that ω_x is pure, so let us suppose that μ is a positive linear functional on $\mathcal{B}(\mathcal{H})$ with $\mu \leq \omega_x$. Then $0 \leq \mu(I-P) \leq \omega_x(I-P) = 0$, hence $\mu(I-P) = 0$, or $\mu(P) = \mu(I)$. In case $\mu(I) = 0$ we immediately see that $\mu = 0$ (by the same reasoning as in the beginning of the proof of the above lemma), so it suffices to consider the case $\mu(I) > 0$.

We put $\mu_1 := \mu/\mu(I)$ and obtain $\mu_1(P) = 1$ and $\mu_1(I-P) = 0$. Hence the Cauchy-Schwarz inequality gives for any $C \in \mathcal{B}(\mathcal{H})$ that $|\mu_1(C(P-I))|^2 \leq \mu_1(C^*C)\mu_1(I-P) = 0$, i.e., $\mu_1(C(P-I)) = 0$, and similarly, $\mu_1((P-I)C) = 0$. Therefore, for arbitrary $B \in \mathcal{B}(\mathcal{H})$,

$$\mu_1(B) = \mu_1(B) + \underbrace{\mu_1(B(P-I))}_{0} + \underbrace{\mu_1((P-I)BP)}_{0} = \mu_1(PBP) = \langle x|Bx\rangle \underbrace{\mu_1(P)}_{1} = \omega_x(B),$$

which implies $\mu = \mu(I)\omega_x$ with $0 < \mu(I) \leq 1$.

Remark: In the context of this example, let us report that in case $\mathcal{H}$ is not finite-dimensional, there are pure states on $\mathcal{B}(\mathcal{H})$ that cannot be vector states. In fact, one can show that there exist pure states ρ on $\mathcal{B}(\mathcal{H})$ that vanish on the subspace of compact operators ([KRII, Corollary 10.4.4 and Remark 10.4.5]), which clearly cannot be true if $\rho(A) = \langle x|Ax\rangle$ with $\|x\| = 1$. (The projection P onto $\operatorname{span}\{x\}$ is a compact operator, since its range is one-dimensional, and we would then obtain the obvious contradiction $1 = \langle x|x\rangle = \langle x|Px\rangle = \rho(P) = 0$.)

1.13. Theorem: Let $\mathcal{A}$ be a C*-algebra. If $A \in \mathcal{A}$ is normal, then there is a pure state $\rho_0 \in \mathcal{P}(\mathcal{A})$ such that $|\rho_0(A)| = \|A\|$.

Proof: By Proposition 1.11, there is a state $\tau \in \mathcal{S}(\mathcal{A})$ such that $|\tau(A)| = \|A\|$. Choose $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\tau(\alpha A) = |\tau(A)| = \|A\|$. Recall that $\mathcal{S}(\mathcal{A})$ is a non-empty, convex, and weak* compact subset of $\mathcal{A}^{\#}$ and that $\mathcal{P}(\mathcal{A})$ is the set of extreme points of $\mathcal{S}(\mathcal{A})$. The linear functional $\theta\colon \mathcal{A}^{\#} \to \mathbb{C}$, given by $\theta(\mu) := \mu(\alpha A)$, is weak* continuous. It is corollary of the Krein-Milman theorem (see, e.g., [KRI, Corollary 1.4.4] or [Hoe23, Corollary 5.11]) that there is some extreme point $\rho_0 \in \mathcal{P}(\mathcal{A})$ of $\mathcal{S}(\mathcal{A})$ such that $\operatorname{Re}\theta(\rho) \leq \operatorname{Re}\theta(\rho_0)$ for all $\rho \in \mathcal{S}(\mathcal{A})$. We obtain

$$\begin{aligned}\|A\| = \tau(\alpha A) = \operatorname{Re}\tau(\alpha A) = \operatorname{Re}\theta(\tau) &\leq \sup\{\operatorname{Re}\theta(\mu) \mid \mu \in \mathcal{S}(\mathcal{A})\} \\ &\leq \operatorname{Re}\theta(\rho_0) \leq |\theta(\rho_0)| = |\rho_0(\alpha A)| = |\rho_0(A)| \leq \|A\|,\end{aligned}$$

therefore, $\|A\| = |\rho_0(A)|$. □

1.14. Corollary: Let $\mathcal{A}$ be a C*-algebra and $A \in \mathcal{A}$ with $A \neq 0$, then there is a pure state $\rho \in \mathcal{P}(\mathcal{A})$ such that $\rho(A) \neq 0$.

Proof: For self-adjoint A the assertion holds by Theorem 1.13. Otherwise we write $A = H_1 + iH_2$ with self-adjoint elements $H_1, H_2 \in \mathcal{A}$. If $\rho(A) = 0$ for every pure state ρ, then also $\rho(H_j) = 0$ $(j = 1, 2)$ for every pure state ρ, which implies $H_1 = H_2 = 0$ and thus $A = 0$. □

2. Commutative C*-algebras and functional calculus

◇ Our main sources for this chapter are [KRI, Con00, Mur90].

The first and fundamental result in this chapter will be an improvement of facts from 0.6 in case of commutative (unital) C*-algebras, in particular, about the Gelfand transform (0.2). Let $\mathcal{A}$ be a unital commutative C*-algebra. Recall that we had considered the set

$$X := \{\rho\colon \mathcal{A} \to \mathbb{C} \mid \rho \text{ is a unital homomorphism}\},$$

i.e., the non-zero multiplicative linear functionals on $\mathcal{A}$, and shown that X is contained in the unit sphere of $\mathcal{A}^{\#}$ and is a weak* compact Hausdorff space. Moreover,

$$\mathrm{sp}(A) = \{\rho(A) \mid \rho \in X\} = \hat{A}(X),$$

where $\hat{A} \in C(X)$ is the Gelfand transform of A, given by $\hat{A}(\rho) = \rho(A)$ for every $\rho \in X$. We had established further that the Gelfand transform is a continuous unital homomorphism $\mathcal{A} \to C(X)$ with (operator) norm 1, since $\|\hat{A}\|_\infty = r(A) \leq \|A\|$. (Note that $\hat{I} = 1$, thus $\|\hat{1}\|_\infty = 1$.)

2.1. Theorem: If $\mathcal{A}$ is a commutative (unital) C*-algebra, then the Gelfand transform $\mathcal{A} \to C(X)$ is an isometric $*$-isomorphism.

Proof: Since every element A in a commutative C*-algebra is normal, we may apply Proposition 1.7(iii) and obtain

$$\|\hat{A}\|_\infty = r(A) = \|A\|,$$

which shows that the Gelfand transform is isometric and hence its image $\hat{\mathcal{A}} = \{\hat{A} \mid A \in \mathcal{A}\}$ is closed in $C(X)$.

We claim that $\widehat{A^*} = \overline{\hat{A}}$ holds for every $A \in \mathcal{A}$, which then proves that the Gelfand transform is also involutive, hence a $*$-homomorphism. Upon writing $A = H_1 + iH_2$ with the self-adjoint real and imaginary parts H_1 and H_2 of A, it suffices to show that $\hat{A}$ is a real-valued function whenever A is self-adjoint. From Proposition 1.7(ii) we know that in this case $\mathrm{sp}(A) \subseteq \mathbb{R}$, thus the fact $\hat{A}(X) = \mathrm{sp}(A)$ settles this question.

It remains to show that $\hat{\mathcal{A}} = C(X)$, which follows from the Stone-Weierstraß theorem ([Con10, Chapter V, 8.1]), since $\hat{\mathcal{A}}$ is a subalgebra of $C(X)$, $1 = \hat{I} \in \hat{\mathcal{A}}$, $\bar{f} = \overline{\hat{A}} = \widehat{A^*} \in \hat{\mathcal{A}}$ whenever $f = \hat{A} \in \hat{\mathcal{A}}$, and $\hat{\mathcal{A}}$ is (point) separating, because for $\rho_1, \rho_2 \in X$ with $\rho_1 \neq \rho_2$ we can find clearly some $A \in \mathcal{A}$ such that $\hat{A}(\rho_1) = \rho_1(A) \neq \rho_2(A) = \hat{A}(\rho_2)$. □

2.2. Remark: A corresponding result holds for the non-unital case ([Mur90, Theorem 2.1.10]), namely, if $\mathcal{A} \neq \{0\}$ is a commutative C*-algebra, then the Gelfand transform is an isometric $*$-isomorphism $\mathcal{A} \to C_0(X)$.

We proceed by considering unital C*-algebras as the default situation.

2.3. Lemma: Let $\mathcal{A}$ be a C*-algebra.

(i) If A is a self-adjoint element of $\mathcal{A}$, then there exist positive elements $A_+, A_- \in \mathcal{A}$ with $A_+A_- = A_-A_+ = 0$ such that $A = A_+ - A_-$.

(ii) Each element in $\mathcal{A}$ can be written as a linear combination of at most four positive elements.

Proof: (i) Let $C^*(A)$ denote the closure of the $*$-subalgebra $\{p(A) \mid p \text{ a polynomial}\}$ in $\mathcal{A}$. Then $C^*(A)$ is a commutative unital C*-subalgebra and $C^*(A) \cong C(X)$ via the Gelfand isomorphism. A $*$-isomorphism respects positivity and self-adjointness, hence the standard way of writing the continuous real-valued function $f := \hat{A}$ as difference $f_+ - f_-$ of the non-negative continuous functions $f_\pm := \max(\pm f, 0)$ translates back into $A = A_+ - A_-$, where $\widehat{A_\pm} = f_\pm$. Moreover, the relation $f_+f_- = f_-f_+ = 0$ in $C(X)$ yields the corresponding property $A_+A_- = A_-A_+ = 0$ in $C^*(A) \subseteq \mathcal{A}$.

(ii) Follows from (i) upon decomposition into real and imaginary parts. □

2.4. Proposition: If $\mathcal{A}$ is a commutative C*-algebra, then the weak* compact Hausdorff space X of non-zero multiplicative linear functionals on $\mathcal{A}$ coincides with the set of pure states $\mathcal{P}(\mathcal{A})$. As a consequence, we obtain the following variant of Theorem 2.1:

$$\mathcal{A} \cong C(\mathcal{P}(\mathcal{A})).$$

Proof: We first show $\mathcal{P}(\mathcal{A}) \subseteq X$: Let ρ be a pure state on $\mathcal{A}$. Given any positive $C \in \mathcal{A}$, we will prove that $\rho(AC) = \rho(A)\rho(C)$ for all $A \in \mathcal{A}$; the claim then follows from linearity and Lemma 2.3(ii). Upon scaling and again by linearity of ρ it suffices to consider $0 \leq C \leq I$. Define a linear functional by $\rho_0(A) := \rho(AC)$ $(A \in \mathcal{A})$. Then ρ_0 is positive, since commutativity implies $0 \leq AC \leq A$ for every $A \geq 0$, and clearly $\rho_0 \leq \rho$ holds. Since ρ is pure, Lemma 1.12 says that there exists $t \in [0,1]$ such that $\rho_0 = t\rho$. Thus for every $A \in \mathcal{A}$,

$$\rho(AC) = \rho_0(A) = t\rho(A) = t\rho(I)\rho(A) = \rho_0(I)\rho(A) = \rho(C)\rho(A).$$

It remains to show $X \subseteq \mathcal{P}(\mathcal{A})$: Let $\rho \in X$. We know that X is a subset of the unit sphere in $\mathcal{A}^\#$, hence we know that ρ is bounded and $\|\rho\| = 1$. Moreover, recall that ρ being multiplicative and non-zero also implies $\rho(I) = 1 = \|\rho\|$. Therefore, an application of the Corollary, part (ii), in 1.10 yields that ρ is positive, in fact, a state. We still have to prove that ρ is pure and will employ Lemma 1.12. Suppose μ is a positive linear functional on $\mathcal{A}$ such that $\mu \leq \rho$. By the Cauchy-Schwarz inequality, $|\mu(A)|^2 = |\mu(I^*A)|^2 \leq \mu(A^*A) \leq \rho(A^*A) = \rho(A^*)\rho(A) = |\rho(A)|^2$, hence $\ker\rho \subseteq \ker\mu$. Since $\ker\rho$ is a closed hyperplane, we must have $\mu = \lambda\rho$ for some scalar λ. Now $0 \leq \mu(I) = \lambda\rho(I) = \lambda$ and $\lambda = \mu(I) \leq \rho(I) = 1$ imply that $\lambda \in [0,1]$. Therefore, we have shown that ρ is pure. □

2.5. The commutative C*-subalgebra generated from a normal element: Let $\mathcal{A}$ be a C*-algebra. If $\mathcal{M}$ is a subset of $\mathcal{A}$, then the C*-algebra *generated* by $\mathcal{M}$ is defined as the smallest C*-subalgebra of $\mathcal{A}$ containing $\mathcal{M}$. It may be written as the intersection of all C*-subalgebras of $\mathcal{A}$ containing $\mathcal{M}$. In case $\mathcal{M} = \{A\}$ with a *normal* element $A \in \mathcal{A}$ we write $\mathrm{C}^*(A)$ and call this the C^*-algebra generated by the normal element A. Note that $\mathrm{C}^*(A)$ is commutative, since $AA^* = A^*A$, and is given as the closure of the $*$-subalgebra $\{p(A, A^*) \mid p \text{ polynomial of } z \text{ and } \bar{z}\}$, where $p(A, A^*) = a_{0,0} + \sum_{k=1}^m a_{k,0}A^k + \sum_{l=1}^m a_{0,l}(A^*)^l + \sum_{1\leq k,l\leq m} a_{k,l}A^k(A^*)^l$ if $p(z, \bar{z}) = \sum_{0\leq k,l\leq m} a_{k,l}z^k\bar{z}^l$. (Observe that any [unital] C^*-subalgebra with $A \in \mathcal{B}$ contains I, A, and A^*, and thus also all limits of polynomials in A and A^*.)

2.6. Theorem (functional calculus for a normal element): Let $\mathcal{A}$ be a C^*-algebra and A be a normal element in $\mathcal{A}$. Then there is a unique $*$-homomorphism $\varphi\colon C(\mathrm{sp}(A)) \to \mathcal{A}$ such that $\varphi(\mathrm{id}_{\mathrm{sp}(A)}) = A$. We have $\|\varphi(f)\| = \|f\|_\infty$ for all $f \in C(\mathrm{sp}(A))$ and $\operatorname{ran}\varphi = \mathrm{C}^*(A)$.

Proof: Since $\mathrm{C}^*(A)$ is commutative we may employ the Gelfand isomorphism $\psi\colon \mathrm{C}^*(A) \to C(X)$, where $X = \mathcal{P}(\mathrm{C}^*(A))$. Recall that $\hat{A}(X) = \mathrm{sp}(A)$.

We will show that $\hat{A}$ is a homeomorphism $X \to \mathrm{sp}(A)$ by reasoning similarly as in the proof of Corollary 0.6: Clearly, $\hat{A}$ is continuous and surjective. If $\rho_1, \rho_2 \in X$ with $\rho_1(A) = \hat{A}(\rho_1) = \hat{A}(\rho_2) = \rho_2(A)$, then recalling that X consists of positive (hence hermitian) multiplicative linear functionals we easily obtain that $\rho_1(p(A, A^*)) = \rho_2(p(A, A^*))$ holds for every polynomial p in z and $\bar{z}$. By continuity of ρ_1 and ρ_2 and the density of $\{p(A, A^*)\}$ in $\mathrm{C}^*(A)$ we obtain $\rho_1 = \rho_2$, thus $\hat{A}$ is injective. As a bijective continuous map between compact Hausdorff spaces $\hat{A}$ is thus a homeomorphism.

The map $\alpha\colon C(\mathrm{sp}(A)) \to C(X)$, $f \mapsto f \circ \hat{A}$, is easily seen to be a $*$-isomorphism, hence the composition $\varphi_0 := \psi^{-1} \circ \alpha$ defines a $*$-isomorphism $C(\mathrm{sp}(A)) \to \mathrm{C}^*(A)$, which we may consider as a $*$-homomorphism $\varphi\colon C(\mathrm{sp}(A)) \to \mathcal{A}$ with $\operatorname{ran}\varphi = \mathrm{C}^*(A)$. (Since both α and ψ map 1 to 1, φ is unital.)

$$\begin{tikzcd} \mathrm{C}^*(A) \arrow[r, "\psi"] & C(X) \\ C(\mathrm{sp}(A)) \arrow[u, "\varphi_0"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

We have $\varphi(\mathrm{id}_{\mathrm{sp}(A)}) = \psi^{-1}(\alpha(\mathrm{id}_{\mathrm{sp}(A)})) = \psi^{-1}(\hat{A}) = A$ and the isometry of φ follows, because both α and ψ are isometric.

As for the uniqueness of φ we simply have to note that due to the Stone-Weierstraß theorem, the C^*-algebra $C(\mathrm{sp}(A))$ is generated from the normal element $\mathrm{id}_{\mathrm{sp}(A)}$. $\square$

In the above theorem we have $\varphi(q) = q(A)$, if q is a polynomial. We therefore introduce the notation

$$f(A) := \varphi(f) \quad (f \in C(\mathrm{sp}(A)))$$

to make the application of the functional calculus more intuitive. Note that the following properties are immediate, because $f \mapsto f(A)$ is a $*$-isomorphism $C(\mathrm{sp}(A)) \to \mathrm{C}^*(A) \subseteq \mathcal{A}$:

- $f(A)$ is normal, since $f(A)f(A)^* = f(A)\bar{f}(A) = (f\bar{f})(A) = (\bar{f}f)(A) = f(A)^*f(A)$;
- $f(A)$ is self-adjoint if and only if f is real-valued;
- if f is non-negative (pointwise as a function), then $f(A)$ is positive. (The reverse implication of this can also be shown easily using the spectral mapping theorem proved in 2.8 below in combination with the (already proven) part (i) $\Rightarrow$ (ii) in Theorem 1.8.)

2.7. Proposition: Let $\beta\colon \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism between C*-algebras, $A \in \mathcal{A}$ be normal, and $f \in C(\operatorname{sp}(A))$. Then $\beta(A)$ is normal in $\mathcal{B}$, $\operatorname{sp}(\beta(A)) \subseteq \operatorname{sp}(A)$, and $f(\beta(A)) = \beta(f(A))$.

Proof: That $\beta(A)$ is normal follows directly from the $*$-homomorphism properties, and the inclusion relation for the spectra follows from Theorem 1.9. We may therefore use the restriction of f to $\operatorname{sp}(\beta(A))$ in the functional calculus for $\beta(A)$.

It remains to show that $f(\beta(A)) = \beta(f(A))$. We have $\beta(A^k(A^*)^l) = \beta(A)^k\beta(A^*)^l$ and therefore $\beta(p(A, A^*)) = p(\beta(A), \beta(A)^*)$ for every polynomial p in z and $\bar{z}$. Therefore, the continuous map $g \mapsto \|g(\beta(A)) - \beta(g(A))\|$, $C(\operatorname{sp}(A)) \to \mathbb{R}$, vanishes on a generating subset of $C(\operatorname{sp}(A))$, hence is the zero map. It follows that $f(\beta(A)) = \beta(f(A))$. $\square$

2.8. Spectral mapping theorem: If A is a normal element of the C*-algebra $\mathcal{A}$ and $f \in C(\operatorname{sp}(A))$, then we have

$$\operatorname{sp}(f(A)) = \{f(\lambda) \mid \lambda \in \operatorname{sp}(A)\} = f(\operatorname{sp}(A)).$$

Moreover, if $g \in C(\operatorname{sp}(f(A)))$, then $(g \circ f)(A) = g(f(A))$.

Proof: We claim that $f(\rho(A)) = \rho(f(A))$ for all $\rho \in \mathcal{P}(\mathrm{C}^*(A))$. Indeed, the maps $h \mapsto h(\rho(A))$ and $h \mapsto \rho(h(A))$ are both $*$-homomorphisms $C(\operatorname{sp}(A)) \to \mathbb{C}$ that agree on $\mathrm{id}_{\operatorname{sp}(A)}$, hence on a dense subset.

We may thus proceed as follows:

$$\begin{aligned}\operatorname{sp}(f(A)) = \widehat{f(A)}(\mathcal{P}(\mathrm{C}^*(A))) &= \{\rho(f(A)) \mid \rho \in \mathcal{P}(\mathrm{C}^*(A))\} = \{f(\rho(A)) \mid \rho \in \mathcal{P}(\mathrm{C}^*(A))\} \\ &= \{f(\hat{A}(\rho)) \mid \rho \in \mathcal{P}(\mathrm{C}^*(A))\} = f(\hat{A}(\mathcal{P}(\mathrm{C}^*(A)))) = f(\operatorname{sp}(A)).\end{aligned}$$

Observe that $\mathrm{C}^*(f(A)) \subseteq \mathrm{C}^*(A)$ and the restriction ρ_0 of any nonzero multiplicative linear functional ρ on $\mathrm{C}^*(A)$ to $\mathrm{C}^*(f(A))$ is again one. We have for all $\rho \in \mathcal{P}(\mathrm{C}^*(A))$,

$$\rho((g \circ f)(A)) = (g \circ f)(\rho(A)) = g(f(\rho(A))) = g(\rho_0(f(A)) = \rho_0(g(f(A))) = \rho(g(f(A))).$$

Therefore $(g \circ f)(A)\hat{} = g(f(A))\hat{}$ for the Gelfand tranforms, which implies $(g \circ f)(A) = g(f(A))$. $\square$

2.9. Examples: 1) Let X be a compact Hausdorff space. Every $h \in C(X)$ is normal and $\operatorname{sp}(h) = h(X)$. The functional calculus for h then boils down to plain composition $f \circ h$ for every $f \in C(h(X))$. This follows from the uniqueness clause in Theorem 2.6.

2) For normal or self-adjoint elements in $\mathcal{B}(\mathcal{H})$, Theorem 2.6 gives the opportunity to establish the usual calculus with continuous functions of Hilbert space operators via abstract C*-algebra theory. In fact, some textbooks (for example, [Con10]) even prefer to quickly develop a sufficient bundle of C*-algebraic methods prior to discussing functional calculus and spectral theory for (non-compact) operators. (An approach independent of the C*-algebraic results is outlined, e.g., in [Hoe23] following [Wer18].)

In the remaining parts of this chapter, we will give a few applications of the functional calculus to C*-algebra theory.

Let $\mathcal{B}$ be a closed subalgebra of a unital Banach algebra $\mathcal{A}$ and suppose that $I \in \mathcal{B}$. Then the notion of spectrum is defined with respect to both algebras and the notation should indicate the dependence on the algebra. We always have for any $A \in \mathcal{B}$

$$\mathrm{sp}_{\mathcal{A}}(A) \subseteq \mathrm{sp}_{\mathcal{B}}(A),$$

since invertibility of an element in $\mathcal{B}$ implies the same in $\mathcal{A}$. In general for Banach algebras, the inclusion may be strict (see, e.g., [KRI, Example 3.2.19]). However, in case of C*-algebras and unital C*-subalgebras the spectrum does not depend on the subalgebra.

2.10. Proposition: If $\mathcal{B}$ is a unital C*-subalgebra of the C*-algebra $\mathcal{A}$ and $A \in \mathcal{B}$, then $\mathrm{sp}_{\mathcal{A}}(A) = \mathrm{sp}_{\mathcal{B}}(A)$.

Proof: It suffices to prove the following: If $C \in \mathcal{B}$ has an inverse C^{-1} in $\mathcal{A}$, then $C^{-1} \in \mathcal{B}$.

In fact, we only need to show this in case C is self-adjoint, since $C^*C \in \mathcal{B}$ is self-adjoint, invertible if and only C is, and in this case $C^{-1} = (C^*C)^{-1}C^* \in \mathcal{B}$ follows from $(C^*C)^{-1} \in \mathcal{B}$.

Let $C \in \mathcal{B}$ be self-adjoint with inverse $C^{-1} \in \mathcal{A}$. Then $0 \notin \mathrm{sp}_{\mathcal{A}}(C)$ and $f(z) = 1/z$ defines a function $f \in C(\mathrm{sp}_{\mathcal{A}}(C))$, $f(C) = C^{-1}$ by functional calculus, and from the statement of Theorem 2.6 we also have $f(C) \in \mathrm{C}^*(C) \subseteq \mathcal{B}$. $\square$

2.11. Completing the proof of Theorem 1.8: Let us now close the gaps left in the proof of Theorem 1.8 characterizing positive elements in a C*-algebra. We recall the statement:
If A is self-adjoint in the C*-algebra $\mathcal{A}$, then the conditions

(i) $A \geq 0$, i.e., there is some self-adjoint $C \in \mathcal{A}$ such that $A = C^2$,

(ii) $\mathrm{sp}(A) \subseteq [0, \infty[$, and

(iii) there is some $C \in \mathcal{A}$ such that $A = C^*C$,

are equivalent. The implication (i) $\Rightarrow$ (ii) has already been noted and (i) $\Rightarrow$ (iii) is obvious.

To show (ii) $\Rightarrow$ (i), we note that the function $f(t) := \sqrt{t}$ belongs to $C(\mathrm{sp}(A))$ and put $C := f(A)$. Since $f \geq 0$, hence real-valued, C is self-adjoint, in fact even positive. The relation $f^2 = \mathrm{id}_{\,\mathrm{sp}(A)}$ implies $C^2 = f^2(A) = \mathrm{id}_{\,\mathrm{sp}(A)}(A) = A$.

(Δ) As an intermediate result we have established that (i) $\Leftrightarrow$ (ii).

Before showing that (iii) implies (i) we need a lemma that is also of interest in its own right.

Lemma Let $\mathcal{A}$ be a C*-algebra.

(a) Let $A \in \mathcal{A}$ be self-adjoint and $c \geq \|A\|$, then A is positive if and only if $\|c - A\| \leq c$.

(b) If $A, B \in \mathcal{A}$ are positive, then also $A + B \geq 0$.

Proof: (a): We have $\mathrm{sp}(A) \subseteq [-c, c]$ and therefore,

$$\|c - A\| = r(c - A) = \max\{|c - a| \mid a \in \mathrm{sp}(A)\} = \max\{c - a \mid a \in \mathrm{sp}(A)\},$$

which shows that $\|c - A\| \leq c$ if and only if $\mathrm{sp}(A) \subseteq [0, \infty[$. The assertion follows from (Δ).

(b): Applying (a) we have $\|\|A\| - A\| \leq \|A\|$ and $\|\|B\| - B\| \leq \|B\|$ and obtain the estimate

$$\|(\|A\|+\|B\|)-(A+B)\| = \|(\|A\|-A)+(\|B\|-B)\| \leq \|\|A\|-A\|+\|\|B\|-B\| \leq \|A\|+\|B\|.$$

Noting $c := \|A\|+\|B\| \geq \|A+B\|$ we use once more (a) and conclude that $A+B$ is positive. $\square$

We finally prove (iii) ⇒ (i): In a first step we claim that for every $B \in \mathcal{A}$,

$$-B^*B \text{ positive} \quad \Rightarrow \quad B = 0.$$

From (Δ) we know that $\mathrm{sp}(-B^*B) \subseteq [0, \infty[$. Recall that from elementary Banach algebra theory, $\mathrm{sp}(-BB^*) \setminus \{0\} = \mathrm{sp}(-B^*B) \setminus \{0\}$, thus $\mathrm{sp}(-BB^*) \subseteq [0, \infty[$, hence the self-adjoint element $-BB^*$ is positive due to (Δ). Using the real and imaginary part of B we may write $B = R + iS$ with self-adjoint $R, S \in \mathcal{A}$. Then $B^*B = R^2 - iSR + iRS + S^2$ and $BB^* = R^2 + iSR - iRS + S^2$, hence $B^*B + BB^* = 2R^2 + 2S^2$, which implies $B^*B = 2R^2 + 2S^2 - BB^*$. Thus, B^*B is a sum of three positive elements and is therefore positive by the above lemma. To summarize, both B^*B and $-B^*B$ are positive, hence again by (Δ) we find that $\mathrm{sp}(B^*B) \subseteq]-\infty, 0] \cap [0, \infty[= \{0\}$. Therefore, $\|B\|^2 = \|B^*B\| = r(B^*B) = 0$, thus $B = 0$.

In the second step now suppose that $A = C^*C$ with some $C \in \mathcal{A}$. By Lemma 2.3(i) we may write $A = A_+ - A_-$ with positive elements A_+ and A_- such that $A_+A_- = A_-A_+ = 0$. Let $B := CA_-$, then

$$-B^*B = -A_-C^*CA_- = -A_-(A_+ - A_-)A_- = (A_-)^3,$$

where $(A_-)^3$ is self-adjoint and $\mathrm{sp}((A_-)^3) = \{\lambda^3 \mid \lambda \in \mathrm{sp}(A_-)\} \subseteq [0, \infty[$, since A_- is positive. Therefore, $-B^*B$ is positive, which implies $B = 0$ by the first step of this proof and then in turn that $(A_-)^3 = 0$. Again by the self-adjointness of A_- and $(A_-)^k = (A_-)^{k-3}(A_-)^3 = 0$ for all $k \geq 3$ we deduce $\|A_-\| = r(A_-) = \lim_{k\to\infty} \|(A_-)^k\|^{1/k} = 0$, i.e., $A_- = 0$. In summary, $A = A_+ - A_- = A_+ \geq 0$.

Square roots and absolute value: As the construction in the proof of (ii) ⇒ (i) above shows, we can have $A = C^2$ *with positive* $C \in \mathcal{A}$ in condition (i). In fact, for each positive $A \in \mathcal{A}$, the positive element $C \in \mathcal{A}$ with $A = C^2$ is even unique and called the *positive square root* of A and denoted by $A^{1/2}$.
The uniqueness of the positive square root can be seen as follows: Let $C := f(A)$ be the root constructed from $f(t) = \sqrt{t}$ via functional calculus as above. Since f is a uniform limit of polynomials on the compact set $\mathrm{sp}(A)$ (Weierstraß theorem), C is a limit of polynomials in A. Let $B \in \mathcal{A}$ be positive and such that $B^2 = A$. Then $BA = BB^2 = B^3 = B^2B = AB$ and therefore B commutes also with every polynomial of A, hence also with C. Let $\mathcal{B}$ denote the commutative C*-algebra generated by B and C, then clearly $A = B^2 = C^2 \in \mathcal{B}$. We have the Gelfand isomorphism $\psi\colon \mathcal{B} \to C(X)$ and both $\psi(B)$ and $\psi(C)$ are positive square roots of $\psi(A)$ in $C(X)$. But in function algebras the positive square root is unique (namely the pointwise square root of the non-negative function), hence $\psi(B) = \psi(C)$ and this implies $B = C$.

If $A \in \mathcal{A}$ is arbitrary, then A^*A is positive by part (iii) of the above theorem. We may thus define the *absolute value* of A by $|A| := (A^*A)^{1/2}$.

We have seen in Theorem 1.9 that a $*$-homomorphisms between C*-algebras is automatically continuous. We will now add general information about the kernel and the range.

2.12. Theorem: Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism between C*-algebras. Then the kernel $\ker\varphi = \{A \in \mathcal{A} \mid \varphi(A) = 0\}$ is a closed self-adjoint ideal in $\mathcal{A}$ and the image $\operatorname{ran}\varphi = \{\varphi(A) \mid A \in \mathcal{A}\}$ is a C*-subalgebra of $\mathcal{B}$. If φ is injective, then φ is an isometry.

Note that $\ker\varphi$ is thus a C*-subalgebra of $\mathcal{A}$ that is non-unital in the typical, nontrivial case $\ker\varphi \neq \mathcal{A}$.

Proof: It follows from the boundedness of φ that $\ker\varphi$ is closed and the ideal properties as well as invariance under the involution follow algebraically from the definition of a $*$-homomorphism.

For pure algebraic reasons, the range is a $*$-subalgebra of $\mathcal{B}$ (and unital, since we consider unital $*$-homomorphisms by default). It remains to show that the range is closed.

Suppose (A_n) is a sequence in $\mathcal{A}$ such that $\lim\varphi(A_n) = B \in \mathcal{B}$. We have to show that $B \in \operatorname{ran}\varphi$. Thanks to real and imaginary parts and linearity of φ we may reduce this to the case where all A_n and B are self-adjoint, which allows for the application of functional calculus.

We may assume that $\|\varphi(A_{n+1}) - \varphi(A_n)\| < 2^{-n}$, because this is certainly true for a suitable subsequence. For every $n \in \mathbb{N}$ choose a continuous function $f_n\colon \mathbb{R} \to \mathbb{R}$ such that $f_n(t) = t$ when $|t| \leq 2^{-n}$ and $f(\mathbb{R}) \subseteq [-2^{-n}, 2^{-n}]$. Note that every $\varphi(A_{n+1}) - \varphi(A_n) = \varphi(A_{n+1} - A_n)$ is self-adjoint and $S_n := \operatorname{sp}(\varphi(A_{n+1}) - \varphi(A_n)) \subseteq [-2^{-n}, 2^{-n}]$, so that $f_n|_{S_n} = \mathrm{id}_{S_n}$. This together with Proposition 2.7 yields

$$\varphi(A_{n+1}) - \varphi(A_n) = f_n(\varphi(A_{n+1} - A_n)) = \varphi(f_n(A_{n+1} - A_n)).$$

The estimate $\|f_n(A_{n+1} - A_n)\| \leq \|f_n\|_\infty \leq 2^{-n}$ proves that the series $A_1 + \sum_{n=1}^{\infty} f_n(A_{n+1} - A_n)$ is absolutely convergent, hence converges in $\mathcal{A}$ to some element A. Then continuity of φ and the above equation implies

$$\begin{aligned}\varphi(A) = \lim_{N\to\infty} \Big(\varphi(A_1) + \sum_{n=1}^{N} \varphi(f_n(A_{n+1} - A_n))\Big) \\ = \lim_{N\to\infty} \Big(\varphi(A_1) + \sum_{n=1}^{N} (\varphi(A_{n+1}) - \varphi(A_n))\Big) = \lim_{N\to\infty} \varphi(A_{N+1}) = B,\end{aligned}$$

which shows that $B \in \operatorname{ran}\varphi$.

Finally, if φ is injective, then φ induces a $*$-isomorphism between the C*-algebras $\mathcal{A}$ and $\operatorname{ran}\varphi$, which is an isometry by Theorem 1.9. Hence φ is an isometric map into $\mathcal{B}$. $\square$

2.13. Remark: One can show that closed ideals in C*-algebras are automatically self-adjoint ([Con00, Chapter 1, Proposition 5.4]) and that the quotient of a C*-algebra by a closed ideal with the standard quotient norm is a C*-algebra ([Con00, Chapter 1, Theorems 5.6]). This opens up a more structure theoretic argument for the properties of the kernel and the image of a $*$-homomorphism φ as established in the previous theorem (cf. [Con00, Chapter 1, Corollary

5.7] or [KRII, Corollary 10.1.8]), because the $*$-homomorphism φ then factors in the usual way, where $\widetilde{\varphi}$ is an injective $*$-homomorphism $\mathcal{A}/\ker\varphi \to \mathcal{B}$ with $\operatorname{ran}\widetilde{\varphi} = \operatorname{ran}\varphi$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\varphi"] \arrow[d] & \mathcal{B} \\
\mathcal{A}/\ker\varphi \arrow[ur, "\widetilde{\varphi}"'] &
\end{tikzcd}$$

3. Representations of C*-algebras

◇ Our main sources for this chapter are [KRI, BR1, KRII, Con00, Mur90]. More background and related material can be found in [Dix82, All11, PII].

One important consequence of the results in this chapter will be that a C*-algebra can always be isometrically embedded as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In this sense, any C*-algebra is an *operator algebra*. This does not hold for Banach $*$-algebras in general: If $\mathcal{A}$ is a Banach $*$-algebra whose norm is not a C*-norm, then the existence of an isometric $*$-homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ would imply $\|A^*A\| = \|\varphi(A^*A)\| = \|\varphi(A)^*\varphi(A)\| = \|\varphi(A)\|^2 = \|A\|^2$ for all $A \in \mathcal{A}$, a contradiction. For a concrete example, consider $\pi\colon L^1(\mathbb{R}) \to \mathcal{B}(L^2(\mathbb{R}))$ with $\pi(f)g = f * g$ ($f \in L^1(\mathbb{R})$, $g \in L^2(\mathbb{R})$); then π is an injective $*$-homomorphism with $\|\pi(f)\| \leq \|f\|_1$ (see, e.g., [KRI, pages 188-190]), but π cannot be isometric, since $\|.\|_1$ is not a C*-norm (Example 1.4.4)). Similarly with $l^1(\mathbb{Z})$, to mention an example of a unital Banach $*$-algebra (cf. [KRI, KRIII, Exercise 3.5.33]).

Recall that we are still using our standard assumptions and consider C*-algebras and $*$-homomorphisms to be unital, unless stated otherwise.

3.1. Definition: Let $\mathcal{A}$ be a C*-algebra.

(i) A *representation* of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a $*$-homomorphism $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$.

(ii) The representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be *faithful*, if π is injective.

(iii) A representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be *cyclic*, if there is a vector $x \in \mathcal{H}$, called a *cyclic vector* for π, such that the subspace $\pi(\mathcal{A})x := \{\pi(A)x \mid A \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

(iv) If $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation and $\mathcal{K} \subseteq \mathcal{H}$ is a closed subspace that is invariant under each operator in $\pi(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H})$, then $A \mapsto \pi(A)|_{\mathcal{K}}$ defines a *subrepresentation* $\pi_1\colon \mathcal{A} \to \mathcal{B}(\mathcal{K})$ of $\mathcal{A}$. In this case, $\mathcal{K}^\perp$ is also closed and invariant under $\pi(\mathcal{A})$ (since $A \in \mathcal{A}$, $x \in \mathcal{K}^\perp$, $y \in \mathcal{K}$ implies $\langle \pi(A)x|y\rangle = \langle x|\pi(A^*)y\rangle = 0$), hence defines a subrepresentation $\pi_2\colon \mathcal{A} \to \mathcal{B}(\mathcal{K}^\perp)$ and $\pi = \pi_1 \oplus \pi_2$ in the sense that $\pi(A)(x_1 + x_2) = \pi_1(A)x_1 + \pi_2(A)x_2$ holds for the unique decomposition of vectors in $\mathcal{H}$ with $x_1 \in \mathcal{K}$ and $x_2 \in \mathcal{K}^\perp$.

(v) Two representations $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\psi\colon \mathcal{A} \to \mathcal{B}(\mathcal{E})$ of $\mathcal{A}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{E}$ are *(unitarily) equivalent*, if there is a unitary linear map $U\colon \mathcal{H} \to \mathcal{E}$ such that $\psi(A) = U\pi(A)U^{-1}$ for every $A \in \mathcal{A}$.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\pi(A)"] \arrow[d, "U"'] & \mathcal{H} \arrow[d, "U"] \\ \mathcal{E} \arrow[r, "\psi(A)"'] & \mathcal{E} \end{tikzcd}$$

Let $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of the C*-algebra $\mathcal{A}$. Then we obtain from Theorems 1.9 and 2.12 that π is continuous, in fact, $\|\pi(A)\| \leq \|A\|$, and furthermore, that $\ker \pi$ is a closed self-adjoint ideal in $\mathcal{A}$ and $\pi(\mathcal{A})$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Moreover, if π is faithful, then π is isometric, i.e., $\|\pi(A)\| = \|A\|$.

Let $\mathcal{A}$ be a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$, then the inclusion map is a faithful representation. If $\mathcal{K} \subseteq \mathcal{H}$ is a closed subspace that is invariant under each $A \in \mathcal{A}$, then $A \mapsto A|_{\mathcal{K}}$ defines a subrepresentation.

3.2. Examples: 1) Let (Ω, μ) be a σ-finite measure space and consider the C*-algebra $L^\infty(\Omega, \mu)$. Let $\pi_0\colon L^\infty(\Omega, \mu) \to \mathcal{B}(L^2(\Omega, \mu))$ with $\pi_0(f) := M_f$, where M_f denotes the multiplication operator $g \mapsto fg$ for every $g \in L^2(\Omega, \mu)$, $f \in L^\infty(\Omega, \mu)$. Then π_0 is a faithful representation. (While it is always true that $\|M_f\| \leq \|f\|_\infty$, the proof of isometry requires σ-finiteness of the measure μ (see [Con10, Chapter II, Theorem 1.5] or [KRI, Example 2.4.11]; in fact, the representation π_0 need not be faithful otherwise, as can be seen from an example on page 28 in [Con10]).)

2) If Ω is a compact Hausdorff space and μ a regular Borel measure, then $C(\Omega)$ is a C*-subalgebra of $L^\infty(\Omega, \mu)$. With π_0 as in 1), $\pi := \pi_0|_{C(\Omega)}$ is a faithful representation $\pi\colon C(\Omega) \to \mathcal{B}(L^2(\Omega, \mu))$ of $C(\Omega)$.

3) If $\mathcal{A}$ is a commutative C*-algebra, $\Omega := \mathcal{P}(\mathcal{A})$, and $\varphi\colon \mathcal{A} \to C(\Omega)$ is the Gelfand isomorphism, then from π as in 2) we obtain a faithful representation $\pi_1 := \pi \circ \varphi\colon \mathcal{A} \to \mathcal{B}(L^2(\Omega, \mu))$.

The representations in 2) and 3) have the function $1 \in L^2(\Omega, \mu)$ as a a cyclic vector, since $\pi(C(\Omega))1 = \{f1 \mid f \in C(\Omega)\} = C(\Omega)$ is dense in $L^2(\Omega, \mu)$.

Note that in the setting of the previous examples, a probability measure μ on Ω corresponds to a state on each algebra; namely, in 1) or 2), via $f \mapsto \int f\, d\mu$, and in 3) via $A \mapsto \int \varphi(A)\, d\mu$. Let us focus on 2) and call the state ρ. Then $\rho(f) = \int f\, d\mu = \langle 1 | \pi(f) 1 \rangle$ and, on the dense subspace $C(\Omega)$ of the representation space Hilbert space $L^2(\Omega, \mu)$, the inner product may be written as $\langle g | h \rangle = \int \bar{g} h\, d\mu = \langle 1 | \pi(g^* h) 1 \rangle = \rho(g^* f)$. An abstract distillation of this is the observation that for any state ρ on a (not necessarily commutative) C*-algebra $\mathcal{A}$, the assignment $(A, B) \mapsto \rho(A^* B)$ defines a positive semi-definite hermitian sesquilinear form on $\mathcal{A}$ (see also the proof of the Cauchy-Schwarz inequality in 1.10). This will be the starting point for the construction of the so-called *GNS representation*, referring to the names Gelfand, Neumark (often transcribed also as Naimark), and Segal, which we will describe in detail shortly (see 3.3 and 3.4).

Another aspect of the examples leads to the following observation: If $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation and $x \in \mathcal{H}$ a unit vector, then we may define a state ρ on $\mathcal{A}$ by $\rho(A) := \langle x | \pi(A) x \rangle$ $(A \in \mathcal{A})$. Employing the notation ω_x for the vector state on $\mathcal{B}(\mathcal{H})$ corresponding to x (see the examples in 1.10), i.e., $\omega_x(L) = \langle x | Lx \rangle$ $(L \in \mathcal{B}(\mathcal{H}))$, we may write $\rho = \omega_x \circ \pi$. It will be one of the results of the GNS construction that, in fact, each state on a C*-algebra arises in this way from a vector state in an appropriate representation.

3.3. Preparations for the GNS construction: Let $\mathcal{A}$ be a C*-algebra and ρ be a state on $\mathcal{A}$. The *left kernel* of ρ is defined by

$$\mathcal{L}_\rho := \{A \in \mathcal{A} \mid \rho(A^* A) = 0\}.$$

We claim that $\mathcal{L}_\rho$ is a *closed left ideal* in $\mathcal{A}$: Boundedness of ρ implies that $\mathcal{L}_\rho$ is closed. Writing $\gamma(A,B) := \rho(A^*B)$ $(A,B \in \mathcal{A})$, noting $\mathcal{L}_\rho = \{A \in \mathcal{A} \mid \gamma(A,A) = 0\}$, and defining $\mathcal{N} := \{A \in \mathcal{A} \mid \forall B \in \mathcal{A} \colon \gamma(A,B) = 0\}$, we see that $\mathcal{N}$ is a vector subspace of $\mathcal{A}$ with $\mathcal{N} \subseteq \mathcal{L}_\rho$. If $A \in \mathcal{L}_\rho$ and $B \in \mathcal{A}$, then we may conclude that $\gamma(A,B) = 0$, since the Cauchy-Schwarz inequality implies

$$|\gamma(A,B)|^2 = |\rho(A^*B)|^2 \leq \rho(A^*A)\rho(B^*B) = 0.$$

Thus $\mathcal{L}_\rho = \mathcal{N}$, in particular, $\mathcal{L}_\rho$ is a vector subspace. The above estimate shows that $\rho(A^*B) = 0$ whenever $A \in \mathcal{L}_\rho$ and $B \in \mathcal{A}$ arbitrary. Since ρ is hermitian, then also

$$(\Delta) \qquad \rho(B^*A) = \rho((A^*B)^*) = \overline{\rho(A^*B)} = 0$$

and further,

$$\rho((BA)^*(BA)) = \rho((B^*BA)^*A) = 0,$$

which implies $BA \in \mathcal{L}_\rho$. We have thus shown that $\mathcal{L}_\rho$ is a left ideal.

