Proseminar 'Advanced functional analysis' — SS 2023

Try to solve the problems on the basis of the corresponding lecture course content (or its prerequisites). (Many of these problems are based on suggestions in Dirk Werner's book 'Funktionalanalysis'.)

1 Let H be a complex Hilbert space.

(a) For any $S \in L(H)$ show that $\operatorname{ran}(S)^\perp = \ker(S^*)$ and therefore also $\ker(S) = \operatorname{ran}(S^*)^\perp$ and $\overline{\operatorname{ran}(S)} = \ker(S^*)^\perp$.

(b) Give an explicit example of a bounded operator on H whose range is not closed.

(c) If $S \in L(H)$ satisfies $S^2 = S$, then $\operatorname{ran}(S)$ is closed. (Hint: Consider also $I - S$.)
Remark: This applies then in particular to any orthogonal projection.

2 Let H be a complex Hilbert space and $E_1, E_2 \in L(H)$ be orthogonal projections. Show that the following are equivalent: (It may be convenient to show (i) $\Leftrightarrow$ (ii) first.)

(i) $\operatorname{ran}(E_1) \subseteq \operatorname{ran}(E_2)$,

(ii) $\ker(E_2) \subseteq \ker(E_1)$,

(iii) $E_1 E_2 = E_2 E_1 = E_1$,

(iv) $E_2 - E_1 \geq 0$.

3 Let H be a complex Hilbert space and $T \in L(H)$. Show that the following are equivalent:

(i) T is normal,

(ii) $\forall x \in H$: $\|Tx\| = \|T^* x\|$. (Hint: $T^*T - TT^*$ is self-adjoint.)

4 Let H be a complex Hilbert space and $T \in L(H)$. Show that the following are equivalent:

(i) T is self-adjoint,

(ii) $\forall x \in H$: $\langle Tx, x\rangle \in \mathbb{R}$. (Hint: Consider $x + \lambda y$ in place of x.)

5 Let H be a complex Hilbert space and consider the resolvent map $R \colon \rho(T) \to L(H)$, $R(\lambda) = R_\lambda := (\lambda - T)^{-1}$ for a given operator $T \in L(H)$.

(a) Prove the resolvent identity

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \quad (\lambda, \mu \in \rho(T)).$$

(b) Based on the Neumann series, show that R is an analytic map, i.e., locally given by a power series with coefficients from $L(H)$,

6 Let H be a complex Hilbert space and $T \in L(H)$. Show that $\lambda \in \rho(T)$ implies

$$\|(\lambda - T)^{-1}\| \geq \frac{1}{d(\lambda, \sigma(T))}.$$

Remark: We see that $\|(\lambda - T)^{-1}\| \to \infty$ as λ approaches $\sigma(T)$.

7 Let $(\alpha_n) \in l^\infty$ and $A \in L(l^2)$ be defined by $Ax = (\alpha_n x_n)$ for every $x = (x_n) \in l^2$. Show that $\sigma(A) = \overline{\{\alpha_n \mid n \in \mathbb{N}\}}$.

Remark: We may thus conclude that for any nonempty compact subset K of $\mathbb{C}$ we can find some $A \in L(l^2)$ such that $\sigma(A) = K$. In fact, upon choosing (α_n) such that $\{\alpha_n \mid n \in \mathbb{N}\}$ is dense in K we may simply put $Ax := (\alpha_n x_n)_{n\in\mathbb{N}}$.

8 We consider the right-shift R on l^2, given by $R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$. Show:

(a) $\sigma(R) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$,

(b) $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\} \subseteq \sigma_r(R)$, (Hint: In what relation is $(1, \overline{\lambda}, \overline{\lambda}^2, \ldots)$ with $\operatorname{ran}(\lambda - R)$?)

(c) $\sigma(R) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$,

(d) $\sigma_p(R) = \emptyset$,

(e) $\sigma_c(R) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, (Hint for $\boxed{\supseteq}$: Show density of $\operatorname{ran}(\lambda - R)$ by looking at $\{(\lambda - R)e_n \mid n \in \mathbb{N}\}^\perp$.)

(f) $\sigma_r(R) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$.

9 We consider the left-shift L on l^2, given by $L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$. Show:

(a) $L = R^*$ and $\sigma(L) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$,

(b) $\sigma_p(L) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$,

(c) $\sigma_c(L) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, (Hint: Use $\boxed{1}$(a).)

