The minimization-of-frustration policy in the *Palio di Siena*

Or: What Michael Grosser discovered in my book gift?

Günther Hörmann

DIANA seminar

November 2010

A book and a paper

AMERICAN MATHEMATICAL SOCIETY
MathSciNet® Mathematical Reviews on the Web

Previous | Up | Next Article

MR1681302 (2000c:91023) 91B12
Götz, Stefan (A-WIEN); **Grosser, Michael** (A-WIEN)
Über das Pferderennen in Siena. (German. German summary) [On the horse race in Siena]
Math. Semesterber. **46** (1999), *no. 1,* 77–92.

Summary (translated from the German): "Twice a year the city of Siena has a horse race, the so-called Palio, at which time ten contradas (city districts) compete against each other. However, Siena has 17 contradas, a situation that requires a just selection procedure. Traditionally, the seven nonparticipating contradas are entitled to participation in the following year, with the remaining three districts needed for the completion of the starter field being determined by a lottery.

"In the following we develop a mathematical model for this problem (i.e., a stochastic process), which answers in particular questions on the distribution of the number of participants, on the average participation rate and on stationary distribution. In these investigations we employ linear difference equations. In the last section we discuss fundamental aspects of the theory of Markov chains, using the example of selection procedures for the Palio."

The rules of participation for Siena's 17 *contrade* (districts)

only 10 out of 17 can participate per race

(races on July 2 and August 16 are independent runs)

mode of selection for next year's race:

- the 7 non-participants plus
- 3 at random from the remaining 10

---

viewpoint of a specific contrada: $X_n := \begin{cases} 1 & \text{contrada is in } n^{\text{th}} \text{ race,} \\ 0 & \text{contrada } \textit{not} \text{ in } n^{\text{th}} \text{ race,} \end{cases}$

$$P(X_{n+1} = 0 \mid X_n = 0) = 0, \qquad P(X_{n+1} = 1 \mid X_n = 0) = 1$$

$$P(X_{n+1} = 0 \mid X_n = 1) = \tfrac{7}{10}, \quad P(X_{n+1} = 1 \mid X_n = 1) = \tfrac{3}{10},$$

$$P(X_{n+1} = \alpha_{n+1} \mid X_n = \alpha_n, \ldots, X_1 = \alpha_1) = P(X_{n+1} = \alpha_{n+1} \mid X_n = \alpha_n)$$

(participation in $(n+1)^{\text{th}}$ race depends only on n^{th} race)

Hitting the road with $P(X_1 = 1) = p_1$, $0 \le p_1 \le 1$ arbitrary

the rate of participation in n races is

$$Z_n := \frac{1}{n} \cdot \sum_{j=1}^{n} X_j$$

? expectations $E(X_n)$ and $E(Z_n)$

? fairness: $E(Z_n) \to \frac{10}{17}$ as $n \to \infty$

$$p_n := P(X_n = 1) \implies E(X_n) = 1 \cdot P(X_n = 1) + 0 \cdot P(X_n = 0) = p_n, \text{ and}$$

$$p_{n+1} = P(X_{n+1} = 1 \mid X_n = 1) \cdot p_n + P(X_{n+1} = 1 \mid X_n = 0) \cdot (1 - p_n)$$

$$= \frac{3}{10} \cdot p_n + 1 \cdot (1 - p_n) = 1 - \frac{7}{10} \cdot p_n$$

$x \mapsto 1 - 7x/10$ is a contraction of $[0, 1]$, fixed point $\rightsquigarrow$ $\exists\, p_\infty := \lim p_n$

hence also $E(Z_n) \to p_\infty$ (sequence of arithmetic means)

$n \to \infty$ in recursion: $p_\infty = 1 - 7p_\infty/10 \implies p_\infty = 10/17$

A variety of solution methods and some consolation

- two further solution methods via difference equations of 1st and 2nd order
- Markoff chain interpretation and yet another solution method
- explicit formulae for expectations in step n as well as for the distribution of $Y_n := X_1 + \ldots + X_n$.

Fairness in the long run is also true for a yearly random selection '10 out of all 17'.

BUT the "Siena method" in addition minimizes the possible frustration for a contrada due to the likelihood of long sequences of years without participation (i.e., 0-runs of X_n).

Victories per Contrada

Contrada	Total Victories	17th century	18th century	19th century	20th century	21st century	Date of last victory
Aquila	24	0	7	6	11	0	03 Jul 1992
Bruco	37	6	7	16	5	3	16 Aug 2008
Chiocciola	51	4	19	14	14	0	16 Aug 1999
Civetta	33	2	7	15	8	1	16 Aug 2009
Drago	36	2	8	10	15	1	16 Aug 2001
Giraffa	33	3	5	9	15	1	02 Jul 2004
Istrice	41	4	11	14	10	2	02 Jul 2008
Leocorno	30	3	3	11	11	2	16 Aug 2007
Lupa	34	1	11	11	11	0	02 Jul 1989
Nicchio	42	5	10	12	15	0	16 Aug 1998
Oca	63	8	14	20	20	1	02 Jul 2007
Onda	38½	4	9½	14	11	0	02 Jul 1995
Pantera	26	3	6	7	9	1	02 Jul 2006
Selva	37	2	11	6	15	3	02 Jul 2010
Tartuca	47½	4	11½	16	12	4	16 Aug 2010
Torre	44	6	12	20	5	1	16 Aug 2005
Valdimontone	43	2	16	10	15	0	16 Aug 1990