The *quotient vector space* $\mathcal{A}/\mathcal{L}_\rho$ can be equipped with the *positive definite inner product*

$$\langle A + \mathcal{L}_\rho | B + \mathcal{L}_\rho \rangle := \rho(A^*B) \quad (A, B \in \mathcal{A}).$$

Indeed, it is well-defined, since $M = A - A' \in \mathcal{L}_\rho$ and $N = B - B' \in \mathcal{L}_\rho$ implies by (Δ) that

$$\rho(A'^*B') = \rho((A^* + M^*)(B + N)) = \rho(A^*B) + \rho(M^*(B+N)) = \rho(A^*B),$$

and positive definite, since $0 = \langle A + \mathcal{L}_\rho | A + \mathcal{L}_\rho \rangle = \rho(A^*A)$ implies $A \in \mathcal{L}_\rho$.

We define the *Hilbert space* $\mathcal{H}_\rho$ to be the completion of the pre-Hilbert space $(\mathcal{A}/\mathcal{L}_\rho, \langle . | . \rangle)$.

Suppose $A, B_1, B_2 \in \mathcal{A}$ such that $B_1 - B_2 \in \mathcal{L}_\rho$. Then $AB_1 - AB_2 = A(B_1 - B_2) \in \mathcal{L}_\rho$ and therefore, $AB_1 + \mathcal{L}_\rho = AB_2 + \mathcal{L}_\rho$. Clearly, for every A fixed, $B + \mathcal{L}_\rho \mapsto AB + \mathcal{L}_\rho$ is linear with respect to $B + \mathcal{L}_\rho$. In other words,

$$\pi_\rho(A)(B + \mathcal{L}_\rho) := AB + \mathcal{L}_\rho \quad (B \in \mathcal{A})$$

is a well-defined *linear map*

$$\pi_\rho(A) \colon \mathcal{A}/\mathcal{L}_\rho \to \mathcal{A}/\mathcal{L}_\rho$$

for every $A \in \mathcal{A}$.

3.4. Theorem (GNS representation): Let ρ be a state on the C*-algebra $\mathcal{A}$ and the notation be as in 3.3. We obtain a cyclic representation π_ρ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_\rho$ with the cyclic unit vector x_ρ corresponding to $I + \mathcal{L}_\rho \in \mathcal{A}/\mathcal{L}_\rho$ such that $\rho = \omega_{x_\rho} \circ \pi_\rho$, i.e.,

$$\forall A \in \mathcal{A} \colon \quad \rho(A) = \langle x_\rho | \pi_\rho(A) x_\rho \rangle.$$

Proof: Step 1: $\pi_\rho(A)$ is bounded and hence has a unique extension to an operator $\pi_\rho(A) \in \mathcal{B}(\mathcal{H}_\rho)$.

Recall from Corollary 1.8(iii) that $\|A\|^2 - A^*A$ is positive and hence by Corollary 1.8(i) also

$$\|A\|^2 B^*B - B^*A^*AB = B^*(\|A\|^2 - A^*A)B \geq 0.$$

We have therefore

$$\begin{aligned}\|A\|^2\|B+\mathcal{L}_\rho\|^2 - \|\pi_\rho(A)(B+\mathcal{L}_\rho)\|^2 &= \|A\|^2\|B+\mathcal{L}_\rho\|^2 - \|AB+\mathcal{L}_\rho\|^2\\ = \|A\|^2\langle B+\mathcal{L}_\rho|B+\mathcal{L}_\rho\rangle - \langle AB+\mathcal{L}_\rho|AB+\mathcal{L}_\rho\rangle &= \|A\|^2\rho(B^*B) - \rho(B^*A^*AB)\\ &= \rho(\|A\|^2B^*B - B^*A^*AB) \geq 0,\end{aligned}$$

so that $\|\pi_\rho(A)\| \leq \|A\|$.

Step 2: $A \mapsto \pi_\rho(A)$ defines a (unital) $*$-homomorphism $\pi_\rho\colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho)$.

From $\pi_\rho(I)(B+\mathcal{L}_\rho) = B+\mathcal{L}_\rho$ we immediately obtain that $\pi_\rho(I) = I$ on $\mathcal{H}_\rho$. Linearity and multiplicativity of $A \mapsto \pi_\rho(A)$ is elementary when checked on the subspace $\mathcal{A}/\mathcal{L}_\rho \subseteq \mathcal{H}_\rho$ in each instance and then follows by density:

$$\begin{aligned}\pi_\rho(\alpha A+\beta B)(C+\mathcal{L}_\rho) &= (\alpha A+\beta B)C+\mathcal{L}_\rho = \alpha(AC+\mathcal{L}_\rho)+\beta(BC+\mathcal{L}_\rho)\\ &= \alpha\,\pi_\rho(A)(C+\mathcal{L}_\rho)+\beta\,\pi_\rho(B)(C+\mathcal{L}_\rho) = (\alpha\,\pi_\rho(A)+\beta\,\pi_\rho(B))(C+\mathcal{L}_\rho)\end{aligned}$$

and

$$\pi_\rho(AB)(C+\mathcal{L}_\rho) = ABC+\mathcal{L}_\rho = \pi_\rho(A)(BC+\mathcal{L}_\rho) = \pi_\rho(A)\pi_\rho(B)(C+\mathcal{L}_\rho).$$

Similarly, we show that $\pi_\rho(A)^* = \pi_\rho(A^*)$ by checking this directly on the dense subspace $\mathcal{A}/\mathcal{L}_\rho$:

$$\begin{aligned}\langle \pi_\rho(A)(B+\mathcal{L}_\rho)|C+\mathcal{L}_\rho\rangle = \langle AB+\mathcal{L}_\rho|C+\mathcal{L}_\rho\rangle = \rho(B^*A^*C) &= \langle B+\mathcal{L}_\rho|A^*C+\mathcal{L}_\rho\rangle\\ &= \langle B+\mathcal{L}_\rho|\pi_\rho(A^*)(C+\mathcal{L}_\rho)\rangle.\end{aligned}$$

In summary, so far we have shown that π_ρ is a representation of $\mathcal{A}$ on $\mathcal{H}_\rho$.

Step 3: x_ρ is a unit vector, cyclic for π_ρ, and $\rho = \omega_{x_\rho} \circ \pi_\rho$.

We have $\|x_\rho\|^2 = \langle I+\mathcal{L}_\rho|I+\mathcal{L}_\rho\rangle = \rho(I^*I) = \rho(I) = 1$ and $\pi_\rho(A)x_\rho = A+\mathcal{L}_\rho$ for every $A \in \mathcal{A}$, hence x_ρ is a unit vector and $\pi_\rho(\mathcal{A})x_\rho = \{\pi_\rho(A)x_\rho \mid A \in \mathcal{A}\}$ is obviously dense in $\mathcal{H}_\rho$. Finally,

$$\langle x_\rho|\pi_\rho(A)x_\rho\rangle = \langle I+\mathcal{L}_\rho|A+\mathcal{L}_\rho\rangle = \rho(I^*A) = \rho(A).$$

□

The additional good news are that the GNS representation is essentially uniquely determined by the requirement $\rho = \omega_{x_\rho} \circ \pi_\rho$ as we will make precise in the following statement.

3.5. Proposition (Uniqueness of the GNS representation): Let ρ be a state on the C*-algebra $\mathcal{A}$ and suppose that π is a cyclic representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ such that $\rho = \omega_x \circ \pi$ holds for some unit cyclic vector $x \in \mathcal{H}$. Then π is (unitarily) equivalent to the GNS representation π_ρ, more precisely, there is a unitary map $U\colon \mathcal{H}_\rho \to \mathcal{H}$ such that

$$Ux_\rho = x \quad \text{and} \quad \forall A \in \mathcal{A}\colon\ U\pi_\rho(A)U^{-1} = \pi(A).$$

Proof: For arbitary $C \in \mathcal{A}$, we have

$$\|\pi(C)x\|^2 = \langle \pi(C)x|\pi(C)x\rangle = \langle x|\pi(C^*C)x\rangle = \rho(C^*C) = \langle \pi_\rho(C)x_\rho|\pi_\rho(C)x_\rho\rangle = \|\pi_\rho(C)x_\rho\|^2,$$

which (upon setting $C = A - B$) shows: $\pi_\rho(A)x_\rho = \pi_\rho(B)x_\rho \quad\Rightarrow\quad \pi(A)x = \pi(B)x$. Therefore, $U\pi_\rho(A)x_\rho := \pi(A)x$ is a well-defined isometric and surjective linear map between the dense subspaces $\mathcal{A}/\mathcal{L}_\rho \subseteq \mathcal{H}_\rho$ and $\pi(\mathcal{A})x \subseteq \mathcal{H}$, thus extends to a unitary map $U\colon \mathcal{H}_\rho \to \mathcal{H}$. By construction, $Ux_\rho = U\pi_\rho(I)x_\rho = \pi(I)x = x$ and on the dense set of vectors $y = \pi_\rho(B)x_\rho$ in $\mathcal{H}_\rho$ we see that

$$\begin{aligned} U\pi_\rho(A)y = U\pi_\rho(A)\pi_\rho(B)x_\rho = U\pi_\rho(AB)x_\rho = \pi(AB)x = \pi(A)\pi(B)x \\ = \pi(A)U\pi_\rho(B)x_\rho = \pi(A)Uy, \end{aligned}$$

hence $U\pi_\rho(A) = \pi(A)U$. □

One immediate consequence of the essential uniqueness of the GNS representation is the following result about the unitary implementation of a $*$-automorphism via invariant states.

3.6. Corollary: Let $\tau\colon \mathcal{A} \to \mathcal{A}$ be a $*$-automorphism of the C*-algebra $\mathcal{A}$ and suppose that ρ is a *τ-invariant state* on $\mathcal{A}$, i.e., $\rho(\tau(A)) = \rho(A)$ holds for all $A \in \mathcal{A}$. Then there exists a unique unitary operator U on the GNS representation Hilbert space $\mathcal{H}_\rho$ such that

$$Ux_\rho = x_\rho \quad \text{and} \quad \forall A \in \mathcal{A}\colon \quad \pi_\rho(\tau(A)) = U\pi_\rho(A)U^*.$$

Proof: Clearly, $\mu := \rho \circ \tau$ is a state on $\mathcal{A}$ and $\mu = \rho$. Thus $\mathcal{H}_\mu = \mathcal{H}_\rho$ and $\pi := \pi_\rho \circ \tau$ is a cyclic representation of $\mathcal{A}$ on $\mathcal{H}_\rho$ with cyclic vector $x = x_\rho$ such that for any $A \in \mathcal{A}$,

$$\rho(A) = \rho(\tau(A)) = (\omega_{x_\rho} \circ \pi_\rho)(\tau(A)) = \omega_x(\pi_\rho(\tau(A))) = \omega_x(\pi(A)) = (\omega_x \circ \pi)(A).$$

The assertion follows from Proposition 3.5 and its proof (since U is determined on a dense subset). □

3.7. Proposition: Let $\mathcal{A}$ be a C*-algebra and $A \in \mathcal{A}$ with $A \neq 0$. Then there is a pure state ρ on $\mathcal{A}$ such that in the corresponding GNS representation, $\pi_\rho(A) \neq 0$.

Proof: By Corollary 1.14, there is a pure state ρ on $\mathcal{A}$ such that $0 \neq \rho(A) = \langle x_\rho | \pi_\rho(A)x_\rho\rangle$. This certainly implies $\pi_\rho(A) \neq 0$. □

3.8. Remark: Let ρ be a state on $\mathcal{A}$ with left kernel $\mathcal{L}_\rho$ as in the construction of the GNS representation π_ρ. Recall that $\ker \pi_\rho$ is a two-sided ideal and $\mathcal{L}_\rho$ is a left ideal.

If $A \in \mathcal{L}_\rho$, then $|\rho(A)|^2 \leq \rho(I^*I)\rho(A^*A) = 0$ implies that $A \in \ker \rho$. Note that the GNS construction gives $\ker \pi_\rho = \{A \in \mathcal{A} \mid \forall B, C \in \mathcal{A}\colon \langle B + \mathcal{L}_\rho | \pi_\rho(A)(C + \mathcal{L}_\rho)\rangle = 0\}$ and using $\langle B + \mathcal{L}_\rho | \pi_\rho(A)(C + \mathcal{L}_\rho)\rangle = \langle I + \mathcal{L}_\rho | \pi_\rho(B^*AC)(I + \mathcal{L}_\rho)\rangle = \rho(B^*AC)$ then implies

$$\ker \pi_\rho = \{A \in \mathcal{A} \mid \forall B, C \in \mathcal{A}\colon \rho(B^*AC) = 0\}.$$

Using $B = A$ and $C = I$ as a special case in the above shows that $A \in \ker \pi_\rho$ implies $A \in \mathcal{L}_\rho$. (In fact, $\ker \pi_\rho \subseteq \mathcal{L}_\rho$ can be seen more directly: If $\pi_\rho(A) = 0$, then $\mathcal{L}_\rho = \pi_\rho(A)x_\rho = A + \mathcal{L}_\rho$ and hence $A \in \mathcal{L}_\rho$.)

In summary, we have the general relations

$$\ker \pi_\rho \subseteq \mathcal{L}_\rho \subseteq \ker \rho.$$

We observe from $\rho(A^*) = \overline{\rho(A)}$ and $\rho(B^*A^*C) = \overline{\rho(C^*AB)}$ that both $\ker\rho$ and $\ker\pi_\rho$ are self-adjoint subsets of $\mathcal{A}$. This does in general not hold for the left kernel $\mathcal{L}_\rho$. (Consider $\mathcal{A} = \mathcal{B}(l^2)$ and let L be the left-shift $L(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$ and recall that $L^* = R$, the right-shift $R(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$, so that $LL^* = I$ and $L^*L(a_1, a_2, a_3, \ldots) = (0, a_2, a_3, \ldots)$; with $e_1 := (1, 0, 0, \ldots)$ and $\rho(A) := \langle e_1 | A e_1 \rangle$ for all $A \in \mathcal{B}(l^2)$ we thus obtain $\rho(L^*L) = 0$ while $\rho(LL^*) = 1$, thus $L \in \mathcal{L}_\rho$, but $L^* \notin \mathcal{L}_\rho$.)

We have learned how to find plenty of nontrivial representations and now we only need methods to combine these into one representation that will be guaranteed to be faithful. The means to achieve this is by the following basic constructions.

3.9. Hilbert space direct sums and direct sum representations: Let J be a nonempty set and suppose that $\mathcal{H}_j$ is a Hilbert space for every $j \in J$. Let $\bigoplus_{j\in J} \mathcal{H}_j$ denote the vector space of all families $(x_j)_{j\in J}$ such that $x_j \in \mathcal{H}_j$ $(j \in J)$ and $\sum_{j\in J} \|x_j\|^2 < \infty$, equipped with the inner product

$$\langle (x_j) | (y_j) \rangle := \sum_{j\in J} \langle x_j | y_j \rangle.$$

It can be shown—we refer to [KRI, Section 2.6] for detailed arguments of all technicalities we skip here—that $\bigoplus_{j\in J} \mathcal{H}_j$ is complete with respect to the norm $\|(x_j)\| := (\sum_{j\in J} \|x_j\|^2)^{1/2} = \sqrt{\langle (x_j) | (x_j) \rangle}$. We call $\bigoplus_{j\in J} \mathcal{H}_j$ the *Hilbert space direct sum*, henceforth often referred to simply as direct sum. Note that $\mathcal{H}_j \perp \mathcal{H}_k$ if $j \neq k$ and that the above direct sum concept can be applied also to a family of mutually orthogonal closed subspaces $\mathcal{H}_j$ of a given Hilbert space $\mathcal{H}$ such that $\operatorname{span} \bigcup_{j\in J} \mathcal{H}_j$ is dense in $\mathcal{H}$.

If $(C_j)_{j\in J}$ is a family of operators $C_j \in \mathcal{B}(\mathcal{H}_j)$ $(j \in J)$ such that $\sup\{\|C_j\| \mid j \in J\} < \infty$, then we may define the direct sum operator $C = \bigoplus_{j\in J} C_j$ on $\bigoplus_{j\in J} \mathcal{H}_j$ by $C(x_j)_{j\in J} := (C_j x_j)_{j\in J}$ and we have

$$\Big\| \bigoplus_{j\in J} C_j \Big\| = \sup\{\|C_j\| \mid j \in J\}.$$

Moreover, with hopefully obvious notation, we have $\bigoplus_{j\in J}(\alpha C_j + \beta D_j) = \alpha\big(\bigoplus_{j\in J} C_j\big) + \beta\big(\bigoplus_{j\in J} D_j\big)$, $\big(\bigoplus_{j\in J} C_j\big)\big(\bigoplus_{j\in J} D_j\big) = \bigoplus_{j\in J} C_j D_j$, and $\big(\bigoplus_{j\in J} C_j\big)^* = \bigoplus_{j\in J} C_j^*$.

We are now ready to define the *direct sum of a family of representations*: Let $\pi_j \colon \mathcal{A} \to \mathcal{B}(\mathcal{H}_j)$ be a representation of $\mathcal{A}$ for every $j \in J$. Recall that for each $A \in \mathcal{A}$, $\|\pi_j(A)\| \leq \|A\|$ $(j \in J)$, so that $\bigoplus_{j\in J} \pi_j(A)$ is defined as bounded operator on $\bigoplus_{j\in J} \mathcal{H}_j$. From the relations for direct sum operators noted in the previous paragraph, it follows that the map $A \mapsto \bigoplus_{j\in J} \pi_j(A)$ defines a representation $\pi \colon \mathcal{A} \to \mathcal{B}(\bigoplus_{j\in J} \mathcal{H}_j)$, which we call the *direct sum representation* and denote it by $\oplus_{j\in J} \pi_j$.

A first application of the concept of direct sum representations and subrepresentations is the following decomposition result.

3.10. Proposition: Every representation of a C*-algebra is equivalent to a direct sum of cyclic representations. In case the representation Hilbert space is separable the direct sum is at most countable.

Sketch of proof: Let $\pi \colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of the C*-algebra $\mathcal{A}$. For any unit vector $h \in \mathcal{H}$ let $\mathcal{E}_h$ denote the closure of $\pi(\mathcal{A})h = \{\pi(A)h \mid A \in \mathcal{H}\}$, then both $\mathcal{E}_h$ and $\mathcal{E}_h^\perp$

define subrepresentations π_h and $\pi_h^\perp$. By construction, the representation π_h is cyclic. We may now proceed in the same way to produce a cyclic subrepresentation of $\pi_h^\perp$.

A routine application of Zorn's lemma allows us to find a maximal set $\mathcal{F} \subseteq \mathcal{H}$ of unit vectors such that $h, g \in \mathcal{F}$ with $h \neq g$ implies $\mathcal{E}_h \perp \mathcal{E}_g$. By maximality, we must have $\mathcal{H} = \bigoplus_{h\in\mathcal{F}} \mathcal{E}_h$ and $\pi = \bigoplus_{h\in\mathcal{F}} \pi_h$.

If $\mathcal{H}$ is separable, then the orthonormal set of vectors $\mathcal{F}$ can be at most countable. □

The second application is one of the main results announced earlier, which implies that any C*-algebra is isometrically $*$-isomorphic to a subalgebra of operators on a Hilbert space.

3.11. Theorem (Gelfand-Neumark): Every C*-algebra has a faithful representation.

Proof: Let $\mathcal{A}$ be a C*-algebra and $\mathcal{S}_0$ a subset of states such that $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{A})$. With π_ρ denoting the GNS representation, let $\pi := \bigoplus_{\rho\in\mathcal{S}_0} \pi_\rho$. We prove that π is faithful: If $A \in \mathcal{A}$ with $\pi(A) = 0$, then $\pi_\rho(A) = 0$ for every pure state ρ. Proposition 3.7 implies $A = 0$. □

3.12. The universal representation: In the proof of the Gelfand-Neumark theorem we have shown that the direct sum representation $\bigoplus_{\rho\in\mathcal{S}_0} \pi_\rho$ of a C*-algebra $\mathcal{A}$ is faithful if $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{A})$. Choosing $\mathcal{S}_0 = \mathcal{S}(\mathcal{A})$ we obtain the so-called *universal representation* of $\mathcal{A}$, given by

$$\Phi := \bigoplus_{\rho\in\mathcal{S}(\mathcal{A})} \pi_\rho,$$

and note that we have thus an isometric $*$-isomorphism $\mathcal{A} \cong \Phi(\mathcal{A})$ of C*-algebras.

If τ is an arbitrary state on $\mathcal{A}$, then we know that $\tau = \omega_{x_\tau} \circ \pi_\tau$ with the standard unit cyclic vector $x_\tau \in \mathcal{H}_\tau$. Define $y = (y_\rho) \in \mathcal{H}_\Phi := \bigoplus_{\rho\in\mathcal{S}(\mathcal{A})} \mathcal{H}_\rho$ by $y_\tau := x_\tau$ and $y_\rho := 0$ if $\rho \neq \tau$. Then we obtain $\tau = \omega_y \circ \Phi$, i.e., the state τ on $\mathcal{A}$ corresponds to the vector state ω_y on $\Phi(\mathcal{A})$. Put in other words, *every state on* $\Phi(\mathcal{A})$ *is a vector state.*

Since $\mathcal{A} \cong \Phi(\mathcal{A})$, taking Banach space duals also gives $\mathcal{A}^\# \cong \Phi(\mathcal{A})^\#$ and $\mathcal{A}^{\#\#} \cong \Phi(\mathcal{A})^{\#\#}$. Recall that $\Phi(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H}_\Phi)$. It is interesting to report (cf. [KRII, Proposition 10.1.21]) that the canonical embedding $\Phi(\mathcal{A}) \hookrightarrow \Phi(\mathcal{A})^{\#\#}$ extends to an isometric isomorphism $\Phi(\mathcal{A})^- \to \Phi(\mathcal{A})^{\#\#}$, where $\Phi(\mathcal{A})^-$ denotes the closure of $\Phi(\mathcal{A})$ in the weak operator topology of $\mathcal{B}(\mathcal{H}_\Phi)$. (Recall that this topology is defined by seminorms on operators T in the form $p(T) = |\langle y|Tz\rangle|$, where y and z are vectors; see, e.g., [Hoe23, 4.12.10)] or [Con10, Chapter IX, Definition 1.2]; we will also introduce it later.) Therefore, we obtain

$$\mathcal{A}^{\#\#} \cong \Phi(\mathcal{A})^-$$

and the latter is a *von Neumann algebra* as we will see in the next chapter. Note that we have an isometric $*$-isomorphic embedding $\mathcal{A} \hookrightarrow \Phi(\mathcal{A})^-$ and $\Phi(\mathcal{A})^-$ is sometimes called the *enveloping von Neumann algebra* of $\mathcal{A}$ (cf. [Ped18, 3.7.6]).

3.13. Remark: (i) The GNS construction can be carried out in essentially the same way for non-unital C*-algebras, but to obtain it as a cyclic representation requires to argue with the help of approximate units (see [BR1, Subsection 2.3.3]). Also the Gelfand-Neumark theorem holds in the non-unital case ([BR1, Theorem 2.1.10 and its proof in Subsection 2.3.4]).

One convenient consequence of our standard requirements in this course of having unital C*-algebras and unital $*$-homomorphisms, hence $\pi(I) = I$ for any representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$, is that $\pi(\mathcal{A})\mathcal{H} := \{\pi(A)x \mid x \in \mathcal{H}, A \in \mathcal{A}\}$ is automatically dense[1] in $\mathcal{H}$. If $\mathcal{A}$ is non-unital, a representation with the latter property is said to be *non-degenerate*.

(ii) In the context of applications to quantum physics, it is being discussed whether the structure of unital $*$-algebras suffices and—with a somewhat weakened version of the GNS representation (cf. [Mor19, Subsections 8.3.2] or [PII, Chapter 9])—could replace the requirement of having unital C*-algebras (see, e.g., [Mor19, Subsection 8.2.2], [Rej16], and [FV15]).

Representations that do not have any nontrivial subrepresentations are certainly of specific interest and will be studied in the remainder of this chapter.

3.14. Definition: A representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C*-algebra $\mathcal{A}$ is called *irreducible*, if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ invariant under all operators from $\pi(\mathcal{A})$.

3.15. Invariant subspaces and projections: Let $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of a C*-algebra $\mathcal{A}$ and suppose $\mathcal{K}$ is a closed invariant subspace, i.e., $\pi(A)\mathcal{K} \subseteq \mathcal{K}$ for all $A \in \mathcal{A}$. Let P be the (orthogonal) projection onto $\mathcal{K}$. Then, by the following standard argument, P *commutes with* $\pi(\mathcal{A})$: Recall that $I - P$ is the projection onto $(\operatorname{ran} P)^\perp$, which is also invariant under $\pi(\mathcal{A})$; for every $x \in \mathcal{H}$ and $A \in \mathcal{A}$, we use the decomposition $x = Px + (I-P)x$ and obtain

$$P\pi(A)x = P\pi(A)(Px + (I-P)x) = P\underbrace{\pi(A)Px}_{\in \operatorname{ran} P} + P\underbrace{\pi(A)(I-P)x}_{\in \operatorname{ran}(I-P)} = \pi(A)Px + 0 = \pi(A)Px.$$

Conversely, if $P \in \mathcal{B}(\mathcal{H})$ is a projection that commutes with $\pi(\mathcal{A})$, then $\operatorname{ran} P$ *is a closed invariant subspace*: Recall that the general relation $\operatorname{ran} P = \ker(I - P)$ shows that the range is closed and the invariance follows immediately from $\pi(A)x = \pi(A)Px = P\pi(A)x \in \operatorname{ran} P$ ($x \in \operatorname{ran} P$, $A \in \mathcal{A}$).

We recall a definition from 0.1, which we need here only in the special case of a subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is some Hilbert space: The commutant of $\mathcal{M}$ is given by $\mathcal{M}' := \{B \in \mathcal{B}(\mathcal{H}) \mid \forall M \in \mathcal{M}\colon MB = BM\}$.

3.16. Proposition: Let $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of a C*-algebra $\mathcal{A}$, then the following conditions are equivalent:

(i) π is irreducible,

(ii) every nonzero vector in $\mathcal{H}$ is cyclic for π,

(iii) $\pi(\mathcal{A})' = \mathbb{C}I$, i.e., only the scalar multiples of the identity I on $\mathcal{H}$ commute with $\pi(\mathcal{A})$.

Proof: (i) $\Rightarrow$ (ii): If there is a nonzero $x \in \mathcal{H}$ such that the subspace $\pi(\mathcal{A})x$ is not dense, then $\mathcal{K} := (\pi(\mathcal{A})x)^\perp \neq \{0\}$ gives rise to a nontrivial subrepresentation, since $\mathcal{K}$ is closed and also invariant under $\pi(\mathcal{A})$ (because $\pi(\mathcal{A})x$ is invariant). Therefore, π cannot be irreducible.

[1] in fact, equal to $\mathcal{H}$

(ii) $\Rightarrow$ (iii): If $\pi(\mathcal{A})' \neq \mathbb{C}I$, then there is some self-adjoint operator $T \in \pi(\mathcal{A})'$ such that $T \notin \mathbb{C}I$. It follows that every spectral projection of T commutes with all the operators in $\pi(\mathcal{A})$ (see [Hoe23, 1.18] or [Con10, 2.2 in Chapter IX]). Let P be any nontrivial spectral projection of T and choose a nonzero vector x in its (closed) range. Then $x = Px$ and we obtain for every $A \in \mathcal{A}$ that $\pi(A)x = \pi(A)Px = P\pi(A)x \in \operatorname{ran} P$ (see also 3.15). Since $\operatorname{ran} P \neq \mathcal{H}$, the vector x cannot be cyclic.

(iii) $\Rightarrow$ (i): If π is not irreducible, then there is some closed invariant subspace $\mathcal{K}$ with $\mathcal{K} \neq \{0\}$ and $\mathcal{K} \neq \mathcal{H}$. Then the projection P onto $\mathcal{K}$ is nontrivial, i.e., $P \neq 0$ and $P \neq I$, and by 3.15 commutes with $\pi(\mathcal{A})$. Therefore, $P \in \pi(\mathcal{A})'$ and clearly, $P \notin \mathbb{C}I$. $\square$

For GNS representations we obtain a particularly nice characterization of irreducibility.

3.17. Theorem: Let ρ be a state on the C*-algebra $\mathcal{A}$ and π_ρ denote the corresponding GNS representation. Then the following conditions are equivalent:

(i) π_ρ is irreducible,

(ii) ρ is pure.

Proof: (i) $\Rightarrow$ (ii): Suppose that ρ is not pure, then by Lemma 1.12 there is some positive linear functional μ on $\mathcal{A}$ with $\mu \leq \rho$ and $\mu \neq t\rho$ for all $t \in [0,1]$. In fact, it then follows that $\mu \neq \lambda\rho$ for all $\lambda \in \mathbb{C}$, since $0 \leq \mu(A^*A) \leq \rho(A^*A)$ for all $A \in \mathcal{A}$.

The Cauchy-Schwarz inequality combined with the above gives that for any $A, B \in \mathcal{A}$,

$$|\mu(A^*B)|^2 \leq \mu(A^*A)\mu(B^*B) \leq \rho(A^*A)\rho(B^*B) = \|\pi_\rho(A)x_\rho\|^2\|\pi_\rho(B)x_\rho\|^2,$$

hence the map $(\pi_\rho(A)x_\rho, \pi_\rho(B)x_\rho) \mapsto \mu(A^*B)$ is a bounded sesquilinear form on $\mathcal{A}/\mathcal{L}_\rho$, which can be extended to such on $\mathcal{H}_\rho$. There exists a unique operator $T \in \mathcal{B}(\mathcal{H}_\rho)$ such that for all $A, B \in \mathcal{A}$,

$$\langle \pi_\rho(A)x_\rho | T\pi_\rho(B)x_\rho \rangle = \mu(A^*B)$$

(cf. [Con10, Chapter II, Theorem 2.2] or [Hoe23, Theorem 0.3(a)]). We obtain $0 \leq T \leq I$ from $0 \leq \mu(A^*A) \leq \rho(A^*A)$ and $T \notin \mathbb{C}I$, since $\mu \neq \lambda\rho$ for all $\lambda \in \mathbb{C}$.

We claim that $T \in \pi_\rho(\mathcal{A})'$, which then completes this part of the proof by appealing to Proposition 3.16(iii). Indeed, we obtain $T\pi_\rho(C) = \pi_\rho(C)T$ for every $C \in \mathcal{A}$ by considering inner products with all vectors of the form $\pi_\rho(A)x_\rho$ and $\pi_\rho(B)x_\rho$ from the dense subset $\mathcal{A}/\mathcal{L}_\rho \subseteq \mathcal{H}_\rho$, in fact,

$$\begin{aligned}\langle \pi_\rho(B)x_\rho | T\pi_\rho(C)\pi_\rho(A)x_\rho \rangle &= \mu(B^*CA) = \mu((C^*B)^*A) \\ &= \langle \pi_\rho(C^*)\pi_\rho(B)x_\rho | T\pi_\rho(A)x_\rho \rangle = \langle \pi_\rho(B)x_\rho | \pi_\rho(C)T\pi_\rho(A)x_\rho \rangle.\end{aligned}$$

(ii) $\Rightarrow$ (i): Suppose that π_ρ is not irreducible, so that by Proposition 3.16 we can find some $T \in \pi_\rho(\mathcal{A})'$ such that $T \neq \mathbb{C}I$. We may assume that T is self-adjoint, since $\pi_\rho(\mathcal{A})'$ is a $*$-algebra and hence the real part $(T + T^*)/2$ as well as the imaginary part $(T - T^*)/(2i)$ belong to $\pi(\mathcal{A})'$ whenever T does. There is a spectral projection P of T such that $P \neq 0$ and $P \neq I$. Recall that also P as well as $(I - P)$ belong to $\pi(\mathcal{A})'$, and $0 \leq P \leq I$, in particular,

$I - P \geq 0$ holds. Therefore the linear functional $\mu\colon \mathcal{A} \to \mathbb{C}$, $\mu(A) := \langle Px_\rho | \pi_\rho(A)x_\rho\rangle$, is easily seen to be positive, since

$$\mu(A^*A) = \langle Px_\rho|\pi_\rho(A)^*\pi_\rho(A)x_\rho\rangle = \langle \pi_\rho(A)Px_\rho|\pi_\rho(A)x_\rho\rangle = \langle P\pi_\rho(A)x_\rho|\pi_\rho(A)x_\rho\rangle \geq 0.$$

We have $\mu \leq \rho$, which can be seen as follows:

$$\begin{aligned}\rho(A^*A) - \mu(A^*A) = \langle x_\rho|\pi_\rho(A^*A)x_\rho\rangle - \langle Px_\rho|\pi_\rho(A^*A)x_\rho\rangle &= \langle (I-P)x_\rho|\pi_\rho(A)^*\pi_\rho(A)x_\rho\rangle \\ &= \langle (I-P)\pi_\rho(A)x_\rho|\pi_\rho(A)x_\rho\rangle \geq 0.\end{aligned}$$

We finally claim that $\mu \neq t\rho$ for all $t \in [0,1]$, which then implies that ρ cannot be pure by Lemma 1.12 and completes the proof.

If $\mu = t\rho$, then we have for all $A, B \in \mathcal{A}$,

$$\langle P\pi_\rho(B)x_\rho|\pi_\rho(A)x_\rho\rangle = \mu(B^*A) = t\rho(B^*A) = t\langle \pi_\rho(B)x_\rho|\pi_\rho(A)x_\rho\rangle,$$

which implies $P = tI$. Since P is a projection, we must have $t = 0$ or $t = 1$, but this contradicts the fact that $P \neq 0$ and $P \neq I$. □

Combining the previous theorem with Proposition 3.7 proves the following statement.

3.18. Corollary: If $A \neq 0$ in the C*-algebra $\mathcal{A}$, then there exists an irreducible representation π of $\mathcal{A}$ such that $\pi(A) \neq 0$.

3.19. Remarks: (i) If $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an irreducible representation, then any unit vector $x \in \mathcal{H}$ induces a pure state on $\mathcal{A}$ by $\rho := \omega_x \circ \pi$. Indeed, x is a cyclic vector by Proposition 3.16 and then the essential uniqueness of the GNS representation shown in Proposition 3.5 implies that π and π_ρ are (unitarily) equivalent. Hence π_ρ is irreducible, thus ρ is pure.

(ii) One can show that in the GNS construction with a pure state ρ, the inner product space $\mathcal{A}/\mathcal{L}_\rho$ is already complete, i.e., $\mathcal{H}_\rho = \mathcal{A}/\mathcal{L}_\rho$ (c.f. [KRII, Theorem 10.2.7]). (The subspace $\mathcal{A}/\mathcal{L}_\rho$ clearly is invariant under $\pi_\rho(\mathcal{A})$ and $\mathcal{A}/\mathcal{L}_\rho \neq \{0\}$; now for C*-algebras it can be shown that topological irreducibility implies algebraic irreducibility ([KRI, Corollary 5.4.4]), hence we must have $\mathcal{A}/\mathcal{L}_\rho = \mathcal{H}_\rho$ even without having to know a priori whether the subspace is closed or not.)

3.20. Example: Let $\mathcal{A}$ be a commutative C*-algebra. If $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an irreducible representation, then $\pi(\mathcal{A})' = \mathbb{C}I$. Since $\mathcal{A}$ is commutative, we have $\pi(\mathcal{A}) \subseteq \pi(\mathcal{A})' = \mathbb{C}I$, which implies that

$$\forall A \in \mathcal{A}\colon \quad \pi(A) = \rho(A)I,$$

where $\rho\colon \mathcal{A} \to \mathbb{C}$ is necessarily a multiplicative linear functional with $\rho(I) = 1$, thus a pure state on $\mathcal{A}$ by Proposition 2.4. We see that $\pi(\mathcal{A}) = \mathbb{C}I$ and, moreover, since every subspace of $\mathcal{H}$ is invariant under $\mathbb{C}I = \pi(\mathcal{A})$, the irreducibility of π requires[2] that $\mathcal{H}$ is one-dimensional. Conversely, every pure state ρ on $\mathcal{A}$ defines a one-dimensional irreducible representation on $\mathbb{C}$ by $A \mapsto \rho(A)$. It is an exercise to show that two irreducible representations given by the pure states ρ_1 and ρ_2 are equivalent if and only if $\rho_1 = \rho_2$.

[2]Note that $\mathcal{H} = \{0\}$, thus $\mathcal{B}(\mathcal{H}) = \{0\}$, is excluded by our conventions about unital normed algebras.

3.21. Remark: We have seen in Proposition 3.10 that every representation of a C*-algebra can be decomposed as a sum of cyclic subrepresentations. The analogue of this result for irreducible subrepresentation is not true. In fact, there are even simple examples of representations that are not irreducible themselves and have no irreducible subrepresentation. Instead of direct sums one may then employ methods from so-called *direct integrals of Hilbert spaces and of operator algebras* (cf. [Dix82, Chapter 8], [TakI, Section IV.8], [Bla10, Section III.5], [KRII, Chapter 14]). As for an example consider $\mathcal{A} = C([0,1])$ with the representation π as multiplication operators on $L^2([0,1])$, i.e., $\pi(f) = M_f$ with $M_f g = fg$ ($f \in C([0,1])$, $g \in L^2([0,1])$). Since $\mathcal{A}$ is commutative, we have seen in the above example that an irreducible representation has to be one-dimensional. Therefore π is not irreducible and an irreducible subrepresentation of π must correspond to some one-dimensional subspace $\mathcal{K}$ of $L^2([0,1])$. In particular, $\mathcal{K} = \operatorname{span}\{h\}$, where $h \in L^2([0,1])$ is an eigenvector for every multiplication operator M_f. But, as is well-known, already $T := \pi(\mathrm{id}) = M_{\mathrm{id}}$ does not have any eigenvector[3] although $\operatorname{sp}(T) = [0,1]$.

Recall that every irreducible representation of $C([0,1])$ has to be given directly by a pure state ρ; and we know from Example 1) in 1.12 that ρ is necessarily of the form $\rho(f) = f(s)$ for some $s \in [0,1]$.

[3]If $\lambda \in \mathbb{C}$ and $th(t) = \lambda h(t)$ for almost all t, then $t = \lambda$ for t in a set of positive measure unless $h = 0$ (a.e.).

4. A glimpse at von Neumann algebras

◇ Our main sources for this chapter are [KRI, KRII, Con00, Mur90, BR1, Sun87, Dix81].

In the current chapter we will study a particular class of C*-subalgebras of $\mathcal{B}(\mathcal{H})$. Thus we focus on algebras of operators on a Hilbert space $\mathcal{H} \neq \{0\}$. In this context, several topologies different from the operator norm topology on $\mathcal{B}(\mathcal{H})$ turn out to be of relevance.

4.1. The weak and strong operator topologies: Recall ([Hoe23, 4.12.10)]) or define the *strong operator topology* (SOT) on $\mathcal{B}(\mathcal{H})$ as the Hausdorff locally convex topology generated by the family of seminorms p_x $(x \in \mathcal{H})$, where

$$p_x(T) := \|Tx\| \quad (T \in \mathcal{B}(\mathcal{H})),$$

and the *weak operator topology* (WOT) on $\mathcal{B}(\mathcal{H})$ as the Hausdorff locally convex topology generated by the family of seminorms $p_{x,y}$ $(x, y \in \mathcal{H})$, where

$$p_{x,y}(T) := |\langle Tx|y\rangle| \quad (T \in \mathcal{B}(\mathcal{H})).$$

Bases of neighborhoods of a given operator $B \in \mathcal{B}(\mathcal{H})$ and convergence of nets to B in these topologies can be described as follows:

(a) SOT: For every $\varepsilon > 0$, $m \in \mathbb{N}$, and $x_1, \ldots, x_m \in \mathcal{H}$,

$$\{T \in \mathcal{B}(\mathcal{H}) \mid j = 1, \ldots, m \colon \|Tx_j - Bx_j\| < \varepsilon\}.$$

Convergence of a net $(T_l)_{l\in\Lambda}$ in $\mathcal{B}(\mathcal{H})$ to B is equivalent to pointwise convergence of (T_l) to B on $\mathcal{H}$, i.e., $T_l x \to Bx$ in $\mathcal{H}$ for every $x \in \mathcal{H}$.

(b) WOT: For every $\varepsilon > 0$, $m \in \mathbb{N}$, $x_1, \ldots, x_m \in \mathcal{H}$, and $y_1, \ldots y_m \in \mathcal{H}$,

$$\{T \in \mathcal{B}(\mathcal{H}) \mid j = 1, \ldots, m \colon |\langle Tx_j - Bx_j|y_j\rangle| < \varepsilon\}.$$

Convergence of a net $(T_l)_{l\in\Lambda}$ in $\mathcal{B}(\mathcal{H})$ to B is equivalent to weak convergence of $(T_l x)$ to Bx in $\mathcal{H}$ for every $x \in \mathcal{H}$, i.e., $\langle y|T_l x\rangle \to \langle y|Bx\rangle$ for all $x, y \in \mathcal{H}$.

The norm topology is finer than the strong operator topology, since sufficiently small balls are contained in any SOT neighborhood. The two topologies are different unless $\mathcal{H}$ is finite-dimensional, since any orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ gives rise to the operator sequence $T_n \in \mathcal{B}(\mathcal{H})$ with $T_n x := \sum_{k=1}^{n} \langle e_k|x\rangle e_k$ $(x \in \mathcal{H})$ that is SOT convergent to the projection P onto the closure of $\operatorname{span}\{e_n \mid n \in \mathbb{N}\}$, but $\|P - T_n\| = 1$ for every $n \in \mathbb{N}$ (clearly $\|P - T_n\| \leq 1$ and, e.g., $\|(P - T_n)e_{n+1}\| = \|e_{n+1}\| = 1$).

The strong operator topology is obviously finer than the weak operator topology. In infinite dimensions the SOT is strictly finer than the WOT, as can be seen easily from the example

$P_n \in \mathcal{B}(\mathcal{H})$ with $P_n x := \langle e_1|x\rangle e_n$ ($x \in \mathcal{H}$, $n \in \mathbb{N}$), where (e_n) is as above. (We have $P_n \to 0$ w.r.t. the WOT, since $|\langle P_n x|y\rangle| = |\langle e_1|x\rangle||\langle e_n|y\rangle|$, but $\|P_n e_1\| = \|e_n\| = 1$ for all $n \in \mathbb{N}$, hence $P_n \not\to 0$ w.r.t. the SOT.) In light of this observation, it is interesting to note that the WOT and SOT closures coincide on convex subsets of $\mathcal{B}(\mathcal{H})$ (cf. [KRI, Theorem 5.1.2]).

It is not difficult to show that the involution, i.e., taking the adjoint $A \mapsto A^*$, $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is WOT continuous (see [KRI, KRIII, Exercise 5.7.1]), but not SOT continuous ([KRI, KRIII, Exercise 2.8.32]).

We recall once more the definition and a few elementary properties of the commutant $\mathcal{M}'$ of a subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$: We have $\mathcal{M}' := \{B \in \mathcal{B}(\mathcal{H}) \mid \forall M \in \mathcal{M}\colon MB = BM\}$ and $\mathcal{M}'$ is automatically a unital subalgebra of $\mathcal{B}(\mathcal{H})$. It is a $*$-subalgebra in case $\mathcal{M}$ is self-adjoint, since $A \in \mathcal{M}'$ and $M \in \mathcal{M}$ then implies $M^* \in \mathcal{M}$ and $A^*M = (M^*A)^* = (AM^*)^* = MA^*$.

We clearly have $\mathcal{M} \subseteq \mathcal{M}'' := (\mathcal{M}')'$ and applying this to $\mathcal{M}'$ in place of $\mathcal{M}$ immediately gives $\mathcal{M}' \subseteq (\mathcal{M}')'' = (\mathcal{M}'')' =: \mathcal{M}'''$. Moreover, $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ always implies $\mathcal{N}' \subseteq \mathcal{M}'$. We therefore obtain $\mathcal{M}''' \subseteq \mathcal{M}' \subseteq \mathcal{M}'''$, which proves $\mathcal{M}' = \mathcal{M}'''$.

Upon defining inductively also $\mathcal{M}^{(4)} := (\mathcal{M}''')'$ and $\mathcal{M}^{(m)} := (\mathcal{M}^{(m-1)})'$ ($m \geq 5$) we may summarize all of this as

$$\mathcal{M} \subseteq \mathcal{M}'' = \mathcal{M}^{(2n)} \quad \text{and} \quad \mathcal{M}' = \mathcal{M}''' = \mathcal{M}^{(2n+1)} \quad (n \in \mathbb{N}).$$

This suggest to single out subsets $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ that satisfy, in addition, $\mathcal{M} = \mathcal{M}''$, and will lead us towards the definition of von Neumann algebras once we have shown a topological characterization of the algebraic construct of the double commutant.