(d) $\sigma_r(L) = \emptyset$.

10 Let H be a complex Hilbert space and $T \in L(H)$ be self-adjoint.

(a) Show that we obtain equality in $\boxed{6}$.

(b) Let $f_s(t) := \exp(ist)$ $(s, t \in \mathbb{R})$ and observe that, for t in a compact subset of $\mathbb{R}$, we have $(f_r(t) - f_s(t))/(r - s) \to itf_s(t)$ uniformly as $r \to s$. Apply this to conclude

$$\frac{d}{ds}e^{isT} := \lim_{r\to s} \frac{e^{irT} - e^{isT}}{r - s} = iTe^{isT}.$$

11 Let H be a complex Hilbert space and $T \in L(H)$.

(a) Consider the numerical range $W(T) = \{\langle Tx, x\rangle \mid x \in H, \|x\| = 1\}$ of T. Show that
(i) $W(T)$ is compact, if H is finite dimensional,
(ii) but $W(T)$ is not necessarily closed, if H is infnite dimensional.

(b) Let T be self-adjoint with $\sigma(T) = \{0, 1\}$. Show that T is an orthogonal projection.

In the following three problems, let h be in $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid \forall \varepsilon > 0\, \exists K \subset \mathbb{R}$ compact, $\forall x \in \mathbb{R} \setminus K \colon |f(x)| < \varepsilon\}$ and M_h denote the operator of multiplication $\varphi \mapsto h\varphi$ on $L^2(\mathbb{R})$.

12 Let $\lambda \in \mathbb{C}$: Find a necessary and sufficient condition for λ to be an eigenvalue of M_h.

13 Show that $\{0\} \cup h(\mathbb{R}) = \overline{h(\mathbb{R})} \supseteq \sigma(M_h)$.

14 Prove that $\sigma(M_h) = \{0\} \cup h(\mathbb{R})$.

(Hint: Since $\sigma(M_h)$ is closed and in view of 13, it suffices to show that $h(\mathbb{R}) \subseteq \sigma(M_h)$. Try to make use of the observation that $\lambda \in \rho(M_h)$ and $\psi \in L^2(\mathbb{R})$ implies that $\varphi := (\lambda - M_h)^{-1}\psi \in L^2(\mathbb{R})$ and $(\lambda - h(t))\varphi(t) = \psi(t)$ holds for almost every t.)

15 Let H be a complex Hilbert space and $E \colon \mathcal{B}(\mathbb{R}) \to L(H)$ be a spectral measure. Show that the following hold for any $A, B \in \mathcal{B}(\mathbb{R})$:

(a) $A \subseteq B \;\Rightarrow\; E_A \leq E_B$, i.e., $E_B - E_A \geq 0$, and $E_A E_B = E_A$,

(b) $A \cap B = \emptyset \;\Rightarrow\; E_A E_B = 0$,

(c) $E_A E_B = E_B E_A = E_{A \cap B}$.

16 Let H be a complex Hilbert space, $\lfloor s \rfloor := \max\{m \in \mathbb{Z} \mid m \leq s\}$ for $s \in \mathbb{R}$ denote the floor function , and $T \in L(H)$ be self-adjoint. Show the following two properties and conclude that the operators with finite spectrum are dense in the subspace of self-adjoint operators:

(a) If $f_n(t) := \lfloor nt \rfloor / n$ ($n \in \mathbb{N}$, $-\|T\| \leq t \leq \|T\|$), then $T = \lim_{n\to\infty} f_n(T)$ (operator norm limit),

(b) $\sigma(f_n(T))$ is a finite subset of $\mathbb{C}$ for every $n \in \mathbb{N}$.

17 Let H be a complex separable Hilbert space with orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$ and let $p_n \in \mathbb{N}$ denote the n^{th} prime number ($n \in \mathbb{N}$).

(a) For $A \in \mathcal{B}(\mathbb{R})$ and $x \in H$ we put $E_A\, x := \sum_{n \in \mathbb{N}, \frac{1}{p_n} \in A} \langle x, e_n \rangle e_n$. Show that $A \mapsto E_A$ defines a spectral measure $E \colon \mathcal{B}(\mathbb{R}) \to L(H)$.

