One or two exotic aspects from our recent work on Schrödinger-type equations

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A) de Hoop-Hörmann-Oberguggenberger: *Evolution systems for paraxial wave equations of Schrödinger-type with non-smooth coefficients*, Jour.Diff.Equs. 2008

B) Hörmann: *The Cauchy problem for Schrödinger-type partial differential operators with generalized functions in the principal part and as data*, arXiv:0909.5672v1 [math.FA]

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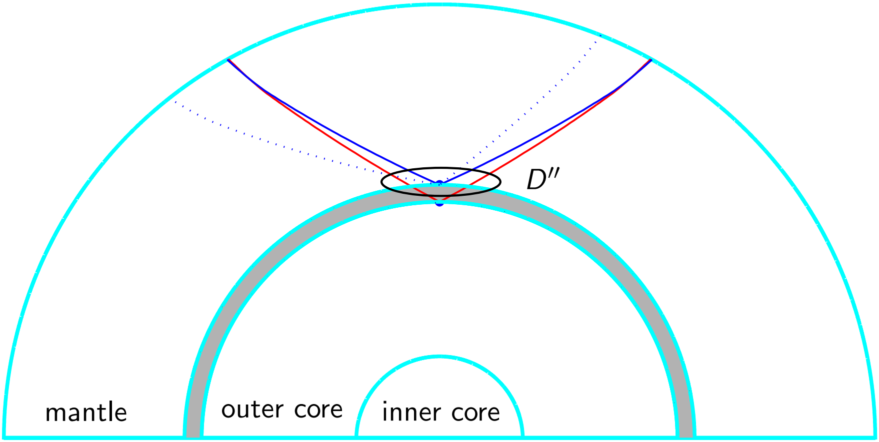
Core reflected seismic waves in the earth



D'' layer above the core-mantle boundary (at approx. 2800km depth)

core-reflected wave and two precursors

locally near D''-layer: tangential direction of propagation z, radial-lateral coordinates $x = (x_1, x_2)$ perpendicular to propagation.

Model equations

- basic seismic scattering: 2nd-order system of elastodymamics

  material properties $\leftrightsquigarrow$ regularity structure of the coefficients

  Remark: (wavelet analysis of) measurements of sound speed [1998]: regularity in volcanic rock $C_*^{r_1}$ and in sandstone $C_*^{r_2}$, where

  $$1 > r_1 > r_2 > 0.$$

- **paraxial approximation** $\rightsquigarrow$ Schrödinger-type equation (depth z as evolution variable and "ΨDO-parameter" time t)

$$\partial_z u - \mathrm{i} \underbrace{\Big(\partial_{x_1}\big(c_1(z,x,D_t)\partial_{x_1} u\big) + \partial_{x_2}\big(c_2(z,x,D_t)\partial_{x_2} u\big)\Big)}_{A(z;x,D_x,D_t)u} = 0,$$

with $c_1(z,x,\tau), c_2(z,x,\tau)$ positive symbols of order -1 in τ, $\mathcal{C}^1$ with respect to z, but
x-regularity non-Lipschitz, typically H^{r+1} or C_*^r with $r < 1$.

Wishes, questions, and aims

? unique solvability of corresponding Cauchy problem for wave component

$$u \in \mathcal{C}([0, \overset{z}{\overset{\downarrow}{\infty}}[, \mathscr{S}'(\overset{t}{\overset{\downarrow}{\mathbb{R}}}; H^s(\overset{x}{\overset{\downarrow}{\mathbb{R}^2}}))$$

with high(est possible) $s \in \mathbb{R}$. (assuming data are suff. reg.)

? estimate bounds for (Hölder- or) Sobolev-regularity s in terms of the coefficient regularity r; e.g., in the form $s \geq f(r)$

Why? **intended Application:** decision support

1. measure u [seismograms]
2. estimate x-regularity $\overline{s}$ [numerical wavelet analysis]
3. materials with C^r_* with $\overline{s} < f(r)$ are ruled out

? wider range of applications: discontinuous or distributional coefficients, initial data, and right-hand side

Solution with spatial H^2-regularity

- elliptic x-regularity for $A(z; x, D_x, \tau)$ via "bootstrap" techniques based on duality products in Sobolev spaces

♣ Blackboard discussion 1 ♣

- establish $(A(z; x, D_x, \tau))_{z \geq 0}$ as a (τ-parametrized) generating family of a strongly-continuous evolution system $(U(\tau; z_1, z_2))_{z_1 \geq z_2 \geq 0}$ on $L^2(\mathbb{R}^2)$ (with additional s-cont. dependence on frequency τ)