Note that a commutant $\mathcal{M}'$ is always WOT closed (hence also SOT closed). Indeed, suppose (A_l) is a net in $\mathcal{M}'$ converging with respect to the WOT to some $A \in \mathcal{B}(\mathcal{H})$. Let $B \in \mathcal{M}$ and $x, y \in \mathcal{H}$ arbitrary, then

$$\langle y|ABx\rangle = \lim \langle y|A_l Bx\rangle = \lim \langle y|BA_l x\rangle = \lim \langle B^*y|A_l x\rangle = \langle B^*y|Ax\rangle = \langle y|BAx\rangle$$

and therefore $A \in \mathcal{M}'$.

4.2. Theorem (Double commutant): Let $\mathcal{M}$ be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then the weak and strong operator closures of $\mathcal{M}$ both coincide with $\mathcal{M}''$.

Proof: Since $\mathcal{M} \subseteq \mathcal{M}''$ and $\mathcal{M}''$ is closed with respect to the WOT and the SOT, we know that both the WOT closure $\mathcal{M}_w^-$ of $\mathcal{M}$ and the SOT closure $\mathcal{M}_s^-$ of $\mathcal{M}$ are contained in $\mathcal{M}''$, in fact, $\mathcal{M}_s^- \subseteq \mathcal{M}_w^- \subseteq \mathcal{M}''$. It remains to show that $\mathcal{M}'' \subseteq \mathcal{M}_s^-$, then the proof will be complete.

Let $T \in \mathcal{M}''$. For arbitrarily given $x_1, \ldots, x_n \in \mathcal{H}$ and $\varepsilon > 0$ we have to find some $T_0 \in \mathcal{M}$ such that $\|(T - T_0)x_j\| < \varepsilon$ ($j = 1, \ldots, n$).

Define $\widetilde{\mathcal{H}} := \bigoplus_{j=1}^n \mathcal{H}$ and, for any $A \in \mathcal{B}(\mathcal{H})$, $\widetilde{A} := \bigoplus_{j=1}^n A$, so that we have $\widetilde{A}(y_1, \ldots, y_n) = (Ay_1, \ldots, Ay_n)$ for all $(y_1, \ldots, y_n) \in \widetilde{\mathcal{H}}$. Put $\widetilde{x} := (x_1, \ldots, x_n)$ and $\widetilde{\mathcal{M}} := \{\widetilde{M} \mid M \in \mathcal{M}\}$. Then $\widetilde{\mathcal{M}}$ is a unital $*$-subalgebra of $\mathcal{B}(\widetilde{\mathcal{H}})$ and the closure $\widetilde{\mathcal{K}}$ of $\{\widetilde{M}\,\widetilde{x} \mid \widetilde{M} \in \widetilde{\mathcal{M}}\}$ is invariant under $\widetilde{\mathcal{M}}$. Let $\widetilde{P}$ be the orthogonal projection onto $\widetilde{\mathcal{K}}$, then $\widetilde{P}$ commutes with all operators from $\widetilde{\mathcal{M}}$ (see 3.15), i.e., $\widetilde{P} \in \widetilde{\mathcal{M}}'$.

We claim that $\widetilde{T} \in \widetilde{\mathcal{M}}''$: The operators in $\mathcal{B}(\widetilde{\mathcal{H}})$ have a natural representation as $(n \times n)$-matrices with entries from $\mathcal{B}(\mathcal{H})$. Then $\widetilde{T}$ is a diagonal matrix with repeated entry T along the diagonal, and similarly for every $\widetilde{M} \in \widetilde{\mathcal{M}}$. It is easy to see that $\widetilde{\mathcal{M}}'$ consists all $(n \times n)$-matrices with arbitrary entries from $\mathcal{M}'$: We simply have to note that for matrices of the form $B = (B_{jk})_{1 \leq j,k \leq n}$ with $B_{jk} \in \mathcal{B}(\mathcal{H})$ and $\widetilde{A} = (\delta_{jk} A)_{1 \leq j,k \leq n}$ with $A \in \mathcal{B}(\mathcal{H})$, we obtain in the resulting (j,k)-entries for the matrix products

$$\big(B \cdot \widetilde{A}\big)_{jk} = B_{jk} A \quad \text{and} \quad \big(\widetilde{A} \cdot B\big)_{jk} = A B_{jk}.$$

Now in turn, $\widetilde{\mathcal{M}}''$ can also be determined by elementary matrix calculations and we claim that it consists of all diagonal matrices of the form

$$(\Delta) \qquad \begin{pmatrix} C & 0 & \cdots & 0 \\ 0 & C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C \end{pmatrix} \quad \text{with } C \in \mathcal{M}''.$$

It is immediate that all these matrices do belong to $\widetilde{\mathcal{M}}''$, and for the reverse inclusion relation one may consider the product of a general matrix $\widetilde{C} = (C_{jk})_{1 \leq j,k \leq n}$, where $C_{jk} \in \mathcal{B}(\mathcal{H})$, with a specific matrix from $\widetilde{\mathcal{M}}'$ of the form $E_{pq}(B) := (\delta_{jp} \delta_{qk} B)_{1 \leq j,k \leq n}$, where $B \in \mathcal{M}'$ and $p, q \in \{1, \ldots, n\}$; this results in

$$\big(\widetilde{C} \cdot E_{pq}(B)\big)_{jk} = \delta_{qk} C_{jp} B \quad \text{and} \quad \big(E_{pq}(B) \cdot \widetilde{C}\,\big)_{jk} = \delta_{jp} B C_{qk};$$

requiring equality for all j, k and B allows us to first to choose $B = I$, which implies $C_{jk} = 0$ when $j \neq k$, and second to choose $j = p$, $k = q$ to obtain the conditions $C_{pp} B = B C_{qq}$ for all p, q; once more choosing $B = I$ in the latter implies $C_{11} = \cdots = C_{nn} =: C$ and then varying $B \in \mathcal{M}'$ yields $C \in \mathcal{M}''$.

Since $\widetilde{\mathcal{M}}''$ is described by (Δ) we see that indeed $\widetilde{T} \in \widetilde{\mathcal{M}}''$.

The properties $\widetilde{T} \in \widetilde{\mathcal{M}}''$ and $\widetilde{P} \in \widetilde{\mathcal{M}}'$ imply that the subspace $\widetilde{\mathcal{K}} = \operatorname{ran} \widetilde{P}$ is invariant under $\widetilde{T}$ (argue as in 3.15). In particular, $\widetilde{T}\,\widetilde{x} \in \widetilde{\mathcal{K}}$, since clearly $\widetilde{x} \in \widetilde{\mathcal{K}}$. By construction, $\{\widetilde{M}\,\widetilde{x} \mid \widetilde{M} \in \widetilde{\mathcal{M}}\}$ is dense in $\widetilde{\mathcal{K}}$, thus there is some $T_0 \in \mathcal{M}$ such that

$$\varepsilon^2 > \|\widetilde{T}\,\widetilde{x} - \widetilde{T_0}\,\widetilde{x}\|^2 = \sum_{j=1}^{n} \|T x_j - T_0 x_j\|^2$$

and we conclude that $\|T x_j - T_0 x_j\| < \varepsilon$ holds for $j = 1, \ldots, n$. □

4.3. Definition: A *von Neumann algebra* (vNA) on a Hilbert space $\mathcal{H}$ is a $*$-subalgebra $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{R} = \mathcal{R}''$. The *center* of $\mathcal{R}$ is the $*$-subalgebra $\mathcal{C} := \mathcal{R} \cap \mathcal{R}'$ of $\mathcal{R}$. If the center is trivial, i.e., $\mathcal{C} = \mathbb{C} I$, then $\mathcal{R}$ is said to be a *factor*.

Alternatively, by the double commutant theorem we could have defined a von Neumann algebra as a unital $*$-subalgebra $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ that is WOT or SOT closed. Since a SOT closed set is also norm closed, every vNA is a C*-algebra. By the way, our defining requirement $\mathcal{R} = \mathcal{R}''$ does certainly imply that $I \in \mathcal{R}$.

4.4. A few elementary constructions and properties: (i) Clearly, $\mathcal{B}(\mathcal{H})$ is a SOT closed unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, hence a von Neumann algebra. It is also a factor, since obviously $\mathbb{C}I$ is a vNA and $(\mathbb{C}I)' = \mathcal{B}(\mathcal{H})$, therefore $\mathcal{B}(\mathcal{H})' = (\mathbb{C}I)'' = \mathbb{C}I$. (Of course, there is also an elementary direct proof that $\mathcal{B}(\mathcal{H})' = \mathbb{C}I$, e.g., noting that any $A \in \mathcal{B}(\mathcal{H})'$ commutes with every projection and hence leaves every closed subspace of $\mathcal{H}$ invariant.)

(ii) For any self-adjoint subset $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$, the commutant $\mathcal{M}'$ is a vNA, and $\mathcal{M}''$ is a vNA containing $\mathcal{M}$. In fact, $\mathcal{M}''$ is the smallest vNA with this property, since for any vNA $\mathcal{A}$ with $\mathcal{M} \subseteq \mathcal{A}$ we obtain $\mathcal{M}'' \subseteq \mathcal{A}'' = \mathcal{A}$.

If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is arbitrary and $\mathcal{M}^* := \{M^* \mid M \in \mathcal{M}\}$, then $(\mathcal{M} \cup \mathcal{M}^*)''$ is a vNA and is called the *von Neumann algebra generated by* $\mathcal{M}$.

(iii) The center $\mathcal{C}$ of a von Neumann algebra $\mathcal{R}$ is itself a vNA, since $\mathcal{C} = \mathcal{R} \cap \mathcal{R}'$ is certainly a unital $*$-subalgebra as well as SOT closed. The center of the von Neumann algebra $\mathcal{R}'$ is also $\mathcal{C} = \mathcal{R} \cap \mathcal{R}' = \mathcal{R}'' \cap \mathcal{R}'$, thus $\mathcal{R}$ is a factor if and only if $\mathcal{R}'$ is a factor.

(iv) We claim that for any von Neumann algebra $\mathcal{R}$,

$$(\mathcal{R} \cup \mathcal{R}')' = \mathcal{R} \cap \mathcal{R}' = \mathcal{C}.$$

The last equality is just the definition of the center and the inclusion $(\mathcal{R}\cup\mathcal{R}')' \supseteq \mathcal{R}'\cap\mathcal{R}'' = \mathcal{R}'\cap\mathcal{R}$ is clear. Since obviously $\mathcal{R} \cup \mathcal{R}' \supseteq \mathcal{R}$ we obtain $(\mathcal{R} \cup \mathcal{R}')' \subseteq \mathcal{R}'$ and, similarly, $\mathcal{R} \cup \mathcal{R}' \supseteq \mathcal{R}'$ implies $(\mathcal{R} \cup \mathcal{R}')' \subseteq \mathcal{R}'' = \mathcal{R}$, therefore $(\mathcal{R} \cup \mathcal{R}')' \subseteq \mathcal{R} \cap \mathcal{R}'$.

We may conclude from the above relation that for a *factor* $\mathcal{R}$ we obtain $(\mathcal{R} \cup \mathcal{R}')' = \mathcal{C} = \mathbb{C}I$ and therefore that the *vNA generated by* $\mathcal{R} \cup \mathcal{R}'$ *is all of* $\mathcal{B}(\mathcal{H})$, since

$$(\mathcal{R} \cup \mathcal{R}')'' = \mathcal{B}(\mathcal{H}).$$

(v) A von Neumann algebra $\mathcal{R}$ is commutative or abelian if and only if $\mathcal{R} \subseteq \mathcal{R}'$. In this case, $\mathcal{R}$ is equal to its center $\mathcal{C}$. An abelian von Neumann algebra $\mathcal{R}$ on a Hilbert space $\mathcal{H}$ is said to be *maximally abelian* if it is not contained in any other abelian vNA on $\mathcal{H}$. We claim that this condition is equivalent to

$$\mathcal{R} = \mathcal{R}'.$$

Indeed, suppose first that the above equality holds and let $\mathcal{A}$ be an abelian vNA with $\mathcal{R} \subseteq \mathcal{A}$. Then we have $\mathcal{R} \subseteq \mathcal{A} \subseteq \mathcal{A}'$, which implies $\mathcal{A} = \mathcal{A}'' \subseteq \mathcal{A}' \subseteq \mathcal{R}' = \mathcal{R}$, hence $\mathcal{R}$ is maximally abelian. Conversely, if $\mathcal{R} \subseteq \mathcal{R}'$ and $\mathcal{R}' \neq \mathcal{R}$, then we may find some self-adjoint element[1] $S \in \mathcal{R}'$ such that $S \notin \mathcal{R}$. Then we have $\mathcal{R} \subseteq (\mathcal{R} \cup \{S\})'' =: \mathcal{A}$ with $\mathcal{A} \neq \mathcal{R}$, where $\mathcal{A}$ is a vNA and commutative, since $\mathcal{R}\cup\{S\} \subseteq (\mathcal{R}\cup\{S\})'$ implies $\mathcal{A} = (\mathcal{R}\cup\{S\})'' \subseteq (\mathcal{R}\cup\{S\})' = (\mathcal{R}\cup\{S\})''' = \mathcal{A}'$. Thus $\mathcal{R}$ is not maximally abelian.

(vi) Suppose $\mathcal{R}$ is a vNA that contains some maximally abelian subalgebra $\mathcal{R}_0 = \mathcal{R}_0'$. Then $\mathcal{R}' \subseteq \mathcal{R}_0' = \mathcal{R}_0 \subseteq \mathcal{R}$ and therefore we have $\mathcal{C} = \mathcal{R}'$ for the center.

4.5. Corollary: A C*-algebra representation $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is irreducible if and only if the von Neumann algebra generated by $\pi(\mathcal{A})$ equals $\mathcal{B}(\mathcal{H})$, i.e., $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$.

[1] Take the real or imaginary part of any element in $\mathcal{R}' \setminus \mathcal{R}$.

Proof: We know that $\pi(\mathcal{A})$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$, hence the vNA generated by $\pi(\mathcal{A})$ is given by the double commutant $\pi(\mathcal{A})''$. By Proposition 3.16, irreducibility of π is equivalent to having $\pi(\mathcal{A})' = \mathbb{C}I$, and clearly the latter implies $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$. Conversely, the relation $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$ implies $\pi(\mathcal{A})' = \pi(\mathcal{A})''' = \mathcal{B}(\mathcal{H})' = \mathbb{C}I$ and thus irreducibility of π. $\square$

4.6. Example: Let (Ω, μ) be a σ-finite measure space, $\mathcal{H} = L^2(\Omega, \mu)$, and $\mathcal{A}$ denote the algebra of multiplication operators M_f with $f \in L^\infty(\Omega, \mu)$ on $\mathcal{H}$, i.e., $M_f\, g = fg$ for all $g \in L^2(\Omega, \mu)$ and $f \in L^\infty(\Omega, \mu)$. We already know that $\mathcal{A}$ is a C*-algebra and that $\|M_f\| = \|f\|_\infty$ (Example 1) in 3.2). Clearly, $\mathcal{A}$ is commutative, i.e., $\mathcal{A} \subseteq \mathcal{A}'$. We will sketch a proof (from [KRI, Example 5.1.6]) that $\mathcal{A}' \subseteq \mathcal{A}$, which then implies

$$\mathcal{A}' = \mathcal{A}$$

and hence that $\mathcal{A}$ is a maximal abelian von Neumann algebra.

Let $T \in \mathcal{A}' \subseteq \mathcal{B}(L^2(\Omega, \mu))$. Choose a sequence of pairwise disjoint measurable subsets Ω_n of Ω $(n \in \mathbb{N})$ such that $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$ and $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$. Let e_n denote the characteristic function of Ω_n and put $f_n := Te_n$.

For any $g \in L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu)$ we may calculate that

$$f_n g = g f_n = M_g\, f_n = M_g\, T e_n = T M_g\, e_n = T(g e_n) = T M_{e_n}\, g,$$

which shows boundedness of the linear map $g \mapsto f_n g$ on the dense subspace of bounded L^2-functions. Denote the unique bounded operator extension by M_{f_n}. One could now reproduce an argument from [KRI, Example 2.4.11] (alternatively, see [Con00, Proposition 1.2.4]) to show that $\operatorname{ess\,sup}|f_n| \le \|M_{f_n}\|$, which gives $f_n \in L^\infty(\Omega, \mu)$ and justifies the notation M_{f_n}. We obtain

$$\forall n \in \mathbb{N}: \quad M_{f_n} = T M_{e_n},$$

where M_{e_n} is the projection onto the subspace of L^2-functions supported in Ω_n. We define the function f on Ω by $f(\omega) := f_n(\omega)$ if $\omega \in \Omega_n$ and note that $\|f_n\|_\infty = \|M_{f_n}\| = \|TM_{e_n}\| \le \|T\|\|M_{e_n}\| \le \|T\|$ yields $\|f\|_\infty \le \|T\|$, thus $f \in L^\infty(\Omega, \mu)$. Since $\sum_{n\in\mathbb{N}} M_{e_n} g = g$ for all $g \in L^2(\Omega, \mu)$ and

$$M_f M_{e_n} = M_{f e_n} = M_{f_n} = T M_{e_n}$$

we finally obtain $M_f g = \sum_{n\in\mathbb{N}} M_f M_{e_n} g = \sum_{n\in\mathbb{N}} T M_{e_n} g = Tg$, i.e., $T = M_f \in \mathcal{A}$.

Note that the special case $\Omega = \mathbb{N}$ with μ the counting measure gives l^∞ acting as the maximal abelian vNA of diagonal or multiplication operators on l^2.

4.7. Remark (Commutative von Neumann algebras): (i) It can be shown that maximal abelian vNA serve as basic building blocks in the description of commutative von Neumann algebras ([KRII, Section 9.4 and Theorem 9.3.2]). Furthermore, any maximal abelian vNA on a *separable* Hilbert space is—modulo unitary isomorphisms of Hilbert spaces—of a form as given in the previous example (cf. [KRII, Theorem 9.4.1] or [Con00, Theorem 14.5]).

(ii) As von Neumann algebras are particular C*-algebras, one might ask what the standard Gelfand isomorphism of an abelian vNA with $C(X)$, X the compact Hausdorff space of pure states, would tell in this case. As it turns out, X is then extremely disconnected (cf. [KRI, Theorem 5.2.1]), which means that the closure of each open subset of X is open[2].

4.8. Measurable functional calculus in von Neumann algebras: (i) We indicate very briefly why a vNA possesses plenty of projections. In fact, we have ([Con00, Proposition 13.3(e)] or [KRI, Throem 5.2.2(v)]) that

a von Neumann algebra always equals the norm closure of the linear span of its projections.

Let A be a self-adjoint operator in a vNA $\mathcal{R}$ on the Hilbert space $\mathcal{H}$ and denote by $\mathcal{A}$ the abelian vNA generated by A, i.e., $\mathcal{A} = \{A\}''$.

In the first variant, consider the Gelfand isomorphism $\mathcal{A} \cong C(X)$, where we noted in the remark above that X is extremely disconnected and hence has a rich supply of so-called clopen subsets (i.e., sets that are at the same time closed and open). The characteristic functions of such sets are therefore continuous, real-valued, and idempotent, hence provide us with projection operators belonging to $\mathcal{A}$.

In the second variant, one may first argue that the standard calculus for A with bounded Borel measurable functions on $\mathrm{sp}(A)$ stays within $\mathcal{A}$, since these functions may be approximated by uniformly bounded sequences of continuous functions $f_n \in C(\mathrm{sp}(A))$ such that $f_n(A)$ converges pointwise on the Hilbert space (see, e.g., [Hoe23, Lemma 0.13 and Theorem 1.8(b)]), i.e., with respect to the strong operator topology, therefore the limit belongs to $\mathcal{A}$. In particular, a characteristic function of a Borel subset defines a projection belonging to $\mathcal{A}$.

In any case, all spectral projections of A are guaranteed to be elements in $\mathcal{A}$ as well. We obtain that A is the norm limit of linear combinations of its spectral projections, which follows, e.g., from its operator integral representation. Since any element R in $\mathcal{R}$ can be expressed as a linear combination of its self-adjoint real and imaginary parts, we conclude that R can be approximated by linear combinations of projections belonging to $\mathcal{R}$.

(ii) The measurable functional calculus can easily be extended to the case of a *normal operator* A acting on a Hilbert space $\mathcal{H}$ by considering the abelian von Neumann algebra $\mathcal{A}$ generated by A and A^*, i.e.,

$$\mathcal{A} = \{A, A^*\}''.$$

We obtain $g(A) \in \mathcal{A}$ *for every bounded Borel measurable function g on* $\mathrm{sp}(A)$ also in this case. If the Hilbert space $\mathcal{H}$ is *separable* and upon passing from Borel functions to classes of such, one can give a structurally very satisfying formulation of this functional calculus ([Con00, 15.10]): There is a Borel measure μ on $\mathrm{sp}(A)$ and a well-defined $*$-isomorphism

$$\varphi\colon L^\infty(\mathrm{sp}(A), \mu) \to \mathcal{A}$$

mapping the class of the bounded Borel function g to the operator $g(A)$. In addition, φ is also a homeomorphism when $L^\infty(\mathrm{sp}(A), \mu) = L^1(\mathrm{sp}(A), \mu)^\#$ is equipped with the weak* topology and $\mathcal{A}$ with the weak operator topology.

[2] This is even stronger than the more well-known property of a totally disconnected space, where each pair of distinct points lie in different connected components.

4.9. Ultraweak topology and normal states: Let $\mathcal{H}$ be a Hilbert space. An net $(H_j)_{j\in J}$ of self-adjoint operators on $\mathcal{H}$ is said to be *increasing* if the map from the directed set J into $\mathcal{B}(\mathcal{H})$ is monotone increasing, i.e., if $j_1, j_2 \in J$ with $j_1 \leq j_2$, then $H_{j_1} \leq H_{j_2}$ in the sense that $H_{j_2} - H_{j_1}$ is a positive operator.

Lemma: (i) If $(H_j)_{j\in J}$ is an increasing net of self-adjoint operators on $\mathcal{H}$ such that

$$\sup\{\|H_j\| \mid j \in J\} < \infty,$$

then (H_j) is SOT convergent to a self-adjoint operator H that is also the least upper bound.

(ii) Let $(P_l)_{l\in\Lambda}$ be a family of pairwise orthogonal projections on $\mathcal{H}$. Then (P_l) is a SOT summable family, i.e, $\sum_{l\in\Lambda} P_l$ is SOT convergent.

Proof: (i): Upon fixing an arbitrary index $j_0 \in J$ and considering $H_j - H_{j_0}$ for $j \geq j_0$ we may suppose, without loss of generality, that every H_j is positive. Put $\gamma := \sup\{\|H_j\| \mid j \in J\}$.

For every $x \in \mathcal{H}$, the net $(\langle x|H_j x\rangle)_{j\in J}$ is increasing and bounded by $\gamma\|x\|^2$, hence convergent. By the polarization identity for complex inner products, we deduce that $\beta(x, y) := \lim\langle x|H_j y\rangle$ exists for all $x, y \in \mathcal{H}$, $(x, y) \mapsto \beta(x, y)$ is sesquilinear, and $|\beta(x, y)| \leq \gamma\|x\|\|y\|$. Thus, there is a bounded operator H on $\mathcal{H}$ such that $\langle x|Hy\rangle = \lim\langle x|H_j y\rangle$ for all $x, y \in \mathcal{H}$.

We immediately obtain that H is positive, $H_j \leq H$ for all $j \in J$, $\|H\| \leq \gamma$, and that $H_j \to H$ with respect to the WOT. If K is a self-adjoint operator such that $H_j \leq K$ for all $j \in J$, then $\langle x|H_j x\rangle \leq \langle x|Kx\rangle$ for all $x \in \mathcal{H}$, hence $H \leq K$ and H is the supremum.

To show that also SOT convergence $H_j \to H$ holds, note first that $f(t) = \sqrt{t}$ is bounded on $\mathrm{sp}(H - H_j) \subseteq [0, \|H\|]$ by $\sqrt{\|H\|}$ and then consider

$$\begin{aligned}\|(H - H_j)x\|^2 &= \|(H - H_j)^{1/2}(H - H_j)^{1/2}x\|^2 \leq \|(H - H_j)^{1/2}\|^2\|(H - H_j)^{1/2}x\|^2\\ &\leq \|H\|\langle(H - H_j)^{1/2}x|(H - H_j)^{1/2}x\rangle = \|H\|\langle x|(H - H_j)x\rangle \to 0.\end{aligned}$$

(ii): For finite subsets $\mathcal{F}$ and $\mathcal{G}$ of Λ with $\mathcal{F} \subseteq \mathcal{G}$, we have $0 \leq \sum_{l\in\mathcal{F}} P_l \leq \sum_{l\in\mathcal{G}} P_l \leq I$. We obtain the uniformly bounded increasing net of positive operators $(\sum_{l\in\mathcal{F}} P_l)_{\mathcal{F}}$, where $\mathcal{F}$ ranges over the finite subsets of Λ with the inclusion relation as partial order. The assertion follows from part (i). $\square$

Since a von Neumann algebra $\mathcal{R}$ is always SOT closed, we see that in the above lemma, the supremum H and the sum $\sum P_l$ exist in $\mathcal{R}$, if the H_j and P_l belong to $\mathcal{R}$.

Definition: (i) A positive linear functional ρ on a von Neumann algebra $\mathcal{R}$ is called *normal*, if $\rho(H_j) \to \rho(H)$ for any increasing net (H_j) of self-adjoint operators in $\mathcal{R}$ with supremum H.

(ii) The *ultraweak topology* on $\mathcal{B}(\mathcal{H})$ is the locally convex Hausdorff topology defined by all seminorms of the form

$$T \mapsto \Big|\sum_{k\in\mathbb{N}}\langle x_k|Ty_k\rangle\Big|,$$

where $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ are sequences in $\mathcal{H}$ with $\sum_{k\in\mathbb{N}}\|x_k\|^2 < \infty$ and $\sum_{k\in\mathbb{N}}\|y_k\|^2 < \infty$.

Obviously, the ultraweak topology is finer than the weak operator topology and coarser than the norm topology. It is certainly inherited on any vNA acting on $\mathcal{H}$, in particular, we may study ultraweak continuity of linear functionals on any vNA.

The following theorem provides an extensive list of properties characterizing normal positive linear functionals or states and hints at their relevance also for quantum physics, where an operator C as in property (vi) is usually called a *density matrix*, if its trace is normalized (cf. [Thi10, Part II, 2.1.2], [BR1, Theorem 2.4.21], [Ara99, Theorem 2.7]). Recall that an operator $B \in \mathcal{B}(\mathcal{H})$ is of *trace class* if there is a complete orthonormal system $\mathcal{S}$ in $\mathcal{H}$ such that $\sum_{e\in\mathcal{S}}\langle e||B|e\rangle$ is finite, where $|B| := (B^*B)^{1/2}$ is the absolute value of B as in 2.11. In this case, the corresponding sum is finite for every choice of complete orthonormal system and its value independent of the choice ([Con00, Section 18]). The *trace* of B is then defined by

$$\operatorname{trace}(B) := \sum_{e\in\mathcal{S}}\langle e|Be\rangle.$$

The set $\mathcal{B}_1(\mathcal{H})$ of trace class operators on $\mathcal{H}$ is an ideal in $\mathcal{B}(\mathcal{H})$ and we obtain a norm by

$$\|B\|_1 := \operatorname{trace}(|B|) \quad (B \in \mathcal{B}_1(\mathcal{H})).$$

We will not give a proof of the theorem here and instead refer to a large reservoir of references: [Con00, Theorem 46.6], [Dix81, Sections 3.3 and 4.2], [BR1, Proposition 2.4.6, Theorem 2.4.21], [TakI, Chapter II, Theorem 2.6], [Bla10, Subsection III.2.1], [KRII, Theorem 7.1.9, Remark 7.1.10, Theorem 7.1.11, Proposition 7.4.5], and [Mur90, Theorem 4.2.10]. We note however that (i) $\Rightarrow$ (ii) follows directly from the shown above lemma and that (iii) $\Rightarrow$ (iv) is immediate, since the SOT is finer than the WOT.

Theorem Let ρ be a positive linear functional on the von Neumann algebra $\mathcal{R}$, then the following statements are equivalent:

(i) ρ is normal,

(ii) ρ is *completely additive*, i.e., $\rho(\sum_{l\in\Lambda} P_l) = \sum_{l\in\Lambda} \rho(P_l)$ for any pairwise orthogonal family of projections $(P_l)_{l\in\Lambda}$ in $\mathcal{R}$,

(iii) ρ is WOT continuous on the unit ball of $\mathcal{R}$,

(iv) ρ is SOT continuous on the unit ball of $\mathcal{R}$,

(v) ρ is ultraweakly continuous,

(vi) there is a positive trace class operator C on $\mathcal{H}$ such that $\rho(A) = \operatorname{trace}(AC)$ for all $A \in \mathcal{R}$.

Remark: A normal state ρ on a von Neumann algebra $\mathcal{R}$ has one further convenient property in terms of its associated GNS representation $\pi_\rho \colon \mathcal{R} \to \mathcal{B}(\mathcal{H}_\rho)$, because it turns out that in this case, the image $\pi_\rho(\mathcal{R})$ is automatically SOT closed in$\mathcal{B}(\mathcal{H}_\rho)$, thus a vNA on $\mathcal{H}_\rho$.

4.10. $\mathcal{B}(\mathcal{H})$ as a dual space: The special case $\mathcal{R} = \mathcal{B}(\mathcal{H})$ in the previous theorem suggests that it might be interesting to study the dual pairing of $\mathcal{B}(\mathcal{H})$ with the trace class operators $\mathcal{B}_1(\mathcal{H})$, defined via the bilinear form $\beta \colon \mathcal{B}_1(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ with

$$\beta(B, A) := \operatorname{trace}(AB) \quad (B \in \mathcal{B}_1(\mathcal{H}), A \in \mathcal{B}(\mathcal{H})).$$

As it turns out, this produces a very good analogy with the classical ([Con10, Theorem 5.6 and Example 5.9]) Banach space dualities $(l^1)^\# \cong l^\infty$ or $L^1(\Omega, \mu)^\# \cong L^\infty(\Omega, \mu)$ for any σ-finite measure space (Ω, μ). Namely, it can be shown (cf. [Con00, Theorem 19.2] or [Mur90, Theorem 4.2.3]) that the linear map $A \mapsto \beta(., A)$ is an isometric isomorphism between $\mathcal{B}(\mathcal{H})$ and the dual $\mathcal{B}_1(\mathcal{H})^\#$ with respect to the trace norm $\|.\|_1$ on $\mathcal{B}_1(\mathcal{H})$. We note in passing that quite similarly and in some analogy with the classical sequence duality $c_0^\# \cong l^1$, one also obtains an interesting isomorphism when β is restricted in its second slot to the compact operators $\mathcal{C}(\mathcal{H})$, thus considering $B \mapsto \beta(B, .)|_{\mathcal{C}(\mathcal{H})}$ produces an isometric isomorphism $\mathcal{B}_1(\mathcal{H}) \to \mathcal{C}(\mathcal{H})^\#$ (cf. [Con00, Theorem 19.1], [Mur90, Theorem 4.2.1], [BR1, Proposition 2.4.3]). In summary,

$$\mathcal{B}(\mathcal{H}) \cong \mathcal{B}_1(\mathcal{H})^\# \quad \text{and} \quad \mathcal{B}_1(\mathcal{H}) \cong \mathcal{C}(\mathcal{H})^\#$$

and from the first isometry, in view of the previous theorem, one might suspect that the weak* topology on $\mathcal{B}(\mathcal{H})$ from its duality with $\mathcal{B}_1(\mathcal{H})$ coincides with the ultraweak topology. This is correct and backed-up, e.g., by [Con00, Proposition 20.2] (and upon sorting out a possible confusion, because the ultraweak topology is called weak* topology in that book; see also [Dix81, Section 3.3], [Mur90, Section 4.2], [BR1, Proposition 2.4.3], [Sun87, Section 0.3]). One immediate consequence in combination with the Banach-Alaoglu theorem is that the closed unit ball $\mathcal{K}_1$ in $\mathcal{B}(\mathcal{H})$ is ultraweakly compact. Since the identity map is continuous from $\mathcal{K}_1$ with the ultraweak topology to $\mathcal{K}_1$ equipped with the weak operator topology, we obtain compactness of both and even a homeomorphism between compact Hausdorff spaces. We learn that both topologies coincide on $\mathcal{K}_1$ and, in particular, that *the closed unit ball of* $\mathcal{B}(\mathcal{H})$ *is WOT compact.*

Some aspects of the observations about duality for $\mathcal{B}(\mathcal{H})$ can be transferred to the general case of a vNA. Moreover, this even gives an opportunity for a more abstract characterization of vNA without recourse to a pre-defined action as operators on some Hilbert space.

4.11. The predual of a von Neumann algebra and W*-algebras:

(i) Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. We collect here a few arguments showing that $\mathcal{R}$ *is the dual space of a Banach space.*

As reported in 4.10, we have the isometric isomorphism $\mathcal{B}(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})^\#$ mapping $A \in \mathcal{B}(\mathcal{H})$ to the functional $C \mapsto \operatorname{trace}(AC)$ $(C \in \mathcal{B}_1(\mathcal{H}))$. In this sense, we may consider $\mathcal{R}$ as a subspace of $\mathcal{B}_1(\mathcal{H})^\#$ and the annihilator of $\mathcal{R}$ is then given by

$$\mathcal{R}_\perp = \{C \in \mathcal{B}_1(\mathcal{H}) \mid \forall A \in \mathcal{R}\colon \operatorname{trace}(AC) = 0\}.$$

This is a closed subspace of the Banach space $\mathcal{B}_1(\mathcal{H})$ and we may construct the quotient Banach space $\mathcal{B}_1(\mathcal{H})/\mathcal{R}_\perp$. Recall that for any closed subspace $\mathcal{Z}$ of a Banach space $\mathcal{X}$, we have the isometric isomorphism $(\mathcal{X}/\mathcal{Z})^\# \cong \mathcal{Z}^\perp$, where $\mathcal{Z}^\perp = \{\mu \in \mathcal{X}^\# \mid \mu(z) = 0 \text{ for all } z \in \mathcal{Z}\}$ ([Con10, Chapter III, Theorem 10.2]). Applying this to our situation with $\mathcal{X} = \mathcal{B}_1(\mathcal{H})$ and $\mathcal{Z} = \mathcal{R}_\perp$ yields

$$(\mathcal{B}_1(\mathcal{H})/\mathcal{R}_\perp)^\# \cong (\mathcal{R}_\perp)^\perp = \mathcal{R},$$

where the last equality follows from the bipolar theorem ([Con10, Chapter V, Theorem 1.8] or [Hoe23, 6.4]) in combination with these facts: On $\mathcal{B}(\mathcal{H})$ the weak* topology coincides with the ultraweak topology and the obviously convex set $\mathcal{R}$ is WOT as well as norm closed, hence ultraweakly closed since this topology lies between the WOT and the norm topology.

(ii) We have seen in (i) that a von Neumann algebra always has a *predual* as a Banach space, i.e., it is the dual space of some Banach space. In general—apart from the case of reflexive Banach spaces (and, as noted in [Sak71, Concluding remarks on 1.13], a reflexive vNA is necessarily finite-dimensional)—a predual need not be unique as the classic example $c_0^\# \cong l^1 \cong c^\#$ illustrates. However, it can been shown ([Bla10, Subsection III.2.4], [Sak71, Corollary 1.13.3], [TakI, Chapter III, Corollary 3.9]) that *for a von Neumann algebra the predual is unique up to isometric isomorphism.*

A somewhat concrete realization of the predual of a von Neumann algebra $\mathcal{R}$ can be given (cf. [KRII, Definition 7.4.1, Theorem 7.4.2, Proposition 7.4.5] or [Mur90, Theorem 4.2.9]) in terms of the set

$$\mathcal{R}_\# := \{\mu \colon \mathcal{R} \to \mathbb{C} \mid \mu \text{ is linear and ultraweakly continuous}\}.$$

In view of the characterization of ultraweakly positive linear functionals outlined in 4.9, the elements in $\mathcal{R}_\#$ are often called *normal linear functionals.* We have naturally that $\mathcal{R}_\# \subseteq \mathcal{R}^\#$, since the norm topology is finer than the ultraweak topology, but the non-obvious result is then that

$$(\mathcal{R}_\#)^\# \cong \mathcal{R}.$$

By uniqueness of the predual we also have $\mathcal{R}_\# \cong \mathcal{B}_1(\mathcal{H})/\mathcal{R}_\perp$. (An independent argument for this last fact can be sketched by the following basic construction: Consider the linear map $\tilde{\theta} \colon \mathcal{B}_1(\mathcal{H}) \to \mathcal{R}_\#$, where $\tilde{\theta}(B)(A) := \operatorname{trace}(AB)$ for all $A \in \mathcal{R}$ and we note that, thanks to a suitable variant of (v) $\Leftrightarrow$ (vi) in Theorem 4.9 for general linear functionals (see, e.g., [Con00, Corollary 54.10] or [Ped18, Theorem 3.6.4] or [TakI, Chapter II, Theorem 2.6]), we do have $\tilde{\theta}(B) \in \mathcal{R}_\#$. The map $\tilde{\theta}$ is seen to be surjective, if we again appeal to the appropriate variant of (v) $\Leftrightarrow$ (vi) in Theorem 4.9, because given any $\mu \in \mathcal{R}_\#$ we then find some $B \in \mathcal{B}_1(\mathcal{H})$ such that $\mu(A) = \operatorname{trace}(AB)$ $(A \in \mathcal{R})$, i.e., $\mu = \tilde{\theta}(B)$. Furthermore, we clearly have $\|\tilde{\theta}(B)(A)| = |\operatorname{trace}(AB)| \leq \|B\|_1 \|A\|$, hence $\tilde{\theta}$ is bounded, in fact it does not increase norm since $\|\tilde{\theta}(B)\| \leq \|B\|_1$. Finally, the kernel of $\tilde{\theta}$ obviously is exactly $\mathcal{R}_\perp$. Therefore, $\tilde{\theta}$ factors to a bijective bounded linear map

$$\theta \colon \mathcal{B}_1(\mathcal{H})/\mathcal{R}_\perp \to \mathcal{R}_\#$$

with $\|\theta\| = \|\tilde{\theta}\| \leq 1$ ([Theorem 1.5.8][KRI]), hence $\|\theta(B + \mathcal{R}_\perp)\| \leq \|B + \mathcal{R}_\perp\|$ for all $B \in \mathcal{B}_1(\mathcal{H})$. It would remain to show that θ is an isometry; I admit that I could not find an elementary proof for this, but it is concluded by other means in the first part of the proof of Theorem 7.4.2 in [KRII].)

(iii) The observations in (i) and (ii) can be used as a basis for an abstract definition of von Neumann algebras without referring to an action as operators on a Hilbert space at the outset. For example, the book [Sak71] develops the theory consequently from this point of view and starts with the definition of a W*-*algebra* as a C*-algebra that is the dual of some Banach space. Later it is shown (cf. [Sak71, Theorem 1.16.7]) that every W*-algebra is $*$-isomorphic to a vNA on some Hilbert space, where the $*$-isomorphism is also continuous with respect to the weak* topologies. Accounts of the relationship between W*-algebras and von Neumann algebras can be found, e.g., in these references: [KRII, KRIV, Exercises 7.6.38, 7.6.41, 7.6.45] [TakI, Chapter III, Section 3], [Ped18, Section 3.9], [Bla10, Subsection III.2.4]. The notion of a W*-algebra is also employed in textbooks that are oriented towards mathematical physics (cf. [Ara99, BR1, BSZ92, Haa96, Thi10]). In these cases, the emphasis occasionally is on an equivalent definition in terms of a C*-algebra that (a) always possesses suprema of norm bounded increasing nets and (b) has a sufficiently rich set of normal states (defined as states that respect suprema of increasing nets).

We review finally a few very brief aspects regarding the types of von Neumann algebras.

4.12. Equivalence of projections and types of factors:

(i) Let $\mathcal{R}$ be a von Neumann algebra on the Hilbert space $\mathcal{H}$. We have noted earlier that a vNA possesses sufficiently many projections so that it is even generated as the norm closure of their linear span in $\mathcal{B}(\mathcal{H})$. We say that two projections $P, Q \in \mathcal{R}$ are *equivalent* (relative $\mathcal{R}$), in notation $P \sim Q$, if there is some $V \in \mathcal{R}$ such that

$$V^*V = P \quad \text{and} \quad VV^* = Q.$$

It follows then automatically ([KRII, Proposition 6.1.1]) that V is a *partial isometry* on $\mathcal{H}$, which means that there is some closed subspace $\mathcal{K}$ of $\mathcal{H}$ such that $\ker V = \mathcal{K}^\perp$ and V maps $\mathcal{K}$ isometrically onto a closed subspace.

In the specific case $\mathcal{R} = \mathcal{B}(\mathcal{H})$ the equivalence $P \sim Q$ corresponds to $\dim \operatorname{ran} P = \dim \operatorname{ran} Q$ (in the sense of cardinality of complete orthonormal systems). In the example of an abelian von Neumann algebra, equivalence always implies equality since $P = V^*V = VV^* = Q$.

It is easy to check that we obtain an equivalence relation on the set

$$\operatorname{Proj}(\mathcal{R}) := \{P \in \mathcal{R} \mid P^* = P, P^2 = P\}$$

of all orthogonal projections on $\mathcal{H}$ that belong to $\mathcal{R}$ (cf. [KRII, Proposition 6.1.5]). Recall that $0 \leq P \leq I$ for any projection P and we have a partial order on the projections, where $P \leq Q$ corresponds to the fact that $\operatorname{ran} P \subseteq \operatorname{ran} Q$. We can make this partial order compatible with the equivalence relation introduced above (cf. [KRII, Propositions 6.2.4 and 6.2.5]) by defining

$$P \precsim Q, \quad \text{if there is some projection } E \leq Q \text{ such that } P \sim E.$$

This partial order turns out to be a total order in case the vNA that is a *factor* (cf. [KRII, Proposition 6.2.6]): If $\mathcal{R}$ is a factor, so that $\mathcal{C} = \mathcal{R} \cap \mathcal{R}' = \mathbb{C}I$ for the center, then any two projections P and Q in $\mathcal{R}$ are comparable, i.e., we have either $P \precsim Q$ or $Q \precsim P$.

A projection P in the von Neumann algebra $\mathcal{R}$ is called *infinite* if it has a subprojection $E \leq P$, $E \in \mathcal{R}$, such that $E \neq P$ and $P \sim E$. Otherwise P is called *finite*.

For example, in $\mathcal{R} = \mathcal{B}(\mathcal{H})$ a projection P is finite if and only if $\dim \operatorname{ran} P < \infty$. As a second example, consider an abelian von Neumann algebra $\mathcal{A} \subseteq \mathcal{A}'$, then every projection is finite.

A von Neumann algebra $\mathcal{R}$ is called *finite*, if I is finite, otherwise $\mathcal{R}$ is said to be *infinite*.

A projection $P \neq 0$ in a von Neumann algebra $\mathcal{R}$ is *minimal*, if it has no subprojection in $\mathcal{R}$ other than 0. A minimal projection is finite, since any projection equivalent to 0 must be zero ($V^*V = 0$ implies $V = 0$ and hence also $VV^* = 0$).

The minimal projections in $\mathcal{R} = \mathcal{B}(\mathcal{H})$ are those with one-dimensional range.

(ii) A von Neumann algebra can be decomposed into a direct sum of vNAs of particular types by way of mutually orthogonal projections in the center that sum up to I ([KRII, Theorem 6.5.2], [Con00, Theorem 48.16], [Bla10, III.1.4.7], [TakI, Chapter V, Theorem 1.19]). Moreover, at least *on separable Hilbert spaces*, any vNA can be expressed in its so-called *central decomposition* as a direct integral of factors ([KRII, Corollary 14.2.3], [Bla10, III.1.6], [TakI, Chpater IV, Theorem 8.21]) and a factor vNA can be shown to be always of one specific type only ([KRII, Corollary 6.5.3], [Bla10, III.1.4.7], [TakI, Chpater V, Corollary 1.20]). These facts serve as our excuse to mention here at the very end of this quick glimpse at von Neumann algebras only a rough distinction into types for factors (following [Sun87, Chapter I]).

Definition: A factor $\mathcal{R}$ is said to be of

type I, if there exist minimal projections in $\mathcal{R}$,

type II, if $\mathcal{R}$ has no minimal projection, but there exist non-zero finite projections,

type III, if there are no non-zero finite projections in $\mathcal{R}$.