(b) Suppose T is the self-adjoint operator on H corresponding to the spectral measure E given in (a). We immediately know that the residual spectrum $\sigma_r(T)$ is empty (from Lemma 0.5 in the lecture notes). Try to determine $\sigma(T)$, $\sigma_p(T)$, and $\sigma_c(T)$ based on the descriptions given in 1.22 of the lecture notes.

In the following two problems, let H be a complex Hilbert space, $T \in L(H)$ be self-adjoint, and $E\colon \mathcal{B}(\mathbb{R}) \to L(H)$ be the spectral measure of T. Consider the map $S\colon \mathbb{R} \to L(H)$, defined by $S(\lambda) := E_{]-\infty,\lambda]}$, which is the analogue of the distribution function (German: Verteilungsfunktion) corresponding to a Borel measure on $\mathbb{R}$.

18 Provide brief sketches of arguments (essentially by identifying the relevant properties of E) indicating that S has the following properties of a so-called *spectral family* (German: *Spektralschar*; historically, this is the "older version" of spectral measures):

(a) $\forall \lambda \in \mathbb{R}$: $S(\lambda)$ is an orthogonal projection,

(b) monotonicity: $\lambda \le \mu \;\Rightarrow\; S(\lambda) \le S(\mu)$,

(c) $\forall x \in H$: $\lim\limits_{\lambda\to\infty} S(\lambda)x = x$ and $\lim\limits_{\lambda\to-\infty} S(\lambda)x = 0$,

(d) pointwise continuity from the right, i.e., $\forall x \in H$: $\lim\limits_{\lambda\to\mu+} S(\lambda)x = S(\mu)x$.

19 The spectral family can be used to characterize spectral points, i.e., show that the following hold for any $\lambda \in \mathbb{R}$:

(i) $\lambda \in \rho(T) \;\Leftrightarrow\;$ S is constant in a neighborhood of λ,

(ii) $\lambda \in \sigma_p(T) \;\Leftrightarrow\;$ S is discontinuous at λ,

(iii) $\lambda \in \sigma_c(T) \;\Leftrightarrow\;$ S is continuous at λ, but not constant in any neighborhood of λ.

In the following two problems, let $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier-Plancherel transform, which is a unitary operator on $L^2(\mathbb{R})$, e.g., obtained by extension of the Fourier transform on the dense subspace $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\sqrt{2\pi}(\mathcal{F}\varphi)(x) = \int_{\mathbb{R}} e^{-ix\xi}\,\varphi(x)\,dx$.

20 (a) Knowing or taking for granted that $(\mathcal{F}^2\varphi)(x) = \varphi(-x)$ holds for every $\varphi \in L^2(\mathbb{R})$ for almost all x, deduce that $\sigma(\mathcal{F}) \subseteq \{-1, 1, -i, i\}$ and that all spectral values of $\mathcal{F}$ are eigenvalues. (Hint: Recall that unitary operators are normal.)

(b) Consider $\varphi(x) = \exp(-x^2/2)$, which is a solution to the ordinary differential equation $\varphi' + x\varphi = 0$. Knowing or taking for granted the exchange formulae $\mathcal{F}(\varphi')(\xi) = i\xi\mathcal{F}\varphi(\xi)$ and $\mathcal{F}(x\varphi) = i(\mathcal{F}\varphi)'$, deduce that $\mathcal{F}\varphi = \lambda\varphi$, where $\lambda = (\mathcal{F}\varphi)(0) = 1$. Therefore, φ is an eigenvector of $\mathcal{F}$ with eigenvalue 1.

21 Define $(Ph)(x) := xh(x) - h'(x)$ $(x \in \mathbb{R})$, if h is a differentiable function on $\mathbb{R}$. Show that P is injective when restricted to functions $h \in \mathscr{S}(\mathbb{R})$. With φ denoting the Gaussian function as in part (b) of the previous problem, let $\varphi_k := P^k\varphi$ $(k \in \mathbb{N}_0)$. Show that φ_k is an eigenvector for $\mathcal{F}$ with eigenvalue $(-i)^k$ and conclude that $\sigma(\mathcal{F}) = \sigma_p(\mathcal{F}) = \{-1, 1, -i, i\}$.

Remark: A spectral respresentation of $\mathcal{F}$ can be obtained in terms of a series involving Hermite functions, but we will not go into details about this here ...

