Fact A: Let $s_1, s_2 \in \mathbb{R}$ such that $\boxed{s_1 + s_2 \geq 0}$. Then $\boxed{H^{s_1}(\mathbb{R}^2) \cdot H^{s_2}(\mathbb{R}^2) \subseteq H^{s_0}(\mathbb{R}^2)}$, provided

$$\boxed{s_0 \leq \begin{cases} \min(s_1, s_2, s_1 + s_2 - 1), & \text{if } s_1 \neq 1, s_2 \neq 1, \text{ and } s_1 + s_2 > 0, \\ \min(s_1, s_2, s_1 + s_2 - 1 - \varepsilon), & \text{with } \varepsilon > 0 \text{ arbitrary, otherwise.} \end{cases}}$$

(Special case of [1, Theorem 8.3.1]: Let $s_1, s_2 \in \mathbb{R}$, $s_1 + s_2 \geq 0$, then $H^{s_1}(\mathbb{R}^n) \cdot H^{s_2}(\mathbb{R}^n) \subseteq H^{s_0}(\mathbb{R}^n)$, provided

$$s_0 \leq \begin{cases} \min(s_1, s_2, s_1 + s_2 - n/2), & \text{if } s_1 \neq n/2, s_2 \neq n/2, \text{ and } s_1 + s_2 > 0, \\ \min(s_1, s_2, s_1 + s_2 - n/2 - \varepsilon), & \text{with } \varepsilon > 0 \text{ arbitrary, otherwise.} \end{cases}$$

The proof also shows that the following estimate holds $\forall w_j \in H^{s_j}(\mathbb{R}^n)$:

$$\|w_1 \cdot w_2\|_{H^{s_0}} \leq C(s_1, s_2, s_0, n) \|w_1\|_{H^{s_1}} \|w_2\|_{H^{s_2}}.$$

Therefore we obtain continuity of the multiplication $H^{s_1} \times H^{s_2} \to H^{s_0}$.¶

Corollary: (i) $s_1 > s_2 > n/2 \implies H^{s_1} \cdot H^{s_2} \subseteq H^{s_2}$. (ii) $\sigma > n/2 \implies H^{\sigma}(\mathbb{R}^n)$ is an algebra.)

Fact B: $\exists F \in \mathcal{C}^\infty(\mathbb{R})$ with $F(0) = 0$, such that

$$\boxed{\frac{1}{c(x)} = \frac{1}{c_0} + F(c_1(x)), \quad \text{where} \quad F \circ c_1 \in H^{r+1}(\mathbb{R}^2)}.$$

(To find F, simply set $F(y) := -y/(c_0(c_0 + y))$ when $y \geq -c_0/2$ and extend it to a smooth function on $\mathbb{R}$. Since $r > 0$ and $H^{1+r}(\mathbb{R}^2) \subseteq L^\infty(\mathbb{R}^2)$, we obtain $F \circ c_1 \in H^{r+1}(\mathbb{R}^2)$ from [1, Theorem 8.5.1].)

Fact C: For any $v \in L^2(\mathbb{R}^2)$ the following equation holds in $\mathscr{S}'(\mathbb{R}^2)$:

(C1) $$\boxed{Av = \operatorname{div}(c \operatorname{grad} v) = \Delta(c\,v) - \operatorname{div}(v \operatorname{grad} c)}.$$

Proof is part of ♣ `Blackboard discussion 1` ♣.

(Under the stronger assumption $v \in H^{r+1}(\mathbb{R}^2)$ we have in addition

(C2) $$Av = \operatorname{grad} c \cdot \operatorname{grad} v + c\,\Delta v$$

in $\mathscr{S}'(\mathbb{R}^2)$. (The meaning of the products on the right-hand side is as follows: since $\Delta v \in H^{r-1}$, Fact A applies and yields $c\Delta v \in H^{r-1}$; furthermore, $\operatorname{grad} c$ and $\operatorname{grad} v$ both lie in H^r, so that a repeated application of Fact A shows that their euclidean inner product belongs to $H^{\min(r, 2r-1)}$.))

References

[1] L. Hörmander. *Lectures on Nonlinear Hyperbolic Differential Equations.* Springer-Verlag, Berlin Heidelberg, 1997.

[2] M. Oberguggenberger. *Multiplication of distributions and applications to partial differential equations.* Longman Scientific & Technical, 1992.

¶Compare this with the *dimension-independent* statement [2, Proposition 5.2], which describes a duality-type continuous multiplication map $W^{m,q}_{\text{loc}} \times W^{l,p}_{\text{loc}} \to W^{k,r}_{\text{loc}}$, where $k = \min(l, m)$ and $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$ under the conditions $l, m \in \mathbb{Z}$, $l + m \geq 0$, $1 \leq p, q \leq \infty$, $\frac{1}{q} + \frac{1}{p} \leq 1$.