If we consider the simple example $\mathcal{R} = \mathcal{B}(\mathcal{H})$, then the equivalence classes of projections are characterized by the dimensions of the ranges and the minimal projections are one-dimensional. Thus we may define a function $D\colon \mathrm{Proj}(\mathcal{B}(\mathcal{H})) \to [0, \infty]$ by setting $D(P) := \dim \operatorname{ran} P = \operatorname{trace}(P)$ if $\operatorname{ran} P$ is finite-dimensional and $D(P) := \infty$ if $\operatorname{ran} P$ is not finite-dimensional. Note that we obtain $D(P) = D(Q)$ if and only if $P \sim Q$, $D(P+Q) = D(P)+D(Q)$ if $\operatorname{ran} P \perp \operatorname{ran} Q$, and $D(P) < \infty$ if and only if P is finite. It can be shown that such a function can be defined on any factor and it will turn out to be very useful in describing the type of a factor. For a proof of the following theorem we refer to [Sun87, Theorem 1.3.1].

Theorem: Let $\mathcal{R}$ be a factor, then there exists a *dimension function* $D\colon \mathrm{Proj}(\mathcal{R}) \to [0, \infty]$ satisfying the following properties for any $P, Q \in \mathrm{Proj}(\mathcal{R})$:

(a) $D(P) = D(Q) \Leftrightarrow P \sim Q$,

(b) $\operatorname{ran} P \perp \operatorname{ran} Q \Rightarrow D(P + Q) = D(P) + D(Q)$,

(c) $D(P) < \infty \Leftrightarrow P$ is finite.

A dimension function is unique up to positive constant multiple.

It is easy to see that $D(P) \leq D(Q)$ holds if $P \precsim Q$, and less trivial to see that D respects sums of sequences of pairwise orthogonal projections P_k $(k \in \mathbb{N})$ in the sense that $D(\sum P_k) = \sum D(P_k)$. If $\mathcal{R}$ has minimal projections, then these are all equivalent and we may normalize D to give value 1 on these. Depending on whether $D(I) < \infty$ or $D(I) = \infty$ we obtain that in this case, which corresponds to type I, $D(\mathrm{Proj}(\mathcal{R}))$ is either a finite set $\{0, 1, \ldots, n\}$ or $\mathbb{N}_0 \cup \{\infty\}$. For type II it turns out that the range of values of D is continuous and, depending again on the value $D(I)$, can normalized to give $[0, 1]$ or $[0, \infty]$, while in case of type III the only choice is $\{0, \infty\}$. In this way, we obtain a characterization and further refinement of types of factors as follows ([Sun87, Proposition 1.3.14]):

Type I_n with $D(\mathrm{Proj}(\mathcal{R})) = \{0, 1, \ldots, n\}$,

type I_∞ with $D(\mathrm{Proj}(\mathcal{R})) = \mathbb{N}_0 \cup \{\infty\}$,

type II_1 with $D(\mathrm{Proj}(\mathcal{R})) = [0, 1]$,

type II_∞ with $D(\mathrm{Proj}(\mathcal{R})) = [0, \infty]$,

type III with $D(\mathrm{Proj}(\mathcal{R})) = \{0, \infty\}$.

The prototypical examples of type I_n ($n \in \mathbb{N}$) are $M(n, \mathbb{C})$ and $\mathcal{B}(\mathcal{H})$ with infinite-dimensional $\mathcal{H}$ for type I_∞. In fact, if $\mathcal{R}$ is a factor of type I_n ($n \in \mathbb{N}$ or $n = \infty$), then $\mathcal{R}$ is $*$-isomorphic to $\mathcal{B}(\mathcal{H})$ with $n = \dim \mathcal{H}$ (cf. [KRII, Theorem 6.6.1]).

Examples of factors of all subtypes do exist (see [Sun87, Section 4.3]) and there is still a finer classification of the type III factors, but this requires more advanced and very different methods (see, e.g., [Sun87, Chapters 2 and 3] for a quick introduction).

5. Canonical commutation relations and the Weyl C*-algebra

◇ Our main sources for this chapter are [Petz90, Thi10, BR2, Mor17, Mor19, DG13, Ott95, Fol08, BLOT].

5.1. The canonical commutation relations: The fundamental observables in elementary quantum mechanics are those corresponding to position and momentum coordinates. These are operators $Q_1, \ldots, Q_N$ and $P_1, \ldots, P_N$ with the *rapidly decreasing Schwartz functions* $\mathscr{S}(\mathbb{R}^N) = \{f \in C^\infty(\mathbb{R}^N) \mid \forall \alpha \in \mathbb{N}_0^N \, \forall m \in \mathbb{N}_0 \colon x \mapsto (1+|x|^2)^m \partial^\alpha f(x) \text{ is bounded on } \mathbb{R}^N\}$ as a common dense and invariant domain in $L^2(\mathbb{R}^N)$. Every Q_j and P_k is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^N)$ ([Mor17, Propositions 5.21, 5.23, 5.29]), hence has a unique self-adjoint extension given by its closure, and the action on any $f \in \mathscr{S}(\mathbb{R}^N)$ is given by

$$(Q_j f)(x) = x_j f(x), \quad (P_k f)(x) = -i\, \partial_k f(x).$$

Depending on the nature of the application, we usually have $N = mn$, where n is the (relevant) spatial dimension (1, 2, or 3) and m is the number of particles involved, and the coordinates of x are relabeled so that $x_{(j-1)n+1}, \ldots, x_{jn}$ refer to particle number j ($1 \leq j \leq m$). Note that we have also followed a widespread habit in mathematical physics by assuming (upon rescaling physical units) that $\hbar = 1$, otherwise the momentum operator action would read $(P_k f)(x) = -i\hbar\, \partial_k f(x)$. Since every Q_j and P_k leaves $\mathscr{S}(\mathbb{R}^N)$ invariant, compositions and hence also commutators $[A, B] := AB - BA$, where A and B are any of these operators, are defined on $\mathscr{S}(\mathbb{R}^N)$. An elementary calculation then gives the *canonical commutation relations*

$$\text{(CCR)} \qquad [Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \quad \text{and} \quad [Q_j, P_k] = i\delta_{jk} I \qquad (1 \leq j, k \leq N).$$

These commutator relations may be summarized conveniently as follows: For any $a, b \in \mathbb{R}^N$ let us write $a \cdot Q := \sum_{j=1}^N a_j Q_j$ and $b \cdot P := \sum_{k=1}^N b_k P_k$, then from bilinearity and antisymmetry of the commutator we have that

$$\forall a, b, c, d \in \mathbb{R}^N\colon \quad [a \cdot Q + b \cdot P, c \cdot Q + d \cdot P] = i \sum_{j=1}^N (a_j d_j - b_j c_j) I = i\big(\langle a|d\rangle - \langle b|c\rangle\big) I.$$

(We (ab)used here the notation $\langle .|. \rangle$ also for the standard euclidean inner product on $\mathbb{R}^N$.) Note that the assigment $((a,b),(c,d)) \mapsto \langle a|d\rangle - \langle b|c\rangle$ defines a symplectic (i.e., antisymmetric, non-degenerate bilinear) form β on $\mathbb{R}^N \times \mathbb{R}^N$ and we may thus abbreviate the above relations further upon introducing the operators $R(a,b) := a \cdot Q + b \cdot P$, which are also essentially self-adjoint on the common domain $\mathscr{S}(\mathbb{R}^N)$, and the $2N$-dimensional symplectic vector space $V := \mathbb{R}^N \times \mathbb{R}^N$ with symplectic form β to obtain

$$\forall u, v \in V\colon \quad [R(u), R(v)] = i\beta(u,v) I.$$

In high energy particle physics or in quantum statistical mechanics the number of particles or of degrees of freedom is not fixed or bounded throughout the processes of interest. This suggests to generalize the above relations to the following situation with potentially infinitely many degrees of freedom, i.e., a possibly infinite dimensional parameter space V:

(a) Let $V \neq \{0\}$ be a real vector space with a bilinear form β that is antisymmetric, $\beta(v, u) = -\beta(u, v)$, and non-degenerate, $\beta(u, v) = 0$ for every $v \in V$ implies that $u = 0$. We call (V, β) a *symplectic vector space*. A quick example is any complex pre-Hilbert space $\mathcal{E}$, considered as a real vector space, with the symplectic form given by $\beta(f, g) := \operatorname{Im}\langle f|g\rangle$ $(f, g \in \mathcal{E})$.

(b) Represent the CCR (canonical commutation relations) on some Hilbert space $\mathcal{H}$ by means of a family $R(u)$ $(u \in V)$ of essentially self-adjoint operators with a common dense and invariant domain $\mathscr{D} \subseteq \mathcal{H}$ such that

$$[R(u), R(v)] = i\beta(u, v)I \quad (u, v \in V).$$

Remark: Recall that a commutation relation of the form $[Q, P] = iI$ cannot be implemented with bounded operators Q and P. Indeed, one easily derives inductively that $Q^{n+1}P - PQ^{n+1} = i(n+1)Q^n$ for every $n \in \mathbb{N}$; supposing that Q and P are bounded and self-adjoint we may apply the C*-property of the operator norm and deduce $\|Q^{2^m}\| = \|Q\|^{2^m}$ as in the proof of Proposition 1.7, (ii); therefore, we have with $n = 2^m$,

$$\begin{aligned}(n+1)\|Q\|^n = (n+1)\|Q^n\| = \|Q^{n+1}P - PQ^{n+1}\| &\leq 2\|P\|\|Q^{n+1}\| \\ &\leq 2\|P\|\|Q\|\|Q^n\| = 2\|P\|\|Q\|\|Q\|^n;\end{aligned}$$

since certainly $\|Q\| \neq 0$ (otherwise $[Q, P] = 0$), we obtain $\|P\|\|Q\| \geq (2^m + 1)/2$ for all $m \in \mathbb{N}$, a contradiction.

5.2. Algebraic formulation of quantum theories: If we are interested mainly in structural and foundational aspects, then the details of concrete Schrödinger operator models with issues about domains and (essential) self-adjointness for certain atomic or molecular potentials should be secondary. In fact, an unbounded self-adjoint operator on a Hilbert space is, thanks to spectral theory, completely determined by the commutative C^*-algebra generated by the bounded Borel measurable functions applied to that operator (all of these give bounded normal operators). It suffices even to consider just the spectral measure that is uniquely associated with the self-adjoint operator.

In this context, it is also worth while to recall the direct correspondence between self-adjoint operators and unitary groups: If A is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then its corresponding unitary group $U(t) := \exp(itA)$ $(t \in \mathbb{R})$ is easily given by functional calculus. Indeed, with the family of continuous bounded functions $e_t\colon \mathbb{R} \to \mathbb{C}$, $e_t(r) := e^{itr}$ $(t \in \mathbb{R})$, we put $U(t) := e_t(A)$ and obtain $\frac{d}{dt}U(t)x|_{t=0} := \lim_{t\to 0}(U(t)x - x)/t = iAx$ for all x in the domain of A, which is obvious in the multiplication operator version of the spectral theorem. Moreover, by a famous theorem due to Stone (cf. [Mor17, Theorem 9.33]), any *strongly continuous unitary group* $U(t)$ $(t \in \mathbb{R})$ on $\mathcal{H}$, i.e., where we have for all $t_0, t_1, t_2 \in \mathbb{R}$,

$$(5.1) \quad U(0) = I, \quad U(t_1 + t_2) = U(t_1)U(t_2), \quad \text{and} \quad \lim_{t \to t_0} U(t)x = U(t_0)x \quad \text{for all } x \in \mathcal{H},$$

is generated from a unique self-adjoint operator as described above. Thanks to the group property, it suffices to require $\lim_{t\to 0} U(t)x = x$ in the last part of (5.1).

We may thus have sketched a little bit of the background motivation for the following basic notion of a quantum theory in the C*-algebraic setting (cf. [Haa96, page 5], [Thi10, Part I, 2.2.32], [Mor19, Subsection 8.2.1], [Mor17, Subection 14.1.1], [BLOT, Section 6.1]):

The observables of a quantum theory are described by self-adjoint elements of a C^**-algebra* $\mathcal{A}$*. The (physical) states of the quantum system are given by the states on* $\mathcal{A}$ *in the sense of Definition 1.10. The possible values of measurements of the observable* $A \in \mathcal{A}$ *in the state* $\rho \in \mathcal{S}(\mathcal{A})$ *are contained in* $\mathrm{sp}(A)$*, with probability distribution corresponding to the measure induced on* $\mathrm{sp}(A)$ *by the pullback of* $\rho|_{\mathrm{C}^*(A)}$ *under the functional calculus* $*$*-isomorphism* $C(\mathrm{sp}(A)) \to \mathrm{C}^*(A)$ *according to Theorem 2.6.*

Let us make the final part about the probability measure associated with an observable A and a state ρ a bit more concrete: Denote the $*$-isomorphism providing the functional calculus by $\varphi\colon C(\mathrm{sp}(A)) \to \mathrm{C}^*(A)$ and $\rho_0 := \rho|_{\mathrm{C}^*(A)} \in \mathrm{C}^*(A)^\#$. Then $\rho_0 \circ \varphi$ belongs to $C(\mathrm{sp}(A))^\#$ and is a state, hence the Riesz representation theorem tells that it is given by a unique Borel probability measure μ on $\mathrm{sp}(A)$ in the form $f \mapsto \int f\, d\mu$. Writing $f(A)$ in place of $\varphi(f)$ for $f \in C(\mathrm{sp}(A))$ we therefore have

$$\rho(f(A)) = \rho_0(f(A)) = (\rho_0 \circ \varphi)(f) = \int_{\mathrm{sp}(A)} f\, d\mu,$$

in particular,

$$\rho(A) = \int_{\mathrm{sp}(A)} r\, d\mu(r).$$

Example or remark: A classical system on a compact Hausdorff configuration space X is modeled by the commutative C*-algebra $\mathcal{A} = C(X)$. A state on $C(X)$ corresponds to a regular Borel probability measure ρ on X and an observable is a real-valued continuous function $h\colon X \to \mathbb{R}$. We have $\mathrm{sp}(h) = h(X)$ and

$$\rho(h) = \int_X h\, d\rho = \int_{h(X)} r\, d\mu(r),$$

where μ is the image measure on $h(X)$ of ρ under h, occasionally denoted by $\mu = h(\rho)$ and meaning that $\mu(Z) = \rho(h^{-1}(Z))$ for every Borel subset Z of $h(X)$.

5.3. A little detour on uncertainty relations: Let A be a self-adjoint element, i.e., an observable, in the C*-algebra $\mathcal{A}$ and ρ be a state on $\mathcal{A}$. From the Cauchy-Schwarz inequality, we have $\rho(A)^2 = |\rho(A)|^2 = |\rho(I^*A)|^2 \le \rho(I^*I)\rho(A^*A) = \rho(A^2)$ and therefore,

$$\rho(A^2) - \rho(A)^2 \ge 0.$$

We may thus define the *mean-square deviation* by

$$\Delta_\rho(A) := \sqrt{\rho(A^2) - \rho(A)^2}.$$

If $\mathcal{A}$ is commutative and ρ is pure, i.e., multiplicative, then ρ is nondispersive for all observables $A \in \mathcal{A}$, since then $\Delta_\rho(A) = 0$. Another example with vanishing mean-square deviation is the case of a self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ and a vector state $\rho = \omega_x$ corresponding to an eigenvector x of A: If $Ax = \lambda x$ with $\lambda \in \mathbb{R}$, then $\rho(A^2) = \langle x|A^2x\rangle = \langle x|\lambda^2 x\rangle = \lambda^2\langle x|x\rangle = \lambda^2 = \langle x|\lambda x\rangle^2 = \langle x|Ax\rangle^2 = \rho(A)^2$.

Proposition: Let $A, B \in \mathcal{A}$ be self-adjoint elements of the C*-algebra $\mathcal{A}$ and ρ be a state on $\mathcal{A}$, then we have the *uncertainty relation*

$$\Delta_\rho(A)\Delta_\rho(B) \geq \frac{|\rho([A,B])|}{2}.$$

Proof: Suppose first that $\Delta_\rho(A) > 0$ and $\Delta_\rho(B) > 0$. We put $C := \frac{1}{\Delta_\rho(A)}(A - \rho(A)) - \frac{i}{\Delta_\rho(B)}(B - \rho(B)) \in \mathcal{A}$ and note that $\rho(A)$, $\rho(B)$, $\Delta_\rho(A)$, and $\Delta_\rho(B)$ are real, hence $C^* = \frac{1}{\Delta_\rho(A)}(A - \rho(A)) + \frac{i}{\Delta_\rho(B)}(B - \rho(B))$. Thus, applying the positive functional ρ to the positive operator C^*C yields

$$\begin{aligned}
0 \leq \rho(C^*C) &= \frac{\rho\big((A-\rho(A))^2\big)}{\Delta_\rho(A)^2} + \frac{\rho\big((B-\rho(B))^2\big)}{\Delta_\rho(B)^2} \\
&\quad + \frac{i\rho\big(-(A-\rho(A))(B-\rho(B)) + (B-\rho(B))(A-\rho(A))\big)}{\Delta_\rho(A)\Delta_\rho(B)} \\
&= \frac{\rho(A^2 - 2\rho(A)A + \rho(A)^2)}{\Delta_\rho(A)^2} + \frac{\rho(B^2 - 2\rho(B)B - \rho(B)^2)}{\Delta_\rho(B)^2} + \frac{i\rho(-AB + BA)}{\Delta_\rho(A)\Delta_\rho(B)} \\
&= \frac{\rho(A^2) - 2\rho(A)\rho(A) + \rho(A)^2}{\Delta_\rho(A)^2} + \frac{\rho(B^2) - 2\rho(B)\rho(B) - \rho(B)^2}{\Delta_\rho(B)^2} - \frac{i\rho([A,B])}{\Delta_\rho(A)\Delta_\rho(B)} \\
&= \frac{\rho(A^2) - \rho(A)^2}{\Delta_\rho(A)^2} + \frac{\rho(B^2) - \rho(B)^2}{\Delta_\rho(B)^2} - \frac{i\rho([A,B])}{\Delta_\rho(A)\Delta_\rho(B)} = 1 + 1 - \frac{i\rho([A,B])}{\Delta_\rho(A)\Delta_\rho(B)},
\end{aligned}$$

hence $0 \leq 2\Delta_\rho(A)\Delta_\rho(B) - i\rho([A,B])$ (note that $[A,B]^* = [B,A] = -[A,B]$, so that $\rho([A,B]) \in i\mathbb{R}$). The analogous calculation with CC^* gives $0 \leq 2\Delta_\rho(A)\Delta_\rho(B) + i\rho([A,B])$, and therefore we arrive at $|\rho([A,B])| = |i\rho([A,B])| \leq 2\Delta_\rho(A)\Delta_\rho(B)$.

Finally, we claim that in case $\Delta_\rho(A) = 0$ or $\Delta_\rho(B) = 0$ we necessarily have $\rho([A,B]) = 0$: Suppose $\Delta_\rho(B) = 0$; if $t > 0$ and $C := t(A - \rho(A)) - i(B - \rho(B))$, then calculations as above yield $0 \leq t^2\Delta_\rho(A)^2 + \Delta_\rho(B)^2 \pm it\rho([A,B]) = t^2\Delta_\rho(A)^2 \pm it\rho([A,B])$, which implies $|\rho([A,B])| \leq t\Delta_\rho(A)^2$; since $t > 0$ may be arbitrarily small we must have $\rho([A,B]) = 0$. $\square$

Remark: In case of (possibly) unbounded self-adjoint operators A and B on a Hilbert space with common dense invariant domain $\mathscr{D}$ of essential self-adjointness, we can show a companion uncertainty relation for any "vector state ω_x" with $x \in \mathscr{D}$, $\|x\| = 1$, in place of ρ by the same calculations as above upon slight adaptations: Writing $\omega_x(A) := \langle x|Ax\rangle$ we have $\omega_x(A^2) - \omega_x(A)^2 = \langle x|(A - \omega_x(A))^2x\rangle = \langle (A - \omega_x(A))x|(A - \omega_x(A))x\rangle \geq 0$, so that we may put

$$\Delta_x(A) := \sqrt{\langle x|A^2x\rangle - \langle x|Ax\rangle^2};$$

we further note that C and C^* also map $\mathscr{D} \to \mathscr{D}$ and that $\langle x|C^*Cx\rangle = \langle Cx|Cx\rangle \geq 0$ holds.

In particular, this remark applies to the position and momentum operators in 5.1, where we now temporarily restore the physics to $\hbar \neq 1$ so that $[Q_j, P_k] = i\hbar\delta_{jk}I$. We therefore obtain the *Heisenberg uncertainty relations*

$$\forall f \in \mathscr{S}(\mathbb{R}^N) \text{ with } \|f\|_2 = 1\colon \quad \Delta_f(Q_j)\Delta_f(P_k) \geq \frac{\hbar}{2}\delta_{jk}.$$

In elementary quantum mechanics, based on experience with the phase space of classical mechanics, one expects the algebra of observables to be constructed from functions of the position and momentum operators. We should therefore try to determine some basic bounded functions of the operators featuring in the canonical commutation relations (CCR) in 5.1.

5.4. From the CCR to the Weyl relations: We consider the position and momentum operators $Q_1, \ldots, Q_N$ and $P_1, \ldots, P_N$ as defined in 5.1 and determine the unitary operators on $L^2(\mathbb{R}^N)$ generated from $R(a,0) = a \cdot Q = \sum_{j=1}^N a_j Q_j$ ($a \in \mathbb{R}^N$) and $R(0,b) = b \cdot P = \sum_{k=1}^N b_k P_k$ ($b \in \mathbb{R}^N$), or rather from the self-adjoint closures of these operators, which we will tacitly denote by the same symbols.

Since every Q_j is already given as a multiplication operator on the Schwartz space $\mathscr{S}(\mathbb{R}^N) \subseteq L^2(\mathbb{R}^N)$ and $[Q_j, Q_l] = 0$, we easily deduce that $(\exp(iQ_j)f)(x) = e^{ix_j}f(x)$ and further that

$$(\exp(iR(a,0))f)(x) = (\exp(ia \cdot Q)f)(x) = e^{i\langle a|x\rangle}f(x) \qquad (f \in L^2(\mathbb{R}^N), \text{ almost all } x \in \mathbb{R}^N).$$

The Fourier transform $\mathcal{F}$ can be normalized to become unitary on $L^2(\mathbb{R}^N)$ and, as operators on $\mathscr{S}(\mathbb{R}^N)$, we have the so-called exchange formulae $\mathcal{F}P_k = Q_k\mathcal{F}$ and therefore $\mathcal{F}R(0,b) = \mathcal{F}(b \cdot P) = (b \cdot Q)\mathcal{F} = R(b,0)\mathcal{F}$. We may thus determine $\exp(iR(0,b))f$ as the inverse Fourier transform of $\exp(iR(b,0))\mathcal{F}f$, which directly gives

$$(\exp(iR(0,b))f)(x) = (\exp(ib \cdot P)f)(x) = f(x+b) \qquad (f \in L^2(\mathbb{R}^N), \text{ almost all } x \in \mathbb{R}^N).$$

Let us introduce the short-hand notation $U(a) := \exp(iR(a,0))$ and $V(b) := \exp(iR(0,b))$, then we obviously have

$$(U(a)V(b)f)(x) = e^{i\langle a|x\rangle}f(x+b),$$

while

$$(V(b)U(a)f)(x) = e^{i\langle a|x+b\rangle}f(x+b),$$

and therefore obtain

$$\text{(WCCR)} \qquad V(b)U(a) = e^{i\langle a|b\rangle}U(a)V(b) \qquad (a,b \in \mathbb{R}^N).$$

These are called the *Weyl form of the canonical commutation relations*. We note in passing that the (CCR) can be derived (quite literally) from (WCCR) by considering the strongly continuous unitary groups $r \mapsto U(re_j)$ and $s \mapsto V(se_k)$ and differentiating at $r = 0$ and $s = 0$ upon action on elements from $\mathscr{S}(\mathbb{R}^N)$.

We may combine U and V into one family of unitary operators with parameters $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$ by setting

$$W(a,b) := e^{\frac{i}{2}\langle a|b\rangle}U(a)V(b),$$

which corresponds to the following formula for the action on functions $f \in L^2(\mathbb{R}^N)$:

$$(W(a,b)f)(x) = e^{\frac{i}{2}\langle a|b\rangle} e^{i\langle a|x\rangle} f(x+b). \tag{5.2}$$

We claim that

$$W(a,b)W(c,d) = e^{\frac{i}{2}(\langle b|c\rangle - \langle a|d\rangle)} W(a+c, b+d), \tag{5.3}$$

where we see again the symplectic form β introduced in 5.1 appearing, since the exponent in the phase factor on the right-hand side equals $-\beta((a,b),(c,d))/2$.

To prove (5.3) we consider $W(a,b)W(c,d) = e^{i(\langle a|b\rangle + \langle c|d\rangle)/2} U(a)V(b)U(c)V(d)$ and apply (WCCR) to the product in the middle, i.e., $V(b)U(c) = e^{i\langle c|bs\rangle} U(c)V(b)$, and then make use of the obvious relations $U(a)U(c) = U(a+c)$ and $V(b)V(d) = V(b+d)$ to obtain $W(a,b)W(c,d) = e^{i(\langle a|b\rangle + \langle c|d\rangle)/2} e^{i\langle c|b\rangle} U(a+c)V(b+d)$. The claim now follows from $\langle a|b\rangle + \langle c|d\rangle + 2\langle c|b\rangle - \langle a+c|b+d\rangle = -\langle a|d\rangle + \langle b|c\rangle$.

With the more compact notation $u = (a,b), v = (c,d) \in \mathbb{R}^N \times \mathbb{R}^N$ and the symplectic form $\sigma(u,v) := -\beta(u,v)/2$ we may rewrite (5.3) in the simpler form

$$W(u)W(v) = e^{i\sigma(u,v)} W(u+v). \tag{5.4}$$

We clearly have the unitarity relations

$$W(0) = I \quad \text{and} \quad W(-u) = W(u)^* \tag{5.5}$$

as well as the non-commutativity

$$W(u)W(v) = e^{2i\sigma(u,v)} W(v)W(u).$$

Remark: One can show that $W(a,b) = \exp(iR(a,b))$ (cf. [Mor17, combining Equations (11.34), (11.47), and (11.48)]). Although we do not need this result in the sequel, let us briefly see why it is very plausible. For arbitrary fixed $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$, $f \in \mathscr{S}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$, and $t \in \mathbb{R}$ we may differentiate the expression for $(W(ta,tb)f)(x)$ with respect to t, evaluate at $t = 0$, and thus get by elementary calculation that

$$\frac{d}{dt}(W(ta,tb)f)(x)|_{t=0} = i\langle a|x\rangle f(x) + \sum_{k=1}^{N} b_k \partial_k f(x) = i((a \cdot Q + b \cdot P)f)(x).$$

It can be checked directly from (5.2) or from Equation (5.3), that $S(t) := W(ta,tb)$ satisfies the relations (5.1) of a unitary group. To make the claim $W(a,b) = \exp(iR(a,b))$ precise, one still has to show strong continuity of $t \mapsto S(t)$ and that the above derivative is correct also in the L^2 norm sense.

In an abstract setting for the CCR such as described in 5.1, (a) and (b), we can in general not expect the analogue of the relations (WCCR) or (5.4) to hold as well ([Thi10, Part I, Remark 3.1.10.2]). The reverse implication is also "not for free" without additional assumptions, i.e., merely starting from a family of unitary operators on some Hilbert space that is parametrized by elements from a symplectic vector space and satisfies (5.4) will not even guarantee that the one-parameter unitary soubgroups $t \mapsto W(tu)$ are strongly continuous and have self-adjoint generators (see [Thi10, Part I, Remark 3.1.6.3] or [DG13, 8.1.8] for examples), which could serve as *field operators* in the physical theory. The latter can be restored with the help of additional technical assumptions, e.g., that the parameter space is a topological vector space, the symplectic form is continuous, and the map $u \mapsto W(u)$ is continuous when the strong operator topology is put on the target space; in this situation, also the CCR can be recovered (cf. [BSZ92, Section 1.2] or [DG13, Section 8.2]). However, such additional assumptions might be somewhat restrictive, unnatural, or too strong to ask for in building concrete models of quantum physics from the outset.

The relations (5.4) are quite elegant and have the advantage of being in better reach from the perspective of general operator algebras. The strategy is therefore to aim for a C*-algebraic implementation of the *Weyl relations*, defined as the abstract version of (5.4), and then put the burden for the question about existence of field operators on the representations of this yet-to-be-found C*-algebra (as discussed in detail, e.g., in [Petz90, Chapter 3]).

5.5. Theorem: Let (V, σ) be a symplectic vector space. There exists a C*-algebra $\mathcal{W}(V, \sigma)$, uniquely determined up to $*$-isomorphism, that is generated by a family of elements $W(u)$ $(u \in V)$ such that for all $u, v \in V$,

$$W(0) = I, \quad W(-u) = W(u)^*, \quad \text{and} \quad W(u)W(v) = e^{i\sigma(u,v)} W(u+v).$$

We call $\mathcal{W}(V, \sigma)$ the *Weyl algebra* over (V, σ).

Remarks: (i) The product relations already imply that $W(0)$ is a unit element. However, we have included the somewhat redundant property $W(0) = I$ here for clarity (and it also emphasizes that the $W(u)$ are nonzero). Combining the second and third property shows that each $W(u)$ is unitary.

(ii) The Weyl relations imply that the $*$-subalgebra generated from $\{W(u) \mid u \in V\}$ coincides with the vector space linear hull $\text{span}\{W(u) \mid u \in V\}$ and thus $\mathcal{W}(V, \sigma)$ is simply the norm closure of this vector subspace.

Proof: Existence: Consider the Hilbert space $l^2(V) = \{F \colon V \to \mathbb{C} \mid \sum_{z \in V} |F(z)|^2 < \infty\}$ with the inner product $\langle F|G\rangle = \sum_{z\in V} \overline{F(z)} G(z)$ $(F, G \in l^2(V))$ and define for any $u \in V$ the obviously unitary operator $W(u) \in \mathcal{B}(l^2(V))$ by

$$(W(u)F)(z) := e^{i\sigma(z,u)} F(z+u) \qquad (F \in l^2(V), z \in V).$$

Clearly, $W(0) = I$, $W(u)^{-1} = W(u)^*$ expresses just the unitarity of $W(u)$, and direct calculation gives

$$\begin{aligned}(W(u)W(v)F)(z) = e^{i\sigma(z,u)} e^{i\sigma(z+u,v)} F(z+u+v) &= e^{i\sigma(u,v)} e^{i\sigma(z,u+v)} F(z+u+v) \\ &= e^{i\sigma(u,v)} (W(u+v)F)(z).\end{aligned}$$

In particular, $W(u)W(-u) = W(-u)W(u) = W(0) = I$, hence $W(-u) = W(u)^{-1} = W(u)^*$. Denote by $\mathcal{A}$ the C*-subalgebra of $\mathcal{B}(l^2(V))$ generated by $\{W(u) \mid u \in V\}$. (As we observed right after the statement of the theorem, it suffices to consider the operator norm closure of $\text{span}\{W(u) \mid u \in V\}$.) Then $\mathcal{A}$ is a Weyl algebra.

Uniqueness: Suppose $\mathcal{B}$ is a C*-algebra generated by the subset $\{W'(u) \mid u \in V\}$ of unitary elements satisfying $W'(0) = I$, $W'(-u) = W'(u)^*$, and $W'(u)W'(v) = e^{i\sigma(u,v)}W'(u+v)$. By the Gelfand-Neumark theorem we may assume that $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Our goal is to construct a $*$-isomorphism $\alpha\colon \mathcal{A} \to \mathcal{B}$ such that $\alpha(W(u)) = W'(u)$ for all $u \in V$.

Step 1: A faithful representation $\pi_0\colon \mathcal{A} \to \mathcal{B}(l^2(V, \mathcal{H}))$.

Define the Hilbert space $l^2(V, \mathcal{H}) := \{\psi\colon V \to \mathcal{H} \mid \sum_{v\in V} \|\psi(v)\|^2 < \infty\}$ with inner product $\langle\varphi|\psi\rangle := \sum_{v\in V}\langle\varphi(v)|\psi(v)\rangle$. (It is isometrically isomorphic to the Hilbert space tensor product $l^2(V) \otimes \mathcal{H}$.)

For $\psi \in l^2(V, \mathcal{H})$ and $u \in V$ let

$$(\pi_0(W(u))\psi)(z) := e^{i\sigma(z,u)}\psi(z+u) \qquad (z \in V).$$

By calculations completely analogous to those in the above existence proof, we obtain that $\pi_0(W(u))$ is unitary on $l^2(V, \mathcal{H})$, $\pi_0(W(u)^*) = \pi_0(W(-u)) = \pi_0(W(u))^{-1} = \pi_0(W(u))^*$, and $(\pi_0(W(u))\pi_0(W(v)) = \pi_0(W(u)W(v))$. We may thus extend π_0 to a $*$-homomorphism from the $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ generated by $\{W(u) \mid u \in V\}$ into $\mathcal{B}(l^2(V, \mathcal{H}))$. Recall that due to the Weyl relations $\mathcal{A}_0 = \operatorname{span}\{W(u) \mid u \in V\}$. We claim that π_0 is isometric: In fact, let $\psi \in l^2(V, \mathcal{H})$, $A \in \mathcal{A}_0$, and $\mathcal{E}$ be a complete orthonormal system in $\mathcal{H}$; note that $\langle e|(\pi_0(A)\psi)(z)\rangle = (A\langle e|\psi\rangle)(z)$ for all $z \in V$ and $e \in \mathcal{E}$, where $\langle e|\psi\rangle \in l^2(V)$ is short-hand for the function $z \mapsto \langle e|\psi(z)\rangle$ (in case $A = W(u)$ this is obvious from $\langle e|(\pi_0(W(u))\psi)(z)\rangle = e^{i\sigma(z,u)}\langle e|\psi(z+u)\rangle$ and follows from linearity for general $A \in \mathcal{A}_0$); we have

$$\begin{aligned}
\|\pi_0(A)\psi\|^2 &= \sum_{z\in V}\sum_{e\in\mathcal{E}} |\langle e|(\pi_0(A)\psi)(z)\rangle|^2 = \sum_{z\in V}\sum_{e\in\mathcal{E}} |(A\langle e|\psi\rangle)(z)|^2 = \sum_{e\in\mathcal{E}}\sum_{z\in V} |(A\langle e|\psi\rangle)(z)|^2 \\
&= \sum_{e\in\mathcal{E}} \|A\langle e|\psi\rangle\|^2 \leq \sum_{e\in\mathcal{E}} \|A\|^2 \|\langle e|\psi\rangle\|^2 = \|A\|^2 \sum_{e\in\mathcal{E}}\sum_{z\in V} |\langle e|\psi(z)\rangle|^2 \\
&= \|A\|^2 \sum_{z\in V}\sum_{e\in\mathcal{E}} |\langle e|\psi(z)\rangle|^2 = \|A\|^2 \sum_{z\in V} \|\psi(z)\|^2 = \|A\|^2\|\psi\|^2,
\end{aligned}$$

which shows that $\|\pi_0(A)\| \leq \|A\|$; specializing in the first part of the above calculation to ψ of the form $\psi(z) = F(z)e$ with $F \in l^2(V)$ and $e \in \mathcal{E}$ gives $\|\pi_0(A)\psi\|^2 = \|AF\|^2$, which upon taking the supremum over $F \in l^2(V)$ with $\|F\| = 1$ and $e \in \mathcal{E}$ implies that $\|\pi_0(A)\| \geq \|A\|$.

We may extend π_0 to an isometric $*$-homomorphism $\mathcal{A} \to \mathcal{B}(l^2(V, \mathcal{H}))$, denoted again by π_0, which is thus a faithful representation of $\mathcal{A}$ and provides a $*$-isomorphism $\mathcal{A} \cong \pi_0(\mathcal{A}) =: \tilde{\mathcal{A}}$. (From the tensor product point of view, $l^2(V, \mathcal{H}) \cong l^2(V) \otimes \mathcal{H}$ and $\tilde{\mathcal{A}} \cong \mathcal{A} \otimes I$.)

Let $W_0(u) := \pi_0(W(u))$, then $\tilde{\mathcal{A}}$ is generated as a C*-algebra by $\{W_0(u) \mid u \in V\}$ and is, thanks to the Weyl relations, the norm closure as a vector space of the $*$-subalgebra $\tilde{\mathcal{A}}_0 := \operatorname{span}\{W_0(u) \mid u \in V\}$.

Step 2: A unitarily equivalent subalgebra $U^*\tilde{\mathcal{A}}U$ of $\mathcal{B}(l^2(V, \mathcal{H}))$.

For $\psi \in l^2(V, \mathcal{H})$ define

$$(U\psi)(z) := W'(z)\psi(z) \qquad (z \in V),$$

then clearly, U is unitary on $l^2(V, \mathcal{H})$ and $(U^*\psi)(z) = W'(-z)\psi(z)$. The assignment $A \mapsto U^*AU$ then defines an isometric $*$-homomorphism $\pi\colon \tilde{\mathcal{A}} \to \mathcal{B}(l^2(V, \mathcal{H}))$ and therefore the C*-algebra $\pi(\tilde{\mathcal{A}}) = U^*\tilde{\mathcal{A}}U$ is $*$-isomorphic with $\tilde{\mathcal{A}}$, hence also $\mathcal{A} \cong \pi(\tilde{\mathcal{A}})$.

A quick calculation using $e^{i\sigma(z,u)}W'(-z)W'(z+u) = W'(u)$ shows that

$$(\pi(W_0(u))\psi)(z) = (U^*W_0(u)U\psi)(z) = W'(u)\psi(z+u). \tag{$*$}$$

Step 3: A $*$-isomorphism $\beta\colon \pi(\tilde{A}) \to \mathcal{B}$.

Let $\mathcal{B}_0 := \operatorname{span}\{W'(u) \mid u \in V\}$, then we know that, thanks to the Weyl relations, $\mathcal{B}_0$ is a $*$-subalgebra and its norm closure is $\mathcal{B}$. We define a $*$-homomorphism $\beta_0\colon \pi(\tilde{\mathcal{A}}_0) \to \mathcal{B}_0$ by linear extension of $\beta_0(\pi(W_0(u))) := W'(u)$ $(u \in V)$ to $\pi(\tilde{\mathcal{A}}_0)$.

We will establish below that β_0 is isometric, which then completes the proof, because β_0 extends to a $*$-isomorphism $\beta\colon \pi(\tilde{A}) \to \mathcal{B}$ and we obtain in summary,

$$\mathcal{A} \cong \tilde{\mathcal{A}} \cong \pi(\tilde{A}) \cong \mathcal{B}.$$

It remains to prove the following assertion.

Claim: Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $u_1, \ldots, u_m \in V$ be arbitrary, then we have

$$\Big\| \sum_{j=1}^{m} \lambda_j \pi(W_0(u_j)) \Big\| = \Big\| \sum_{j=1}^{m} \lambda_j W'(u_j) \Big\|.$$

The strategy is to find some kind of diagonalization of the operators appearing on the left-hand side, so that the norm becomes computable similarly to those of multiplication operators. The means to achieve this is to apply some Fourier analysis for the discrete commutative additive group V that is underlying the construction of $l^2(V)$. In this case the dual group of all characters on V is[1] $\hat{V} := \{\chi\colon V \to S^1 \mid \forall u, v \in V\colon \chi(u+v) = \chi(u)\chi(v)\} \subseteq (S^1)^V = \prod_{z \in V} S^1$ equipped with the product topology, also known as topology of pointwise convergence. Obviously $\hat{V}$ is closed and hence compact, since Tychonoff's theorem guarantees that $\prod_{z \in V} S^1$ is a compact Hausdorff space. Pointwise multiplication of characters thus turns $\hat{V}$ into a commutative compact group with continuous group operations.

The Fourier transform can be given explicitly on the dense subset of $l^2(V)$ consisting of all functions with finite support, namely we have for any such function $F\colon V \to \mathbb{C}$ the Fourier transform $\hat{F}\colon \hat{V} \to \mathbb{C}$, given by

$$\hat{F}(\chi) = \sum_{z \in V} \overline{\chi(z)}\, F(z) \qquad (\chi \in \hat{V}).$$

We may thus calculate similarly for any $\psi \in l^2(V, \mathcal{H})$ with $\psi(z) \neq 0$ for at most finitely many $z \in V$, upon defining $(\mathcal{F}\psi)(\chi) := \sum_{z \in V} \overline{\chi(z)}\, \psi(z)$, and using $(*)$ to obtain

$$\begin{aligned} \mathcal{F}(\pi(W_0(u))\psi)(\chi) &= \sum_{z \in V} \overline{\chi(z)}\, (\pi(W_0(u))\psi)(z) = \sum_{z \in V} \overline{\chi(z)}\, W'(u)\psi(z+u) \\ &= \sum_{y \in V} \overline{\chi(y-u)}\, W'(u)\psi(y) = \chi(u)W'(u) \sum_{y \in V} \overline{\chi(y)}\, \psi(y) = \chi(u)W'(u)(\mathcal{F}\psi)(\chi). \end{aligned}$$

This is "our diagonalization" of $\pi(W_0(u))$, but we have to ask for more substantial help from harmonic analysis to establish $\mathcal{F}$ as a unitary transformation ([Fol16, Chapters 2 and 4]):

[1] $S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$

Plancherel's theorem yields that the Fourier transform extends to a unitary map from $l^2(V)$ to $L^2(\hat{V}, \mu)$, where μ denotes the suitably normalized Haar measure on $\hat{V}$, which is a regular Borel measure. It follows then that the transformation $\mathcal{F}$ we introduced above also extends to a unitary map from $l^2(V, \mathcal{H})$ to the Hilbert space $L^2(\hat{V}, \mathcal{H})$, which is the completion of the continuous maps $\phi \colon \hat{V} \to \mathcal{H}$ with respect to the norm $\big(\int_{\hat{V}} \|\phi(\chi)\|^2 \, d\mu(\chi) \big)^{1/2}$ (or realized as Hilbert space tensor product $L^2(\hat{V}, \mu) \otimes \mathcal{H}$).

We have therefore reached the intermediate result

$$(**) \qquad \Big\| \sum_{j=1}^{m} \lambda_j \pi(W_0(u_j)) \Big\| = \Big\| \sum_{j=1}^{m} \lambda_j \hat{W}(u_j) \Big\|,$$

where

$$(\hat{W}(u)\phi)(\chi) := \chi(u) W'(u) \phi(\chi) \qquad (\phi \in L^2(\hat{V}, \mathcal{H}), \chi \in \hat{V}, u \in V).$$

We have $\| \big(\sum_{j=1}^{m} \lambda_j \hat{W}(u_j) \big) \phi \|^2 = \int_{\hat{V}} \| \big(\sum_{j=1}^{m} \lambda_j \chi(u_j) W'(u_j) \big) \phi(\chi) \|^2 \, d\mu(\chi)$, which implies, by arguments essentially as in the scalar case $\mathcal{H} = \mathbb{C}$ (see [Sla72, Equation (3.7)]), that

$$(***) \qquad \| \sum_{j=1}^{m} \lambda_j \hat{W}(u_j) \| = \sup \Big\{ \| \sum_{j=1}^{m} \lambda_j \chi(u_j) W'(u_j) \| \mid \chi \in \hat{V} \Big\}.$$

By continuity of $\chi \mapsto \chi(u_j) W'(u_j)$, thus also of $\chi \mapsto \| \sum_{j=1}^{m} \lambda_j \chi(u_j) W'(u_j) \|$, we may evaluate the supremum on a suitable dense subset of $\hat{V}$.

Note that for any $z \in V$ we have the character $\chi_z(u) := e^{2i\sigma(u,z)}$ $(u \in V)$ and we may again draw from the resources of harmonic analysis ([Fol16, Chapter 4]) to argue that

$$T := \{ \chi_z \mid z \in V \}$$

is dense in $\hat{V}$: By Pontrjagin duality, the dual group of $\hat{V}$ is isomorphic to V via the map $v \mapsto \tilde{v}$, where $\tilde{v}$ is the character on $\hat{V}$ defined by $\tilde{v}(\chi) := \chi(v)$ $(\chi \in \hat{V})$; note that T is a subgroup of $\hat{V}$ and so is its closure $\overline{T}$; for any closed subgroup H of a locally compact abelian group G, one can show ([Fol16, Proposition 4.39]) that $(H^\perp)^\perp = H$, where $H^\perp := \{ \xi \in \hat{G} \mid \forall h \in H \colon \xi(h) = 1 \}$ etc.; we have $H = \overline{T}$ in $G := \hat{V}$ with $\hat{G} \cong V$ and determine

$$\overline{T}^\perp = \{ \tilde{v} \mid \forall \chi \in \overline{T} \colon \chi(v) = 1 \} \subseteq T^\perp = \{ \tilde{v} \mid \forall z \in V \colon \chi_z(v) = 1 \};$$

thus, $\tilde{v} \in \overline{T}^\perp$ implies that $e^{2i\sigma(v,z)} = 1$ for all $z \in V$, which means that $\sigma(v, z) \in \pi\mathbb{Z}$ for all $z \in V$; since V is a vector space, the latter can only hold, if

$$\forall z \in V \colon \quad \sigma(v, z) = 0,$$

which implies $v = 0$ by non-degeneracy of σ; therefore, $\overline{T}^\perp = \{\tilde{0}\}$ and hence $\overline{T} = \{\tilde{0}\}^\perp = \hat{V}$.