22 Let T be the unitary operator on $L^2(\mathbb{R})$, given by translation $(T\varphi)(x) := \varphi(x-1)$. Can you find a multiplication operator representation of T? Determine $\sigma(T)$.

23 Let H be a complex Hilbert space and $\mathcal{U}(H)$ denote the group of unitary operators on H, considered as a topological space with the metric topology induced by the operator norm on $L(H)$. Make use of the observations in Example 1.26 of the lecture notes to show that $\mathcal{U}(H)$ is pathwise connected by proving that any $U \in \mathcal{U}(H)$ is the endpoint of a continuouns path $\gamma\colon [0,1] \to \mathcal{U}(H)$ with $\gamma(0) = I$.

24 (Impossibility of implementing *Heisenberg's uncertainty relation* with bounded operators) Show that there cannot be a complex Hilbert space H and two bounded operators $P, Q \in L(H)$ such that

$$[Q, P] := QP - PQ = iI.$$

(Hint: Derive an expression for $Q^{n+1}P - PQ^{n+1}$ and use it to produce an estimate showing that $\|P\|\|Q\|$ cannot be finite.)

25 Let H be a complex Hilbert space and T be a linear operator on H with domain $\mathrm{dom}(T)$. Show the equivalence of the following statements:

(i) T is closed, i.e., $\mathrm{gr}(T)$, the graph of T, is closed in $H \times H$,

(ii) $\mathrm{dom}(T)$ is complete with respect to the *graph norm* $\|x\|_T := \sqrt{\|x\|^2 + \|Tx\|^2}$,

(iii) if (x_n) is a sequence in $\mathrm{dom}(T)$ such that $x_n \to x$ and $Tx_n \to y$ hold in H as $n \to \infty$, then $x \in \mathrm{dom}(T)$ and $y = Tx$.

26 Consider the operator T on $L^2([-1,1])$ with dense domain $\mathrm{dom}(T) := C([-1,1])$ and given by $T\varphi := \varphi(0) \cdot 1$ (constant function on $[-1,1]$).

(a) Show that T is unbounded and not closed.

(b) Show that $\mathrm{dom}(T^*) = \{1\}^\perp$. Hence T^* is not densely defined.

27 Let H be a complex Hilbert space and T, S be a densely defined linear operators on H. Show that the following hold:

(a) $T \subseteq S \;\Rightarrow\; S^* \subseteq T^*$.

(b) If T is essentially self-adjoint, then T possesses a unique self-adjoint extension.

(c) If T is self-adjoint, then there is no symmetric extension of T distinct from T.

28 Let H be a complex Hilbert space and T and S be densely defined linear operators on H. Suppose that $\mathrm{dom}(ST) := \{x \in \mathrm{dom}(T) \mid Tx \in \mathrm{dom}(S)\}$ is dense and define $ST\colon \mathrm{dom}(ST) \to H$ by $x \mapsto S(Tx)$. Analogously, define $\mathrm{dom}(T^*S^*) := \{x \in \mathrm{dom}(S^*) \mid S^*x \in \mathrm{dom}(T^*)\}$ (no density assumed) with $T^*S^*x := T^*(S^*x)$. Show that

(a) $T^*S^* \subseteq (ST)^*$, (b) $T^*S^* = (ST)^*$, if $S \in L(H)$.

In the following two problems, let H be a complex Hilbert space and T be a densely defined *closed* operator on H. Consider $V\colon H \times H \to H \times H$, given by $V(x,y) := (-y,x)$. Prove:

29 (a) V is unitary with respect to the inner product $\langle (x,y),(u,v)\rangle_2 := \langle x,u\rangle + \langle y,v\rangle$,

(b) $\mathrm{gr}(T) = (V(\mathrm{gr}(T^*)))^\perp$. (Hint: Showing '$\supseteq$' can be achieved via $\mathrm{gr}(T)^\perp \subseteq V(\mathrm{gr}(T^*))$.)

30 (a) $z \in \mathrm{dom}(T^*)^\perp \;\Rightarrow\; (0,z) \in \mathrm{gr}(T)$,

(b) T^* is densely defined and $T = T^{**}$.

31 Let H be a complex Hilbert space and T be an operator on H with domain $\mathrm{dom}(T)$. The operator T is called *closable*, if it possesses a closed extension.