Thm. A: $J \subseteq [0, \infty[$ compact interval. The Cauchy problem

$$\partial_z u - \mathrm{i}\, A(z; x, D_x, D_t) u = f \in \mathcal{C}^1(J, L^1(\mathbb{R}, L^2(\mathbb{R}^2)))$$
$$u \,|_{z=0} = u_0 \in L^1(\mathbb{R}, H^2(\mathbb{R}^2)).$$

has a unique solution $u \in \mathcal{C}^1_w(J, H^{-1/2}(\mathbb{R}; H^2(\mathbb{R}^2))$.

More general coefficients and data, more general form of the operator, and in arbitrary space dimension n

$$\partial_t u - \mathrm{i} \sum_{k=1}^{n} \partial_{x_k}(c_k \partial_{x_k} u) - \sum_{k=1}^{n} a_k \partial_{x_k} u - Vu = f, \quad u\,|_{t=0} = u_0.$$

Regularization idea: e.g. $\mathcal{C}^\infty \ni u_{0\varepsilon} \to u_0$ $(\varepsilon \to 0)$ etc ...
or, more generally, replace u_0 by a net $(u_{0\varepsilon})_{0<\varepsilon\leq 1}$ of $\mathcal{C}^\infty$ functions, convergent or not, but with moderate asymptotics:

$$\forall k\, \exists m: \quad \|u_{0\varepsilon}\|_{H^k} = O(\varepsilon^{-m}) \quad (\varepsilon \to 0).$$

— similarly for right-hand side and coefficients.
Example: $u_0 = \mu$ probability measure on $\mathbb{R}^n$ (quantum mechanics)
? construct $u_{0\varepsilon}$ moderate $\mathcal{C}^\infty$-net with $u_{0\varepsilon}^2 \to \mu$ $(\varepsilon \to 0)$
— in this sense $(u_{0\varepsilon})_{0<\varepsilon\leq 1}$ represents a *square root of* μ.
♣ Blackboard discussion 2 ♣

Generalized function interpretation

Moderate net of solutions u_ε $(0 < \varepsilon \leq 1)$: constructed from variational methods (energy estimates, Galerkin approximation); smoothness shown step by step and using mild solution concept; asymptotic estimates of higher derivatives from differentiated PDE and corresponding energy inequalities imply moderateness,

provided $\|\partial_t c_{j\varepsilon}\|_{L^\infty} = O(\log(\frac{1}{\varepsilon})) \quad (\varepsilon \to 0)$. [log-type condition]

Asymptotic uniqueness: negligible errors in data, i.e.,

$$\forall k \, \forall p : \quad \|u_{0\varepsilon} - \widetilde{u_{0\varepsilon}}\|_{H^k} = O(\varepsilon^p) \quad (\varepsilon \to 0)$$

(and similarly for right-hand side and coefficients) $\rightsquigarrow$ **negligible errors in solutions.**

Interpretation alias **Thm. B:** There exists a unique generalized function solution, namely

$$u = \text{class of } (u_\varepsilon)_{0<\varepsilon\leq 1} \text{ modulo negligible nets}$$

in the Colombeau algebra $\mathcal{G}_{H^\infty}$.

Coherence, convergence, and regularity

Prop: $\mathcal{C}^\infty$-coefficients , $u_0 \in H^1(\mathbb{R}^n)$, $f \in \mathcal{C}^1([0,T], L^2(\mathbb{R}^n))$,
unique "classical" solution $w \in \mathcal{C}([0,T], H^1(\mathbb{R}^n))$ [Lions-Magenes]

$\Longrightarrow$ (solution Thm. B) $u \approx w$, i.e., $u_\varepsilon \to w$ in $\mathcal{D}'$ $(\varepsilon \to 0)$

A $\leftrightarrow$ B $a_k = V = 0$ and $n = 2$; c_j, u_0, f as in Thm. A and
$u_{0\varepsilon} \to u_0$ in $H^1(\mathbb{R}^2)$, $f_\varepsilon \to f$ in $H^1(J \times \mathbb{R}^2)$

$\Longrightarrow$ (repr. of solution Thm. B) $u_\varepsilon \longrightarrow u$ (solution Thm. A) in $\mathcal{C}(J, H^1(\mathbb{R}^2))$