Let $z \in V$ and recall that by unitarity of $W'(z)$ we obviously have $\|W'(z) R W'(-z)\| = \|R\|$

for any $R \in \mathcal{B}(L^2(\hat{V}, \mathcal{H}))$, so that we obtain

$$
\begin{aligned}
\| \sum_{j=1}^{m} \lambda_j \chi_z(u_j) W'(u_j) \| &= \| W'(z) \Big(\sum_{j=1}^{m} \lambda_j \chi_z(u_j) W'(u_j) \Big) W'(-z) \| \\
&= \| \sum_{j=1}^{m} \lambda_j e^{2i\sigma(u_j,z)} W'(z) W'(u_j) W'(-z) \| = \| \sum_{j=1}^{m} \lambda_j e^{2i\sigma(u_j,z)} e^{i\sigma(z,u_j)} W'(z+u_j) W'(-z) \| \\
&= \| \sum_{j=1}^{m} \lambda_j e^{2i\sigma(u_j,z)} e^{i\sigma(z,u_j)} e^{i\sigma(z+u_j,-z)} W'(u_j) \| = \| \sum_{j=1}^{m} \lambda_j e^{2i\sigma(u_j,z)} e^{-i\sigma(u_j,z)} e^{-i\sigma(u_j,z)} W'(u_j) \| \\
&= \| \sum_{j=1}^{m} \lambda_j W'(u_j) \|.
\end{aligned}
$$

Calling on $(**)$ and $(***)$ now proves the claim and completes the proof. □

Note that the uniqueness proof of the previous theorem showed, in fact, that for any two models of the Weyl algebra over the symplectic vector space (V, σ), say $\mathcal{A}$ generated by $W(u)$ and $\mathcal{B}$ generated by $W'(u)$ with $u \in V$, there is a *unique $*$-isomorphism* $\alpha \colon \mathcal{A} \to \mathcal{B}$ such that $\alpha(W(u)) = W'(u)$.

We collect a few direct consequences of Theorem 5.5. Recall that we learned from the existence proof that $\operatorname{span}\{W(u) \mid u \in V\}$ is a norm dense $*$-subalgebra of the Weyl algebra $\mathcal{W}(V, \sigma)$. In particular, a continuous linear functional on the Weyl algebra is determined once the values on all linear combinations $\sum_{u \in V} \lambda(u) W(u)$ are known, where $\lambda \colon V \to \mathbb{C}$ is a function vanishing for all but finitely many points in V. It can also be shown that the set $\{W(u) \mid u \in V\}$ is linearly independent ([Mor17, Theorem 11.48, (b)]).

5.6. Corollary: Let (V, σ) be a symplectic vector space and $\mathcal{W}(V, \sigma)$ be the Weyl algebra.

(i) For every $u \in V$ such that $u \neq 0$ we have $\operatorname{sp}(W(u)) = S^1$ and $\|W(u) - I\| = 2$.
If $u, v \in V$ are distinct, then $\|W(u) - W(v)\| = 2$.

(ii) $\mathcal{W}(V, \sigma)$ is not separable.

(iii) Every representation of $\mathcal{W}(V, \sigma)$ is faithful.

(iv) If $T \colon V \to V$ is a *symplectic* linear map, i.e., T is invertible and $\sigma(Tu, Tv) = \sigma(u, v)$ for all $u, v \in V$, then there is unique $*$-automorphism α of $\mathcal{W}(V, \sigma)$ such that

$$\alpha(W(u)) = W(Tu) \qquad (u \in V).$$

The $*$-automorphism α is called the *Bogoliubov*[2] *transform* corresponding to T.

(v) Let M be a subspace of V and $\mathcal{W}_M$ the C*-subalgebra of $\mathcal{W}(V, \sigma)$ generated by $\{W(u) \mid u \in M\}$. We have $\mathcal{W}_M = \mathcal{W}(V, \sigma)$ if and only if $M = V$.

Remark: In case M is symplectic, i.e., σ restricted to M is also non-degenerate, then $\mathcal{W}_M \cong \mathcal{W}(M, \sigma)$ by Theorem 5.5. Hence we obtain $\mathcal{W}(M, \sigma)$ as C*-subalgebra of $\mathcal{W}(V, \sigma)$.

[2]Transcriptions also as Bogolubov, Bogoljubov, Bogolyubov, Bogoljubow, Bogoliouboff ...

Proof: (i): We have $W(v)W(u)W(v)^* = e^{i\sigma(v,u)}W(v+u)W(-v) = e^{i(\sigma(v,u)-\sigma(v+u,v))}W(u) = e^{2i\sigma(v,u)}W(u)$ and $\mathrm{sp}(W(v)W(u)W(v)^*) = \mathrm{sp}(W(u))$, hence $\mathrm{sp}(W(u))$ is a subset of S^1 (since $W(u)$ is unitary) and must be invariant under rotations, if $u \neq 0$. This implies $\mathrm{sp}(W(u)) = S^1$ in this case, and further we obtain from $\mathrm{sp}(W(u) - I) = \{\lambda - 1 \mid \lambda \in S^1\}$ that the spectral radius of $W(u) - I$ is 2, hence $\|W(u) - I\| = 2$, since $W(u) - I$ is normal.

Consider now $u, v \in V$ such that $u \neq v$. A simple calculation using the Weyl relations gives

$$(W(u) - W(v))^*(W(u) - W(v)) = (W(-u) - W(-v))(W(u) - W(v))$$
$$= I - e^{-i\sigma(v,u)}W(u-v) - e^{-i\sigma(u,v)}W(v-u) + I = 2 - e^{i\sigma(u,v)}W(u-v) - (e^{i\sigma(u,v)}W(u-v))^*.$$

We know from the previous paragraph that $\mathrm{sp}(e^{i\sigma(u,v)}W(u-v)) = S^1$ and may now apply the spectral mapping theorem to the normal element $A = e^{i\sigma(u,v)}W(u-v)$ to deduce

$$\mathrm{sp}\Big((W(u) - W(v))^*(W(u) - W(v))\Big) = \mathrm{sp}\Big(2 - e^{i\sigma(u,v)}W(u-v) - (e^{i\sigma(u,v)}W(u-v))^*\Big)$$
$$= \{2 - \lambda - \overline{\lambda} \mid \lambda \in S^1\} = \{2 - 2\,\mathrm{Re}\,\lambda \mid \lambda \in S^1\} = [0,4]$$

and therefore, $\|W(u) - W(v)\|^2 = \|(W(u) - W(v))^*(W(u) - W(v))\| = 4$.

(ii): The set V is uncountable (since $V \neq \{0\}$ by our basic assumption for a symplectic vector space in 5.1, (a)), hence by (i) the subset $\{W(u) \mid u \in V\}$ cannot be approximated to arbitrary precision by any countable subset in $\mathcal{W}(V,\sigma)$.

(iii): Let $\pi\colon \mathcal{W}(V,\sigma) \to \mathcal{B}(\mathcal{H})$ be a representation. Then the C*-algebra $\pi(\mathcal{W}(V,\sigma))$ is a Weyl algebra generated by $\{\pi(W(u)) \mid u \in V\}$, hence is $*$-isomorphic to $\mathcal{W}(V,\sigma)$ via π (as remarked above just before the statement of the corollary). Therefore, π is isometric, in particular, injective.

(iv): By considering $W'(u) := W(Tu)$ $(u \in V)$ this follows also from the uniqueness of the Weyl algebra in addition with the remark above about uniqueness of $*$-isomorphisms mapping generators to generators.

(v): Clearly, $M = V$ implies $\mathcal{W}_M = \mathcal{W}(V,\sigma)$. If $M \neq V$, then pick $v \in V \setminus M$ so that by (i), $\|W(v) - W(u)\| = 2$ for all $u \in M$, hence $\mathcal{W}_M \neq \mathcal{W}(V,\sigma)$. □

Property (iii) in the corollary can be used to show that the Weyl algebra is *simple*, i.e., does not possess any nontrivial closed ideals. The argument uses the fact that any closed ideal $\mathcal{J} \neq \mathcal{A}$ in a C*-algebra $\mathcal{A}$ is the kernel of some representation: As mentioned earlier in 2.13, the quotient $\mathcal{A}/\mathcal{J}$ can be shown to be a C*-algebra, which can be faithfully represented as an operator algebra by the Gelfand-Neumark theorem, say by $\pi\colon \mathcal{A}/\mathcal{J} \to \mathcal{B}(\mathcal{H})$; if $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{J}$ denotes the canonical surjection, then $\pi \circ q$ defines a representation of $\mathcal{A}$ with kernel $\mathcal{J}$. If every representation of $\mathcal{A}$ has to be faithful, as is the case with $\mathcal{A} = \mathcal{W}(V,\sigma)$, then we necessarily have $\mathcal{J} = \{0\}$.

We will study a bit of representation theory for the Weyl algebra. The first focus will be on systems with finitely many degrees of freedom, i.e., $\mathcal{W}(V,\sigma)$ with finite-dimensional V. Note that in this case the existence of the non-degenerate antisymmetric form σ has the consequence that $\dim V$ is an even integer: Indeed, let $e_1, \ldots, e_d$ be a basis of V and $S := (\sigma(e_j, e_k))_{1 \leq j,k \leq d}$ be the matrix representing σ; non-degeneracy and antisymmetry of σ imply that S is invertible and $S^T = -S$; hence $0 \neq \det S = \det S^T = \det(-S) = (-1)^d \det S$ and therefore, $(-1)^d = 1$.

For any symplectic form σ on $\mathbb{R}^N \times \mathbb{R}^N$ we can find a so-called *symplectic basis*, where the matrix representing σ is of the form $\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ (see, e.g., [Wald2, Satz 4.35]). This corresponds to an invertible linear map L on $\mathbb{R}^{2N}$ such that $\sigma(Lu, Lv) = \beta(u, v)$ for all $u, v \in \mathbb{R}^{2N}$, where $\beta((a, b), (c, d)) = \langle a|d\rangle - \langle b|c\rangle$ for $u = (a, b)$ and $v = (c, d)$ with $a, b, c, d \in \mathbb{R}^N$ is the same standard symplectic form as in 5.1. However, the corresponding Weyl algebras $\mathcal{W}(\mathbb{R}^{2N}, \sigma)$ and $\mathcal{W}(\mathbb{R}^{2N}, \beta)$ should be distinguished. (Although the natural assignment $\alpha(W(u)) := W(Lu)$ can be linearly extended to a bijective linear map α between the linear spans of the corresponding generators, this does not define an algebra homomorphism, since, e.g., $\alpha(W(u)W(v)) = e^{i\sigma(u,v)}W(L(u + v))$, while $\alpha(W(u))\alpha(W(v)) = e^{i\beta(u,v)}W(L(u + v))$.)

In the case of *finitely many degrees of freedom*, we may thus assume that $V = \mathbb{R}^{2N}$. We will prove below that, up to unitary equivalence, $\mathcal{W}(\mathbb{R}^{2N}, \sigma)$ possesses only *one irreducible representation* $\pi_S\colon \mathcal{W}(\mathbb{R}^{2N}, \sigma) \to \mathcal{B}(\mathcal{H})$ such that $u \mapsto \pi_S(W(u))$ is *continuous* $\mathbb{R}^{2N} \to \mathcal{B}(\mathcal{H})$ with respect to the *strong operator topology* on $\mathcal{B}(\mathcal{H})$. The equivalence class of π_S is described by an analogue of the *Schrödinger representation* given in 5.1 and 5.4 for the special case $\sigma = -\beta/2$. (Schrödinger representations for different symplectic forms on $\mathbb{R}^{2N}$ are not unitarily equivalent, because a transformation of the form $\pi_S(W(u)) \mapsto U^*\pi_S(W(u))U$ cannot change the phase factors from $e^{i\sigma(u,v)}$ to $e^{i\beta(u,v)}$ in the Weyl relations, unless $\sigma = \beta$. Recall that σ also depends on whether one uses the convention $\hbar = 1$ or not in the defining CCR.)

In the general situation, the additional continuity condition for representations mentioned above is replaced by the following notion, which is equivalent in the finite-dimensional case. Note that for any $u \in V$ and $t, s \in \mathbb{R}$ we have from the Weyl relations

$$W(su)W(tu) = e^{ist\sigma(u,u)}W(su + tu) = W((s + t)u)$$

and $W(0u) = I$, thus $t \mapsto W(tu)$ is a homomorphism from the additive group $\mathbb{R}$ into the group of unitary elements of the C*-algebra $\mathcal{W}(V, \sigma)$. Note that we learn from Corollary 5.6, (i), that the map $t \mapsto W(tu)$ cannot be norm continuous unless $u = 0$.

5.7. Definition: (i) A representation $\pi\colon \mathcal{W}(V, \sigma) \to \mathcal{B}(\mathcal{H})$ of the Weyl algebra is said to be *regular*, if for every $u \in V$ the map $t \mapsto \pi(W(tu))$ is strongly continuous, i.e., continuous as a map $\mathbb{R} \to \mathcal{B}(\mathcal{H})$ with respect to the strong operator topology on $\mathcal{B}(\mathcal{H})$. (By the group homomorphims property, it suffices to check continuity at $t = 0$.) In this case, every $u \in V$ defines a strongly continuous unitary group $\pi(W(tu))$ $(t \in \mathbb{R})$ on $\mathcal{H}$, whose self-adjoint generator $B_\pi(u)$ is called a *field operator*.

(ii) A state ρ on the Weyl algebra $\mathcal{W}(V, \sigma)$ is said to be *regular*, if the corresponding GNS representation π_ρ is regular.

5.8. Lemma: Let $\mathcal{H}$ be a Hilbert space.

(i) If $U(t) \in \mathcal{B}(\mathcal{H})$ $(t \in \mathbb{R})$ is a family of unitary operators, then WOT continuity of $t \mapsto U(t)$ implies SOT continuity. In particular, a one-parameter unitary group of operators is strongly continuous, if and only if it is weakly continuous, i.e., with respect to WOT.

(ii) Let $t \mapsto C(t)$ and $t \mapsto D(t)$ be SOT continuous maps $\mathbb{R} \to \mathcal{B}(\mathcal{H})$ and suppose that $\gamma := \sup\{\|C(t)\| \mid t \in \mathbb{R}\} < \infty$, then $t \mapsto C(t)D(t)$ is SOT continuous.

(iii) Suppose $\mathcal{E}$ is a dense subspace of $\mathcal{H}$, $C \in \mathcal{B}(\mathcal{H})$, the sequence $(C_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{B}(\mathcal{H})$ is uniformly bounded, and $\lim_{n\to\infty} C_n y = Cy$ for every $y \in \mathcal{E}$. Then $C_n \to C$ with respect to the SOT.

Proof: Let $x \in \mathcal{H}$ and $t, t_0 \in \mathbb{R}$.

(i): We have $\|U(t)x - U(t_0)x\|^2 = \langle (U(t) - U(t_0))x | (U(t) - U(t_0))x\rangle = \langle U(t)x|U(t)x\rangle - \langle U(t_0)x|U(t)x\rangle - \langle U(t)x|U(t_0)x\rangle + \langle U(t_0)x|U(t_0)x\rangle = 2\|x\|^2 - 2\operatorname{Re}\langle U(t)x|U(t_0)x\rangle$, which converges to $2\|x\|^2 - 2\operatorname{Re}\langle U(t_0)x|U(t_0)x\rangle = 2\|x\|^2 - 2\|x\|^2 = 0$ as $t \to t_0$.

(ii): Estimate $\|C(t)D(t)x - C(t_0)D(t_0)x\| = \|C(t)(D(t) - D(t_0))x + (C(t) - C(t_0))D(t_0)x\| \leq \gamma\|(D(t) - D(t_0))x\| + \|(C(t) - C(t_0))D(t_0)x\|$ to see that this converges to 0 as $t \to t_0$.

(iii): Put $\gamma := \sup(\{\|C_n\| \mid n \in \mathbb{N}\} \cup \{\|C\|\})$ and let $\varepsilon > 0$, $x \in \mathcal{H}$ arbitrary. Choose $y \in \mathcal{E}$ with $\|x - y\| < \varepsilon/(3\gamma)$ and $n_0 \in \mathbb{N}$ such that $\|C_n y - Cy\| < \varepsilon/3$ for all $n \geq n_0$. Then we have for such n also

$$\begin{aligned}\|C_n x - Cx\| &= \|C_n x - C_n y + C_n y - Cy + Cy - Cx\| \leq \|C_n(x-y)\| + \|C_n y - Cy\| + \|C(y-x)\| \\ &\leq \|C_n\|\|x-y\| + \|C_n y - Cy\| + \|C\|\|y-x\| \leq \gamma\frac{\varepsilon}{3\gamma} + \frac{\varepsilon}{3} + \gamma\frac{\varepsilon}{3\gamma} = \varepsilon.\end{aligned}$$

Thus $(C_n x)_{n\in\mathbb{N}}$ converges to Cx. Since $x \in \mathcal{H}$ was arbitrary, we have $C_n \to C$ in the SOT. $\square$

If we consider a regular state ρ on the Weyl algebra $\mathcal{W}(V,\sigma)$, then in its GNS representation π_ρ the map $t \mapsto \pi_\rho(W(tu))$ is WOT continuous for every $u \in V$. In particular, for every $v, w \in V$ the map $t \mapsto \langle \pi_\rho(W(w))x_\rho | \pi_\rho(W(tu))\pi_\rho(W(v))x_\rho\rangle$ is continuous. Noting that $\langle x_\rho | \pi_\rho(W(-w)W(tu)W(v))x_\rho\rangle = \rho(W(-w)W(tu)W(v))$ and applying the Weyl relations, one easily deduces that we obtain continuity of the map $t \mapsto \rho(W(tu + z))$ for all $u, z \in V$.

5.9. Proposition: Suppose ρ is a state on $\mathcal{W}(V,\sigma)$ such that for every $u \in V$, the function $t \mapsto \rho(W(tu))$ is continuous at $t = 0$. Then ρ is regular.

Proof: Regularity of the representation π_ρ means SOT continuity of $t \mapsto \pi_\rho(W(tu))$ at $t = 0$ for every $u \in V$. Let $u \in V$ and $x_\rho \in \mathcal{H}_\rho$ be the cyclic vector from the GNS construction, so that $\mathcal{E} := \operatorname{span}\{\pi_\rho(W(v))x_\rho \mid v \in V\}$ is dense[3] in $\mathcal{H}_\rho$. SOT continuity at $t = 0$ may be checked with sequences $\pi_\rho(W(t_n u))$ given $t_n \to 0$ and part (iii) of the above lemma allows us to check pointwise convergence of these sequences on the dense subspace $\mathcal{E}$. We write again t in place of t_n and consider

$$\begin{aligned}\|(\pi_\rho(W(tu)) - I)\pi_\rho(W(v))x_\rho\|^2 &= \langle (\pi_\rho(W(tu)) - I)\pi_\rho(W(v))x_\rho | (\pi_\rho(W(tu)) - I)\pi_\rho(W(v))x_\rho\rangle \\ &= \langle x_\rho | \pi_\rho(W(v))^*(\pi_\rho(W(tu)) - I)^*(\pi_\rho(W(tu)) - I)W(v)x_\rho\rangle \\ &= \langle x_\rho | \pi_\rho(W(-v)(W(-tu) - I)(W(tu) - I)W(v))x_\rho\rangle \\ = \rho(W(-v)(W(-tu) - I)(W(tu) - I)W(v)) &= \rho(W(-v)(I - W(tu) - W(-tu) + I)W(v)) \\ &= \rho(I - W(-v)W(tu)W(v) - W(-v)W(-tu)W(v) + I) \\ &= 2 - \rho(W(-v)W(tu)W(v)) - \rho((W(-v)W(tu)W(v))^*) \\ = 2 - 2\operatorname{Re}\rho(W(-v)W(tu)W(v)) &= 2 - 2\operatorname{Re}(e^{-it\sigma(v,u)}e^{i\sigma(tu-v,v)}\rho(W(tu))) \\ &= 2 - 2\operatorname{Re}(e^{2it\sigma(u,v)}\rho(W(tu))).\end{aligned}$$

[3]The density of $\operatorname{span}\{W(v) \mid v \in V\}$ in $\mathcal{W}(V,\sigma)$ together with the density of $\{\pi_\rho(A)x_\rho \mid A \in \mathcal{W}(V,\sigma)\}$ in $\mathcal{H}_\rho$ implies that $\operatorname{span}\{\pi_\rho(W(v))x_\rho \mid v \in V\}$ is dense in $\mathcal{H}_\rho$.

Since $\rho(W(0)) = 1$, the hypothesis of continuity of $t \mapsto \rho(W(tu))$ at $t = 0$ implies that $\|(\pi_\rho(W(tu)) - I)\pi_\rho(W(v))x_\rho\| \to 0$ as $t \to 0$. Hence we obtain regularity of π_ρ, thus of ρ. $\square$

5.10. Examples: 1) Recall the Schrödinger representation in 5.1 and 5.4 with $V = \mathbb{R}^N \times \mathbb{R}^N$ and $\sigma = -\beta/2$, where $\beta((a,b),(c,d)) = \langle a|d\rangle - \langle b|c\rangle$ for $u = (a,b)$, $v = (c,d)$ with $a,b,c,d \in \mathbb{R}^N$ and $S(a,b) := \pi_S(W(a,b))$ acts on a function $f \in L^2(\mathbb{R}^N)$ by

$$S(a,b)f(x) = e^{i\langle a|b\rangle/2} e^{i\langle a|x\rangle} f(x+b). \tag{5.6}$$

In fact, in 5.4 we had arrived at these concrete Weyl operators in the form $S(a,b) = e^{i\langle a|b\rangle/2}U(a)V(b)$ with $U(a) = \exp(ia\cdot Q)$ and $V(b) = \exp(ib\cdot P)$ generated from the self-adjoint operators $a \cdot Q$ and $b \cdot P$. Thus the maps $t \mapsto U(ta)$ and $t \mapsto V(tb)$ are strongly continuous unitary groups, in particular also uniformly bounded. By part (ii) in the above lemma, the product $U(ta)V(tb)$ is SOT continuous, hence also $t \mapsto e^{it^2\langle a|b\rangle/2}U(ta)V(tb) = S(ta,tb)$.

To summarize, the Schrödinger representation π_S is regular. According to the remark in 5.4, the field operators are $B(a,b) = a\cdot Q + b\cdot P$. We will show below that the Schrödinger representation π_S is irreducible. Therefore, the von Neumann algebra generated in this representation by the $*$-isomorphic image of the Weyl algebra $\pi_S(\mathcal{W}(\mathbb{R}^{2N}, -\beta/2))$ is all of $\mathcal{B}(L^2(\mathbb{R}^N))$, hence a factor of type I_∞.

As for the announced irreducibility of π_S, we will show that every nonzero function $f \in L^2(\mathbb{R}^N)$ is cyclic for π_S and then appeal to Proposition 3.16. It suffices to show that for any $g \in L^2(\mathbb{R}^N)$, $\langle g|S(a,b)f\rangle = 0$ for every $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$ implies $g = 0$.

Since $\langle g|S(a,b)f\rangle = e^{i\langle a|b\rangle/2}\langle U(a)^* g|V(b)f\rangle$, we have that $\langle U(-a)g|V(b)f\rangle = 0$ for all (a,b). Recalling that $(V(b)f)(x) = f(x+b)$, writing $f = \mathcal{F}\mathcal{F}^{-1}f$ with the Fourier transform $\mathcal{F}$ and applying the exchange formula $\mathcal{F}(-b\cdot Q) = (b\cdot P)\mathcal{F}$, hence $V(b)\mathcal{F} = \mathcal{F}U(-b)$, gives $V(b)f = V(b)\mathcal{F}(\mathcal{F}^{-1}f) = \mathcal{F}U(-b)\mathcal{F}^{-1}f$. We obtain that for all $(a,b) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$\begin{aligned} 0 = \langle U(-a)g|V(b)f\rangle &= \langle \mathcal{F}\mathcal{F}^{-1}U(-a)g|\mathcal{F}U(-b)\mathcal{F}^{-1}f\rangle = \langle \mathcal{F}^{-1}U(-a)g|U(-b)\mathcal{F}^{-1}f\rangle \\ &= \langle \mathcal{F}^{-1}g(.-a)|U(-b)\mathcal{F}^{-1}f\rangle = \int e^{-i\langle b|\xi\rangle}\,\overline{\mathcal{F}^{-1}g(\xi-a)}\,\mathcal{F}^{-1}f(\xi)\,d\xi. \end{aligned}$$

This tells that for all $a \in \mathbb{R}^N$, the Fourier transform of the function $\xi \mapsto \overline{\mathcal{F}^{-1}g(\xi-a)}\,\mathcal{F}^{-1}f(\xi) = \mathcal{F}\overline{g}(\xi-a)\,\mathcal{F}^{-1}f(\xi)$ vanishes, hence

$$\mathcal{F}\overline{g}(\xi-a)\,\mathcal{F}^{-1}f(\xi) = 0 \qquad \text{for all } a \in \mathbb{R}^N \text{ and almost all } \xi \in \mathbb{R}^N.$$

Since $f \neq 0$ we know that also $\mathcal{F}^{-1}f(\xi)$ is nonzero for ξ in some set of positive measure. Translations $\xi - a$ by all $a \in \mathbb{R}^N$ then tell that $0 = \mathcal{F}\overline{g}(\eta)$ holds for almost all $\eta \in \mathbb{R}^N$, so uniqueness of the Fourier transform implies that $g = 0$ in $L^2(\mathbb{R}^N)$.

2) The representation employed in the proof of existence of a Weyl algebra $\mathcal{W}(V,\sigma)$ (Theorem 5.5) is not regular: Recall that $W(u)F(z) = e^{i\sigma(z,u)}F(z+u)$ for $F \in l^2(V) = \{G\colon V \to \mathbb{C} \mid \sum_{z\in V}|G(z)|^2 < \infty\}$. Let $u \neq 0$ and consider $F = \delta_0$, where $\delta_0(z) := \delta_{z,0}$, then $W(tu)F(z) = e^{i\sigma(z,tu)}\delta_{z+tu,0} = \delta_{z+tu,0}$ (since $\sigma(-u,u) = 0$). Introducing also δ_v ($v \in V$) with $\delta_v(z) := \delta_{z,v}$ we may write $W(tu)\delta_0 = \delta_{-tu}$ and get for all $t \neq 0$,

$$\langle F|W(tu)F\rangle = \langle \delta_0|\delta_{-tu}\rangle = 0 \neq 1 = \langle \delta_0|\delta_0\rangle = \langle F|W(0u)F\rangle.$$

Note that the function $F = \delta_0$ is a cyclic vector for this representation, since $\operatorname{span}\{W(u)\delta_0 \mid u \in V\} = \operatorname{span}\{\delta_v \mid v \in V\}$ is dense in $l^2(V)$. We have here therefore a representation that is unitarily equivalent to the GNS representation associated with a, necessarily non-regular, state ρ on $\mathcal{W}(V,\sigma)$ such that

$$\rho(W(u)) = \langle \delta_0 | W(u)\delta_0 \rangle = \langle \delta_0 | \delta_{-u} \rangle = \delta_{u,0}.$$

On the dense subset $\operatorname{span}\{W(u) \mid u \in V\}$ the values of ρ are given by $\rho\big(\sum_{z\in V} \lambda(z)W(z)\big) = \lambda(0)$, where $\lambda\colon V \to \mathbb{C}$ is a function of finite support. Since $\rho(W(u)W(v)) = e^{i\sigma(u,v)}\rho(W(u+v)) = e^{i\sigma(u,v)}\delta_{u+v,0} = \delta_{u+v,0} = e^{i\sigma(v,u)}\delta_{u+v,0} = e^{i\sigma(v,u)}\rho(W(v+u)) = \rho(W(v)W(u))$ we obtain that ρ is a so-called *tracial state*, i.e,

$$\forall A, B \in \mathcal{W}(V,\sigma)\colon \quad \rho(AB) = \rho(BA).$$

It can be shown ([Sla72]) that the von Neumann algebra generated in this representation by the $*$-isomorphic image of the Weyl algebra $\mathcal{W}(V,\sigma)$ within $\mathcal{B}(l^2(V))$ is a factor of type II_1.

5.11. Remark: In Proposition 5.9 the regularity of a state ρ on $\mathcal{W}(V,\sigma)$ was guaranteed by properties of the scalar functions $t \mapsto \rho(W(tu))$ for every $u \in V$. One could also ask to what extent any state on the Weyl algebra is determined by the values of the function $u \mapsto \rho(W(u)) =: g(u)$. It is not difficult to show ([Petz90, Proposition 3.1]) that given a function $g\colon V \to \mathbb{C}$, there exists a state ρ on $\mathcal{W}(V,\sigma)$ such that $\rho(W(u)) = g(u)$ for all $u \in V$, if and only if $g(0) = 1$ and g is positive definite in the sense that

$$\sum_{1\le j,k\le N} c_j \overline{c_k}\, g(u_j - u_k) e^{-i\sigma(u_j,u_k)} \ge 0$$

for every possible choice of $N \in \mathbb{N}$ and $c_1, \dots, c_N \in \mathbb{C}$ and $u_1, \dots, u_N \in V$.

For example ([Petz90, Theorem 3.4 and Chapter 4]), if V is the real vector space underlying a complex Hilbert space $\mathcal{H}$ and $\sigma(u,v) := -\operatorname{Im}\langle u|v\rangle/2$, then it can be shown that $u \mapsto g(u) := e^{-\langle u|u\rangle/4}$ is positive definite and hence there exists a state ρ on the corresponding Weyl algebra such that

$$\rho(W(u)) = e^{-\langle u|u\rangle/4} \qquad (u \in V).$$

States of this form are called *Fock states*. The GNS representation corresponding to a Fock state is said to be a *Fock representation* and its cyclic vector is referred to as *vacuum vector*.

For any symplectic form σ on $\mathbb{R}^{2N}$ we may introduce the corresponding *Schrödinger representation* $\pi_S\colon \mathcal{W}(\mathbb{R}^{2N},\sigma) \to \mathcal{B}(L^2(\mathbb{R}^N))$ as follows: Upon suitably rescaling and rearranging a symplectic basis we may pick an invertible linear map $L\colon \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ such that $\sigma(u,v) = -\beta(Lu,Lv)/2$ for all $u, v \in \mathbb{R}^{2N}$; referring to the operators S in (5.6) we put

$$\pi_S(W(u)) := S(Lu) \qquad (u \in V)$$

and indeed obtain the correct Weyl relations, since $\pi_S(W(u))\pi_S(W(v)) = S(Lu)S(Lv) = e^{-i\beta(Lu,Lv)/2}S(Lu+Lv) = e^{i\sigma(u,v)}\pi_S(W(u+v))$; The Weyl algebra generated from $\{\pi_S(W(u)) \mid u \in V\}$ is $*$-isomorphic to $\mathcal{W}(\mathbb{R}^{2N},\sigma)$ and hence π_S can be extended to the representation corresponding to this $*$-isomorphism. We obtain from Example 5.10, 1), that π_S is regular and irreducible.

We will now show that π_S is the unique irreducible regular representation of $\mathcal{W}(\mathbb{R}^{2N}, \sigma)$ up to unitary equivalence.

5.12. Theorem (Stone-von Neumann): Let $\pi\colon \mathcal{W}(\mathbb{R}^{2N}, \sigma) \to \mathcal{B}(\mathcal{H})$ be a regular representation of the Weyl algebra. Then π can be decomposed as the direct sum $\pi = \oplus_{j\in J}\pi_j$ of subrepresentations π_j ($j \in J$) such that π_j is equivalent to the Schrödinger representation π_S for each $j \in J$. In particular, if π is irreducible then π is equivalent to π_S.

Proof: It suffices to prove the theorem for the case $\sigma = -\beta/2$, since the same method used above to define Schrödinger representations for different σ can be applied to any given representation and provides a perfect correspondence.

Step 1: The "integrated representation" $\tilde{\pi}\colon L^1(\mathbb{R}^{2N}) \to \mathcal{B}(\mathcal{H})$.

For any given function $F \in L^1(\mathbb{R}^{2N})$ we may define the sesquilinear form s_F on $\mathcal{H}$, defined by

$$s_F(x, y) := \int F(u)\langle x|\pi(W(u))y\rangle \, du \qquad (x, y \in \mathcal{H}).$$

Note that the regularity of π provides continuity of $u \mapsto \langle x|\pi(W(u))y\rangle$ (hence Borel measurability) and that $|\langle x|\pi(W(u))y\rangle| \leq \|x\|\|\pi(W(u))y\| = \|x\|\|y\|$ guarantees the convergence of the integral while also implying $|s_F(x, y)| \leq \|F\|_{L^1}\|x\|\|y\|$. Therefore, s_F is bounded and there is a unique operator $\tilde{\pi}(F) \in \mathcal{B}(\mathcal{H})$ with $\|\tilde{\pi}(F)\| \leq \|F\|_{L^1}$ such that

$$s_F(x, y) = \langle x|\tilde{\pi}(F)y\rangle \qquad (x, y \in \mathcal{H}).$$

We thus obtained a bounded linear map $\tilde{\pi}\colon L^1(\mathbb{R}^{2N}) \to \mathcal{B}(\mathcal{H})$. It is easy to see that

$$\tilde{\pi}(F)^* = \tilde{\pi}(F^*),$$

where $F^*(u) := \overline{F(-u)}$. Furthermore, if $F, G \in L^1(\mathbb{R}^{2N})$ then the Weyl relations yield[4]

$$\tilde{\pi}(F)\tilde{\pi}(G) = \tilde{\pi}(F \star G),$$

where $F \star G$ is given by the *twisted convolution*

$$(F \star G)(u) = \int e^{i\sigma(u,v)}F(u - v)G(v) \, dv.$$

Indeed, for arbitrary $x, y \in \mathcal{H}$, we have

$$\begin{aligned}
\langle x|\tilde{\pi}(F)\tilde{\pi}(G)y\rangle &= \int F(w)\langle x|\pi(W(w))\tilde{\pi}(G)y\rangle \, dw = \int F(w)\langle \pi(W(w))^*x|\tilde{\pi}(G)y\rangle \, dw \\
&= \iint F(w)G(v)\langle \pi(W(w))^*x|\pi(W(v))y\rangle \, dv \, dw = \iint F(w)G(v)\langle x|\pi(W(w)W(v))y\rangle \, dw \, dv \\
&= \iint F(w)G(v)e^{i\sigma(w,v)}\langle x|\pi(W(w + v))y\rangle \, dw \, dv \\
&= \iint F(u - v)G(v)e^{i\sigma(u-v,v)}\langle x|\pi(W(u))y\rangle \, du \, dv \\
&= \int \langle x|\pi(W(u))y\rangle \int e^{i\sigma(u,v)}F(u - v)G(v) \, dv \, du.
\end{aligned}$$

[4]Strictly speaking, this aspect of $\tilde{\pi}$ about the twisted convolution is not required logically for the current proof. However, we decided to add the information about it here, because it seems to be the best context for this elegant relation.

To summarize, $\tilde{\pi}$ is a $*$-representation of the Banach $*$-algebra $(L^1(\mathbb{R}^{2N}), \star)$.

Step 2: The representation $\tilde{\pi}$ of $L^1(\mathbb{R}^{2N})$ is faithful, i.e., injective.

It is easily seen from the Weyl relations that for any $F \in L^1(\mathbb{R}^{2N})$, $x, y \in \mathcal{H}$, and $w \in \mathbb{R}^{2N}$,

$$\langle x|\pi(W(w))\tilde{\pi}(F)\pi(W(-w))y\rangle = \int e^{2i\sigma(w,u)} F(u)\langle x|\pi(W(u))y\rangle \, du.$$

Suppose $\tilde{\pi}(F) = 0$, then we obtain for all $w = (a, b) \in \mathbb{R}^N \times \mathbb{R}^N$ (writing also $u = (c, d)$) that

$$0 = \int e^{i(\langle b|c\rangle - \langle a|d\rangle)} F(c, d)\langle x|\pi(W(c, d))y\rangle \, d(c, d) = \int e^{i\langle (b,-a)|u\rangle} F(u)\langle x|\pi(W(u))y\rangle \, du.$$

We conclude from Fourier analysis that $F(u)\langle x|\pi(W(u))y\rangle = 0$ for almost all $u \in \mathbb{R}^{2N}$ and for all $x, y \in \mathcal{H}$, which in turn implies that $F(u) = 0$ for almost all $u \in \mathbb{R}^{2N}$.

Step 3: An orthogonal projection P from the Gaussian function $G_0(u) := e^{-|u|^2/4}$.

Clearly, $0 \neq G_0 \in L^1(\mathbb{R}^{2N})$ and from $G_0^*(u) = \overline{G_0(-u)} = G_0(u)$ we obtain that $P_0 := \tilde{\pi}(G_0)$ is a nonzero self-adjoint operator on $\mathcal{H}$. We claim that for every $w \in \mathbb{R}^{2N}$,

$$P_0\pi(W(w))P_0 = (2\pi)^N G_0(w)P_0.$$

Arguing similarly as above, we have for arbitrary $x, y \in \mathcal{H}$,

$$\begin{aligned}\langle x|P_0\pi(W(w))P_0y\rangle &= \iint G_0(u)G_0(v)\langle x|\pi(W(u)W(w)W(v))y\rangle \, du \, dv \\ &= \iint G_0(u)G_0(v)e^{i\sigma(u,w)}e^{i\sigma(u+w,v)}\langle x|\pi(W(u + w + v)y\rangle \, du \, dv.\end{aligned}$$

Now a change of integration variables from (u, v) to (ξ, η) defined by $u = (\xi - \eta - w)/2$ and $v = (\xi + \eta - w)/2$ gives, upon some trivial simplifications,

$$\langle x|P_0\pi(W(w))P_0y\rangle = \frac{1}{2^{2N}} \iint G_0\Big(\frac{\xi - \eta - w}{2}\Big)G_0\Big(\frac{\xi + \eta - w}{2}\Big)e^{\frac{i}{2}\sigma(\xi+w,\eta)} \, d\eta \, \langle x|\pi(W(\xi)y\rangle \, d\xi.$$

Noting that $G_0((\xi - \eta - w)/2)G_0((\xi + \eta - w)/2) = e^{-|\xi-w|^2/8}e^{-|\eta|^2/8}$ we have the intermediate result

$$\langle x|P_0\pi(W(w))P_0y\rangle = \int F_w(\xi)e^{-\frac{|\xi-w|^2}{8}}\langle x|\pi(W(\xi)y\rangle \, d\xi,$$

where $F_w(\xi) := \frac{1}{2^{2N}} \int e^{\frac{i}{2}\sigma(\xi+w,\eta)}e^{-|\eta|^2/8} \, d\eta$. Writing $\xi = (r, s)$, $w = (a, b)$, and $\eta = (q, p)$ and using $\sigma(\xi + w, \eta)/2 = -\beta((r + a, s + b),(q, p))/4 = -(\langle r + a|p\rangle - \langle s + b|q\rangle)/4$ we have

$$F_{(a,b)}(r, s) = \frac{1}{2^{2N}} \int e^{-\frac{i}{4}\langle(-s-b,r+a)|(q,p)\rangle}e^{-\frac{|q|^2+|p|^2}{8}} \, d(q, p),$$

which we may interpret as the $2N$-dimensional Fourier transform of the Gaussian function $u \mapsto e^{-\langle u|u/4\rangle/2}/2^{2N}$ evaluated at $(-s - b, r + a)/4 \in \mathbb{R}^N \times \mathbb{R}^N$. The result is $F_{(a,b)}(r, s) = (2\pi)^N e^{-(|s+b|^2+|r+a|^2)/8} = (2\pi)^N e^{-|\xi+w|^2/8}$ (using, e.g., [Hoe90, Theorem 7.6.1]). Noting that $e^{-|\xi-w|^2/8}e^{-|\xi+w|^2/8} = e^{-|\xi|^2/4}e^{-|w|^2/4}$ we arrive at

$$\langle x|P_0\pi(W(w))P_0y\rangle = (2\pi)^N e^{-|w|^2/4} \int e^{-|\xi|^2/4}\langle x|\pi(W(\xi)y\rangle \, d\xi = (2\pi)^N G_0(w)\langle x|P_0y\rangle$$

and thus the claim $P_0\pi(W(w))P_0 = (2\pi)^N G_0(w)P_0$ is verified. In particular, with $w = 0$ we obtain $P_0^2 = (2\pi)^N P_0$ so that

$$P := (2\pi)^{-N} P_0 = (2\pi)^{-N}\tilde{\pi}(G_0)$$

is an orthogonal projection on the representation Hilbert space $\mathcal{H}$ with the property

$$P\pi(W(w))P = e^{-|w|^2/4}P.$$

Step 4: Decomposing $\mathcal{H}$ and π with the help of P.

Let $\mathcal{E} := \operatorname{ran} P$. We have for any $x, y \in \mathcal{E}$ and $u, v \in \mathbb{R}^{2N}$,

$$(*) \quad \langle \pi(W(u))x|\pi(W(v))y\rangle = \langle \pi(W(u))Px|\pi(W(v))Py\rangle = \langle P\pi(W(-v)W(u))Px|y\rangle$$
$$= e^{i\sigma(v,u)}\langle P\pi(W(u-v))Px|y\rangle = e^{i\sigma(v,u)}e^{-|u-v|^2/4}\langle Px|y\rangle = e^{i\sigma(v,u)}e^{-|u-v|^2/4}\langle x|y\rangle,$$

hence $x \perp y$ implies $\pi(W(u))x \perp \pi(W(v))y$ for all $u, v \in \mathbb{R}^{2N}$.

Let $\{e_j \mid j \in J\}$ be a complete orthonormal system of $\mathcal{E}$ and define $\mathcal{H}_j$ to be the closure of $\operatorname{span}\{\pi(W(u))e_j \mid u \in \mathbb{R}^{2N}\}$ for every $j \in J$. Then each $\mathcal{H}_j$ is invariant under π and $\mathcal{H}_j \perp \mathcal{H}_k$ when $j \neq k$. We define π_j as the subrepresentation of π on $\mathcal{H}_j$ ($j \in J$).

We claim that $\bigoplus_{j\in J}\mathcal{H}_j = \mathcal{H}$. Let $\mathcal{K} := (\bigoplus_{j\in J}\mathcal{H}_j)^{\perp}$, then $\mathcal{K}$ defines a subrepresentation. In case $\mathcal{K} \neq \{0\}$ we could argue as above to obtain $P|_{\mathcal{K}}$ as a nonzero orthogonal projection. But this would contradict the fact that $\mathcal{E} = \operatorname{ran} P \subseteq \mathcal{K}^{\perp}$. Thus $\mathcal{K} = \{0\}$ and indeed $\pi = \oplus_{j\in J}\pi_j$.

Step 5: It remains to show that each π_j is equivalent to the Schrödinger representation π_S.