(a) Suppose T is closable. Show that the following holds:

($*$) for every sequence (x_n) in $\mathrm{dom}(T)$ with $x_n \to 0$ and such that (Tx_n) is a Cauchy sequence, it follows that $Tx_n \to 0$.

Remark: One can show that ($*$) implies the property

($**$) the closure of $\mathrm{gr}(T)$ in $H \times H$ is the graph of a closed operator $\overline{T}$ extending T.

Obviously, $\overline{T}$ is the smallest closed extension of T. It is therefore called its *closure*. Moreover, the properties of being closable, ($*$), ($**$), and that T^* is densely defined are all equivalent.

(b) Show that the following operator on l^2 is not closable: With $a := (\frac{1}{l})_{l\in\mathbb{N}} = \sum_{l=1}^{\infty} \frac{1}{l} e_l \in l^2$ let $\mathrm{dom}(T) := \mathrm{span}(\{a\} \cup \{e_l \mid l \in \mathbb{N}\})$ and define $T(\lambda a + \mu_1 e_{l_1} + \cdots + \mu_m e_{l_m}) := \lambda a$.

In the following two problems, we consider the operator T with $\mathrm{dom}(T) := \mathscr{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ and given by $T\varphi := i\varphi'$. (Your reasoning here should be independent of the reasoning in Example 3.2 of the lecture notes.)

32 By techniques similar to those in Example 2.7 of the lecure notes, show that $\mathrm{dom}(T^*)$ consists of all $\psi \in L^2(\mathbb{R})$ such that the restriction of ψ to every bounded interval is absolutely continuous and $\psi' \in L^2(\mathbb{R})$. In your argument you may use (without proof) the (true) fact that $\mathrm{ran}(T)$ is dense in $L^2(\mathbb{R})$.

33 Show that T is essentially self-adjoint and its closure is given by $\psi \mapsto i\psi'$ on the domain described in 32.

34 Let H be a complex *infinite dimensional Hilbert space* and T be a self-adjoint operator on H. Suppose there exists $\lambda_0 \in \rho(T)$ such that $(\lambda_0 - T)^{-1}$ is a compact operator. T is said to possess a *compact resolvent* in this case.

(a) Show that $(\lambda - T)^{-1}$ is compact for every $\lambda \in \rho(T)$.

(b) Let $\lambda \in \rho(T)$. The operator $R_\lambda = (\lambda - T)^{-1}$ is compact, normal, and injective. We know therefore (from the basic course on functional analysis and noting the injectivity of R_λ on an infinite dimensional space) that $0 \in \sigma(R_\lambda) \setminus \sigma_p(R_\lambda)$ and that $\sigma(R_\lambda) \setminus \{0\} = \sigma_p(R_\lambda) = \{\mu_k \in \mathbb{C} \mid k \in \mathbb{N}\}$ with pairwise distinct $\mu_k \to 0$ $(k \to \infty)$ and mutually orthogonal finite dimensional eigenspaces V_k for each μ_k. A spectral representation for R_λ is obtained as follows (see, e.g., Chapter II, §7 in Conway's book from the bibliography in the lecture notes): Let E_k denote the orthogonal projection onto V_k; note that $E_l E_k = \delta_{lk} E_l$ and $x = \sum_{l=1}^{\infty} E_l x$ holds for every $x \in H$; we then have the operator norm convergent series representation

$$(*) \qquad (\lambda - T)^{-1} = R_\lambda = \sum_{k=1}^{\infty} \mu_k E_k.$$

Now finally the formulation of the problem: Deduce from $(*)$ a spectral respresentation of T and conclude that $\sigma(T)$ is countable and has no accumulation point.

In the following two problems let H be a complex Hilbert space and A be a self-adjoint operator on H. Define $U(t) := \exp(itA)$ via functional calculus for every $t \in \mathbb{R}$.

35 Show the following: (Hint: Multiplication operator variant of the spectral theorem and dominated convergence.)

(a) The family $(U(t))_{t\in\mathbb{R}}$ is a *strongly continuous unitary group of operators* on H, i.e.,

$$\forall s, t \in \mathbb{R}\colon\ U(s+t) = U(s)\,U(t)$$

and $\forall x \in H\colon\ \lim_{t\to 0} U(t)\,x = x$.