We obtain from $(*)$ that the vectors $\pi(W(u))e_j = \pi_j(W(u))e_j$ ($u \in \mathbb{R}^{2N}$) satisfy

$$\langle \pi_j(W(u))e_j|\pi_j(W(v))e_j\rangle = e^{i\sigma(v,u)}e^{-|u-v|^2/4}.$$

We compare this with inner products of actions on Gaussian functions in the Schrödinger representation as given in (5.6): Let $S(u) := \pi_S(W(u))$, write $u = (a, b)$, $v = (c, d)$, and consider $\varphi(r) := e^{-|r|^2/2}/\pi^{N/4}$ ($r \in \mathbb{R}^N$); then $\varphi \in L^2(\mathbb{R}^N)$ with $\|\varphi\|_{L^2} = 1$ and

$$S(u)\varphi(r) = S(a,b)\varphi(r) = \frac{1}{\pi^{N/4}}e^{i\langle a|b\rangle/2}e^{i\langle a|r\rangle}e^{-|r+b|^2/2},$$

so that we may start calculating the L^2 inner products

$$\langle S(u)\varphi|S(v)\varphi\rangle = \langle S(a,b)\varphi|S(c,d)\varphi\rangle = \frac{e^{i(\langle c|d\rangle - \langle a|b\rangle)/2}}{\pi^{N/2}}\int e^{-i\langle a-c|r\rangle}e^{-(|r+b|^2+|r+d|^2)/2}\,dr;$$

changing of variables according to $r = s - \frac{1}{2}(b+d)$ gives

$$\langle S(u)\varphi|S(v)\varphi\rangle = \frac{e^{i(\langle c|d\rangle - \langle a|b\rangle)/2}}{\pi^{N/2}}\int e^{-i\langle a-c|s-\frac{1}{2}(b+d)\rangle}e^{-(|s+\frac{1}{2}(b-d)|^2+|s-\frac{1}{2}(b-d)|^2)/2}\,ds$$
$$= \frac{e^{i(\langle c|d\rangle - \langle a|b\rangle + \langle a-c|b+d\rangle)/2}}{\pi^{N/2}}\int e^{-i\langle a-c|s\rangle}e^{-|s|^2}e^{-\frac{|b-d|^2}{4}}\,ds$$
$$= \frac{e^{i(\langle a|d\rangle - \langle b|c\rangle)/2}}{\pi^{N/2}}e^{-\frac{|b-d|^2}{4}}\int e^{-i\langle a-c|s\rangle}e^{-|s|^2}\,ds = \frac{e^{i\sigma(v,u)}}{\pi^{N/2}}e^{-\frac{|b-d|^2}{4}}\int e^{-i\langle a-c|s\rangle}e^{-|s|^2}\,ds,$$

where the integral is the N-dimensional Fourier transform of the function $s \mapsto e^{-|s|^2}$ evaluated at $a-c$, which gives $\pi^{N/2}e^{-|a-c|^2/4}$; therefore, we obtain

$$(**) \qquad \langle S(u)\varphi | S(v)\varphi\rangle = e^{i\sigma(v,u)}e^{-|u-v|^2/4} = \langle \pi_j(W(u))e_j | \pi_j(W(v))e_j\rangle.$$

Recall that $\mathcal{H}_j$ is the closure of $\{\pi_j(W(u))e_j \mid u \in \mathbb{R}^{2N}\}$ and observe that by $(**)$, we have for any linear combination $z = \sum_{l=1}^m \lambda_l \pi_j(W(u_l))e_j$ the equation $\|z\|^2 = \|\sum_{l=1}^m \lambda_l S(u_l)\varphi\|_{L^2}^2$, which tells us that the linear map $z \mapsto Uz := \sum_{l=1}^m \lambda_l S(u_l)\varphi$ is isometric, thus extends to an isometric linear map $U\colon \mathcal{H}_j \to L^2(\mathbb{R}^N)$ such that

$$U\pi_j(W(u)) = S(u)U = \pi_S(W(u))U$$

for all $u \in \mathbb{R}^{2N}$. Since the Schrödinger representation is irreducible, the nonzero vector φ is cyclic for π_S, hence U is surjective and therefore unitary ([KRI, Proposition 2.4.6, (ii)]). $\square$

5.13. Remark: (i) Any regular representation π of the Weyl algebra $\mathcal{W}(\mathbb{R}^{2N}, \sigma)$ on a Hilbert space $\mathcal{H}$ can be interpreted as a unitary representation of the *Heisenberg group* $H_N := \mathbb{R}^{2N} \times \mathbb{R}$, where the group multiplication is given by $(u,s)\circ(v,t) := (u+v, s+t+\sigma(u,v))$. We may then define the strongly continuous group homomorphism $\theta\colon H_N \to \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators on $\mathcal{H}$, by

$$\theta(u,s) := e^{is}\pi(W(u)) \qquad ((u,s) \in H_N).$$

We quickly check that indeed

$$\theta(u,s)\theta(v,t) = e^{i(s+t)}\pi(W(u))\pi(W(v)) = e^{i(s+t)}e^{i\sigma(u,v)}\pi(W(u+v)) = \theta((u,s)\circ(v,t)).$$

Note that θ has the specific property $\theta(0,s) = e^{is}I$, thus maps the center $\{0\} \times \mathbb{R}$ of H_N onto the center $S^1 I$ of $\mathcal{U}(\mathcal{H})$, while the family ζ_z $(z \in \mathbb{R}^{2N})$ of irreducible one-dimensional group representations $\zeta_z(u,s) := e^{i\langle z|u\rangle}$ acts trivially on the center and cannot arise from a Weyl algebra representation (the latter are necessarily faithful, hence always infinite-dimensional). An analogue of the Stone-von Neumann theorem can be proved also by advanced methods from the theory of unitary representations of locally compact groups (cf. [Fol16, Theorem 6.50]).

(ii) We learn from Equation $(**)$ in the above proof that $\langle \varphi | \pi_S(W(u))\varphi\rangle = e^{-|u|^2/4}$ for the Gaussian function $\varphi(x) = e^{-|x|^2/2}/\pi^{N/4}$ in the Schrödinger representation. Therefore, $A \mapsto \rho(A) := \langle \varphi | \pi_S(A)\varphi\rangle$ defines a Fock state (see Remark 5.11) on the Weyl algebra $\mathcal{W}(\mathbb{R}^{2N}, \sigma)$. By Proposition 3.5, the Schrödinger representation is equivalent to the corresponding GNS representation and the irreducibility of π_S implies that ρ is pure. (See also [BR2, Corollary 5.2.15 and Example 5.2.16] and [Petz90, Theorem 4.7, Corollary 4.8, and Theorems 9.3 and 9.4] for more on this context with Fock states.)

(iii) In a certain sense, the Stone-von Neumann theorem "justifies" the focus on representations built up from the Schrödinger representation for quantum models with finitely many degrees of freedom. The situation is radically different with infinitely many degrees of freedom: If (V, σ) is an infinite-dimensional symplectic vector space, then there are always uncountably many inequivalent regular irreducible representations of the Weyl algebra $\mathcal{W}(V, \sigma)$; this can be considered one of the reasons for describing basic models of quantum systems to some extent in the language of C*-algebras rather than solely in terms of concrete representations. For a first selection of brief discussions, including partial results and many further references, we may recommend the sources [Petz90, Chapter 9, in particular, e.g., Theorem 9.10], [BR2, on page 218], [Emch, Chapter 3, Subsection 1.f], and [Haa96, Subsection II.1.1].

5.14. The Boson and Fermion Fock spaces: We briefly sketch the basic constructions here and refer to [KRI, Section 2.5], [RSI, Section II.4], [Ott95, Section 2.1], [Fol08, Section 4.5] for more details and proofs.

A preparation—Hilbert space tensor product: Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces and denote by $\mathcal{K}$ their algebraic tensor product. There is a unique inner product on $\mathcal{K}$ such that on splitting tensors $x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \in \mathcal{K}$ with $x_j, y_j \in \mathcal{H}_j$ $(j = 1, \ldots, n)$ we have

$$\langle x_1 \otimes \cdots \otimes x_n | y_1 \otimes \cdots \otimes y_n \rangle = \langle x_1 | y_1 \rangle \cdots \langle x_n | y_n \rangle.$$

The *Hilbert space tensor product* $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is the completion of $\mathcal{K}$ with respect to the norm defined from the above inner product. If we have a complete orthonormal system S_j of $\mathcal{H}_j$ for each $j = 1, \ldots, n$, then the set $\{z_1 \otimes \cdots \otimes z_n \mid z_j \in S_j \, (j = 1, \ldots, n)\}$ is a complete orthonormal system in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$.

Hilbert space tensor products $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ arise in quantum mechanical models describing systems with n particles, where the particle number j has vector state space $\mathcal{H}_j$.

Symmetric and antisymmetric tensor products: In case of n particles of the same species we have $\mathcal{H}_j = \mathcal{H}$ for all $j = 1, \ldots, n$. We denote by $\otimes^n \mathcal{H}$ the n-fold Hilbert space tensor product $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$. If $S = \{e_l \mid l \in M\}$ is a complete orthonormal system in $\mathcal{H}$, then $\{e_{l_1} \otimes \cdots \otimes e_{l_n} \mid l_1, \ldots, l_n \in M\}$ is a complete orthonormal system in $\otimes^n \mathcal{H}$.

Since quantum particles of the same species are indistinguishable, certain configurations of splitting tensors $x_1 \otimes \cdots \otimes x_n$ of one-particle state vectors $x_j \in \mathcal{H}$ will yield the same state of the entire n-particle system.

Particles whose multiparticle states are invariant under any permutation of the indices j in each splitting tensor vector state are called *Bosons*. Their modeling Hilbert space is given by a subspace of $\otimes^n \mathcal{H}$ corresponding to the *symmetric tensor product* $Ⓢ^n \mathcal{H}$, which can be defined as follows: Let $\mathcal{S}_n$ denote the permutation group for n elements, then it can be shown that there is a uniquely determined orthogonal projection $P_{n,s}$ on $\otimes^n \mathcal{H}$ satisfying

$$P_{n,s}(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)};$$

we then define

$$Ⓢ^n \mathcal{H} := \operatorname{ran} P_{n,s}.$$

Fermions are particles whose multiparticle states are built up from the subspace of $\otimes^n \mathcal{H}$ corresponding to the *antisymmetric tensor product* $\bigwedge^n \mathcal{H}$, which can be defined in terms of the uniquely determined orthogonal projection $P_{n,a}$ on $\otimes^n \mathcal{H}$ satisfying

$$P_{n,a}(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} (\operatorname{sgn} \tau) \, x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}$$

and putting

$$\bigwedge^n \mathcal{H} := \operatorname{ran} P_{n,a}.$$

Note that $\bigwedge^n \mathcal{H} = \{0\}$, if $\dim \mathcal{H} < n$.

In elementary quantum mechanics or non-relativistic quantum physics it is an additional postulate that particles with integer spin are described as Bosons (with Bose-Einstein statistics) and particles with spin that is the half of an odd number are described as Fermions (with Fermi-Dirac statistics). In relativistic quantum theories, this *connection between spin and statistics* can be turned into a theorem (cf. [Thi10, Part I, 3.1.16], [Fol08, pages 89, 116, and 152-153] [Haa96, Subsection II.5.1], and [SW00, Section 4.4]).

The Boson Fock space: To allow for models of large systems in quantum statistical mechanics or for relativistic theories where particles may be annihilated or created in scattering processes, we strive for a basic Hilbert state space that can "accommodate" an arbitrary number of particles. Let us add here the conventions $\otimes^0 \mathcal{H} := \mathbb{C}$, $\circledS^0 \mathcal{H} := \mathbb{C}$, and $\bigwedge^0 \mathcal{H} := \mathbb{C}$, to potentially have observables also acting on a nontrivial Hilbert space when no particle is present (vacuum).

As a basic arena in case of particles with integer spin and one-particle Hilbert space $\mathcal{H}$ one can then take the *Boson Fock space*

$$\mathcal{F}_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \circledS^n \mathcal{H}.$$

(Recall that the Hilbert space direct sum is the completion of the algebraic direct sum and that the summands are mutually orthogonal subspaces.)

Upon putting $P_{0,s} := I$, we may define $P_s := \oplus_{n=0}^{\infty} P_{n,s}$ as operator on the general *Fock space* $\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \otimes^n \mathcal{H}$ and obtain $\mathcal{F}_s(\mathcal{H}) = \operatorname{ran} P_s$. Let $\mathcal{F}^0(\mathcal{H})$ denote the algebraic direct sum of the spaces $\otimes^n \mathcal{H}$ $(n \in \mathbb{N}_0)$, then $\mathcal{F}^0(\mathcal{H})$ is dense in $\mathcal{F}(\mathcal{H})$ and

$$\mathcal{F}_s^0(\mathcal{H}) := P_s(\mathcal{F}^0(\mathcal{H})) \tag{5.7}$$

is dense in the Boson Fock space $\mathcal{F}_s(\mathcal{H})$. Each state vector in $\mathcal{F}_s^0(\mathcal{H})$ or $\mathcal{F}^0(\mathcal{H})$ involves only a bounded total number of particles and is produced from a finite linear combination of splitting tensors with a bounded number of factors.

The Fermion Fock space: For an arbitrary number of particles with half-odd (non-integer) spin we define the *Fermion Fock space*

$$\mathcal{F}_a(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \bigwedge^n \mathcal{H}.$$

We can again put $P_{0,a} := I$ and $P_a := \oplus_{n=0}^{\infty} P_{n,a}$ to obtain $\mathcal{F}_a(\mathcal{H}) = \operatorname{ran} P_a$ and define

$$\mathcal{F}_a^0(\mathcal{H}) := P_a(\mathcal{F}^0(\mathcal{H})). \tag{5.8}$$

5.15. A Weyl algebra representation on the Boson Fock space: Let $\mathcal{E}$ be a pre-Hilbert space with completion $\mathcal{H}$. For an example used frequently in physics, one could consider the situation where $\mathcal{E}$ is a space of test functions like $\mathscr{D}(\mathbb{R}^N)$ or $\mathscr{S}(\mathbb{R}^N)$ and $\mathcal{H} = L^2(\mathbb{R}^N)$.

Define V as the real vector space underlying $\mathcal{E}$ and

$$\sigma(u,v) := -\frac{1}{2}\operatorname{Im}\langle u|v\rangle \qquad (u,v \in V).$$

We will sketch the basics for a construction of a specific representation of the Weyl algebra $\mathcal{W}(V,\sigma)$ on the Boson Fock space $\mathcal{F}_s(\mathcal{H})$. For more details we may refer to [BR2, Subsections 5.2.1-2], [Ott95, Section 3.1], and [Fol08, Section 4.5].

General prototypes for annihilation and creation operators: For any given $u \in V$ we define operators $A_0(u)$ and $C_0(u)$ with dense domain $\mathcal{F}^0(\mathcal{H})$ in the Fock space $\mathcal{F}(\mathcal{H})$ by linear extension of the following maps given on splitting tensors in the form

$$A_0(u)(v_1 \otimes \cdots \otimes v_n) := \sqrt{n}\,\langle u|v_1\rangle\, v_2 \otimes \cdots \otimes v_n \qquad (n \geq 2),$$

$A_0(u)\lambda := 0$ for $\lambda \in \mathbb{C} = \otimes^0\mathcal{H}$, $A_0(u)v_1 := \langle u|v_1\rangle \in \mathbb{C} = \otimes^0\mathcal{H}$, and

$$C_0(u)(v_1 \otimes \cdots \otimes v_n) := \sqrt{n+1}\; u \otimes v_1 \otimes \cdots \otimes v_n \qquad (n \geq 1),$$

$C_0(u)\lambda := \lambda u$ for $\lambda \in \mathbb{C} = \otimes^0\mathcal{H}$.

We see that $A_0(u)(\otimes^{n+1}\mathcal{H}) \subseteq \otimes^n\mathcal{H}$, $C_0(u)(\otimes^n\mathcal{H}) \subseteq \otimes^{n+1}\mathcal{H}$, and easily check, using the fact $\otimes^k\mathcal{H} \perp \otimes^n\mathcal{H}$ if $k \neq n$ within the Fock space and by calculating on splitting tensors, that we have for all $\varphi, \psi \in \mathcal{F}^0(\mathcal{H})$,

$$\langle A_0(u)\varphi|\psi\rangle = \langle \varphi|C_0(u)\psi\rangle \tag{5.9}$$

This relation means in the language of adjoints of densely defined unbounded operators that $C_0(u) \subseteq A_0(u)^*$ and $A_0(u) \subseteq C_0(u)^*$.

We claim that $A_0(u)$ has both $\mathcal{F}^0_s(\mathcal{H})$ and $\mathcal{F}^0_a(\mathcal{H})$ as invariant subspaces, i.e.,

$$A_0(u)\mathcal{F}^0_s(\mathcal{H}) \subseteq \mathcal{F}^0_s(\mathcal{H}) \quad \text{and} \quad A_0(u)\mathcal{F}^0_a(\mathcal{H}) \subseteq \mathcal{F}^0_a(\mathcal{H}). \tag{5.10}$$

Indeed, for the symmetric case, we note that

$$\begin{aligned} A_0(u)P_s(v_1 \otimes \cdots \otimes v_n) &= \frac{\sqrt{n}}{n!}\sum_{\tau \in \mathcal{S}_n} \langle u|v_{\tau(1)}\rangle\, v_{\tau(2)} \otimes \cdots \otimes v_{\tau(n)} \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\langle u|v_k\rangle \frac{1}{(n-1)!}\sum_{\tau \in \mathcal{S}_n, \tau(1)=k} v_{\tau(2)} \otimes \cdots \otimes v_{\tau(n)} \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\langle u|v_k\rangle P_s(v_1 \otimes \cdots \widehat{v_k} \cdots \otimes v_n) \in \mathcal{F}^0_s(\mathcal{H}), \end{aligned} \tag{5.11}$$

where $\widehat{v_k}$ indicates that v_k is omitted; for the antisymmetric analogue we have

$$
\begin{aligned}
(5.12)\quad A_0(u)P_a(v_1 \otimes \cdots \otimes v_n) &= \frac{\sqrt{n}}{n!} \sum_{\tau \in \mathcal{S}_n} \langle u|v_{\tau(1)}\rangle (\operatorname{sgn} \tau)\, v_{\tau(2)} \otimes \cdots \otimes v_{\tau(n)} \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \langle u|v_k\rangle \frac{1}{(n-1)!} \sum_{\tau \in \mathcal{S}_n, \tau(1)=k} (\operatorname{sgn} \tau)\, v_{\tau(2)} \otimes \cdots \otimes v_{\tau(n)} \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-1)^{k-1} \langle u|v_k\rangle P_a(v_1 \otimes \cdots \widehat{v_k} \cdots \otimes v_n) \in \mathcal{F}_a^0(\mathcal{H}).
\end{aligned}
$$

(In the last equality we used $\operatorname{sgn} \tau = (-1)^{k-1} \operatorname{sgn} \tau^{(k)}$, where the factor $(-1)^{k-1}$ arises from moving v_k back from place 1 to place k and $\tau^{(k)}$ is the permutation corresponding to τ but now leaving k fixed.)

Bosonic annihilation and creation operators: For any $u \in V$ we may project the above objects to the symmetric tensor products and thus define the operators

$$B_0(u) := P_s A_0(u) \quad \text{and} \quad B_0^\dagger(u) := P_s C_0(u)$$

on the common dense domain $\mathcal{F}_s^0(\mathcal{H}) = P_s(\mathcal{F}^0(\mathcal{H}))$, with the operators $A_0(u)$ and $C_0(u)$ considered to be restricted to that subspace of their original domain. We call $B_0(u)$ the *annihilation operator* and $B_0^\dagger(u)$ the *creation operator* for $u \in V$. Recall that by the invariance noted in (5.10) we have

$$B_0(u)P_s = P_s A_0(u) P_s = A_0(u) P_s.$$

Furthermore, it follows directly from the definition of $C_0(u)$ that

$$B_0^\dagger(u)P_s = P_s C_0(u) P_s = P_s C_0(u).$$

If $\varphi, \psi \in \mathcal{F}_s^0(\mathcal{H})$ and $u \in V$ then we immediately get from (5.9) that

$$\langle B_0(u)\varphi|\psi\rangle = \langle \varphi|B_0^\dagger(u)\psi\rangle. \tag{5.13}$$

Therefore, the *Segal field operator* for each $u \in V$

$$\Phi_0(u) := \frac{1}{\sqrt{2}}(B_0(u) + B_0^\dagger(u))$$

is densely defined and symmetric.

We claim that the bosonic annihilation and creation operators satisfy the following variant of the *canonical commutation relations*

$$[B_0(u), B_0(v)] = 0 = [B_0^\dagger(u), B_0^\dagger(v)], \quad [B_0(u), B_0^\dagger(v)] = \langle u|v\rangle I \qquad (u, v \in V). \tag{5.14}$$

Before discussing their proof, we quickly observe that the relations (5.14) immediately imply by elementary calculation these commutation relations for the Segal field operators:

$$[\Phi_0(u), \Phi_0(v)] = i \operatorname{Im}\langle u|v\rangle I \qquad (u, v \in V). \tag{5.15}$$

Due to linearity it suffices to show (5.14) when acting on an arbitrary element of the form $P_s(v_1 \otimes \cdots \otimes v_n)$. To show the first relation in (5.14) we start by noting that $B_0(w)B_0(u)P_s = A_0(w)P_sA_0(u)P_s = A_0(w)A_0(u)P_s$, whose value on a typical splitting tensor is

$$\begin{aligned} A_0(w)A_0(u)P_s(v_1 \otimes \cdots \otimes v_n) &= \frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} A_0(w)A_0(u)\, v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)} \\ &= \frac{\sqrt{n}}{n!} \sum_{\tau \in \mathcal{S}_n} \langle u|v_{\tau(1)}\rangle A_0(w)\, v_{\tau(2)} \otimes \cdots \otimes v_{\tau(n)} \\ &= \frac{\sqrt{n(n-1)}}{n!} \sum_{\tau \in \mathcal{S}_n} \langle u|v_{\tau(1)}\rangle \langle w|v_{\tau(2)}\rangle\, v_{\tau(3)} \otimes \cdots \otimes v_{\tau(n)} \end{aligned}$$

and, similarly,

$$A_0(u)A_0(w)P_s(v_1 \otimes \cdots \otimes v_n) = \frac{\sqrt{n(n-1)}}{n!} \sum_{\tau \in \mathcal{S}_n} \langle w|v_{\tau(1)}\rangle \langle u|v_{\tau(2)}\rangle\, v_{\tau(3)} \otimes \cdots \otimes v_{\tau(n)}.$$

We obtain $A_0(w)A_0(u)P_s(v_1 \otimes \cdots \otimes v_n) = A_0(u)A_0(w)P_s(v_1 \otimes \cdots \otimes v_n)$ since the summations are over all permutations. We further obtain the second relation in (5.14) using the fact established above that $B_0^\dagger(u)$ is a formal adjoint of $B_0(u)$, i.e., $B_0^\dagger(u) \subseteq B_0(u)^*$.

It remains to check the third relation in (5.14). We note that $B_0(u)B_0^\dagger(v)P_s = A_0(u)P_sC_0(v)$ and let this act on splitting tensors $w_1 \otimes \cdots \otimes w_n$ using (5.11):

$$\begin{aligned} A_0(u)P_sC_0(v)(w_1 \otimes \cdots \otimes w_n) &= \sqrt{n+1}\, A_0(u)P_s(v \otimes w_1 \otimes \cdots \otimes w_n) \\ &= \langle u|v\rangle P_s(w_1 \otimes \cdots \otimes w_n) + \sum_{k=1}^{n} \langle u|w_k\rangle\, P_s(v \otimes w_1 \otimes \cdots \widehat{w_k} \cdots \otimes w_n); \end{aligned}$$

now calculate using $B_0^\dagger(v)B_0(u)P_s = P_sC_0(v)A_0(u)P_s$ and again (5.11), which gives

$$P_sC_0(v)A_0(u)P_s(w_1 \otimes \cdots \otimes w_n) = \frac{1}{\sqrt{n}} P_sC_0(v) \sum_{k=1}^{n} \langle u|w_k\rangle P_s(w_1 \otimes \cdots \widehat{w_k} \cdots \otimes w_n)$$

$$= \frac{\sqrt{n}}{\sqrt{n}} P_s \sum_{k=1}^{n} \langle u|w_k\rangle\, v \otimes P_s(w_1 \otimes \cdots \widehat{w_k} \cdots \otimes w_n) = \sum_{k=1}^{n} \langle u|w_k\rangle P_s(v \otimes w_1 \otimes \cdots \otimes \widehat{w_k} \otimes \cdots \otimes w_n);$$

we therefore obtain

$$A_0(u)P_sC_0(v)(w_1 \otimes \cdots \otimes w_n) - P_sC_0(v)A_0(u)P_s(w_1 \otimes \cdots \otimes w_n) = \langle u|v\rangle P_s(w_1 \otimes \cdots \otimes w_n),$$

which proves $B_0(u)B_0^\dagger(v) - B_0^\dagger(v)B_0(u) = \langle u|v\rangle I$.

Weyl operators from the Segal fields: The operator $\Phi_0(u)$ can be shown to be essentially self-adjoint([BR2, Proposition 5.2.3,(1)] or [Ott95, page 70]), hence we have a unique self-adjoint extension that we denote by $\Phi(u)$. The domain of $\Phi(u)$ certainly contains the domain of $\Phi_0(u)$, which was the dense subspace $\mathcal{F}_s^0(\mathcal{H})$.

We are now ready to define the Weyl operators by

$$W(u) := \exp(i\Phi(u)) \qquad (u \in V).$$

To keep the notation simple we skipped an extra symbol π for the prospective representation of $\mathcal{W}(V,\sigma)$. We immediately obtain that $W(0) = \exp(i\Phi(0)) = I$, since $\Phi(0) = 0$, and that $W(u)$ is unitary with $W(-u) = \exp(-i\Phi(u)) = W(u)^*$ by definition. One other obvious consequence of the definition is that $t \mapsto W(tu)$ is strongly continuous for every $u \in V$, because $\Phi(tu) = t\Phi(u)$ for real t. Thus we may claim to have a regular representation of the Weyl algebra once the Weyl relations have been established, which we undertake now.

For every $u, v \in V$, it can be shown ([BR2, Proposition 5.2.4, (1)]) that the domain of $\Phi(v)$ is invariant under $W(u)$; furthermore, based on the power series expansion $W(v)\psi = \exp(i\Phi(v))\psi = \sum_{k=0}^{\infty}(i^k/k!)\Phi(v)^k\psi$, which can be shown to be valid for $\psi \in \mathcal{F}_s^0(\mathcal{H})$, and using the commutation relation for the Segal fields (5.15) in expressions like

$$\begin{aligned}\Phi(u)\Phi(v)^2 &= (\Phi(v)\Phi(u) + i\operatorname{Im}\langle u|v\rangle)\Phi(v) = \Phi(v)\Phi(u)\Phi(v) + i\operatorname{Im}\langle u|v\rangle\Phi(v)\\ &\Phi(v)(\Phi(v)\Phi(u) + i\operatorname{Im}\langle u|v\rangle) + i\operatorname{Im}\langle u|v\rangle\Phi(v) = \Phi(v)^2\Phi(u) + 2i\operatorname{Im}\langle u|v\rangle\Phi(v)\end{aligned}$$

and inductively, $\Phi(u)\Phi(v)^k = \Phi(v)^k\Phi(u) + ik\operatorname{Im}\langle u|v\rangle\Phi(v)^{k-1}$, we may calculate

$$\begin{aligned}\Phi(u)W(v)\psi &= \sum_{k=0}^{\infty}\frac{i^k}{k!}\Phi(u)\Phi(v)^k\psi = \sum_{k=0}^{\infty}\frac{i^k}{k!}\Phi(v)^k\Phi(u)\psi + \sum_{k=0}^{\infty}\frac{i^k}{k!}ik\operatorname{Im}\langle u|v\rangle\Phi(v)^{k-1}\psi\\ &= W(v)\Phi(u)\psi + i^2\operatorname{Im}\langle u|v\rangle\sum_{k=1}^{\infty}\frac{i^{k-1}}{(k-1)!}\Phi(v)^{k-1}\psi = W(v)\Phi(u)\psi - \operatorname{Im}\langle u|v\rangle W(v)\psi\end{aligned}$$

and thus derive the relation

$$\Phi(u)W(v) = W(v)\Phi(u) - \operatorname{Im}\langle u|v\rangle W(v).$$

Let $\psi \in \mathcal{F}_s^0(\mathcal{H})$ and consider the differentiable function $F\colon \mathbb{R} \to \mathcal{H}$, given by $F(t) := W(tu)W(tv)W(t(u+v))^*\psi$ ($t \in \mathbb{R}$). Noting that a generator commutes with its unitary group and $\Phi(u+v) = \Phi(u) + \Phi(v)$ we calculate the derivative as

$$\begin{aligned}F'(t) &= i\Phi(u)W(tu)W(tv)W(t(u+v))^*\psi + W(tu)(i\Phi(v))W(tv)W(t(u+v))^*\psi\\ &\qquad + W(tu)W(tv)(-i\Phi(u+v))W(t(u+v))^*\psi\\ &= iW(tu)\Big(\Phi(u)W(tv) + \Phi(v)W(tv) - W(tv)\Phi(u) - W(tv)\Phi(v)\Big)W(t(u+v))^*\psi\\ &= iW(tu)\,[\Phi(u), W(tv)]\,W(t(u+v))^*\psi = itW(tu)(-\operatorname{Im}\langle u|tv\rangle W(tv))W(t(u+v))^*\psi\\ &= -it\operatorname{Im}\langle u|v\rangle W(tu)W(tv)W(t(u+v))^*\psi = -it\operatorname{Im}\langle u|v\rangle F(t).\end{aligned}$$

Therefore, $F(t) = e^{-it^2\operatorname{Im}\langle u|v\rangle/2}F(0) = e^{-it^2\operatorname{Im}\langle u|v\rangle/2}\psi$ and considering $t = 1$ gives $e^{i\sigma(u,v)}\psi = e^{-i\operatorname{Im}\langle u|v\rangle/2}\psi = F(1) = W(u)W(v)W(u+v)^*\psi$. Since $\mathcal{F}_s^0(\mathcal{H})$ is dense, we conclude that the Weyl relations

$$W(u)W(v) = e^{i\sigma(u,v)}W(u+v)$$

are indeed satisfied.

Vacuum expectation values: Let $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}_s^0(\mathcal{H})$, i.e., Ω has vanishing vector or tensor components of order $n \geq 1$ and entry 1 in the component $\circledS^0\mathcal{H} = \mathbb{C}$. We obviously have $B_0(u)\Omega = 0$ and $B_0^\dagger(u)\Omega = u$ (the state u is created out of the vacuum) and further $B_0(u)^2\Omega = 0$, $B_0(u)B_0^\dagger(u)\Omega = \langle u|u\rangle\Omega$, and $B_0^\dagger(u)B_0^\dagger(u)\Omega = \sqrt{2}P_s(u \otimes u) = \sqrt{2}\,u \otimes u$. It is apparent that only products with an equal number of annihilators and creators acting on Ω can give any nontrivial contribution in the 0-component of $\mathcal{F}_s(\mathcal{H})$.

Recalling the definition $\Phi_0(u) = (B_0(u) + B_0^\dagger(u))/\sqrt{2}$, we therefore see that unavoidably $\langle\Omega|\Phi(u)^k\Omega\rangle = 0$ in case k is odd. If $k = 2m$ then only the terms with exactly m factors of $B_0(u)$ and $B_0^\dagger(u)$ each can contribute at all—the order of factors matters too, because there have to be more creators at work before too many annihilators act. In any case, we certainly obtain a result proportional to $\langle u|u\rangle^m$ from creating and annihilating u successively and in a "fortunate" order. It turns out ([Ott95, page 74]) that the precise combinatorics gives

$$\langle\Omega|\Phi(u)^{2m}\Omega\rangle = \frac{(2m)!}{4^m m!}\langle u|u\rangle^m$$

and therefore,

$$\langle\Omega|W(u)\Omega\rangle = \sum_{m=0}^{\infty} \frac{i^{2m}}{(2m)!}\langle\Omega|\Phi(u)^{2m}\Omega\rangle = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\frac{\langle u|u\rangle^m}{4^m} = e^{-\langle u|u\rangle/4}.$$

We give a plausibility argument—maybe it is in fact a proof—that Ω *is a cyclic vector* for the representation of the Weyl algebra defined by the operators $W(u)$ $(u \in V)$ on the Boson Fock space. (The assertion is definitely true, because the representation constructed above is known to be irreducible for independent reasons, hence every nonzero vector is cyclic.)

Recall that due to the Weyl relations, it suffices to show that $\mathcal{G} := \mathrm{span}\{W(u)\Omega \mid u \in V\}$ is a dense subspace of $\mathcal{F}_s(\mathcal{H})$. The closure $\overline{\mathcal{G}}$ is an invariant closed subspace and thus defines a subrepresentation in which the field operators exist as strong derivatives of the Weyl operators and have to agree with the restriction of $\Phi(u)$ and leave $\overline{\mathcal{G}}$ invariant. Then also the annihilation and creation operators can be obtained as operators on $\overline{\mathcal{G}}$, namely in the form $B_0(u) = (\Phi(u) + i\Phi(iu))/\sqrt{2}$ and $B_0^\dagger(u) = (\Phi(u) - i\Phi(iu))/\sqrt{2}$ (one may check that $\Phi_0(iu) = -i(B_0(u) - B_0^\dagger(u))/\sqrt{2}$ holds for the operators as introduced above).

If $\overline{\mathcal{G}} \neq \mathcal{F}_s(\mathcal{H})$, then also $\mathrm{span}\{B_0^\dagger(v_1)\cdots B_0^\dagger(v_n)\Omega \mid v_1,\ldots,v_n \in V\}$ cannot be dense in $\mathcal{F}_s(\mathcal{H})$. But this produces a contradiction, since we observe that $B_0^\dagger(v_1)\cdots B_0^\dagger(v_m)\Omega$ is a scalar multiple of $P_s(v_1 \otimes \cdots \otimes v_m)$ and hence we can clearly generate the dense subspace $\mathcal{F}_s^0(\mathcal{H})$.

Finally, we remark that having Ω as a cyclic vector implies that the Fock space representation of the Weyl algebra is unitarily equivalent to the GNS representation corresponding to the Fock state ρ with the property

$$\rho(W(u)) = e^{-\langle u|u\rangle/4} = \langle\Omega|W(u)\Omega\rangle \qquad (u \in V).$$

(Recall that strictly speaking we should have written $\pi(W(u))$ in place of $W(u)$ in the rightmost term, if π denotes the Fock representation of the abstract Weyl algebra constructed here.)

The Fock representation is irreducible. This is shown in the approach from the GNS construction in [Petz90, Theorem 4.7] and in the concrete context of Fock space, e.g., in [DG13, Theorem 9.5, (2)] or [BR2, Proposition 5.2.4, (3)] (although, it seems that according to [Ott95, Corollary 12, page 75] there might be a problem with the proof by Bratteli-Robinson, so Ottensen refers to [RSII, page 232, Lemma 1]).

6. Fermions and the canonical anticommutation relations

◇ Our main sources for this chapter are [BR2, Ott95, Fol08, Ara87, PR94, DG13].

In this chapter we start from concrete operators on the Fermion Fock space that will lead us to the canonical anticommutation relations (CAR), which are typical of fermionic fields in quantum physics. This will establish the existence of a corresponding C*-algebra of the CAR.

Throughout the current chapter, let $\mathcal{E}$ be a complex pre-Hilbert space with completion $\mathcal{H}$. The Hilbert space $\mathcal{H}$ serves as the one-particle Hilbert space for the Fermion Fock space $\mathcal{F}_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \bigwedge^n \mathcal{H}$ defined in 5.14.

6.1. Annihilation and creation operators on the Fermion Fock space: Recalling the prototypes in 5.15 and the projection P_a from the general Fock space $\mathcal{F}(\mathcal{H})$ to the Fermion Fock space built up from antisymmetric tensor products, we may now define the operators

$$a(u) := P_a A_0(u) \quad \text{and} \quad a^\dagger(u) := P_a C_0(u) \qquad (u \in \mathcal{E})$$

on the dense domain $\mathcal{F}_a^0(\mathcal{H}) = P_a(\mathcal{F}^0(\mathcal{H}))$ (recall that $\mathcal{F}^0(\mathcal{H})$ denoted the algebraic direct sum over $\otimes^n \mathcal{H}$, $n \in \mathbb{N}_0$), where we may again consider $A_0(u)$ and $C_0(u)$ to be restricted to this subspace of their original domain. We have seen in (5.10) that $A_0(u)$ leaves $\mathcal{F}_a(\mathcal{H})$ invariant, hence

$$a(u) P_a = P_a A_0(u) P_a = A_0(u) P_a,$$

while the definition of $C_0(u)$ implies

$$a^\dagger(u) P_a = P_a C_0(u) P_a = P_a C_0(u).$$

As in the bosonic case, the general symmetry relation (5.9) directly implies also a corresponding one for the fermionic annihilation and creation operators:

$$\langle a(u)\varphi | \psi \rangle = \langle \varphi | a^\dagger(u) \psi \rangle \qquad (\varphi, \psi \in \mathcal{F}_a^0(\mathcal{H})). \tag{6.1}$$

Recall from (5.12) that for the annihilation operators we have already determined the action on the antisymmetric Fock space as

$$A_0(u) P_a(w_1 \otimes \cdots \otimes w_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-1)^{k-1} \langle u | w_k \rangle P_a(w_1 \otimes \cdots \widehat{w_k} \cdots \otimes w_n),$$

and for the creation operator it is simply

$$P_a C_0(u) P_a(w_1 \otimes \cdots \otimes w_n) = P_a C_0(u)(w_1 \otimes \cdots \otimes w_n) = \sqrt{n+1}\, P_a(u \otimes w_1 \otimes \cdots \otimes w_n).$$

We can obtain even simpler descriptions by using the notation of the exterior or wedge product

$$w_1 \wedge \cdots \wedge w_n = P_a(w_1 \otimes \cdots \otimes w_n) = \frac{1}{n!} \sum_{\tau \in \mathcal{S}_n} (\operatorname{sgn} \tau)\, w_{\tau(1)} \otimes \cdots \otimes w_{\tau(n)} \qquad (w_j \in \mathcal{H}).$$

In fact, the two resulting equations could also serve as a convenient shortcut to the definition:

$$(6.2) \qquad a(u)(w_1 \wedge \cdots \wedge w_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-1)^{k-1} \langle u|w_k\rangle\, w_1 \wedge \cdots \widehat{w_k} \cdots \wedge w_n,$$

$$(6.3) \qquad a^\dagger(u)(w_1 \wedge \cdots \wedge w_n) = \sqrt{n+1}\, P_a(u \otimes w_1 \otimes \cdots \otimes w_n) = \sqrt{n+1}\; u \wedge w_1 \wedge \cdots \wedge w_n.$$

We will draw a few immediate, though important, conclusions.

(i) The *Pauli exclusion principle* reads

$$a^\dagger(u)a^\dagger(u) = 0$$

and is verified quickly from (6.3), since for any $\varphi \in \bigwedge^n \mathcal{H}$,

$$a^\dagger(u)(a^\dagger(u)\varphi) = \sqrt{n+1}\, a^\dagger(u)(u \wedge \varphi) = \sqrt{(n+1)(n+2)}\, (u \wedge u \wedge \varphi) = 0.$$

(ii) Using the expression of an *anticommutator* of A and B, given by $\{A, B\} := AB + BA$, we have the following *canonical anticommutation relations* (CAR):

$$(6.4) \qquad \{a(u), a(v)\} = 0 = \{a^\dagger(u), a^\dagger(v)\}, \quad \{a(u), a^\dagger(v)\} = \langle u|v\rangle I \qquad (u, v \in \mathcal{E}).$$

Indeed, recalling $u \wedge v = -v \wedge u$ we get for any $\varphi \in \bigwedge^n \mathcal{H}$ that

$$a^\dagger(u)a^\dagger(v)\varphi = \sqrt{(n+1)(n+2)}\, (u \wedge v \wedge \varphi) = -\sqrt{(n+1)(n+2)}\, (v \wedge u \wedge \varphi) = -a^\dagger(v)a^\dagger(u)\varphi$$

and $a(u)a(v) = -a(v)a(u)$ then follows from the symmetry relation (6.1). To show the third relation in (6.4) we first calculate

$$\begin{aligned} a(u)a^\dagger(v)(w_1 \wedge \cdots \wedge w_n) &= \sqrt{n+1}\, a(u)(v \wedge w_1 \wedge \cdots \wedge w_n) \\ &= \langle u|v\rangle w_1 \wedge \cdots \wedge w_n + \sum_{k=1}^{n} (-1)^k \langle u|w_k\rangle\, v \wedge w_1 \wedge \cdots \widehat{w_k} \cdots \wedge w_n \end{aligned}$$

and then also

$$\begin{aligned} a^\dagger(v)a(u)(w_1 \wedge \cdots \wedge w_n) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-1)^{k-1} \langle u|w_k\rangle\, a^\dagger(v)(w_1 \wedge \cdots \widehat{w_k} \cdots \wedge w_n) \\ &= -\sum_{k=1}^{n} (-1)^k \langle u|w_k\rangle\, v \wedge w_1 \wedge \cdots \widehat{w_k} \cdots \wedge w_n. \end{aligned}$$

(iii) *The fermionic annihilation and creation operators are bounded:* Let $\varphi \in \mathcal{F}_a^0(\mathcal{H})$, then

$$\|a^\dagger(u)\varphi\|^2 + \|a(u)\varphi\|^2 = \langle \varphi|a(u)a^\dagger(u)\varphi\rangle + \langle \varphi|a^\dagger(u)a(u)\varphi\rangle = \langle \varphi|\{a(u), a(u)^\dagger\}\varphi\rangle = \|u\|^2\|\varphi\|^2$$

implies that $\|a^\dagger(u)\varphi\| \le \|u\|\|\varphi\|$ and $\|a(u)\varphi\| \le \|u\|\|\varphi\|$. It follows that $a^\dagger(u)$ and $a(u)$ can be extended to bounded operators on $\mathcal{F}_a(\mathcal{H})$, which we denote by the same symbols, and that

$$a^\dagger(u) = a(u)^* \qquad (u \in \mathcal{E}). \tag{6.5}$$

Moreover, we can show that the real linear maps $u \mapsto a(u)$ and $u \mapsto a^\dagger(u)$ are isometric, i.e,

$$\|a(u)\| = \|u\| = \|a^\dagger(u)\| \qquad (u \in \mathcal{E}). \tag{6.6}$$

Indeed, the CAR and the Pauli principle imply

$$\begin{aligned}(a^\dagger(u)a(u))^2 = a^\dagger(u)(a(u)a^\dagger(u))a(u) &= a^\dagger(u)(\|u\|^2 - a^\dagger(u)a(u))a(u) \\ &= \|u\|^2 a^\dagger(u)a(u) - a^\dagger(u)a^\dagger(u)a(u)a(u) = \|u\|^2 a^\dagger(u)a(u)\end{aligned}$$

and then the C*-property of the operator norm yields

$$\|a(u)\|^4 = \|a^\dagger(u)a(u)\|^2 = \|(a^\dagger(u)a(u))^2\| = \|u\|^2\|a^\dagger(u)a(u)\| = \|u\|^2\|a(u)\|^2.$$

If $u = 0$ then $a(u) = 0$ and the assertion is trivial; in case $u \neq 0$ we have $a(u) \neq 0$ as is seen by (6.2), therefore the above equation yields $\|a(u)\|^2 = \|u\|^2$ in this case. Clearly also $\|a^\dagger(u)\| = \|a(u)^*\| = \|a(u)\| = \|u\|$.

(iv) Finally, we remark that the map $u \mapsto a(u)$, $\mathcal{E} \to \mathcal{B}(\mathcal{F}_a(\mathcal{H}))$, is conjugate linear as is obvious by (6.2).

6.2. Examples for the smallest dimensions of $\mathcal{E} = \mathcal{H}$: Let us consider the simplest examples for the constructions in 6.1 with finite $n = \dim \mathcal{E} = \dim \mathcal{H}$.

1) The most basic case is $n = 1$, i.e., $\mathcal{H} = \mathbb{C}$, with the standard inner product $\langle \lambda | \mu \rangle = \overline{\lambda}\mu$. We have $\mathcal{F}_a(\mathcal{H}) = \mathbb{C} \oplus \mathbb{C}$ and may choose the basis $(1,0)$, $(0,1)$, which yields the following matrix representations of the annihilation and creation operators:

$$a(\lambda) = \begin{pmatrix} 0 & \overline{\lambda} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad a^\dagger(\mu) = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix} \qquad (\lambda, \mu \in \mathbb{C} = \mathcal{E}).$$

Elementary calculations verify not only the CAR, but on the way also give $a(u)a^\dagger(v) = \left(\begin{smallmatrix} \overline{\lambda}\mu & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $a^\dagger(v)a(u) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \overline{\lambda}\mu \end{smallmatrix}\right)$, from which we may deduce that the $*$-algebra generated by the set $\{a(u) \mid u \in \mathcal{E}\}$ is all of $M(2, \mathbb{C})$.

2) Let us take $n = 2$ so that $\mathcal{H} = \mathbb{C}^2$, equip it again with the standard inner product $\langle u|v \rangle = \overline{u_1}v_1 + \overline{u_2}v_2$, and denote the standard unit vectors by $e_1 = (1,0)$ and $e_2 = (0,1)$. We now have the four dimensional Fermion Fock space $\mathcal{F}_a(\mathcal{H}) = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}$ with basis $(1,0,0)$, $(0,e_1,0)$, $(0,e_2,0)$, $(0,0,\sqrt{2}\,e_1 \wedge e_2)$, which is easily checked to give the following matrix representations of the annihilation and creation operators:

$$a(u) = \begin{pmatrix} 0 & \overline{u_1} & \overline{u_2} & 0 \\ 0 & 0 & 0 & -\overline{u_2} \\ 0 & 0 & 0 & \overline{u_1} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad a^\dagger(v) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 \\ 0 & -v_2 & v_1 & 0 \end{pmatrix} \qquad (u, v \in \mathbb{C}^2 = \mathcal{E}).$$

Although it is superfluous (and already a bit tedious) to verify the CAR here, formulae for the products involved on the way will again be helpful in determining the $*$-algebra $\mathcal{A}_0$ generated from $\{a(u) \mid u \in \mathcal{E}\}$. We have

$$a(u)a^\dagger(v) = \begin{pmatrix} \langle u|v\rangle & 0 & 0 & 0 \\ 0 & \overline{u_2}v_2 & -\overline{u_2}v_1 & 0 \\ 0 & -\overline{u_1}v_2 & \overline{u_1}v_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad a^\dagger(v)a(u) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{u_1}v_1 & \overline{u_2}v_1 & 0 \\ 0 & \overline{u_1}v_2 & \overline{u_2}v_2 & 0 \\ 0 & 0 & 0 & \langle u|v\rangle \end{pmatrix}$$

and therefore also

$$I - 2a^\dagger(v)a(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2\overline{u_1}v_1 & -2\overline{u_2}v_1 & 0 \\ 0 & -2\overline{u_1}v_2 & 1-2\overline{u_2}v_2 & 0 \\ 0 & 0 & 0 & 1-2\langle u|v\rangle \end{pmatrix}.$$

Let E_{jk} denote the standard basis (2×2)-matrix with entry 1 at row j, column k, and 0 otherwise. Further, denote by I_2 and 0_2 the unit and zero (2×2)-matrices, respectively. We may thus identify the following block matrices all as elements of $\mathcal{A}_0$:

$$E^{(1)}_{11} := a(e_1)a^\dagger(e_1) = \begin{pmatrix} E_{11} & 0_2 \\ 0_2 & E_{11} \end{pmatrix}, \qquad E^{(1)}_{12} := a(e_1) = \begin{pmatrix} E_{12} & 0_2 \\ 0_2 & E_{12} \end{pmatrix},$$
$$E^{(1)}_{21} := a^\dagger(e_1) = \begin{pmatrix} E_{21} & 0_2 \\ 0_2 & E_{21} \end{pmatrix}, \qquad E^{(1)}_{22} := a^\dagger(e_1)a(e_1) = \begin{pmatrix} E_{22} & 0_2 \\ 0_2 & E_{22} \end{pmatrix},$$

We want to find a second set of four linearly independent matrices $E^{(2)}_{pq}$ in $\mathcal{A}_0$ commuting with every $E^{(1)}_{jk}$. Upon introducing $V_1 := I - 2a^\dagger(e_1)a(e_1) = \mathrm{diag}(1,-1,1,-1)$ we consider

$$E^{(2)}_{11} := a(e_2)a^\dagger(e_2) = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \qquad E^{(2)}_{12} := V_1 a(e_2) = \begin{pmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{pmatrix},$$
$$E^{(2)}_{21} := V_1 a^\dagger(e_2) = \begin{pmatrix} 0_2 & 0_2 \\ I_2 & 0_2 \end{pmatrix}, \qquad E^{(2)}_{22} := a^\dagger(e_2)a(e_2) = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix},$$

which are obviously elements in $\mathcal{A}_0$. They have the nice property that $E^{(1)}_{jk}E^{(2)}_{lm} = E^{(2)}_{lm}E^{(1)}_{jk}$ for all $j,k,l,m \in \{1,2\}$ and that the matrix $B_{jklm} := E^{(1)}_{jk}E^{(2)}_{lm}$ has the (2×2)-block E_{jk} put in the corner corresponding to place (l,m) and 0_2 otherwise. In other words, the elements $B_{jklm} \in \mathcal{A}_0$ give all the members of the standard (4×4)-matrix basis and we conclude that the $*$-algebra generated from $\{a(u) \mid u \in \mathcal{E}\}$ is all of $M(4,\mathbb{C})$.

6.3. Remark: In view of Example 2) above, but also as a preparation for the proof of the following theorem, we consider a higher dimensional analogue. Suppose $\mathcal{A}$ is a finite-dimensional C*-algebra that is generated from mutually commuting subsets $E^{(1)},\dots,E^{(n)}$, such that each $E^{(r)} = \{E^{(r)}_{jk} \mid j,k \in \{1,2\}\}$ $(r = 1,\dots,n)$ is structurally like a (2×2)-matrix basis or, in the terminology of [KRII], a *self-adjoint system of matrix units* in $\mathcal{A}$, i.e., we have

$$E^{(r)*}_{jk} = E^{(r)}_{kj}, \quad E^{(r)}_{jk}E^{(r)}_{pq} = \delta_{kp}E^{(r)}_{jq}, \quad E^{(r)}_{11} + E^{(r)}_{22} = I. \tag{6.7}$$

(Remark: Elementary, though slightly tedious, arguments show that (6.7) implies the linear independence of $E^{(r)}$; furthermore, these relations together with the requirement that $E^{(r)}$ and $E^{(s)}$ commute for $r \neq s$ guarantee that these sets are, in fact, also mutually disjoint.)

It can be shown that, in this situation, we necessarily have

$$\mathcal{A} \cong M(2^n, \mathbb{C}).$$

We sketch two routes to a proof of this fact. (See also [KRII, pages 759-760].)

One is to first note that the subalgebra $\mathcal{A}_r$ of $\mathcal{A}$ generated from $E^{(r)}$ is equal to $\operatorname{span} E^{(r)}$ and $*$-isomorphic to $M(2, \mathbb{C})$, say via $\varphi_r \colon M(2, \mathbb{C}) \to \mathcal{A}_r$; then one can construct a direct $*$-isomorphism of $\bigotimes^n M(2, \mathbb{C})$ with $\mathcal{A}$, essentially by $B_1 \otimes \cdots \otimes B_n \mapsto \varphi_1(B_1) \cdots \varphi_n(B_n)$, and finally use $\bigotimes^n M(2, \mathbb{C}) \cong M(2^n, \mathbb{C})$.

The other way is as in Example 2) above and consists in showing that the $2^n \cdot 2^n$ elements produced from the commuting products $E^{(1)}_{j_1 k_1} \cdot E^{(2)}_{j_2 k_2} \cdots E^{(n)}_{j_n k_n}$ ($j_l, k_l = 1, 2$ and $l = 1, \ldots, n$) produce a self-adjoint system of ($2^n \times 2^n$)-matrix units in $\mathcal{A}$ that generates a subalgebra $\mathcal{A}_0 \subseteq \mathcal{A}$ with $\mathcal{A}_0 \cong M(2^n, \mathbb{C})$; since $I = E^{(s)}_{11} + E^{(s)}_{22}$ for all $s = 1 \ldots, n$, we may write $E^{(r)}_{jk} = E^{(r)}_{jk} \prod_{s \neq r} (E^{(s)}_{11} + E^{(s)}_{22}) \in \mathcal{A}_0$; it follows that $\mathcal{A} \subseteq \mathcal{A}_0$, since $\mathcal{A}$ is generated from $\bigcup_{r=1}^n E^{(r)}$; in summary, $\mathcal{A} \cong M(2^n, \mathbb{C})$.

We will now show that the canonical anticommutation relations determine a unique C*-algebra so that we do not rely on particular Hilbert space operator representations in their analysis.

6.4. Theorem: Let $\mathcal{E}$ be a complex pre-Hilbert space with completion $\mathcal{H}$. There exists a (unital) C*-algebra $\mathcal{A}(\mathcal{E})$, uniquely determined up to $*$-isomorphism, that is generated from a family of elements $a(u)$ ($u \in \mathcal{E}$) such that $u \mapsto a(u)$ is conjugate linear and for all $u, v \in \mathcal{E}$ the following *canonical anticommutation relations* (CAR) are satisfied:

$$\{a(u), a(v)\} = 0, \qquad \{a(u), a(v)^*\} = \langle u|v\rangle I \qquad (u, v \in \mathcal{E}). \tag{6.8}$$

We call $\mathcal{A}(\mathcal{E})$ the *CAR algebra* or *Fermion algebra* over $\mathcal{E}$. It has the following properties:

(i) The map $u \mapsto a(u)$ is an isometry, i.e., $\|a(u)\| = \|u\|$ holds for all $u \in \mathcal{E}$,

(ii) $\mathcal{A}(\mathcal{H}) = \mathcal{A}(\mathcal{E})$,

(iii) $\mathcal{A}(\mathcal{E})$ is separable, if and only if $\mathcal{E}$ is separable.

Proof: The *existence* of $\mathcal{A}(\mathcal{E})$ is established by considering the annihilation and creation operators on the Fermion Fock space as described in 6.1 and the unital C*-subalgebra of $\mathcal{B}(\mathcal{F}_a(\mathcal{H}))$ generated from these.

For the proof of *uniqueness* and properties (i)-(iii) let $\mathcal{A}$ be a C*-algebra satisfying the hypothesis of the theorem. We will show in course of the proof that (a) in case $\dim \mathcal{E}$ is finite, $\mathcal{A}$ is determined as a full matrix algebra, and (b) for infinite-dimensional $\mathcal{E}$, $\mathcal{A}$ is generated by an increasing net of full matrix algebras.

Step 1: We show that (6.8) implies that the map $u \mapsto a(u)$ is isometric. This therefore holds in the abstract setting of any CAR algebra over $\mathcal{E}$ and proves (i).

Note that $\{a(u), a(u)\} = 0$ implies $a(u)^2 = 0$ (hence also the Pauli principle $a(u)^* a(u)^* = (a(u)^2)^* = 0$) and therefore we may repeat the following calculation used in 6.1 for the proof of (6.6):

$$\begin{aligned}(a(u)^* a(u))^2 &= a(u)^*(a(u)a(u)^*)a(u) = a(u)^*(\|u\|^2 - a(u)^* a(u))a(u) \\ &= \|u\|^2 a(u)^* a(u) - a(u)^* a(u)^* a(u) a(u) = \|u\|^2 a(u)^* a(u),\end{aligned}$$

and therefore

$$\|a(u)\|^4 = \|a(u)^* a(u)\|^2 = \|(a(u)^* a(u))^2\| = \|u\|^2 \|a(u)^* a(u)\| = \|u\|^2 \|a(u)\|^2;$$

if $u \neq 0$ then necessarily $a(u) \neq 0$, for otherwise the CAR yield $\langle u|v\rangle I = \{a(u), a(v)^*\} = 0$ for all $v \in \mathcal{E}$, thus $u = 0$; hence in this case we conclude $\|a(u)\|^2 = \|u\|^2$ from the above; if $u = 0$ then $a(0) = 0$ by conjugate linearity of $u \mapsto a(u)$ and the equality of norms is trivial.

Step 2: We consider the case $\dim \mathcal{E} = n \in \mathbb{N}$ and will prove that $\mathcal{A} \cong M(2^n, \mathbb{C})$.

Choose an orthonormal basis $e_1, \ldots, e_n$ of $\mathcal{E}$ and use the abbreviations $a_j := a(e_j)$ $(j = 1, \ldots, n)$. We observe that $U_j := I - 2a_j^* a_j$ $(j = 1, \ldots, n)$ is self-adjoint and unitary in $\mathcal{A}$. Self-adjointness is obvious and by the CAR (recall in particular $a(u)^2 = 0$ etc),

$$\begin{aligned}U_j^2 &= (I - 2a_j^* a_j)^2 = I - 4a_j^* a_j + 4a_j^* a_j a_j^* a_j \\ &= I - 4a_j^* a_j + 4a_j^*(-a_j^* a_j + \langle e_j|e_j\rangle I)a_j = I - 4a_j^* a_j - 4a_j^* a_j^* a_j a_j + 4a_j^* a_j = I.\end{aligned}$$

Moreover, $U_j U_k = U_k U_j$ for $j \neq k$, since

$$\begin{aligned}U_j U_k &= (I - 2a_j^* a_j)(I - 2a_k^* a_k) = I - 2a_j^* a_j - 2a_k^* a_k + 4a_j^* a_j a_k^* a_k \\ &= I - 2a_j^* a_j - 2a_k^* a_k + 4a_j^*(-a_k^* a_j + \langle e_j|e_k\rangle)a_k = I - 2a_j^* a_j - 2a_k^* a_k - 4a_k^* a_j^* a_k a_j \\ &= I - 2a_j^* a_j - 2a_k^* a_k - 4a_k^*(-a_k a_j^* + \langle e_j|e_k\rangle)a_j \\ &= I - 2a_j^* a_j - 2a_k^* a_k + 4a_k^* a_k a_j^* a_j = U_k U_j.\end{aligned}$$

A very simple calculation also shows that U_j commutes with a_k and a_k^* in case $j \neq k$, while U_j anticommutes with a_j and a_j^*. Setting $V_0 := I$ and $V_{r-1} := U_1 \cdots U_{r-1}$ $(r = 2, \ldots, n)$ we now define for each $r = 1, \ldots, n$ the subset $E^{(r)} = \{E_{jk}^{(r)} \mid j, k \in \{1, 2\}\}$ of $\mathcal{A}$ with the elements

$$\begin{aligned}E_{11}^{(r)} &:= a_r a_r^*, & E_{12}^{(r)} &:= V_{r-1} a_r \\ E_{21}^{(r)} &:= V_{r-1} a_r^*, & E_{22}^{(r)} &:= a_r^* a_r.\end{aligned}$$

We claim that each $E^{(r)}$ is a self-adjoint system of (2×2)-matrix units as in (6.7). The verification is by straightforward calculations: Self-adjointness of $E_{11}^{(r)}$ and $E_{22}^{(r)}$ is obvious and the properties of U_j and a_k established above immediately yield

$${E_{12}^{(r)}}^* = (V_{r-1} a_r)^* = a_r^* V_{r-1} = V_{r-1} a_r^* = E_{21}^{(r)};$$

we have $E_{11}^{(r)} + E_{22}^{(r)} = a_r a_r^* + a_r^* a_r = \langle e_r|e_r\rangle I = I$; the relations $E_{j1}^{(r)} E_{2q}^{(r)} = 0$ and $E_{j2}^{(r)} E_{1q}^{(r)} = 0$ are clear, since the occurring products all involve terms $a_r^* a_r^* = 0$ or $a_r a_r = 0$; it remains to check the products $E^{(r)}_{j1}E^{(r)}_{1q}$ and $E^{(r)}_{j2}E^{(r)}_{2q}$; the special cases $E^{(r)}_{11}E^{(r)}_{11} = E^{(r)}_{11}$ and $E^{(r)}_{22}E^{(r)}_{22} = E^{(r)}_{22}$ follow by applying $a_r^* a_r = -a_r a_r^* + I$ and $a_r a_r = 0$; similarly, we have

$$E^{(r)}_{11}E^{(r)}_{12} = a_r a_r^* V_{r-1} a_r = V_{r-1} a_r a_r^* a_r = V_{r-1} a_r(-a_r a_r^* + I) = V_{r-1} a_r = E^{(r)}_{12}$$

and hence $E^{(r)}_{21}E^{(r)}_{11} = (E^{(r)}_{11}E^{(r)}_{12})^* = (E^{(r)}_{12})^* = E^{(r)}_{21}$; finally,

$$E^{(r)}_{12}E^{(r)}_{22} = V_{r-1} a_r a_r^* a_r = V_{r-1} a_r(-a_r a_r^* + I) = V_{r-1} a_r = E^{(r)}_{12}$$

and thus also $E^{(r)}_{22}E^{(r)}_{21} = (E^{(r)}_{12}E^{(r)}_{22})^* = (E^{(r)}_{12})^* = E^{(r)}_{21}$.

If $r \neq s$ then $E^{(r)}$ and $E^{(s)}$ commute: It suffices to suppose $r < s$; let $\tilde{a_k}$ denote either a_k or a_k^*; note first that $V_{s-1}\tilde{a_r} = -\tilde{a_r}V_{s-1}$, while $V_{r-1}\tilde{a_s} = \tilde{a_s}V_{r-1}$, and $V_{s-1}V_{r-1} = V_{r-1}V_{s-1}$ by the properties of U_j established above; each matrix unit in $E^{(r)}$ is the product of two elements that commute or anticommute with each of the two factors defining any matrix unit in $E^{(s)}$; we have to check that one always needs an even number of anticommuting swaps in rearranging the terms of $E^{(r)}_{jk}E^{(s)}_{pq}$ to get to the expression for $E^{(s)}_{pq}E^{(r)}_{jk}$; for $E^{(r)}_{11}$ each factor anticommutes with both factors in every $E^{(s)}_{pq}$, hence four anticommuting swaps suffice; the same holds for $E^{(r)}_{22}$; for $E^{(r)}_{12}$ or $E^{(r)}_{21}$ the factor V_{r-1} commutes with all factors in $E^{(s)}_{pq}$ while the factor $\tilde{a_r}$ anticommutes with all the factors in $E^{(s)}_{pq}$, hence two anticommuting swaps suffice.

Let $\mathcal{A}_0$ be the $*$-subalgebra of $\mathcal{A}$ generated from $\bigcup_{r=1}^n E^{(r)}$. As we saw in Remark 6.3, we may deduce from the n commuting subsets of (2×2)-matrix units constructed above, that $\mathcal{A}_0$ is $*$-isomorphic to $M(2^n, \mathbb{C})$. We claim that, in fact, $\mathcal{A}_0 = \mathcal{A}$, which then completes Step 2.

We note that $E^{(j)}_{11} - E^{(j)}_{22} = a_j a_j^* - a_j^* a_j = I - 2a_j^* a_j = U_j$, $a_1 = E^{(1)}_{12}$, and obtain for $2 \leq r \leq n$,

$$a_r = V_{r-1}(V_{r-1}a_r) = U_1 \cdots U_{r-1}(V_{r-1}a_r) = \prod_{j=1}^{r-1}(E^{(j)}_{11} - E^{(j)}_{22})E^{(r)}_{12}.$$

We thus see that $a_r \in \mathcal{A}_0$ for all $r \in \{1, \dots, n\}$ and therefore, $\mathcal{A} \subseteq \mathcal{A}_0$.

Step 3: We show uniqueness in the case where $\mathcal{E}$ is infinite-dimensional.

Let S be a maximal[1] orthonormal system in $\mathcal{E}$. For each finite subset $F \subseteq S$ let $\mathcal{E}_F := \operatorname{span} F$ and let $\mathcal{A}_F$ be the finite-dimensional C*-subalgebra of $\mathcal{A}$ generated from $\{a(u) \mid u \in \mathcal{E}_F\}$. We have

$$\mathcal{A}_{F_1} \subseteq \mathcal{A}_{F_2}, \quad \text{if } F_1 \subseteq F_2.$$

Define $\mathcal{A}_0$ as the union of all $\mathcal{A}_F$, where F ranges over the finite subsets of S. Observe that $\mathcal{A}_0$ is a $*$-subalgebra of $\mathcal{A}$, because any two elements of $\mathcal{A}_0$ lie in some joint finite-dimensional $*$-subalgebra $\mathcal{A}_F$ for sufficiently large F. In fact, $\mathcal{A}_0$ is generated as a $*$-algebra from the set $\{a(u) \mid u \in \mathcal{E}\}$ and $\mathcal{A}$ is the norm closure of $\mathcal{A}_0$. For any finite subset $F \subseteq S$, let $|F|$ denote the cardinality of F. We clearly have $|F| = \dim \mathcal{E}_F$ and, by the result of Step 2, $\mathcal{A}_F$ is the unique CAR algebra over $\mathcal{E}_F$ and satisfies

$$\mathcal{A}_F \cong M(2^{|F|}, \mathbb{C}).$$

[1]The existence of a maximal orthonormal system does not require completeness, but the convergence of the corresponding Fourier-type series expansions is guaranteed only in the completion.

If $\mathcal{B}$ is another CAR algebra over $\mathcal{E}$, then we have the analogous construction with finite-dimensional $*$-subalgebras $\mathcal{B}_F \cong \mathcal{A}_F$, due to Step 2, for every finite subset F of S; let $\varphi_F : \mathcal{A}_F \to \mathcal{B}_F$ denote the $*$-isomorphism obtained from $\mathcal{A}_F \cong M(2^{|F|}, \mathbb{C}) \cong \mathcal{B}_F$ and mapping generators to corresponding generators. We may suppose that $\varphi_{F_2}|_{\mathcal{A}_{F_1}} = \varphi_{F_1}$, if $F_1 \subseteq F_2$, since $\mathcal{A}_{F_1} \subseteq \mathcal{A}_{F_2}$ and $\mathcal{B}_{F_1} \subseteq \mathcal{B}_{F_2}$. We may consistently define a $*$-isomorphism $\varphi_0 : \mathcal{A}_0 \to \mathcal{B}_0$, such that $\varphi_0|_{\mathcal{A}_F} = \varphi_F$, where $\mathcal{B}_0$ denotes the $*$-subalgebra of $\mathcal{B}$ obtained as the union of all $\mathcal{B}_F$, where F ranges over the finite subsets of S. Recall that $\mathcal{B}$ is the norm closure of $\mathcal{B}_0$, exactly as it was the case with $\mathcal{A}$ and $\mathcal{A}_0$ above. The $*$-isomorphism φ_0 is isometric on each $\mathcal{A}_F$, since the finite-dimensional $*$-subalgebras $\mathcal{A}_F$ and $\mathcal{B}_F$ are C*-algebras. Hence φ_0 is isometric $\mathcal{A}_0 \to \mathcal{B}_0$ and has by continuity a unique extension $\varphi : \mathcal{A} \to \mathcal{B}$ which automatically becomes a $*$-isomorphism.

Step 4: We show (ii) and (iii).

By Step 1, we know that the conjugate linear map $a : u \mapsto a(u)$ is isometric $\mathcal{E} \to \mathcal{A}$. Hence the map is uniformly continuous and has thus a unique continuous extension $\hat{a} : \mathcal{H} \to \mathcal{A}$ (e.g., as map between metric spaces). By considering norm limits, it is clear that $\hat{a}$ is also conjugate linear, isometric, and satisfies the CAR. By uniqueness, we therefore obtain $\mathcal{A}(\mathcal{H})$ as a C*-subalgebra of $\mathcal{A}(\mathcal{E})$. Since the $*$-subalgebras corresponding to the finite-dimensional subspaces of $\mathcal{E}$ are certainly contained in $\mathcal{A}(\mathcal{H})$, the norm closure of their union, which is $\mathcal{A}(\mathcal{E})$, must be contained in $\mathcal{A}(\mathcal{H})$. Therefore $\mathcal{A}(\mathcal{H}) = \mathcal{A}(\mathcal{E})$ and (ii) is proved.

It remains to show (iii). If $\mathcal{A}(\mathcal{E})$ is separable then its subspace $a(\mathcal{E})$ is separable[2] and hence its isometric image $\mathcal{E}$ is separable. Finally, let $\mathcal{E}$ be separable and choose a countable orthonormal system $D = \{e_m \mid m \in \mathbb{N}\}$ such that $\operatorname{span} D$ is dense in $\mathcal{E}$. Introduce the notation $\mathcal{E}_n := \operatorname{span}\{e_m \mid m = 1, \ldots, n\}$ and $\mathcal{A}_n := \mathcal{A}_{\{e_1,\ldots,e_n\}}$. For any finite subset $F \subseteq D$, we can find $n \in \mathbb{N}$ such that $\mathcal{E}_F = \operatorname{span} F \subseteq \mathcal{E}_n$ and therefore, $\mathcal{A}_F \subseteq \mathcal{A}_n$. We obtain $\mathcal{A}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ and recall that $\mathcal{A}_0$ is a dense subspace of $\mathcal{A}$. Each $\mathcal{A}_n$ is generated by the finite set of $(2^n \times 2^n)$-matrix units constructed in Step 2, hence $\mathcal{A}_0$ is generated as a $*$-subalgebra by a countable set G of elements from $\mathcal{A}_0$. Linear combinations of finite products of elements in G with scalars that have rational real and imaginary parts now provide a countable dense subset of $\mathcal{A}_0$, thus $\mathcal{A}_0$ is separable. It follows that its norm closure $\mathcal{A}$ is separable. $\square$

6.5. Remark: (i) An inspection of the second paragraph in Step 3 of the proof of Theorem 6.4 shows that for any two CAR algebras $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{E}$, say $\mathcal{A}$ generated from $\{a(u) \mid u \in \mathcal{E}\}$ and $\mathcal{B}$ generated from $\{a'(u) \mid u \in \mathcal{E}\}$, there is a unique $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(a(u)) = a'(u)$ for all $u \in \mathcal{E}$. (Make also use of the observation near the end of Step 2, where it is shown how the generators can be expressed in terms of the matrix units.)

(ii) The proof we gave for Theorem 6.4 is quite technical and long. It is based on the proof in [BR2, Theorem 5.2.5], which is very sketchy. We got some support also from a slightly different approach, but still focusing on the unions of matrix algebras, that has been described in detail in [Ara87], also taken up and described in [DG13, Chapter 12]. However, one has to be aware of the fact that the latter two sources employ also a different convention about CAR relations—a translation between the formalisms is provided in [Ott95, Section 2.6]–and, in particular, a real odd dimensional case does not occur in our approach considering only complex pre-Hilbert spaces $\mathcal{E}$ as parameter space for the generators instead of real euclidean

[2] Recall: This is true in metric spaces (though not in topological spaces in general).

vector spaces. (Tempting as it was to work out the very short uniqueness proof for the CAR algebra sketched in [Ott95, page 19], it seems to me that this misses an argument about a well-defined extension from $\alpha(a(f)) := a'(f)$ to products; unique existence of such an extension could be built into a definition of Clifford-algebra type [see (ii)] in terms of a universal property; in our context, this would then complicate the existence proof or bring up the need to develop some Clifford theory prior to going into CAR.)

(iii) As indicated, e.g., in [Ott95, page 24], there is an equivalent approach to CAR algebras in terms of C* Clifford algebras and details on this can be found in [PR94, Section 1.2]. The starting point there is a real vector space V with a positive definite inner product and a so-called *Clifford map* $f\colon V \to \mathcal{B}$, a real linear map into a unital associative complex algebra $\mathcal{B}$ such that $f(v)^2 = \|v\|^2 I$. The connection with our context would be to take V as the real vector space underlying $\mathcal{E}$, $\mathrm{Re}\langle u|v\rangle/2$ as the inner product, and $f(u) = (a(u) + a(u)^*)/\sqrt{2}$.

(iv) The proof of Theorem 6.4 shows that CAR algebras are norm closures of a union over matrix algebras $\mathcal{A}_F$ with the compatibility property $\mathcal{A}_{F_1} \subseteq \mathcal{A}_{F_2}$ if $F_1 \subseteq F_2$. In case of *increasing sequences* of nested matrix algebras these are known and extensiveley studied in the literature under the names of uniformly matricial algebras ([KRII, Section 10.4]) or Glimm algebras [Ped18, Section 6.4] or UHF (uniformly hyperfinite) algebras ([BR1, Example 2.6.12], referred to from [BR2, page 16]; more in [Mur90, Section 6.2]).

6.6. Corollary: Let $\mathcal{E}$ be a complex pre-Hilbert space and $\mathcal{A}(\mathcal{E})$ be the CAR algebra over $\mathcal{E}$.

(i) Every representation of $\mathcal{A}(\mathcal{E})$ is faithful. (Thus the CAR algebra is simple.)

(ii) Let U be bounded linear and V bounded conjugate linear on $\mathcal{E}$ such that[3]

$$V^*U + U^*V = 0 = UV^* + VU^*,$$
$$U^*U + V^*V = I = UU^* + VV^*,$$

then there is a unique $*$-automorphism, a *Bogoliubov transformation*, α of $\mathcal{A}(\mathcal{E})$ satisfying

$$\alpha(a(z)) = a(Uz) + a(Vz)^* \qquad (z \in \mathcal{E}).$$

Proof: (i): Let $\pi\colon \mathcal{A}(\mathcal{E}) \to \mathcal{B}(\mathcal{K})$ be a representation on the Hilbert space $\mathcal{K}$. Then $\pi(a(u)) \neq 0$ for every $0 \neq u \in \mathcal{E}$, since $0 \neq \|u\|^2 I = \pi(\|u\|^2 I) = \pi(\{a(u), a(u)\})$ by the CAR. Therefore, the C*-algebra $\pi(\mathcal{A}(\mathcal{E}))$ is a CAR algebra and, as observed in Remark 6.5, (i), π is the unique $*$-isomorphism $\mathcal{A}(\mathcal{E}) \to \pi(\mathcal{A}(\mathcal{E}))$ mapping $a(u) \mapsto \pi(a(u))$, hence π is faithful.

(ii): Let $a'(u) := a(Uu) + a(Vu)^*$ $(u \in \mathcal{E})$, then $u \mapsto a'(u)$ is conjugate linear and the set

[3] A special case is certainly $V = 0$ and U unitary. The adjoint V^* of the conjugate linear operator V is characterized by the requirement $\langle Vy|z\rangle = \langle V^*z|y\rangle = \overline{\langle y|V^*z\rangle}$ for all $y, z \in \mathcal{E}$.

$\{a'(u) \mid u \in \mathcal{E}\}$ satisfies the CAR. Indeed, we have

$$\begin{aligned} a'(y)a'(z) &= a(Uy)a(Uz) + a(Uy)a(Vz)^* + a(Vy)^*a(Uz) + a(Vy)^*a(Vz)^* \\ &= -a(Uz)a(Uy) - a(Vz)^*a(Uy) + \langle Uy|Vz\rangle - a(Uz)a(Vy)^* + \langle Uz|Vy\rangle - a(Vz)^*a(Vy)^* \\ &= -a'(z)a'(y) + \langle y|U^*Vz\rangle + \langle y|V^*Uz\rangle = -a'(z)a'(y) + \langle y|(U^*V + V^*U)z\rangle = -a'(z)a'(y) \end{aligned}$$

and

$$\begin{aligned} a'(y)a'(z)^* &= a(Uy)a(Uz)^* + a(Uy)a(Vz) + a(Vy)^*a(Uz)^* + a(Vy)^*a(Vz) \\ &= -a(Uz)^*a(Uy) + \langle Uy|Uz\rangle - a(Vz)a(Uy) - a(Uz)^*a(Vy)^* - a(Vz)a(Vy)^* + \langle Vz|Vy\rangle \\ &= -a'(z)^*a'(y) + \langle y|U^*Uz\rangle + \langle y|V^*Vz\rangle = -a'(z)^*a'(y) + \langle y|(U^*U + V^*V)z\rangle \\ &= -a'(z)^*a'(y) + \langle y|z\rangle. \end{aligned}$$

Let $\mathcal{B}$ be the C*-subalgebra of $\mathcal{A}(\mathcal{E})$ generated from $\{a'(u) \mid u \in \mathcal{E}\}$. We know by the uniqueness that $\mathcal{B}$ is $*$-isomorphic to $\mathcal{A}(\mathcal{E})$ via a unique (Remark 6.5, (i)) $*$-isomorphism φ such that $\varphi(a(u)) = a'(u)$ for all $u \in \mathcal{E}$. Since also

$$\begin{aligned} a'(U^*u) + a'(V^*u)^* &= a(UU^*u) + a(VU^*u)^* + a(UV^*u)^* + a(VV^*u) \\ &= a((UU^* + VV^*)u) + a((VU^* + UV^*)u)^* = a(u) + a(0) = a(u), \end{aligned}$$

we must have $\mathcal{B} = \mathcal{A}(\mathcal{E})$ and therefore φ is a $*$-automorphism. □

6.7. A few further properties of the CAR algebra:

(i) The Fock representation 6.1 is irreducible (see, e.g., [BR2, Proposition 5.2.2, (3)] or [Ott95, Section 2.2, Theorem 1]), hence the von Neumann algebra generated from the CAR algebra on the Fermion Fock space is all of $\mathcal{B}(\mathcal{F}_a(\mathcal{H}))$. This is obvious for finite $n = \dim \mathcal{E}$, since then the Fock space has dimension 2^n and the CAR algebra is $M(2^n, \mathbb{C})$ as we have shown in the uniqueness proof. We obtain in this case a von Neumann algebra of type I_{2^n}. If $\mathcal{E}$ is infinite-dimensional then the von Neumann algebra generated from the CAR algebra in the Fock space representation is of type I_∞.

It is very easy to see independently that the Fock representation is cyclic. Consider the vacuum vector $\Omega = (1, 0, 0, \ldots)$ and let several creation operators act on it successively, then we get according to (6.3) that for any $u_1, \ldots, u_m$,

$$a^\dagger(u_m) \cdots a^\dagger(u_2)a^\dagger(u_1)\,\Omega = \sqrt{m!}\; u_m \wedge \cdots \wedge u_2 \wedge u_1,$$

so that we clearly obtain a dense subspace by their linear combinations.

Let ρ denote the vector state on $\mathcal{A}(\mathcal{E})$ corresponding to the vacuum Ω, i.e., $\rho(A) = \langle \Omega|A\Omega\rangle$ for all $A \in \mathcal{A}(\mathcal{E})$, then we have

$$\rho(a(u)a^\dagger(v)) = \langle a^\dagger(u)\Omega|a^\dagger(v)\Omega\rangle = \langle u|v\rangle.$$

Moreover, every $a(z)$ ($z \in \mathcal{E}$) belongs to the left kernel of ρ, since

$$\rho(a^\dagger(z)a(z)) = \langle a(z)\Omega|a(z)\Omega\rangle = 0.$$

(ii) For finitely many degrees of freedom, i.e., $\dim \mathcal{E} < \infty$, we have seen that the CAR algebra $\mathcal{A}(\mathcal{E})$ is finite-dimensional, in fact, $*$-isomorphic to the full matrix algebra $M(2^{\dim \mathcal{E}}, \mathbb{C})$. We will argue that in this case, $\mathcal{A}(\mathcal{E})$ possesses only one equivalence class of irreducible representations, since this is true of any full matrix algebra $M(d, \mathbb{C})$.

Suppose π is an irreducible representation of $M(d, \mathbb{C})$ on a Hilbert space $\mathcal{K}$. Then every nonzero vector in $\mathcal{K}$ is cyclic for π by Proposition 3.16. We may choose some $0 \neq z \in \mathcal{K}$, then Proposition 3.5 implies that π is equivalent to the GNS representation π_ρ associated with the state $\rho(A) := \langle z|\pi(A)z\rangle$ on $M(d, \mathbb{C})$. According to Theorem 3.17, ρ must be pure. It is elementary to show ([KRI, Exercise 4.6.18], with detailed solution in [KRIII, pages 152-153]) that the pure states on $\mathcal{M}(d, \mathbb{C})$ are the vector states, hence $\rho(A) = \overline{x_0}^T \cdot A \cdot x_0 = \langle x_0 | \operatorname{id}(A) x_0\rangle$ for some unit vector $x_0 \in \mathbb{C}^d$, where id denotes the identity representation. The latter is clearly cyclic, thus Proposition 3.5 now in turn implies that π_ρ and id are equivalent.

The uniqueness of irreducible representations for finitely many degrees of freedom is somewhat similar to the Stone-von Neumann theorem for the CCR or Weyl algebra, though much simpler because the representations here are also finite-dimensional.

(iii) As already noted in case of the CCR, it happens also with the CAR that the situation changes drastically for infinitely many degrees of freedom, where $\mathcal{E}$ is not finite-dimensional. There exist uncountably many inequivalent irreducible representations of $\mathcal{A}(\mathcal{E})$. We may compare this with the corresponding Remark 5.13, (iii), for the Weyl algebra and again just hint at a few sources containing further information or references: [BR2, on page 218], [Emch, Chapter 3, Section 2], [Ara87, Theorem 6.14], [DG13, Subsection 16.4.3, Theorem 16.58], and [Haa96, Subsection II.1.1].

(iv) Note that additional regularity of representations was not an issue with CAR algebras, since the conjugate linear map $u \mapsto a(u)$ is a norm continuous, even isometric, map $\mathcal{E} \to \mathcal{A}(\mathcal{E})$. Therefore also $u \mapsto \pi(a(u))$ is norm continuous in any representation π of $\mathcal{A}(\mathcal{E})$.

(v) It is a convenient property that states ρ on the CAR algebra $\mathcal{A}(\mathcal{E})$ are determined by their values on the norm dense subspace $\mathcal{A}_0$, defined as the $*$-subalgebra generated from $\{a(u) \mid u \in \mathcal{E}\}$. Linearity of ρ reduces everything to knowing the values of ρ on arbitrary finite products of elements and their adjoints from $\{a(u) \mid u \in \mathcal{E}\}$, but the CAR allow further reduction to products in the form $a(u_1)^* \cdots a(u_l)^* a(v_1) \cdots a(v_m)$ with $l, m \in \mathbb{N}_0$, $l + m \geq 1$, and $u_j, v_k \in u$. This is called a *Wick ordered* or *normally ordered product*—all creation operators occur to the left of all annihilation operators. In summary, a state ρ is determined by its values on all Wick ordered products

$$\rho(a(u_1)^* \cdots a(u_l)^* a(v_1) \cdots a(v_m)).$$

(vi) The *tracial state on the CAR algebra:* We recall that $\mathcal{A}(\mathcal{E})$ is the norm completion of the $*$-subalgebra $\mathcal{A}_0$ which is the union over finite-dimensional subalgebras $\mathcal{A}_F \cong M(2^{|F|}, \mathbb{C})$, where F ranges over finite subsets of some maximal orthonormal system in $\mathcal{E}$. On each $\mathcal{A}_F$ we inherit a state τ_F which corresponds to the trace on the matrix algebra $M(2^{|F|}, \mathbb{C})$. Recall from linear algebra that $\operatorname{trace}(MN) = \operatorname{trace}(NM)$ for square matrices M and N, hence we also have $\tau_F(AB) = \tau_F(BA)$ for all $A, B \in \mathcal{A}_F$. Thanks to the compatibility $\mathcal{A}_{F_1} \subseteq \mathcal{A}_{F_2}$ if $F_1 \subseteq F_2$, we can define a continuous linear functional $\tau_0 \colon \mathcal{A}_0 \to \mathbb{C}$ consistently by $\tau(A) = \tau_F(A)$ if $A \in \mathcal{A}_F$. The unique extension $\tau \colon \mathcal{A}(\mathcal{E}) \to \mathbb{C}$ is a state that inherits also the property

$$\tau(AB) = \tau(BA) \qquad (A, B \in \mathcal{A}(\mathcal{E})).$$

This property of a state is usually referred to as *tracial*, though it typically belongs more to the context of von Neumann algebras. It can be shown that in the corresponding GNS representation π_τ, one obtains as the generated von Neumann algebra $\pi_\tau(\mathcal{A}(\mathcal{E}))''$ a factor of type II_1, if $\mathcal{E}$ is of infinite dimension (cf. [Bla10, II.8.2.2, (iv), III.3.1.4, III.3.1.5, III.3.4.6], [PR94, Theorems 1.3.6 and 1.3.7], or [DG13, Theorem 12.59]).

6.8. Remark on Fock space constructions for free particles of several species: Perturbative methods in standard quantum field theory take noninteracting systems as a starting point. To incorporate several different species of particles relevant for some scattering process, one can take a tensor product of corresponding bosonic and fermionic Fock spaces for each particle species involved (see, e.g., [Fol08, pages 95-96] and also [Haa96, Section II.3]). The creation and annihilation operators for each species act like the identity operator on all other tensor factors, so informally we have the tensor product of CAR or Weyl algebras that contain also the observables. Tensor product operators acting in different factors usually commute, but one adds also a "fermionic anticommutativity", reflecting by an overall sign whether an even or odd number of Fermions is being acted on. There is a certain dilemma: On the one hand, depending too much on particular representations for models in quantum physics is problematic, on the other hand, C*-algebraic approaches face the difficulty to describe particles in a more abstract setting ([Haa96, Sections VI.1 and VI.2]).

7. A brief outlook on quasi-local algebras

◇ Our main sources for this chapter are [BR1, BR2, Emch, Haa96].

In many instances the C*-algebra and the states describing a quantum model are generated from somewhat localized fields, observables, and states associated for example with bounded regions in space (and time) or with finite parts of a chain or lattice arrangement of spins. The structure of the idealized infinite or infinitely extended quantum system is often sufficiently determined by the information on the local structures and this is the basic idea behind the notion of *quasi-local algebras*, which we will now describe mathematically. To keep the notion flexible for a variety of applications, we introduce rather abstract structures as index sets. To support our intuition with these definitions, we may imagine each index representing some bounded region in space, subsets being partially ordered by inclusion, and the orthogonality relation to mean disjointness of sets.

7.1. Directed sets with an additional orthogonality relation: Recall that a set Λ is said to be *directed* if there is a partial order[1] relation $\leq$ on Λ satisfying in addition that for each $\alpha, \beta \in \Lambda$ we can find some $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

An *orthogonality relation* on the directed set Λ is a symmetric relation $\perp$ on Λ with the following properties ($\alpha, \beta, \gamma, \delta \in \Lambda$):

(a) For each $\alpha \in \Lambda$ there is some $\beta \in \Lambda$ with $\alpha \perp \beta$,

(b) $\alpha \leq \beta$ and $\beta \perp \gamma$ imply $\alpha \perp \gamma$,

(c) if $\alpha \perp \beta$ and $\alpha \perp \gamma$ then there is some $\delta \in \Lambda$ with $\alpha \perp \delta$ and $\beta \leq \delta$ and $\gamma \leq \delta$.

7.2. Quasi-local algebras: One ingredient of the definition in this context will be an self-inverse $*$-automorphism θ on a C*-algebra $\mathcal{A}$, which may be used mainly to implement particle statistics, for example, $\theta = \mathrm{id}$ for purely bosonic models and $\theta(a(u)) = -a(u)$ on the annihilation operators of a Fermion algebra.

In this case, elements $A \in \mathcal{A}$ with $\theta(A) = A$ are called *even*, and *odd* if $\theta(A) = -A$. The subset $\mathcal{A}^+$ of even elements is easily seen to form a C*-subalgebra of $\mathcal{A}$, while the subset $\mathcal{A}^-$ of odd elements is a closed subspace of $\mathcal{A}$. Every element $A \in \mathcal{A}$ can be uniquely decomposed into odd and even parts by $A^\pm := (A \pm \theta(A))/2$ in the form $A = A^+ + A^-$.

Definition: A C*-algebra $\mathcal{A}$ is called *quasi-local*, if there is an self-inverse $*$-automorphism θ, a directed set Λ with an orthogonality relation, and a family $\mathcal{A}_\alpha$ ($\alpha \in \Lambda$) of C*-subalgebras of $\mathcal{A}$ with the following poperties:

[1]Partial order means that $\leq$ is reflexive ($\alpha \leq \alpha$), transitive ($\alpha \leq \beta$ and $\beta \leq \gamma$ imply $\alpha \leq \gamma$), and antisymmetric ($\alpha \leq \beta$ and $\beta \leq \alpha$ imply $\alpha = \beta$). In many definitions of directed sets, the antisymmetry of $\leq$ is not required, so that $\leq$ is a so-called pre-order; technically, introducing the equivalence relation $\alpha \sim \beta$, if $\alpha \leq \beta$ and $\beta \leq \alpha$, then yields a partial order on $\Lambda/\sim$.

(i) All algebras have the common unit $I \in \mathcal{A}$,

(ii) $\alpha \leq \beta$ implies $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$,

(iii) $\mathcal{A}$ is the norm closure of $\bigcup_{\alpha\in\Lambda} \mathcal{A}_\alpha$,

(iv) $\theta(\mathcal{A}_\alpha) = \mathcal{A}_\alpha$ for all $\alpha \in \Lambda$ and in case $\alpha, \beta \in \Lambda$ are such that $\alpha \perp \beta$, then[2]

$$[\mathcal{A}_\alpha^+, \mathcal{A}_\beta^+] = \{0\}, \quad [\mathcal{A}_\alpha^+, \mathcal{A}_\beta^-] = \{0\}, \quad \{\mathcal{A}_\alpha^-, \mathcal{A}_\beta^-\} = \{0\}.$$

The C*-subalgebras $\mathcal{A}_\alpha$ ($\alpha \in \Lambda$) are called the *local subalgebras* of $\mathcal{A}$.

In the special case $\theta = \mathrm{id}$ we have $\mathcal{A}_\alpha^+ = \mathcal{A}_\alpha$, so that $[\mathcal{A}_\alpha, \mathcal{A}_\beta] = \{0\}$ holds whenever $\alpha \perp \beta$.

7.3. Remark: (i) Many properties considered and proved for quasi-local algebras do not depend on the existence of an self-inverse $*$-automorphism satisfying (iv) in the above definition. We will therefore occasionally drop or ignore the requirement (iv) and somewhat loosely speak of quasi-local algebras if (i)-(iii) are satisfied.

(ii) Important aspects, which we omit in this course, that are central in many physical models based on a quasi-local algebra $\mathcal{A}$ generated from local algebras $\mathcal{A}_\alpha$, are so-called *clustering properties* of a state ρ on $\mathcal{A}$. Very roughly speaking, this investigates whether for any given $A \in \mathcal{A}$ there is some $\alpha \in \Lambda$ such that the values $\rho(AB)$ may be approximated well by the products $\rho(A)\rho(B)$ whenever $B \in \mathcal{A}_\beta$ with $\beta \perp \alpha$ (or B close to some $\mathcal{A}_\beta$ with $\beta \perp \alpha$).

The following result indicates that representations of quasi-local algebras are well described by their restrictions to all local algebras.

7.4. Proposition: Let $\mathcal{A}$ be a quasi-local algebra generated from the local algebras $\mathcal{A}_\alpha$ ($\alpha \in \Lambda$). If π is a representation of $\mathcal{A}$ such that for every $\alpha \in \Lambda$, $\pi_\alpha := \pi|_{\mathcal{A}_\alpha}$ is a faithful representation of $\mathcal{A}_\alpha$, then π is faithful.