(b) $\forall x \in \operatorname{dom}(A)\colon\ \lim\limits_{t\to 0} \dfrac{U(t)\,x - x}{t} = iAx$.

36 (a) Show that, if in addition $A \in L(H)$, then $\lim\limits_{t\to 0} \|U(t) - I\| = 0$.

(b) Let A be the self-adjoint extension of the operator discussed in problems 32 and 33. Determine $U(t)$ explicitly in this example.

37 Recall that the right-shift R on l^2 is given by $R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$ for every $x = (x_k) \in l^2$. Check whether $\lim_{n\to\infty} R^n$ exists

(a) in the operator norm topology on $L(l^2)$,

(b) in the strong operator topology on $L(l^2)$,

(c) in the weak operator topology on $L(l^2)$.

38 Consider the sequence $(\sqrt{n} \cdot e_n)_{n\in\mathbb{N}}$ in l^2. Show that 0 is in the weak closure of $A := \{\sqrt{n} \cdot e_n \mid n \in \mathbb{N}\}$, but there is no subsequence of $(\sqrt{n_m} \cdot e_{n_m})_{m\in\mathbb{N}}$ converging weakly to 0.

39 Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$, P be the set of *all seminorms* on X, and τ_P denote the locally convex topology generated by P. Show the following:

(a) (X, τ_P) is a Hausdorff space,

(b) every linear map T from X to another locally convex vector space Y is τ_P-continuous,

(c) every subspace E of X is τ_P-closed.

(It is easy to see that τ_P is the finest locally convex vector space topology that exists on X. As we know from Remark 4.8(i) in the lecture notes, τ_P cannot be the discrete topology.)

40 Let X be a locally convex vector space. A subset $B \subseteq X$ is said to be *bounded*, if it is absorbed by every neighborhood U of 0, i.e., there exists some $\alpha > 0$ such that $B \subseteq \alpha U$.
(In fact, this notion is used in the above sense also in general topological vector spaces.)
Show that the following properties of a subset $B \subseteq X$ are equivalent:

(i) B is bounded,

(ii) every continuous seminorm is bounded on B,

(iii) for every sequence (x_n) in B and for every null sequence (α_n) in $\mathbb{K}$, the sequence $(\alpha_n x_n)$ converges to 0 in X.

Remark: A locally convex Hausdorff space can be shown to be *normable*, if and only if it possesses a bounded neighborhood of 0; see, e.g., Proposition 14.4 in F. Trèves, *Topological vector spaces, distributions and kernels*, Academic Press 1967.

41 Let X be a Banach space, Y be a normed space, and $T\colon X \to Y$ be linear. Show that the following properties are equivalent: (Hint: 5.9., Example 2) in the lecture notes; uniform boundedness principle.)

(i) T is norm continuous,

(ii) T is $\sigma(X, X')$-$\sigma(Y, Y')$-continuous.

42 Let X be an *infinite dimensional* normed vector space and $\sigma := \sigma(X, X')$. Show that the unit sphere $S := \{x \in X \mid \|x\| = 1\}$ is σ-dense in the σ-closed unit ball $B := \{x \in X \mid \|x\| \leq 1\}$, more precisely, that the σ-closure of S equals B.

(Hints: 1. Show that $X \setminus B$ is σ-open by applying the Hahn-Banach theorem for normed spaces and the fact that $(X_\sigma)' = X'$ [see lecture notes, 5.9., Example 2)]. 2. For any $x_0 \in B$, consider a typical σ-neighborhood of the form $\{x_0\} + U_{\{f_1,\ldots,f_m\},\varepsilon}$ and prove that $\{x_0\} + L_0$ does meet S, where $L_0 := \bigcap_{j=1}^m \ker(f_j)$.)

Remark: On $\mathbb{R}^n$ or $\mathbb{C}^n$ every locally convex Hausdorff topology is equivalent to the Euclidean norm topology. Therefore, S is always $\sigma(X, X')$-closed in a finite dimensional normed space X.

43 (a) Let (X, Y) be a dual pair and $A \subseteq X$. Show that $A^{\circ\circ\circ} = A^{\circ}$.