Proof: Suppose $0 \neq A \in \mathcal{A}$ is such that $\pi(A) = 0$. There is a sequence (A_n) in $\bigcup_{\alpha\in\Lambda} \mathcal{A}_\alpha$ such that $A_n \to A$ as $n \to \infty$. For every $n \in \mathbb{N}$ choose $\alpha(n) \in \Lambda$ such that $A_n \in \mathcal{A}_{\alpha(n)}$.

Since $\|A\| > 0$ and $A_n \to A$ there is some $N \in \mathbb{N}$ such that $\|A_n\| \geq \|A\|/2$ for all $n \geq N$. Recall that injective $*$-homomorphisms are isometric and this applies to each π_β ($\beta \in \Lambda$). Therefore we obtain that $\|\pi(A_n)\| = \|\pi_{\alpha(n)}(A_n)\| = \|A_n\| \geq \|A\|/2$, which contradicts the fact that $\pi(A_n) \to \pi(A) = 0$ by continuity of π. $\square$

Of course the result just stated also has an algebraic sibling, namely that $\mathcal{A}$ is simple if every A_α is simple ([BR1, Corollary 2.6.19]).

7.5. The CAR algebra as a quasi-local algebra: We consider the CAR algebra $\mathcal{A}$ over some Hilbert space $\mathcal{H}$.

Let Λ be a subset of the set of all closed subspaces of $\mathcal{H}$, directed by set inclusion, such that $\mathcal{E} := \operatorname{span} \bigcup_{M\in\Lambda} M$ is dense in $\mathcal{H}$ and the notion of orthogonality inherited from the Hilbert

[2]Here $[\mathcal{B}, \mathcal{C}] := \{[B, C] \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ and $\{\mathcal{B}, \mathcal{C}\} := \{\{B, C\} \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ for subsets $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$.

space satisfies[3] the relevant properties of an orthogonality relation: (a) If $M \in \Lambda$ then there is some $N \in \Lambda$ with $N \subseteq M^\perp$; (b) is no issue, since $M, N, L \in \Lambda$ with $M \subseteq N$ and $N \perp L$ always implies $M \perp L$; (c) is satisfied, if for any $N, L \in \Lambda$ also the closure K of $N + L$ belongs to Λ, since in this case having $M, N, L \in \Lambda$ with $M \perp N$ and $M \perp L$ implies $N \subseteq K$, $L \subseteq K$, and $M \perp K$.

A nontrivial example of Λ is provided by the set of all finite-dimensional subspaces of $\mathcal{H}$.

For every $M \in \Lambda$ let $\mathcal{A}_M$ be the C*-subalgebra of $\mathcal{A}$ generated by the set $\{a(u) \mid u \in M\}$.

Finally, we may define the $*$-automorphism θ of $\mathcal{A}$ with the property $\theta(a(u)) = -a(u)$ for all $u \in \mathcal{H}$ as the special case of a Bogoliubov transformation in Corollary 6.6, (ii), corresponding to the choice $U = -I$ and $V = 0$.

We certainly have $I \in \mathcal{A}_M$ for all $M \in \Lambda$ and $\mathcal{A}_M \subseteq A_N$ if $M, N \in \Lambda$ with $M \subseteq N$, hence properties (i) and (ii) in the definition of a quasi-local algebra hold. To prove property (iii), we first recall that $\mathcal{A}$ is the norm closure of the $*$-subalgebra generated by the subspace $a(\mathcal{H}) := \{a(u) \mid u \in \mathcal{H}\}$, and then use the assumption that $\mathcal{E} = \operatorname{span} \bigcup_{M \in \Lambda} M$ is dense in $\mathcal{H}$ and the isometry of the map $u \mapsto a(u)$ due to Theorem 6.4, (i), to conclude that $a(\mathcal{E}) := \{a(u) \mid u \in \mathcal{E}\}$ is dense in $a(\mathcal{H})$.

It remains to check property (iv) in the definition of a quasi-local algebra. We clearly have $\theta(\mathcal{A}_M) = \mathcal{A}_M$ for all $M \in \Lambda$, since θ is a linear bijection on the set $\{a(u) \mid u \in M\}$ of generators for $\mathcal{A}_M$. We argue that it suffices to show the commutation relation stated in condition (iv) for polynomial expressions in $a(u)$ and $a(v)^*$: Any $A \in A_M$ is the norm limit of a sequence (Q_n) of such polynomials with $u, v \in M$; if A is even, then $A = (A + \theta(A))/2 = \lim(Q_n + \theta(Q_n))/2$ and each $(Q_n + \theta(Q_n))/2$ is a polynomial in $a(u)$ and $a(v)^*$ belonging to $\mathcal{A}_M^+$, let us say even polynomial for brevity; similarly, an odd element in $\mathcal{A}_M$ can be approximated by odd polynomials. Finally, for given $M, N \in \Lambda$ with $M \perp N$ the CAR for $u \in M$ and $v \in N$ give that $a(u)$ or $a(u)^*$ always anticommute with $a(v)^*$ or $a(v)$. Therefore, even powers of one type commute with any power of the other type, while odd powers of one type anticommute with odd powers of the other.

7.6. The Weyl algebra as a quasi-local algebra: Let $\mathcal{W} := \mathcal{W}(V, \sigma)$ be the Weyl algebra over the symplectic vector space (V, σ).

We consider the orthogonality relation on subspaces of V, where we write $M \perp N$ for subspaces M and N, if and only if $\sigma(u, v) = 0$ for all $u \in M$ and $v \in N$.

Let Λ be a subset of the set of all subspaces of V, directed by set inclusion, with the following properties of an orthogonality relation: (a) If $M \in \Lambda$ then there is some $N \in \Lambda$ with $M \perp N$, i.e., we need $N \subseteq M^\perp := \{v \in V \mid \forall u \in M \colon \sigma(u, v) = 0\}$; (b) is automatically satisfied, since $M, N, L \in \Lambda$ with $M \subseteq N$ and $N \perp L$ implies $M \perp L$; (c) if $M, N, L \in \Lambda$ are such that $M \perp N$ and $M \perp L$ then there is some $K \in \Lambda$ such that $M \perp K$, $N \subseteq K$, and $L \subseteq K$ (for example, if $N + L \in \Lambda$, then $K := N + L$ would work). Examples for such Λ are provided

[3]Taking Λ plainly to be the set of all closed subspaces of $\mathcal{H}$ has the required properties, but gives a somewhat trivial example in the construction to follow; in fact, because then $\mathcal{H} \in \Lambda$ would already imply $A_{\mathcal{H}} = \mathcal{A}$ for the "local algebra" $A_{\mathcal{H}}$.

by the set of all subspaces or just the finite-dimensional subspaces of V. The last additional requirement on Λ is

$$\bigcup_{M \in \Lambda} M = V.$$

We note that mere density of the union on the left-hand side could produce some unwanted artefacts for the quasi-local construction of Weyl algebras due to Corollary 5.6, (iv). For example, compactly supported L^2 functions on $\mathbb{R}^d$ are dense in L^2, but the Weyl algebras defined by either the dense subspace or the entire L^2 (with the symplectic form extracted from the imaginary part of the inner product) are not $*$-isomorphic.

For $M \in \Lambda$ let $\mathcal{W}_M$ be the C*-subalgebra of $\mathcal{W}(V, \sigma)$ generated by the set $\{W(u) \mid u \in M\}$.

As a self-inverse $*$-automorphism we now take simply $\theta = \mathrm{id}$, so that condition (iv) in the definition of a quasi-local algebra reduces to $[A, B] = 0$, if $A \in \mathcal{W}_M$ and $B \in \mathcal{W}_N$ with $M \perp N$, which clearly holds by the Weyl relations for the generators and is therefore established.

Properties (i) and (ii) in the definition of a quasi-local algebra are immediate, i.e., obviously $I \in \mathcal{W}_M$ for all $M \in \Lambda$ and $\mathcal{W}_M \subseteq \mathcal{W}_N$ if $M, N \in \Lambda$ with $M \subseteq N$. As for property (iii), we only have to note that $\bigcup_{M \in \Lambda} \mathcal{W}_M$ contains all generators of the Weyl algebra $\mathcal{W}(V, \sigma)$, since $\bigcup_{M \in \Lambda} M = V$, hence $\bigcup_{M \in \Lambda} \mathcal{W}_M$ is norm dense in $\mathcal{W}(V, \sigma)$.

7.7. Examples of quasi-local algebras in quantum statistical mechanics:

1) *Spin systems:* The basic idea is that at each vertex z of an infinitely extended lattice $\mathbb{Z}^d$ we have the spin algebra $\mathcal{R}_z$ which is $*$-isomorphic to $M(2, \mathbb{C})$ and generated by I and the Pauli matrices $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The local algebras are built with the set of all finite subsets of $\mathbb{Z}^d$ as index set Λ, where we define for any $F \in \Lambda$,

$$\mathcal{A}_F := \bigotimes_{z \in F} \mathcal{R}_z.$$

It is mathematically a bit delicate in its details to construct and identify an appropriate completion of $\bigcup_{F \in \Lambda} \mathcal{A}_F$, which may be described as an infinite tensor product of C*-algebras or von Neumann algebras. (A few introductory references for more physical and mathematical aspects are [Emch, Chapter 4, Section 2(a)], [BR1, Example 2.6.12] and [BR2, Section 6.2], [Thi10, Part II, Section 1.4], [KRII, Chapter 11].)

2) *Fermi and Bose gases:* Basically these are built up from CAR or CCR with a parameter vector space $L^2(\mathbb{R}^d)$. For the local subalgebras, one typically uses subspaces consisting of functions, often also smooth, that are supported in bounded subsets of $\mathbb{R}^d$. For the Fermi gas this suffices to describe the entire quasi-local structure, but for the Bose gas one has to escape the rigidity of Weyl algebras illustrated by Corollary 5.6, (iv); the definition of a global algebra then takes the detour via the von Neumann algebra generated in the GNS representation of an equilibrium state. We refer to [Emch, Chapter 4, Section 2(b), and Chapter 2, Section 1(c)] and [BR2, Subsections 5.2.4 and 5.2.5] for their constructions and properties.

Before we will attempt at the end of this course a brief outlook on the very basics regarding quasi-local C*-algebras for relativistic theories, it does make sense to invest some time and energy into a detour to visit the older Hilbert space based quantum field theory. Again our emphasis is entirely on the mathematical objects and structures.

7.8. Intermezzo with the Wightman axioms for a quantum field theory: We describe here the essence of an early attempt to mathematically formalize a special relativistic quantum theory (and borrow a lot from the corresponding appendix in [HK21]). A classical tool to model localized interactions is by the concept of fields, which are functions on space or spacetime with values in some vector space, or sections of vector bundles in a more differential geometric context. In special relativity these objects are defined over Minkowski space, that is, $\mathbb{R}^4$ with the standard symmetric non-degenerate bilinear form η with signature according to $(+,-,-,-)$ (the so-called West Coast signature) given by

$$\eta(x,y) = x_0 y_0 - \sum_{j=1}^{3} x_j y_j \qquad \text{for all } x = (x_0, x_1, x_2, x_3), y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4.$$

Recall that $\mathrm{O}(1,3)$ is the group of invertible linear maps on $\mathbb{R}^4$ that leave the Minkowski metric η invariant, $\mathrm{SO}(1,3)$ is the subgroup of $\mathrm{O}(1,3)$ with elements having determinant 1, and $\mathrm{SO}^+(1,3)$ is the proper orthochronous Lorentz group consisting of those elements in $\mathrm{SO}(1,3)$ that leave $\{x \in \mathbb{R}^4 \mid \eta(x,x) > 0 \text{ and } x_0 > 0\}$ invariant. Thus, the transformations belonging to $\mathrm{SO}^+(1,3)$ preserve both the time orientation and the spatial orientation in Minkowski space. It can be shown that $\mathrm{SO}^+(1,3)$ is the connected component of the identity in $\mathrm{O}(1,3)$.

The transition to *quantum fields* is more than delicate, because the fields neither have values in finite-dimensional vector spaces nor can they directly be considered as maps defined on Minkowski space, but are instead distributions (generalized functions). Further complications arise: The distributional fields have their "values" in the set of unbounded densely defined operators on an infinite-dimensional Hilbert space; the latter is not even a vector space, since addition of operators can only be defined on a common domain and this only gets worse when we need compositions; still more severe is the fact that distribution theory is purely linear (defined in terms of dual spaces) and does not allow easily for nonlinear combinations. This is to indicate a little of the difficulties even in formulating the list of the (Gårding-)Wightman axioms given below. No completely mathematical treatment is known as soon as one wants to discuss truly interacting fields and one has to make a lot of "compromises with rigor" in approaching results that are often so astonishingly coherent with experimental facts.

The two-fold covering group of the Poincaré group: Every element of the invariance group of Minkowski space, the Poincaré group, should give rise to a Hilbert space transformation leaving the vector states invariant, hence is determined apart from a phase factor by a unitary operator. Requiring also compatibility of the respective group multiplications leads to the concept of a so-called *projective* unitary group representation, which in the case at hand can be shown to be in exact correspondence with the usual unitary representations of the simply connected covering group (see [Var07, Chapters VII and IX]).

The Poincaré group $\mathcal{P}$ is the semidirect product $\mathbb{R}^4 \ltimes \mathrm{O}(1,3)$, i.e., $(a,A)\cdot(b,B) = (a+Ab, AB)$ for $a, b \in \mathbb{R}^4$ and $A, B \in \mathrm{O}(1,3)$. Let $\mathcal{P}_0$ denote the connected component of the identity $(0, I)$ in $\mathcal{P}$, then we have $\mathcal{P}_0 = \mathbb{R}^4 \ltimes \mathrm{SO}^+(1,3)$. The universal covering group of $\mathrm{SO}^+(1,3)$ is $\mathrm{Spin}^+(1,3) = \mathrm{SL}(2,\mathbb{C})$ (see, e.g., [HK21]) with the 2-fold covering map $\kappa\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$, which is constructed as follows: If $H(2,\mathbb{C})$ denotes the set of all Hermitian complex (2×2)-matrices, then we have the isomorphism of real vector spaces $M\colon \mathbb{R}^4 \to H(2,\mathbb{C})$, given conveniently in terms of $\sigma_0 := \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and the Pauli matrices $\sigma_1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right), \sigma_2 = \left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right), \sigma_3 = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ by

$$M(x) := \sum_{\nu=0}^{3} x_\nu \sigma_\nu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \qquad \text{for all } x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4.$$

A simple calculation shows that $\det M(x) = \eta(x,x)$. For any $A \in \mathrm{SL}(2,\mathbb{C})$ we have[4]

$$\forall x \in \mathbb{R}^4\colon \quad AM(x)A^\dagger \in H(2,\mathbb{C}) \quad \text{and} \quad \det(AM(x)A^\dagger) = \det M(x) = \eta(x,x),$$

hence we can find a unique $\kappa(A) \in \mathrm{SO}^+(1,3)$ such that

$$\forall x \in \mathbb{R}^4\colon \quad AM(x)A^\dagger = M(\kappa(A)x).$$

We observe that $\kappa(\pm I) = I$ and obtain the two-fold covering map $\widetilde{\mathcal{P}_0} := \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C}) \to \mathcal{P}_0$ by the assignment $(a, A) \mapsto (a, \kappa(A))$.

Fields as operator-valued distributions on Minkowski space: The following list introduces all basic elements and properties used below for the formulation of the Gårding-Wightman axioms.

(i) Let $(\mathcal{F}, \langle .|.\rangle)$ be a complex Hilbert space with a distinguished dense subspace $\mathcal{D}$ of $\mathcal{F}$.

(ii) Let $\tilde{U}\colon \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathcal{F}) = \{T \in \mathcal{B}(\mathcal{F}) \mid \exists T^{-1} \in \mathcal{B}(\mathcal{F})\}$ be a unitary strongly continuous[5] representation such that $\mathcal{D}$ is invariant under $\tilde{U}$, i.e. $\tilde{U}(a,A)\mathcal{D} \subseteq \mathcal{D}$ for all $(a, A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C})$.

(iii) We suppose to have $K \in \mathbb{N}$ different *types of particles* and for each $j = 1, \ldots, K$ to have $n_j \in \mathbb{N}$ *field components*. If $n_j = 1$, then the field corresponding to particle type j is scalar; with $n_j > 1$ we have vector, tensor, or spinor fields. The number of *inner degrees of freedom* is the total number of field components $N := n_1 + \ldots + n_K$. For every particle type $l = 1, \ldots, K$ we are given an irreducible representation $S_l\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathbb{C}^{n_l})$ (specifying also the so-called particle statistics in the sense of Fermionic type [half-integer spin] or Bosonic type [integer spin]) and we denote by $S := S_1 \oplus \cdots \oplus S_K$ the direct sum representation $S\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(\mathbb{C}^N)$.

(iv) The set of bounded operators on $\mathcal{F}$ given by $\{\tilde{U}(a,I) \mid a \in \mathbb{R}^4\}$ is a commutative subgroup of the group of all unitary operators on $\mathcal{F}$ and is generated by the following four one-parameter unitary groups: For each of the standard basis vectors e_0, e_1, e_2, e_3 in $\mathbb{R}^4$ we have the strongly continuous unitary group $t \mapsto \tilde{U}(te_\nu, I)$ and thus, by Stone's theorem, also a unique self-ajoint operator P_ν as generator such that $\tilde{U}(te_\nu, I) = \exp(itP_\nu)$. Spectral theory ([RSI, Theorems VIII.12 and VIII.13]) provides us with a joint spectral measure E for the (commuting) *momentum operators* P_0, P_1, P_2, P_3, which is a (pointwise or SOT) σ-additive map from the Borel σ-algebra of $\mathbb{R}^4$ to the set of orthogonal projections in $\mathcal{F}$, normalized by $E(\emptyset) = 0$ and $E(\mathbb{R}^4) = I$, such that

$$\tilde{U}(a,I) = \int\limits_{\mathbb{R}^4} e^{-i\eta(p,a)}\, dE(p).$$

[4]writing here and later $A^\dagger$ for the adjoint of A

[5]That is, $\tilde{U}(a,A)$ is a unitary operator for every $(a,A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C})$ and, for every $\xi \in \mathcal{F}$, the map $(a,A) \mapsto \tilde{U}(a,A)\xi$ is continuous $\mathbb{R}^4 \ltimes \mathrm{SL}(2,\mathbb{C}) \to \mathcal{F}$ (in other words, $(a,A) \mapsto \tilde{U}(a,A)$ is SOT continuous).

(v) *Fields* (in fact, field components) $\phi_1, \ldots, \phi_N$ are linear maps from the test function space $\mathscr{S}(\mathbb{R}^4)$ to the set of unbounded operators on $\mathcal{F}$ having $\mathscr{D}$ contained in their domain as well as in the domain of their adjoint operators[6] with the following two additonal properties:
(a) Invariance of the common domain, i.e., for all $f \in \mathscr{S}(\mathbb{R}^4)$ and $n \in \{1, \ldots, N\}$ we have $\phi_n(f)\mathscr{D} \subseteq \mathscr{D}$ and $\phi_n(f)^\dagger \mathscr{D} \subseteq \mathscr{D}$.
(b) Distributional continuity in the sense that for every $\xi, \eta \in \mathscr{D}$, the map $\mathscr{S}(\mathbb{R}^4) \to \mathbb{C}$, $f \mapsto \langle \xi | \phi_n(f)\eta \rangle$ defines an element in $\mathscr{S}'(\mathbb{R}^4)$, i.e., is a temperate distribution on $\mathbb{R}^4$.

The (Gårding-)Wightman axioms: Suppose we have all the objects with properties as specified above, then the axioms read as follows (cf. [Ara99, Fol08, Haa96, RSII, SW00]):

(1) There exists a vector $\Omega \in \mathscr{D}$, $\|\Omega\| = 1$, unique up to a phase factor, such that

$$\forall (a, A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2, \mathbb{C}): \quad \tilde{U}(a, A)\Omega = \Omega.$$

We call Ω the *vacuum state* of the theory.

(2) *Completeness:* The vacuum vector is a cyclic vector for the field algebra, i.e., the linear hull of $\{\phi_{l_1}(f_1) \cdots \phi_{l_m}(f_m)\Omega \mid m \in \mathbb{N}_0, 1 \leq l_1, \ldots, l_m \leq N, f_1, \ldots, f_m \in \mathscr{S}(\mathbb{R}^4)\}$ is dense in $\mathcal{F}$. (In case $m = 0$ we define the empty product of field operators as the identity on $\mathcal{F}$.)

(3) For arbitary $f \in \mathscr{S}(\mathbb{R}^4)$, $(a, A) \in \mathbb{R}^4 \ltimes \mathrm{SL}(2, \mathbb{C})$, $1 \leq n \leq N$, $\xi \in \mathscr{D}$, and upon defining $((a, A)f)(x) := f(\kappa(A)^{-1}(x - a))$ for every $x \in \mathbb{R}^4$, we have

$$\tilde{U}(a, A)\phi_n(f)\tilde{U}(a, A)^{-1}\xi = \sum_{m=1}^{N} S(A^{-1})_{nm}\, \phi_m((a, A)f)\xi.$$

Pretending that distributions were functions, the latter could be rewritten in the form

$$\tilde{U}(a, A)\phi_n(x)\tilde{U}(a, A)^{-1} = \sum_{m=1}^{N} S(A^{-1})_{nm}\, \phi_m(\kappa(A)x + a).$$

(4) *Spectral condition:* The support of the spectral measure E is inside the forward light cone, i.e., contained in $\{(p_0, \vec{p}) \in \mathbb{R}^4 \mid p_0 \geq |\vec{p}|\}$. Equivalently, with the self-adjoint momentum operators P_ν ($\nu = 0, 1, 2, 3$), both P_0 and $P_0^2 - P_1^2 - P_2^2 - P_3^2$ are positive operators.

(5) *Causality:* If $f, g \in \mathscr{S}(\mathbb{R}^4)$ have their supports *space-like separated*, i.e., $\eta(x - y, x - y) < 0$ if $f(x)g(y) \neq 0$, and $m, n \in \{1, \ldots, N\}$, then as operators on the domain $\mathscr{D}$ the fields $\phi_m(f)$ and $\phi_n(g)$ or $\phi_m(g)^\dagger$, either *commute*

$$[\phi_m(f), \phi_n(g)] = 0 = [\phi_m(f), \phi_n(g)^\dagger]$$

or *anticommute*

$$\{\phi_m(f), \phi_n(g)\} = 0 = \{\phi_m(f), \phi_n(g)^\dagger\}$$

(depending on m, n, and on the particle species).

Remarks: (i) We add a quote by Rudolf Haag showing how he phrases the causality condition in a discussion preparing for the translation of the above axioms into a structure with quasi-local C^*-algebras ([Haa96, page 107]):"Two observables associated with space-like separated

[6] which we denote here in this context by a superscript † instead of $*$

regions are compatible. The measurement of one does not disturb the measurement of the other. The operators representing these observables must commute." And then he immediately adds a long comment starting with the following sentence: "To avoid possible confusion it must be stressed that this has nothing to do with the discussion around the Einstein-Podolsky-Rosen paradox and Bell's inequality."

(ii) The spin-statistics theorem (cf. [SW00, Section 4.4] or [Haa96, Section II.5]) implies the following: Let the Schwartz functions f and g have space-like separated supports as in the causality axiom. Particles of integer spin must be *Bosons*, that is, each of their field components ϕ_n satisfies the commutation relation $[\phi_n(f), \phi_n(g)^\dagger] = 0$, while particles of half-integer spin are *Fermions*, which means that each of their field components ϕ_n satisfies the anticommutator relation $\{\phi_n(f), \phi_n(g)^\dagger\} = 0$.

(iii) The theorem on invariance under PCT ([SW00, Section 4.3]) or CPT ([Haa96, Section II.5]) or TCP ([Ara99, Section 5.6]) states that a QFT is not influenced by the combined operation of space inversion or parity P, time inversion T, and charge conjugation C. (The invariance is in general not true for a single operation or an arbitrary pairwise combination, as is illustrated by the non-axiomatic theory of weak interaction, which is known to be not P-invariant, and evidence from meson decays about violation of CP-invariance.)

(iv) The reconstruction theorem ([SW00, Section 3.4] or [BLOT, Section 8.3]) shows that a theory can be "recovered from knowing all its vacuum expectation values". It means that the so-called *hierarchy of Wightman distributions* $w^r_{l_1,\ldots,l_r} : f_1 \otimes \cdots \otimes f_r \mapsto \langle \Omega | \phi_{l_1}(f_1) \cdots \phi_{l_r}(f_r)\Omega\rangle$ on $\mathbb{R}^{4r}$ $(r \in \mathbb{N})$ determines $\mathcal{F}$ and the action of all field operators $\phi_n(f)$ uniquely (up to a unitary isomorphism).

(v) The rigorous construction of some theories with nontrivial interactions (cf. [SW00, Appendix] or [GJ87]) has been successful in space-time dimensions 2 and 3, although none so far in dimension 4, where at least many *rigorous free quantum field theories* do exist and thus show that the axioms are consistent (see, e.g., [Fol08, Chapter 5]).

7.9. Quasi-local algebras over Minkowski space: The Haag-Kastler axioms sketched out below can be viewed as an attempt to transfer the structural essence of the Wightman axioms to a C*-algebraic setting (cf. [Haa96, Section III.1], [Ara99, Chapter 4], [Fred15, Section 1.1]). They are formulated for a family of (unital) C*-algebras $\mathcal{A}(\mathcal{O})$, where $\mathcal{O}$ is from the set Λ of bounded subsets of Minkowski space, directed by set inclusion. An orthogonality relation is introduced on Λ by writing $\mathcal{O}_1 \perp \mathcal{O}_2$, if and only if $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated (recall that this means $\eta(x-y, x-y) < 0$ if $x \in \mathcal{O}_1$ and $y \in \mathcal{O}_2$); it is easily checked by elementary Minkowski geometry that properties (a)-(c) of an orthogonality relation on a directed set are indeed satisfied.

It can be shown that, for a net of C*-algebras as above with all the additional properties listed below—a so-called *Haag-Kastler net*—there is a unique (up to $*$-isomorphism) C*-algebra $\mathcal{A}$ (given by an inductive limit construction) with compatible embeddings $\mathcal{A}(\mathcal{O}) \to \mathcal{A}$ such that $\bigcup_{\mathcal{O}\in\Lambda} \mathcal{A}(\mathcal{O})$ is norm dense in $\mathcal{A}$. However, we slightly reformulate it partially by already supposing that $\mathcal{A}$ is a C*-algebra with C*-subalgebras $\mathcal{A}(\mathcal{O})$ $(\mathcal{O} \in \Lambda)$ such that the following properties hold:

(A) $I \in \mathcal{A}(\mathcal{O})$ for all $\mathcal{O} \in \Lambda$ and $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subseteq \mathcal{O}_2$,

(B) there is a group homomorphism α from the covering group of the identity component $\mathcal{P}_0$ of the Poincaré group to the group of $*$-automorphisms of $\mathcal{A}$ such that

$$\alpha_{(b,L)}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\kappa(L)\mathcal{O} + b) \qquad (\mathcal{O} \in \Lambda, b \in \mathbb{R}^4, L \in \mathrm{SL}(2, \mathbb{C})),$$

(C) if $\mathcal{O}_1, \mathcal{O}_2 \in \Lambda$ are space-like separated and $\mathcal{O} \in \Lambda$ is such that $\mathcal{O}_1 \cup \mathcal{O}_2 \subseteq \mathcal{O}$, then

$$[A, B] = 0 \quad \text{holds in } \mathcal{A}(\mathcal{O}) \text{ for all } A \in \mathcal{A}(\mathcal{O}_1) \text{ and } B \in \mathcal{A}(\mathcal{O}_2).$$

(This is supposed to model causality for the local observable algebras $\mathcal{A}(\mathcal{O})$ and not directly for Wightman fields, hence there is no variant with anticommutation and it does not mean here that Bose statistics is implied.)

The spectral condition (4) of the Wightman axioms can be replaced by the requirement that there exists a(n irreducible) representation of $\mathcal{A}$, where each $\alpha_{(b,L)}$ is unitarily implementable in a strongly continuous way and satisfies the analogue of the spectral condition on the representation Hilbert space. Recall that cyclic representations correspond to GNS representations associated with states on the C*-algebra. So the spectrum condition could be read as a requirement that a particular kind of vacuum state exists on $\mathcal{A}$.

Remark: As indicated in [Haa96, Page 106] and [Ara99, Sections 4.8 and 4.9], neither do the Wightman axioms imply the existence of a Haag-Kastler net nor is the converse true in general, although the relationship seems close enough for most purposes.

7.10. Quasi-local algebras over curved spacetimes: To build an algebraic theory of quantum fields on a general relativistic background requires to reformulate nets of C*-algebras parametrized by certain subsets of an appropriate spacetime model. The basic class of spacetimes considered in this context is that of time-oriented connected Lorentzian manifolds, in particular globally hyperbolic Lorentzian manifolds (cf. [BGP07]). A Lorentzian metric on a smooth manifold is given by a symmetric non-degenerate $(0, 2)$ tensor field that induces on each tangent space a symmetric bilinear form of signature $(+, -, \dots, -)$ (like the Minkowski metric on $\mathbb{R}^4$). Roughly speaking, a Lorentzian manifold has tangent spaces isomorphic to the Minkowski space (or its analogue in $d + 1$ dimensions) and it is time-oriented, if we can choose the time-orientations in the tangent spaces in a smooth consistent way.

Global hyperbolicity of a connected time-oriented Lorentzian manifold M can be defined as a combination of causality with topological conditions: Recall that in Minkowski space a nonzero vector x is said to be time-like, light-like, or space-like, if $\eta(x, x) > 0$, $\eta(x, x) = 0$, or $\eta(x, x) < 0$, respectively; in addition, the zero vector may be considered as space-like; a curve in M is called causal, if its tangent vectors are all time-like or light-like; the condition of strong causality requires that there are no almost closed causal curves in M and the precise formulation of this is part of the definition[7] of global hyperbolicity; the other requirement in the definition is compactness of the intersections of the causal future and the causal past of any two points (events) in M; here, the causal future is the subset of M that can be reached from a given point by causal curves; likewise for causal past. The notion of global hyperbolicity can be seen as a certain guarantee that wave-type equations, e.g., also the Klein-Gordon equation, on such manifolds are always well-posed.

[7] As it turned out (see [BS07]), in combination with the second condition, the plain causality assumption suffices, i.e., there are no closed causal curves.

Finally a few, still sketchy, words on the definition of quasi-local algebras on a given globally hyperbolic, or more generally, connected time-oriented, Lorentzian manifold M (cf. [Fred15, Section 1.2], [BGP07, Chapter 4], [FV15]). The index set Λ is directed by set inclusion and consists of subsets $\mathcal{O}$ of M that are open, relatively compact, and have locally essentially the same causality structure as M in a compatible way. The orthogonality relation $\mathcal{O}_1 \perp \mathcal{O}_2$ is then introduced so as to model the idea that there are no causal curves connecting any events in $\overline{\mathcal{O}_1}$ with events in $\overline{\mathcal{O}_2}$. This relation can be shown to satisfy the usual properties (a)-(c) of an orthogonality relation, if M is globally hyperbolic, and at least properties (a) and (b), if M is just connected time-oriented Lorentzian. Then the adapted Haag-Kastler-type axioms for a C*-algebra $\mathcal{A}$ with C*-subalgebras $\mathcal{A}(\mathcal{O})$ ($\mathcal{O} \in \Lambda$) such that $\bigcup_{\mathcal{O}\in\Lambda} \mathcal{A}(\mathcal{O})$ is norm dense in $\mathcal{A}$ read as follows:

(A') All algebras have the common unit I and $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subseteq \mathcal{O}_2$,

(C') if $\mathcal{O}_1 \perp \mathcal{O}_2$ then $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\}$.

There is no analogue of (B), since general relativity is not Poincaré invariant. As in the special case of Minkowski space, further and refined requirements can be formulated in terms of properties of representations or states. In particular, this is possible with a generally covariant replacement of the spectral condition (see [FV15, Subsection 4.5.4]).

It can be shown that $\mathcal{A}$ is simple, if M is globally hyperbolic.

The construction of these algebras is often formulated and carried out in the context of a functor from a category of Lorentzian manifolds to one of quasi-local algebras.

Bibliography

[All11] G. Allan: *Introduction to Banach Spaces and Algebras.*
Oxford University Press 2011. 1, 27

[Ara87] H. Araki: *Bogoliubov automorphisms and Fock representations of canonical anticommutation relations.*
Contemp. Math. **62**, 23-141, 1987. 81, 88, 91

[Ara99] H. Araki: *Mathematical Theory of Quantum Fields.*
Oxford University Press 1999. 46, 48, 99, 100, 101

[BGP07] C. Bär and N. Ginoux and F. Pfäffle: *Wave equations on Lorentzian manifolds and quantization.* European Mathematical Society 2007. 101, 102

[BS07] A. N. Bernal and M. Sánchez: *Globally hyperbolic spacetimes can be defined as 'causal' instead of 'strongly causal'.*
Class. Quant. Grav. **24**, 745-749, 2007. 101

[BSZ92] J. C. Baez and I. E. Segal and Z. Zhou: *Introduction to Algebraic and Constructive Quantum Field Theory.*
Princeton University Press 1992. 48, 59

[Bla10] B. Blackadar: *Operator Algebras. Theory of* C^**-Algebras and von Neumann Algebras.*
Springer-Verlag 2010. 37, 46, 48, 49, 50, 92

[BLOT] N. N. Bogolubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov:
General principles of Quantum Field Theory.
Kluwer Academic Publishers 1990. 53, 55, 100

[BR1] O. Bratteli and D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics 1.* C^**- and* W^**-Algebras, Symmetry Groups, Decomposition of States.*
Springer-Verlag, 2nd edition 2010. 9, 10, 27, 33, 39, 46, 47, 48, 89, 93, 94, 96

[BR2] O. Bratteli and D. W. Robinson: *Operator Algebras and Quantum Statistical Mechanics 2. Equilibrium States, Models in Quantum Statistical Mechanics.*
Springer-Verlag, 2nd edition 1997. 53, 72, 73, 75, 77, 78, 79, 81, 88, 89, 90, 91, 93, 96

[Con10] J. B. Conway: *A Course in Functional Analysis.*
Springer-Verlag, 2nd edition 2010. i, 1, 5, 7, 8, 15, 16, 19, 22, 28, 33, 35, 46, 47

[Con00] J. B. Conway: *A Course in Operator Theory.* American Mathematical Society 2000. 9, 19, 25, 26, 27, 39, 43, 44, 46, 47, 48, 49

[DG13] J. Dereziński and C. Gérard: *Mathematics of Quantization and Quantum Fields.* Cambridge University Press 2013. 53, 59, 79, 81, 88, 91, 92

[Dix81] J. Dixmier: *Von Neumann Algebras.* North-Holland Publishing Company 1981. 39, 46, 47

[Dix82] J. Dixmier: C*-*Algebras.* North-Holland Publishing Company, revised edition 1982. 10, 27, 37

[Emch] G. Emch: *Algebraic Methods in Statistical Mechanics and Quantum Field Theory.* Wiley 1972, reprinted Dover edition 2009. 73, 91, 93, 96

[FV15] C. J. Fewster and R. Verch: *Algebraic Quantum Field Theory in Curved Spacetimes.* Chapter 4 in the collection *Advances in Algebraic Quantum Field Theory*, edited by R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason, Springer-Verlag 2015. 34, 102

[Fred15] K. Fredenhagen: *An Introduction to Algebraic Quantum Field Theory.* Chapter 1 in the collection *Advances in Algebraic Quantum Field Theory*, edited by R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason, Springer-Verlag 2015. 100, 102

[Fol08] G. B. Folland: *Quantum Field Theory. A Tourist Guide for Mathematicians.* American Mathematical Society 2008. 53, 73, 74, 75, 81, 92, 99, 100

[Fol16] G. B. Folland: *A Course in Abstract Harmonic Analysis.* CRC Press, second edition 2016. 61, 62, 72

[GJ87] J. Glimm and A. Jaffe: *Quantum Physics. A Functional Integral Point of View.* Springer-Verlag, 2nd edition 1987. 100

[Haa96] R. Haag: *Local Quantum Physics. Fields, Particles, Algebras.* Springer-Verlag, 2nd edition 1996. 48, 55, 73, 74, 91, 92, 93, 99, 100, 101

[Hoe90] L. Hörmander: *The Analysis of Linear Partial Differential Operators I.* Springer-Verlag, second edition 1990. 70

[Hoe23] G. Hörmann: *Advanced Functional Analysis.* Lecture notes, University of Vienna 2023. http://www.mat.univie.ac.at/~gue/lehre/23AFA/AFA.pdf i, 12, 16, 18, 22, 33, 35, 39, 44, 47

[HK21] G. Hörmann and M. Kunzinger: *Gauge Theory, Lagrangians, and Symmetries.* Lecture notes, University of Vienna 2021. http://www.mat.univie.ac.at/~mike/teaching/ws2021/LNGHMK.pdf 97

[KRI] R. V. Kadison and J. R. Ringrose: *Fundamentals of the theory of operator algebras. Volume I. Elementary Theory.* Academic Press 1983. 1, 2, 5, 6, 7, 9, 12, 13, 15, 16, 17, 18, 19, 23, 27, 28, 32, 36, 39, 40, 43, 44, 48, 72, 73, 91

[KRII] R. V. Kadison and J. R. Ringrose: *Fundamentals of the theory of operator algebras. Volume II. Advanced Theory.* Academic Press 1986. 13, 18, 26, 27, 33, 36, 37, 39, 43, 46, 48, 49, 51, 84, 85, 89, 96

[KRIII] R. V. Kadison and J. R. Ringrose: *Fundamentals of the theory of operator algebras. Volume III, Special Topics. Elementary Theory—An Exercise Approach.* Birkhäuser 1991. 27, 40, 91

[KRIV] R. V. Kadison and J. R. Ringrose: *Fundamentals of the theory of operator algebras. Volume IV, Special Topics. Advanced Theory—An Exercise Approach.* Birkhäuser 1992. 48

[Mor17] V. Moretti: *Spectral Theory and Quantum Mechanics. Mathematical Foundations of Quantum Theories, Symmetries and Introduction to the Algebraic Formulation.* Springer-Verlag, 2nd edition 2017. 53, 54, 55, 58, 63

[Mor19] V. Moretti: *Fundamental Mathematical Structures of Quantum Theory. Spectral Theory, Foundational Issues, Symmetries, Algebraic Formulation.* Springer-Verlag 2019. 34, 53, 55

[Mur90] G. J. Murphy: C^**-Algebras and Operator Theory.* Academic Press 1990. 1, 5, 6, 7, 8, 9, 10, 19, 27, 39, 46, 47, 48, 89

[Ott95] J. T. Ottesen: *Infinite Dimensional Groups and Algebras in Quantum Physics.* Springer-Verlag 1995. 53, 73, 75, 77, 79, 81, 88, 89, 90

[PI] T. W. Palmer: *Banach Algebras and the General Theory of* **-Algebras. Volume I: Algebras and Banach Algebras.* Cambridge University Press 1994. 1, 2

[PII] T. W. Palmer: *Banach Algebras and the General Theory of* **-Algebras. Volume II:* **-Algebras.* Cambridge University Press 2001. 9, 27, 34

[Ped18] G. K. Pedersen: C^**-algebras and Their Automorphism Groups.* Academic Press, second edition 2018. 33, 48, 89

[Petz90] D. Petz: *An Invitation to the Algebra of Canonical Commutation Relations.* Leuven University Press 1990. 53, 59, 68, 72, 73, 79

[PR94] R.J. Plymen and P.L. Robinson: *Spinors in Hilbert Space.* Cambridge University Press 1994. 81, 89, 92

[RSI] M. Reed and B. Simon: *Methods of modern mathematical physics. Volume I: Functional Analysis.* Academic Press, revised and enlarged edition 1980. 73, 98

[RSII] M. Reed and B. Simon: *Methods of modern mathematical physics. Volume II: Fourier Analysis, Self-Adjointness.* Academic Press 1975. 79, 99

[Rej16] K. Rejzner: *Perturbative Algebraic Quantum Field Theory. An Introduction for Mathematicians.* Springer-Verlag 2016. 34

[Sak71] S. Sakai: *C^*-Algebras and W^*-Algebras.* Springer-Verlag 1971. 48

[Sun87] V. S. Sunder: *An Invitation to von Neumann Algebras.* Springer-Verlag 1987. 39, 47, 50, 51

[Sla72] J. Slawny: *On Factor Representations and the C^*-Algebra of Canonical Commutation Relations.* Comm. Math. Phys. **24**, 151-170, 1972. 62, 68

[SW00] R. F. Streater and A. S. Wightman: *PCT, Spin and Statistics, and All That.* Princeton University Press, revised printing with corrections 2000. 74, 99, 100

[TakI] M. Takesaki: *Theory of Operator Algebras I.* Springer-Verlag 1979. 9, 37, 46, 48, 49, 50

[Thi10] W. Thirring: *Quantum Mathematical Physics: Atoms, Molecules and Large Systems.* Springer-Verlag, 2nd edition 2010. 46, 48, 53, 55, 58, 59, 74, 96

[Var07] V.S.. Varadarajan: *Geometry of Quantum Theory.* Springer-Verlag, 2nd edition 2007. 97

[Wald2] S. Waldmann: *Lineare Algebra 2.* Springer-Verlag 2017. 65

[Wer18] D. Werner: *Funktionalanalysis.* Springer-Verlag, 8. Auflage 2018. 17, 22

Index

absolute value, 24
adjoint, 11
annihilation operator, 76
anticommutator, 82
antisymmetric tensor product, 73

Banach algebra, 2
Banach $*$-algebra, 10
Bogoliubov transform, 63
Boson Fock space, 74
Bosons, 73

canonical anticommutation relations, 82, 85
canonical commutation relations, 53, 76
CAR algebra, 85
Cauchy-Schwarz-inequality, 15
CCR, 53, 54
center, 41
central decomposition of a vNA, 49
Clifford map, 89
commutant, 1
completely additive positive functional, 46
creation operator, 76
C*(A) or C*-algebra generated by the normal element A, 21
C*-algebra, 9
C*-subalgebra, 9
cyclic representation, 27
cyclic vector, 27

density matrix, 46
dimension function, 50
direct integral, 37
direct sum
  of Hilbert spaces, 32
  of representations, 32
directed set, 93

enveloping von Neumann algebra, 33
equivalence of representations
  unitary, 27
equivalent projections, 49

factor, 41
faithful representation, 27
Fermion algebra, 85
Fermion Fock space, 74
Fermions, 73
field operator, 65
field operators, 59
finite projection, 49
finite von Neumann algebra, 49
Fock representation, 68
Fock space, 74
Fock states, 68

Gelfand transform, 7
generated
  C*-algebra, 20
  von Neumann algebra, 42
GNS representation, 28, 29

Haag-Kastler net, 100
Heisenberg group, 72
Heisenberg uncertainty relations, 57
hermitian, 11
  linear functional, 14
Hilbert space tensor product, 73

imaginary part, 11
increasing net of self-adjoint operators, 45
infinite projection, 49
infinite von Neumann algebra, 49
invariant state, 31
involution, 9
irreducible representation, 34

left kernel of a state, 28

local subalgebras, 94

maximal abelian von Neumann algebra, 42
mean-square deviation, 56
minimal projection, 49
multiplicative linear functional, 5

non-degenerate
    representation, 34
normal, 11
normal linear positive functional, 45
normally ordered product, 91

operator algebra, 27
orthogonality relation on a directed set, 93

partial isometry, 49
Pauli principle, 82
positive, 12
    linear functional, 14
predual of a vNA, 47
pure state, 16

quasi-local C*-algebra, 93
quasi-local algebras, 93

rapidly decreasing Schwartz functions, 53
real part, 11
regular representation, 65
regular state, 65
representation, 27
resolvent set, 4

Schrödinger representation, 65, 68
Schwartz functions, 53
Segal field operator, 76
self-adjoint, 11
    subset, 11
self-adjoint system of matrix units, 84
simple, 64
space-like separated, 99
spectral radius, 5
spectrum, 4
square root
    positive, 24
$*$-algebra, 9
$*$-homomorphism, 9
$*$-subalgebra, 9
state, 14
state space, 16
strong operator topology, 39
strongly continuous unitary group, 54
subrepresentation, 27
symmetric tensor product, 73
symplectic basis, 65
symplectic linear map, 63
symplectic vector space, 54

tensor product of Hilbert spaces, 73
trace, 46
trace class operator, 46
tracial state, 68
tracial state on the CAR algebra, 91
twisted convolution, 69
type I factor, 50
type II factor, 50
type III factor, 50

ultraweak topology, 45
uncertainty relation, 56
unitary, 11
universal representation, 33

vacuum vector, 68
vector states, 14
von Neumann algebra, 41

weak operator topology, 39
Weyl algebra, 59
Weyl form of the canonical commutation
    relations, 57
Weyl relations, 59
Wick ordered product, 91
W*-algebra, 48