(b) Let X be a normed vector space and B_X, $B_{X'}$ denote the closed unit ball in X, X', respectively. Show that in the dual pair (X, X') we have $(B_X)^{\circ} = B_{X'}$ and $(B_{X'})^{\circ} = B_X$. Furthermore, if W is a subspace of X, then the polar W° coincides with the annihilator $W^{\perp} := \{x' \in X' \mid \forall w \in W \colon x'(w) = 0\}$.

44 (a) Let X be a normed vector space and W be a dense subspace of X. Show that on bounded subsets of X', the topologies $\sigma(X', X)$ and $\sigma(X', W)$ coincide.

(b) Apply (a) to show that a net converges in the closed unit ball B of l^{∞} with respect to the $\sigma(l^{\infty}, l^1)$-topology if and only if it convergences in every coordinate. (Therefore, on B the topology $\sigma(l^{\infty}, l^1)$ coincides with the topology of pointwise convergence inherited from $\mathbb{C}^{\mathbb{N}}$.)

In course of the following sequence of four problems we will prove the theorem of Mackey-Arens, and in the fifth problem we will discuss the special case of normed spaces. Let (E, F) be a dual pair. A locally convex topology τ on E is said to be *compatible* with the duality, if $(E_{\tau})' = F$ (upon the canonical identification $F \hookrightarrow E^*$). The *Mackey topology* $\mu(E, F)$ on E is generated by the family of seminorms

$$p_K(x) := \sup_{y \in K} |\langle x, y \rangle|,$$

where $K \subseteq F$ is a $\sigma(F, E)$-compact absolutely convex subset.

The **theorem of Mackey-Arens** states: A locally convex topology τ on E is compatible with the duality (E, F), if and only if $\sigma(E, F) \subseteq \tau \subseteq \mu(E, F)$.

45 (a) Let X be a topological vector space and $K_1, \ldots, K_m$ be compact absolutely convex subsets of X. Show that the *absolutely convex hull* $\operatorname{aco}(K_1 \cup \cdots \cup K_m) := \{\lambda_1 x_1 + \ldots + \lambda_m x_m \mid \lambda_j \in \mathbb{K}, x_j \in K_j \ (j = 1, \ldots, m) \text{ and } \sum_{j=1}^m |\lambda_j| \leq 1\}$ is compact.

(b) Show that $\mathcal{U} := \{B^{\circ} \mid B \subseteq F \text{ is absolutely convex and } \sigma(F, E)\text{-compact}\}$ is a basis of $\mu(E, F)$-neighborhoods of 0 in E.

46 (a) Let $l \colon E \to \mathbb{K}$ be linear and $\mu(E, F)$-continuous. Show that there exists an absolutely convex $\sigma(F, E)$-compact subset $K \subseteq F$ such that

$$\forall x \in E: \quad |l(x)| \leq p_K(x) = \sup_{y \in K} |\langle x, y \rangle|.$$

(The point is here to obtain a continuity estimate with a single seminorm.)

Show that $l \in K$ in the sense that $F \hookrightarrow E^*$. (Hint: Apply a Hahn-Banach separation theorem.)

(b) Show that $(E_{\mu(E,F)})' = F$.

47 Let τ be a locally convex topology on E such that $(E_{\tau})' = F$. Show that τ is finer than $\sigma(E, F)$. Let U be a closed absolutely convex τ-neighborhood of 0 in E. Show that U equals its $\sigma(E, F)$-closure and belongs to the neighborhood basis $\mathcal{U}$ given in 45(b).

48 Prove the theorem of Mackey-Arens.

49 Let E be a Banach space. What is the Mackey topology $\mu(E, E')$ on E?
(It will be useful to observe that $\sigma(E', E)$-compact subsets are bounded in norm, as can be seen from the uniform boundedness principle.)

50 Do the following assignments (with $\varphi \in \mathscr{D}(\Omega)$) define distributions $T \in \mathscr{D}'(\Omega)$?

(a) $\Omega = \mathbb{R}^2, \quad T(\varphi) = \int\limits_0^{2\pi} \varphi(\cos(s), \sin(s))\, ds,$

(b) $\Omega =]0,1[, \quad T(\varphi) = \sum\limits_{n=2}^{\infty} \varphi^{(n)}(\frac{1}{n}),$

(c) $\Omega = \mathbb{R}, \quad T(\varphi) = \sum\limits_{n=1}^{\infty} \varphi^{(n)}(\frac{1}{n}).$